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SUMMARY

An application of Richardson's extrapolation to the numerical evaluation of the sonic-boom integrals occurring in the theory of Whitham (Communications on Pure and Applied Mathematics, August 1952) has been considered. Equations have been developed for evaluating the sonic-boom integrals, by using second derivatives which are obtained from the central difference formulas of finite difference techniques. It has been shown by error analysis that these equations have an order of error proportional to the square of the solution point interval spacing for the far-field sonic-boom integral, and to the three-halves power of the interval spacing for the near-field sonic-boom integral. Extrapolation of the numerical results to zero interval spacing on the basis of the predicted order of error is demonstrated, and some numerical examples confirm the essential validity of this procedure. Some of the numerical results, however, show that the extrapolation technique must be used with caution.

INTRODUCTION

The purpose of this report is to present an application of Richardson's extrapolation as a method of improving the accuracy of a numerical evaluation of the sonic-boom integrals. These integrals are encountered in using the theory of reference 1 for the calculation of the sonic-boom overpressure fields due to the flow pattern of a supersonic aircraft or projectile. In general, numerical methods must be used in the calculation since the integrals are a function of the effective area distribution which is usually known only numerically. The lift as well as the volume contribution to the overpressure field of an aircraft configuration may be included in the sonic-boom theory through the use of an equivalent body concept. Supersonic-area-rule cuts may be used to determine the contribution of the aircraft volume to the effective area distribution of the equivalent body; thus, the effect of Mach number on these contributions is introduced. These extensions of the basic theory of reference 1, as applied to the calculation of sonic-boom overpressure fields, have been summarized in reference 2.

The procedure referred to as Richardson's extrapolation was first presented in reference 3, and an expanded version was later presented in reference 4. As stated in reference 4, the philosophy of this process is to reverse the order of passing to the limit in the solution of a problem. In an analytical solution involving an application of the methods of the calculus, the passage to the limit is taken at the earliest possible stage in the formulation of the problem. In a numerical solution, however, this procedure is unworkable, and the solution must therefore be obtained in finite differences. In applying Richardson's extrapolation, the approach to the limit is deferred until after the problem has been solved for a finite number of discrete increments. If the variation of the solution error with increment size is known, the results may then be extrapolated to what they would be for vanishingly small increments. In references 3 and 4, the theory of the deferred approach to the limit is applied only to problems which are solved by central difference techniques, resulting in what is called "h^2 extrapolation." However, the philosophy itself is not necessarily restricted to problems of this class. It is the general theory of the deferred approach to the limit which is applied in this report, and the particular form of the extrapolation used is derived from an analysis of the discretization error.

A number of numerical methods of evaluating the sonic-boom integrals are available. (See, for example, refs. 2 and 5 to 7.) The method of reference 2 involves the use of the second derivatives of the effective area distribution of an equivalent body in a straightforward application of the theory of reference 1 for both the near- and far-field solutions. Reference 5 presents a method of evaluating the near-field overpressure integral involving only first derivatives. The far-field overpressure may be readily evaluated once the near-field solution is obtained. References 6 and 7 present methods of evaluating the far-field overpressure integrals where again only first derivatives are required. The numerical procedure for evaluation of the sonic-boom integrals which is utilized in this report is essentially the same as that of reference 2, the main difference being in the method by which the derivatives of the equivalent body effective area distribution are obtained. For this step, the standard central difference formulas are used. This method was chosen because it appeared most readily amenable to an error analysis and to the subsequent application of Richardson's extrapolation.

In the following sections of this report, after a description of the symbol notation which is used, the near- and far-field overpressure expressions are introduced from reference 1 to show the significance of the sonic-boom integrals. Next the three-point formulas of the standard central difference equations are used to obtain the required second derivatives of the equivalent body effective area distribution. The second derivatives are then integrated twice to obtain a check of the original area distribution. Following this the second derivatives are used to obtain expressions for the sonic-boom integrals in a form which is suitable for an error analysis. The presentation of the foregoing material is essentially a preface to the analysis of the discretization error of the given sonic-boom integral solution and the subsequent application of Richardson's extrapolation. The error is obtained as a function of the solution interval spacing and only the lowest order terms are retained. Finally some numerical examples are presented to show a possible application of the extrapolation technique.

SYMBOLS

All length dimensions are for a body of unit length.

$A(x)$ effective cross-sectional area of equivalent body which combines effects of volume and lift, nondimensionalized with respect to square of equivalent body length

a_j coefficient in jth order term of polynomial used to represent $A(x)$

$$F(x) = \frac{1}{2\pi} \int_0^x \frac{A''(\xi)}{\sqrt{x - \xi}} \, d\xi$$

h width of interval in x between regularly spaced body stations

$$I(x) = \int_0^x F(\xi) \, d\xi$$

i integer designating ith derivative

J integer representing maximum order of polynomial

j integer representing the order of polynomial terms

K reflection factor (equals unity if pressure wave does not contact reflective surface)

k arbitrary exponent

M Mach number

m,n integers designating body stations starting from zero at nose

N integer representing maximum station number at base or rear extremity of equivalent body

p reference pressure for uniform atmosphere

Q quantity representing either F or I

$R_1(x),R_2(x)$ radii, nondimensionalized with respect to equivalent body length

r perpendicular distance from flight path of equivalent body to field point at which the sonic-boom overpressure is evaluated, nondimensionalized with respect to equivalent body length

x, ξ longitudinal distance from nose along equivalent body axis, positive downstream, nondimensionalized with respect to equivalent body length

x_o value of x for which integral $I(x)$ is maximum

$\beta = \sqrt{M^2 - 1}$

γ ratio of specific heats, 1.4 for air

Δp incremental pressure (sonic-boom overpressure) due to flow field of equivalent body at supersonic speeds

$\delta = x - x_n$

Subscripts:

check first or second integral of numerical approximation of $A''(x)$

error difference between exact value and numerical approximation of a function

exact value of function evaluated analytically

extrap value of function obtained by Richardson's extrapolation

far field asymptotic value of Δp

m,n,N function evaluated at station m,n,N

max maximum positive value of a function

Primes, Roman numerals, and (i) are used as superscripts to indicate derivatives with respect to x.

SONIC-BOOM OVERPRESSURE EQUATIONS

At a sufficiently large distance from the body the incremental pressure at a specific point in the flow field of a projectile at supersonic speeds is given, to a first approximation, by the theory of reference 1 as

$$\frac{\Delta p}{p} = \frac{K\gamma M^2 F(x)}{\sqrt{2\beta r}} \tag{1}$$

where

$$F(x) = \frac{1}{2\pi} \int_0^x \frac{A''(\xi)}{\sqrt{x - \xi}}\, d\xi$$

The configuration of the shock waves in the supersonic flow field and the magnitude of the pressure increments across the shock waves are determined from area balancing requirements which must be applied to the $F(x)$ function. A complete description of the influence of the $F(x)$ function with regard to the strength and location of the shock waves is given in reference 1. A review and application of this theory is given in reference 8, and a numerical method of accomplishing the area balancing technique as applied to the $F(x)$ function to obtain pressure signatures is presented in reference 9. Under far-field conditions, reference 1 shows that the pressure increment due to the front shock asymptotically approaches the value

$$\left(\frac{\Delta p}{p}\right)_{\text{far field}} = \frac{\gamma(2\beta)^{1/4}K\left[I(x_o)\right]^{1/2}}{\sqrt{\gamma + 1}\, r^{3/4}} \tag{2}$$

where $I(x_o)$ is the maximum positive value of the integral

$$I(x) = \int_0^x F(\xi)\, d\xi$$

The functions $F(x)$ and $I(x)$ represent the sonic-boom integrals which are to be evaluated. It can be seen from equations (1) and (2) that when the values of the functions $F(x)$ and $I(x)$ are known, the incremental pressures in the flow field may then be determined, both under asymptotic and nonasymptotic conditions. As mentioned previously, reference 2 has presented a numerical method (involving second derivatives) for evaluating the functions $F(x)$ and $I(x)$. An alternative method for evaluating these functions is developed in a form considered suitable for the subsequent analysis of the discretization error.

NUMERICAL SECOND DERIVATIVE OF AREA DISTRIBUTION

The second derivatives of the effective area distribution can be approximated numerically by using the three-point formula of the standard central difference equations

$$A_n'' \approx \left.\frac{\Delta^2 A}{\Delta x^2}\right|_n = \frac{1}{h^2}\left(A_{n-1} - 2A_n + A_{n+1}\right) \tag{3}$$

which is equivalent to taking the second derivative of a parabolic arc which is passed through three adjacent points. This expression is used to represent the value of the second derivative over an interval h in width centered about the nth station or in the range of x

$$\left(n - \frac{1}{2}\right)h \leqq x \leqq \left(n + \frac{1}{2}\right)h$$

At the nose, the second derivative is obtained by

$$A_0'' \approx \left.\frac{\Delta^2 A}{\Delta x^2}\right|_0 = \frac{2}{h^2} A_1 \qquad \left(0 \leqq x \leqq \frac{h}{2}\right) \tag{4}$$

Equation (4) is strictly applicable only to configurations with pointed noses. However, this is not necessarily a restrictive condition to the numerical method developed herein since the theory of reference 1 to which it is applied is also similarly restricted. At the base of the body, the second derivative in the interval at the base is assumed equal to the second derivative in the interval immediately preceding it

$$A_N'' = A_{N-1}'' \qquad \left(\left(1.0 - \frac{h}{2}\right) \leqq x \leqq 1.0\right) \tag{5}$$

where the normalized configuration length is unity. This assumption at the base is equivalent to applying the forward difference formula of finite difference techniques at the base instead of the central difference formula which is applied elsewhere on the configuration. A check of the $A''(x)$ approximation is obtained by integrating twice to obtain $A(x)_{check}$. The first integral for the check function is

$$A'(x)_{check} = \int_0^x A''(\xi)\, d\xi \tag{6}$$

which may be integrated with the approximations of equations (3), (4), and (5) to yield

$$A'(x)_{check} \approx A_0'' x \qquad \left(0 \leqq x \leqq \frac{h}{2}\right) \tag{7}$$

The derivation of equation (8) is shown in detail since the same method is used in the summations for $A(x)_{check}$, $F(x)$, and $I(x)$. This method yields expressions in a form which are well suited to the analysis for the discretization error. Equation (6) may be rewritten

$$A'(x)_{check} = \int_0^{h/2} A''(\xi)\ d\xi + \sum_{n=1}^{m-1} \int_{\left(n-\frac{1}{2}\right)h}^{\left(n+\frac{1}{2}\right)h} A''(\xi)\ d\xi + \int_{\left(m-\frac{1}{2}\right)h}^{x} A''(\xi)\ d\xi$$

The second derivative approximations of equations (3), (4), and (5) are used to obtain

$$A'(x)_{check} \approx \int_0^{h/2} A_0''\ d\xi + \sum_{n=1}^{m-1} \int_{\left(n-\frac{1}{2}\right)h}^{\left(n+\frac{1}{2}\right)h} A_n''\ d\xi + \int_{\left(m-\frac{1}{2}\right)h}^{x} A_m''\ d\xi$$

The intervals over which the integrals are taken are changed as follows

$$A'(x)_{check} \approx \int_0^{x} A_0''\ d\xi - \int_{h/2}^{x} A_0''\ d\xi + \sum_{n=1}^{m-1} \left(\int_{\left(n-\frac{1}{2}\right)h}^{x} A_n''\ d\xi - \int_{\left(n+\frac{1}{2}\right)h}^{x} A_n''\ d\xi \right) + \int_{\left(m-\frac{1}{2}\right)h}^{x} A_m''\ d\xi$$

Integration and rearrangement of terms yields the following form:

$$A'(x)_{check} \approx A_0''x + \sum_{n=1}^{m} \left(A_n'' - A_{n-1}''\right)\left[x - \left(n - \frac{1}{2}\right)h\right] \qquad \left(\left(m - \frac{1}{2}\right)h \leqq x \leqq \left(m + \frac{1}{2}\right)h\right) \tag{8}$$

The second integral

$$A(x)_{check} = \int_0^{x} A'(\xi)_{check}\ d\xi \tag{9}$$

yields

$$A(x)_{check} \approx \frac{1}{2} A_0''x^2 \qquad \left(0 \leqq x \leqq \frac{h}{2}\right) \tag{10}$$

and

$$A(x)_{check} \approx \frac{1}{2}\left\{A_0''x^2 + \sum_{n=1}^{m}\left(A_n'' - A_{n-1}''\right)\left[x - \left(n - \frac{1}{2}\right)h\right]^2\right\}$$

$$\left(\left(m - \frac{1}{2}\right)h \leqq x \leqq \left(m + \frac{1}{2}\right)h\right) \qquad (11)$$

Values at station or half-station points are obtained by letting $x = mh$ or $x = \left(m + \frac{1}{2}\right)h$, respectively, in equation (11). The foregoing check is performed to determine how closely the area distribution for which the numerical solution is obtained matches the area distribution for which the solution is desired.

NUMERICAL INTEGRATION FOR $F(x)$ AND $I(x)$

The integral for the function $F(x)$ for area distributions possessing continuous first derivatives is

$$F(x) = \frac{1}{2\pi}\int_0^x \frac{A''(\xi)}{\sqrt{x - \xi}}\,d\xi \qquad (12)$$

Now using the same stepwise procedure in equation (12) as was used for the area check function

$$F(x) \approx \frac{1}{\pi} A_0'' \sqrt{x} \qquad \left(0 \leqq x \leqq \frac{h}{2}\right) \qquad (13)$$

and

$$F(x) \approx \frac{1}{\pi}\left[A_0'' \sqrt{x} + \sum_{n=1}^{m}\left(A_n'' - A_{n-1}''\right)\sqrt{x - \left(n - \frac{1}{2}\right)h}\right] \qquad \left(\left(m - \frac{1}{2}\right)h \leqq x \leqq \left(m + \frac{1}{2}\right)h\right) \qquad (14)$$

To evaluate $F(x)$ at a station point, let $x = mh$ in equation (14)

$$F_m \approx \frac{\sqrt{h}}{\pi}\left[A_0'' \sqrt{m} + \sum_{n=1}^{m}\left(A_n'' - A_{n-1}''\right)\sqrt{m - n + \frac{1}{2}}\right] \qquad (15)$$

and at a half-station point, let $x = \left(m + \frac{1}{2}\right)h$

$$F_{m+\frac{1}{2}} \approx \frac{\sqrt{h}}{\pi}\left[A_0''\sqrt{m + \frac{1}{2}} + \sum_{n=1}^{m} \left(A_n'' - A_{n-1}''\right)\sqrt{m - n + 1}\right] \tag{16}$$

The second integration for the function $I(x)$ is defined as follows

$$I(x) = \int_0^x F(\xi)\, d\xi \tag{17}$$

The integration for $I(x)$ is accomplished with the same stepwise procedure as before

$$I(x) \approx \frac{2}{3\pi} A_0'' x^{3/2} \qquad \left(0 \leqq x \leqq \frac{h}{2}\right) \tag{18}$$

and

$$I(x) \approx \frac{2}{3\pi}\left\{A_0'' x^{3/2} + \sum_{n=1}^{m} \left(A_n'' - A_{n-1}''\right)\left[x - \left(n - \frac{1}{2}\right)h\right]^{3/2}\right\}$$

$$\left(\left(m - \frac{1}{2}\right)h \leqq x \leqq \left(m + \frac{1}{2}\right)h\right) \tag{19}$$

Equation (19) for $I(x)$ may be evaluated at a station point by substituting $x = mh$

$$I_m \approx \frac{2}{3}\,\frac{h^{3/2}}{\pi}\left[A_0'' m^{3/2} + \sum_{n=1}^{m} \left(A_n'' - A_{n-1}''\right)\left(m - n + \frac{1}{2}\right)^{3/2}\right] \tag{20}$$

and at a half-station point by substituting $x = \left(m + \frac{1}{2}\right)h$

$$I_{m+\frac{1}{2}} \approx \frac{2}{3}\,\frac{h^{3/2}}{\pi}\left[A_0''\left(m + \frac{1}{2}\right)^{3/2} + \sum_{n=1}^{m} \left(A_n'' - A_{n-1}''\right)(m - n + 1)^{3/2}\right] \tag{21}$$

ERROR ANALYSIS

Among the important sources of error in the numerical solution is the discretization error due to the finite-size increments that are used to obtain solutions. This error occurs because the second derivative of the effective area distribution is approximated numerically and is assumed constant over a discrete station interval. At any given station point, the second derivative error is of order h^2 but in the $\pm h/2$ interval about the station point, for which the second derivative is considered constant, the order of error degrades from order h^2 of the central difference formulas to order h of the forward difference formulas. The object of the error analysis is to find the lowest order of h on which the discretization error of the F and I functions depends. All higher orders of h are considered negligible in Richardson's extrapolation as h is made vanishingly small.

The order of error of the second derivative may be shown as follows. The station areas A_{n-1}, A_n, and A_{n+1} are expanded in a Taylor's series centered at the arbitrary point x which lies within the interval $x_n - \frac{h}{2} \leqq x \leqq x_n + \frac{h}{2}$. Letting $\delta = x - x_n$, there is obtained

$$A_{n-1} = A(x) - (h + \delta)A'(x) + (h + \delta)^2 \frac{A''(x)}{2!} - (h + \delta)^3 \frac{A'''(x)}{3!}$$

$$+ (h + \delta)^4 \frac{A^{IV}(x)}{4!} - (h + \delta)^5 \frac{A^{V}(x)}{5!} + \dots$$

$$A_n = A(x) - \delta A'(x) + \delta^2 \frac{A''(x)}{2!} - \delta^3 \frac{A'''(x)}{3!} + \delta^4 \frac{A^{IV}(x)}{4!} - \delta^5 \frac{A^{V}(x)}{5!} + \dots$$

and

$$A_{n+1} = A(x) + (h - \delta)A'(x) + (h - \delta)^2 \frac{A''(x)}{2!} + (h - \delta)^3 \frac{A'''(x)}{3!}$$

$$+ (h - \delta)^4 \frac{A^{IV}(x)}{4!} + (h - \delta)^5 \frac{A^{V}(x)}{5!} + \dots$$

The above expressions are combined in the central difference formula (eq. (3)) to obtain

$$\left.\frac{\Delta^2 A}{\Delta x^2}\right|_n = \frac{1}{h^2}\left\{\frac{A''(x)}{2!}\left[(h+\delta)^2 - 2\delta^2 + (h-\delta)^2\right] + \frac{A'''(x)}{3!}\left[-(h+\delta)^3 + 2\delta^3\right.\right.$$

$$\left.\left. + (h-\delta)^3\right] + \frac{A^{IV}(x)}{4!}\left[(h+\delta)^4 - 2\delta^4 + (h-\delta)^4\right] + \ldots\; 0\left(h^5\right)\right\}$$

which may be simplified to

$$\left.\frac{\Delta^2 A}{\Delta x^2}\right|_n = A''(x) - \left(x - x_n\right)A'''(x) + \frac{h^2 + 6\left(x - x_n\right)^2}{12} A^{IV}(x) + \ldots\; 0\left(h^3\right) \tag{22}$$

In equation (22), the term $A''(x)$ is the true second derivative and the rest of the expression is the error. Equation (22) may be rewritten as

$$\left.\frac{\Delta^2 A}{\Delta x^2}\right|_n = A''(x) + A''(x)_{error}$$

where

$$A''(x)_{error} = -\left(x - x_n\right)A'''(x) + \frac{h^2 + 6\left(x - x_n\right)^2}{12} A^{IV}(x) + \ldots\; 0\left(h^3\right) \tag{23}$$

The discretization error in the F and I functions may then be written as

$$F(x)_{error} = \frac{1}{2\pi}\int_0^x \frac{A''(\xi)_{error}}{\sqrt{x - \xi}}\, d\xi \tag{24}$$

and

$$I(x)_{error} = \int_0^x F(\xi)_{error}\, d\xi \tag{25}$$

In order to proceed with the discretization error analysis it is necessary to make some assumption regarding the shape or form of the $A(x)$ curve. A polynomial representation of $A(x)$ is used to illustrate the order of error of the numerical solution. Use of a 4th degree polynomial retains error terms of order h^2 in equation (22) and eliminates terms of order h^3 and higher. This assumption is considered adequate for the purposes of the error analysis. The general polynomial can be written

$$A(x) = \sum_{j=0}^{J} \frac{a_j x^j}{j!}$$

and the ith derivative is

$$A^{(i)}(x) = \sum_{j=i}^{J} \frac{a_j x^{j-i}}{(j - i)!}$$

For $J = 4$, the polynomial expression becomes

$$A(x) = \frac{a_4}{4!} x^4 + \frac{a_3}{3!} x^3 + \frac{a_2}{2!} x^2 + a_1 x + a_0 \tag{26}$$

and the third and fourth derivatives are

$$A'''(x) = a_4 x + a_3 \tag{27}$$

and

$$A^{IV}(x) = a_4 \tag{28}$$

Equations (27), (28), and (23) may be substituted into equation (24) to obtain

$$F(x)_{error} \approx \frac{1}{\pi} \left\{ -\frac{2}{3} a_3 x^{3/2} + \frac{a_4}{12} h^2 x^{1/2} - \frac{4}{15} a_4 x^{5/2} + \sum_{n=1}^{m} \left[a_3 h + \left(n - \frac{1}{2}\right) a_4 h^2\right] \left[x - \left(n - \frac{1}{2}\right) h\right]^{1/2} \right\}$$

$$\left(\left(m - \frac{1}{2}\right) h \leqq x \leqq \left(m + \frac{1}{2}\right) h\right) \tag{29}$$

Equation (29) may be evaluated at a station point by letting $x = mh$

$$F_{m,error} \approx \frac{h}{\pi} \left\{ a_3 h^{1/2} \left[-\frac{2}{3} m^{3/2} + \sum_{n=1}^{m} \left(m - n + \frac{1}{2}\right)^{1/2} \right] \right.$$

$$\left. + a_4 h^{3/2} \left[-\frac{4}{15} m^{5/2} + \frac{1}{12} m^{1/2} + \sum_{n=1}^{m} \left(n - \frac{1}{2}\right)\left(m - n + \frac{1}{2}\right)^{1/2} \right] \right\} \quad (30)$$

and at a half-station point by letting $x = \left(m + \frac{1}{2}\right)h$

$$\left(F_{m+\frac{1}{2}}\right)_{error} \approx \frac{h}{\pi} \left\{ a_3 h^{1/2} \left[-\frac{2}{3} \left(m + \frac{1}{2}\right)^{3/2} + \sum_{n=1}^{m} (m - n + 1)^{1/2} \right] \right.$$

$$+ a_4 h^{3/2} \left[-\frac{4}{15} \left(m + \frac{1}{2}\right)^{5/2} + \frac{1}{12} \left(m + \frac{1}{2}\right)^{1/2} \right.$$

$$\left. \left. + \sum_{n=1}^{m} \left(n - \frac{1}{2}\right)(m - n + 1)^{1/2} \right] \right\} \quad (31)$$

To evaluate the error expression for the I function, equation (29) may be used in equation (25) to obtain

$$I(x)_{error} = \frac{2}{3\pi} \left(a_3 \left\{ -\frac{2}{5} x^{5/2} + h \sum_{n=1}^{m} \left[x - \left(n - \frac{1}{2}\right)h \right]^{3/2} \right\} \right.$$

$$\left. + a_4 \left\{ -\frac{4}{35} x^{7/2} + \frac{h^2}{12} x^{3/2} + h^2 \sum_{n=1}^{m} \left(n - \frac{1}{2}\right) \left[x - \left(n - \frac{1}{2}\right)h \right]^{3/2} \right\} \right)$$

$$\left(\left(m - \frac{1}{2}\right)h \leqq x \leqq \left(m + \frac{1}{2}\right)h \right) \quad (32)$$

When $I(x)_{error}$ is evaluated at a station point $x = mh$, equation (32) becomes

$$I_{m,error} \approx \frac{2}{3\pi} h^2 \left\{ a_3 h^{1/2} \left[-\frac{2}{5} m^{5/2} + \sum_{n=1}^{m} \left(m - n + \frac{1}{2}\right)^{3/2} \right] \right.$$

$$\left. + a_4 h^{3/2} \left[-\frac{4}{35} m^{7/2} + \frac{1}{12} m^{3/2} + \sum_{n=1}^{m} \left(n - \frac{1}{2}\right)\left(m - n + \frac{1}{2}\right)^{3/2} \right] \right\} \tag{33}$$

and at a half-station point, where $x = \left(m + \frac{1}{2}\right)h$, equation (32) becomes

$$\left(I_{m+\frac{1}{2}}\right)_{error} \approx \frac{2}{3\pi} h^2 \left\{ a_3 h^{1/2} \left[-\frac{2}{5} \left(m + \frac{1}{2}\right)^{5/2} + \sum_{n=1}^{m} (m - n + 1)^{3/2} \right] \right.$$

$$+ a_4 h^{3/2} \left[-\frac{4}{35} \left(m + \frac{1}{2}\right)^{7/2} + \frac{1}{12} \left(m + \frac{1}{2}\right)^{3/2} \right.$$

$$\left. \left. + \sum_{n=1}^{m} \left(n - \frac{1}{2}\right)(m - n + 1)^{3/2} \right] \right\} \tag{34}$$

If approximations for the sums of the series are introduced in equations (30), (31), (33), and (34) then, for m large, on an order of magnitude basis, the F and I error expressions are reduced to the following expressions:

$$F_{m,error} \approx \frac{\sqrt{2}}{24} \frac{h^{3/2}}{\pi} \left(a_3 + a_4 x\right) \tag{35}$$

$$\left(F_{m+\frac{1}{2}}\right)_{error} \approx -\frac{5}{24} \frac{h^{3/2}}{\pi} \left(a_3 + a_4 x\right) \tag{36}$$

$$I_{m,error} \approx -\frac{x^{1/2} h^2}{24\pi} \left(a_3 - 2a_4 x\right) \tag{37}$$

$$\left(I_{m+\frac{1}{2}}\right)_{error} \approx - \frac{x^{1/2}h^2}{24\pi}\left(a_3 - 2a_4x\right) \tag{38}$$

Equations (35) and (37) show that the F and I functions, as numerically evaluated at a station point using equations (15) and (20), have discretization errors essentially of order $h^{3/2}$ and h^2, respectively. By reducing h, the station interval width (and increasing N, the number of station points in a solution), the discretization error can be reduced but only at the expense of increased computing time and accumulated error. Another method of reducing the discretization error is to use Richardson's extrapolation to extrapolate the computed values to vanishingly small values of h if the order of error of the solution is known.

To use Richardson's extrapolation, the F or the I functions are evaluated at common station points for two different values of h $\left(h_1 \text{ and } h_2\right)$. Then extrapolated values may be computed by

$$F_{m,extrap} = \frac{F_{m,1}{h_2}^{3/2} - F_{m,2}{h_1}^{3/2}}{{h_2}^{3/2} - {h_1}^{3/2}} \tag{39}$$

$$I_{m,extrap} = \frac{I_{m,1}{h_2}^2 - I_{m,2}{h_1}^2}{{h_2}^2 - {h_1}^2} \tag{40}$$

Use of this extrapolation should give an improved approximation to the correct value provided that round-off and other accumulated errors are small and that the station intervals are sufficiently small so as to minimize the effect of the higher orders of h which have been neglected in the derivation of the error terms. The half-station error results (eqs. (36) and (38)) are of little interest with regard to extrapolation since it is usually difficult to obtain solutions at very many matching stations. It may be observed here that the $A(x)_{check}$ equation (eq. (11)) also has a discretization error of order h^2.

STEPS IN NUMERICAL SOLUTION

A possible procedure for obtaining numerical solutions for the sonic-boom equations may be as follows:

(1) Determine the effective equivalent body area distribution including both the volume and lift effects following the methods of reference 2.

(2) Obtain A_n values at the required station points selecting an interval spacing which will give the smallest desired value of h. Select multiples of the smallest interval spacing which will give common station points for larger values of h (at least two spacings h_1 and h_2 where $h_2 = 2h_1$, for instance).

(3) Compute numerical approximations of the second derivative $A''(x)$ by using equations (3), (4), and (5).

(4) Check the resulting equivalent area distribution $A(x)_{check}$ by using equations (10) and (11).

(5) Compute the F_m and I_m values at all required station points for all desired station interval spacings by using equations (15) and (20).

(6) At common station points for the various station interval spacings selected, apply the extrapolation formulas of equations (39) and (40) to the F_m and I_m values, respectively, or plot the results against $h^{3/2}$ and h^2, respectively, and extrapolate graphically to $h = 0$. In the event that computed results are obtained which do not follow the order of error shown here, determine the order of error h^k by graphical means, if possible, and complete the extrapolation graphically or use the formula

$$Q_{m,extrap} = \frac{Q_{m,1}h_2{}^k - Q_{m,2}h_1{}^k}{h_2{}^k - h_1{}^k} \tag{41}$$

where Q is a quantity representing either the F or the I values.

(7) Use methods such as those presented in references 8 or 9 to compute the desired pressure fields.

(8) If the maximum far-field overpressure is desired from equation (2), use graphical or analytical interpolation to determine $I(x)_{max}$. One simple analytical interpolation scheme is to fit a parabola through the three adjacent largest positive computed values of I_m. If these three consecutive values are designated I_1, I_2, and I_3, then $I(x)_{max}$ is given approximately by

$$I(x)_{max} \approx I_1 - \frac{1}{8}\frac{\left(-3I_1 + 4I_2 - I_3\right)^2}{\left(I_1 - 2I_2 + I_3\right)} \tag{42}$$

In general, better results are obtained from equation (42) if it is applied to extrapolated results at the necessarily wider spacing rather than to the original computed results at the closest spacing. The computed values of $I(x)_{max}$

cannot be extrapolated very well since they will not necessarily be at common station points or x values.

NUMERICAL EXAMPLES

Parabolic Bodies

Two numerical examples for a parabolic body are shown. The first is for a simple smooth body with a parabolic distribution of radii

$$R_1(x) = 0.7725\left[1 - 4(x - 0.5)^2\right] \tag{43}$$

The second is for a body similar to the first one but with a bump equivalent to adding the amount of cross-sectional area contained in a body with parabolic distribution of radii

$$R_2(x) = 0.252\left[1 - 100(x - 0.4)^2\right] \qquad \left(0.3 \leqq x \leqq 0.5\right) \tag{44}$$

Only normal or cross-sectional areas (corresponding to $M = 1.0$ cuts) were used for the equivalent body area distribution in the numerical examples. The resulting normal area distributions are for the smooth parabolic body

$$A_1(x) = \pi R_1^2(x) \tag{45}$$

and for the parabolic body with bump

$$\left.\begin{aligned} A_2(x) &= \pi R_1^2(x) && \left(0 \leqq x \leqq 0.3\right) \\ A_2(x) &= \pi\left[R_1^2(x) + R_2^2(x)\right] && \left(0.3 \leqq x \leqq 0.5\right) \\ A_2(x) &= \pi R_1^2(x) && \left(0.5 \leqq x \leqq 1.0\right) \end{aligned}\right\} \tag{46}$$

The enclosed volume of the parabolic body with bump is approximately 2 percent greater than the volume of the smooth parabolic body. These bodies were chosen only for illustrative purposes and do not necessarily resemble any realistic configuration. It should be observed that the smooth parabolic body area distribution function (eq. (45)) is a 4th degree polynomial in x and therefore corresponds to what was assumed in the error analysis. The results of the error analysis (eqs. (35) to (38)) are therefore most nearly applicable to the solution for this body. The solution for the parabolic body with bump however will contain higher order terms. For the smooth parabolic body, the constants in equation (26) are:

$$a_0 = 0.0$$

$$a_1 = 0.0$$

$$a_2 = 19.0962\pi$$

$$a_3 = -114.5772\pi$$

$$a_4 = 229.1544\pi$$

The A_{check}, F_m, and I_m values, computed with equations (11), (15), and (20), are compared with the exact values in figure 1 for both bodies for an interval spacing of $h = 0.1$ and $h = 0.05$. It may be seen from figure 1 that increasing the number of station points from 10 to 20 improved the solution accuracy. A further illustration of the improvement in solution accuracy with number of station points ($N = 1/h$) is shown in figure 2. Here, F_m and I_m are shown in an extrapolation against $h^{3/2}$ and h^2, respectively, at a body station point of $x = 0.4$ in figure 2(a) and at $x = 0.8$ in figure 2(b) for both bodies. All but the F function for the smooth parabolic body at $x = 0.4$ show an orderly progression. In this one plot, the F scale is greatly expanded and even the $N = 10$ ($h = 0.1$) solution is quite accurate (error less than 1 percent). Here it is possible that errors of order other than $h^{3/2}$ or sources of error other than the discretization error are influencing the behavior of the solutions.

Logarithmic plots of the absolute value of the error fractions $\left|\frac{F_{m,error}}{F_{exact}}\right|$ and $\left|\frac{I_{m,error}}{I_{exact}}\right|$, obtained by subtracting the exact values from the approximate computed values, are presented in figure 3 as a function of h for selected body station points. These data show, by the nearly constant slope of most of the lines, that the order of error is approximately as predicted by equations (35) and (37). A few notable exceptions are for $x = 0.3$ and $x = 0.5$ for the parabolic body with bump, and for $x = 1.0$ for both bodies. The first two stations correspond to the points where the bump starts and ends, and although the first derivative of the area distribution is continuous here, the second derivative is not. The same is true at the base ($x = 1.0$) of both bodies. At such points, apparently, the order of error is not predictable. However, graphically at the bump discontinuities, the order of error seems to approach $h^{1/2}$ for the F values and $h^{3/2}$ for the I values. Error fractions for $I_{m+\frac{1}{2}}$ are presented in figure 4 for data computed by using the half-station formula of equation (21) and verify the h^2 order of error in the half-station error expression of equation (38).

Computed values of the I function at body stations of $x = 0.32$, $x = 0.34$, and $x = 0.36$ for h intervals of 0.01 and 0.02 are shown in the following table for the smooth parabolic body:

h	I_m (smooth parabolic body) for -		
	x = 0.32	x = 0.34	x = 0.36
0.02	0.85643415	0.86271865	0.86095690
.01	.85781819	.86407731	.86228505

These values were used in the extrapolation formula of equation (40) and the extrapolated results are also shown as follows:

I_{extrap} for -		
x = 0.32	x = 0.34	x = 0.36
0.85827954	0.86453020	0.86272777

The value $I_{max} = 0.86483733$ at $x_o = 0.3455236$ was obtained by interpolation with equation (42) and the values of I_{extrap} given. This compares with an exact value of $I_{max,exact} = 0.86483645$ at $x = 0.3454915$ with an error in I_{max} of about one part in a million.

Practical Airplane Shape

Figure 5 shows a comparison of the F_m and I_m functions computed by the central differences method used in this report (circular symbols) and by the method of reference 2 (square symbols). The area distribution chosen was one for a practical airplane shape similar to the one presented in figure 2 of reference 2. The comparison shows that the two methods, both of which involve integrals containing the second derivative explicitly, predict substantially the same results for both the F and I functions. In the comparison of A_{check}, also in figure 5, the square symbols correspond to the input values since a feature of the method of reference 2 is that, at the station points, the input area is recovered exactly when the second derivative is integrated twice for the area. However, departures from the desired area distribution would occur in between the station points because of the oscillatory nature of the second derivatives in that method.

The convergence of the solutions with decreasing interval spacing is shown in figure 6 for body stations $x = 0.4$ and $x = 0.8$. Extrapolation of the I function against h^2 appears quite regular but extrapolation of the F function against $h^{3/2}$ apparently indicates that convergence is not obtained until the interval spacing is quite small ($h = 0.01$, $h = 0.02$). Convergence would not be expected until the interval spacing is sufficiently small so as to define the waviness or bumpiness of the area distribution correctly. Probably a minimum of four stations within a given wave or bump would be required to give adequate definition and only solutions with this interval spacing and closer would be expected to show convergence in an extrapolation. It is therefore apparent that the extrapolation technique must be used with caution, and cannot be applied indiscriminately or without some prior knowledge of the nature of convergence of a solution with decreasing interval spacing.

CONCLUDING REMARKS

An application of Richardson's extrapolation to the numerical evaluation of the sonic-boom integrals occurring in the theory of Whitham (Communications on Pure and Applied Mathematics, August 1952) has been considered. Equations have been developed for evaluating the sonic-boom integrals, using second derivatives which are obtained from the central difference formulas of finite difference techniques. It has been shown by error analysis that these equations have an order of error proportional to the square of the solution point interval spacing for the far-field sonic-boom integral, and to the three-halves power of the interval spacing for the near-field sonic-boom integral. Extrapolation of the numerical results to zero interval spacing on the basis of the predicted order of error is demonstrated, and some numerical examples confirm the essential validity of the procedure. Some of the numerical results however show that the extrapolation technique must be used with caution.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., September 6, 1966,
126-16-03-04-23.

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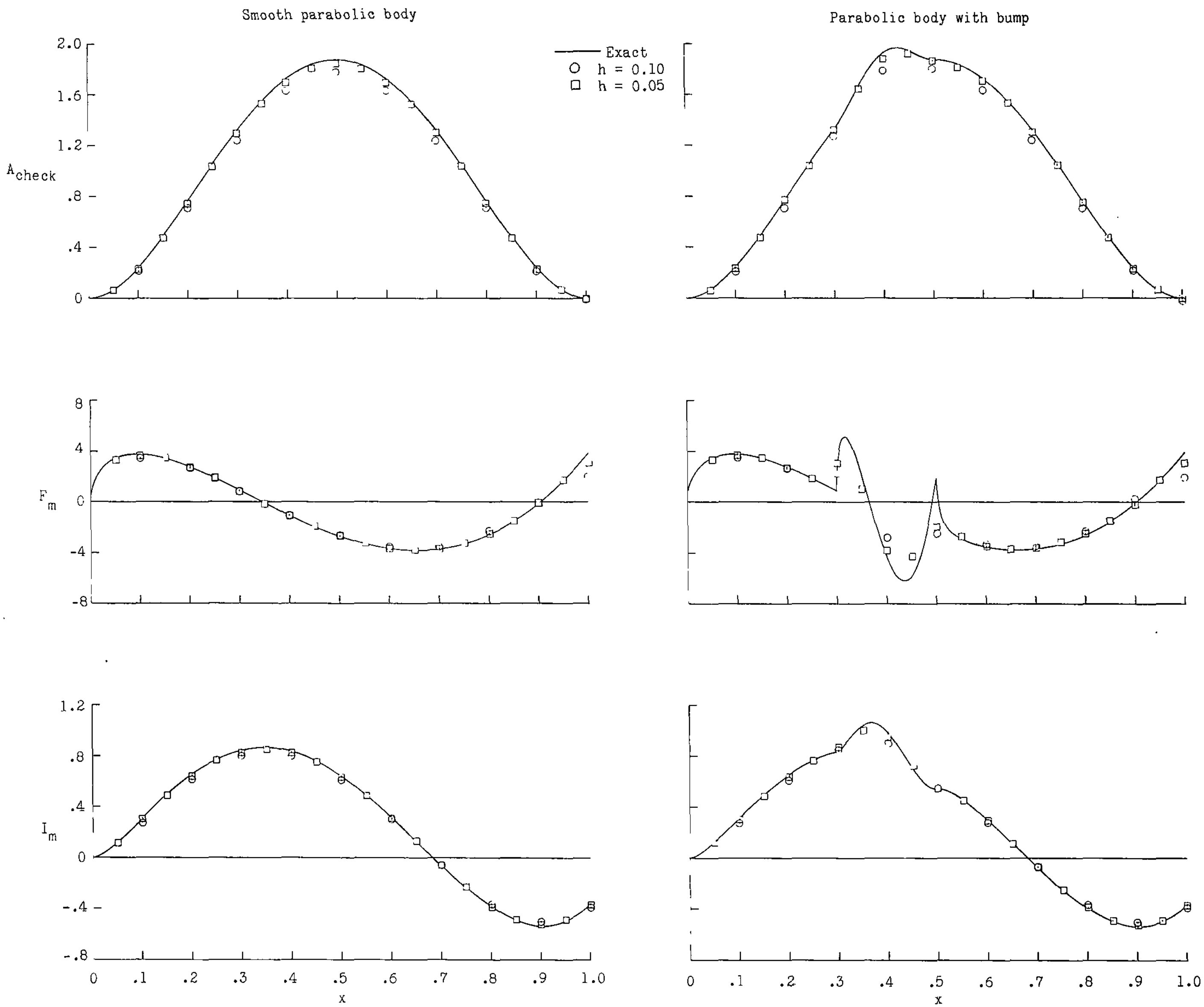
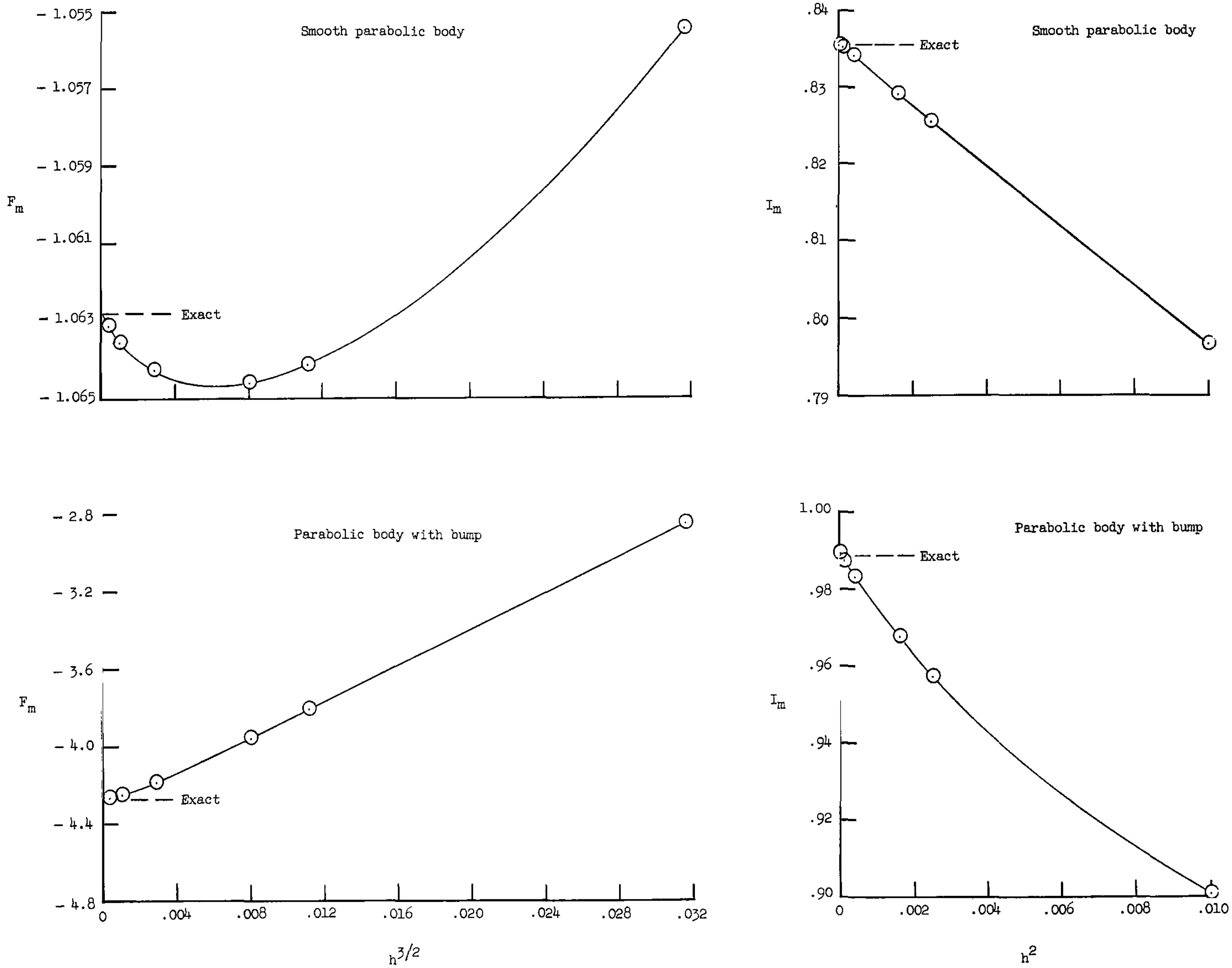
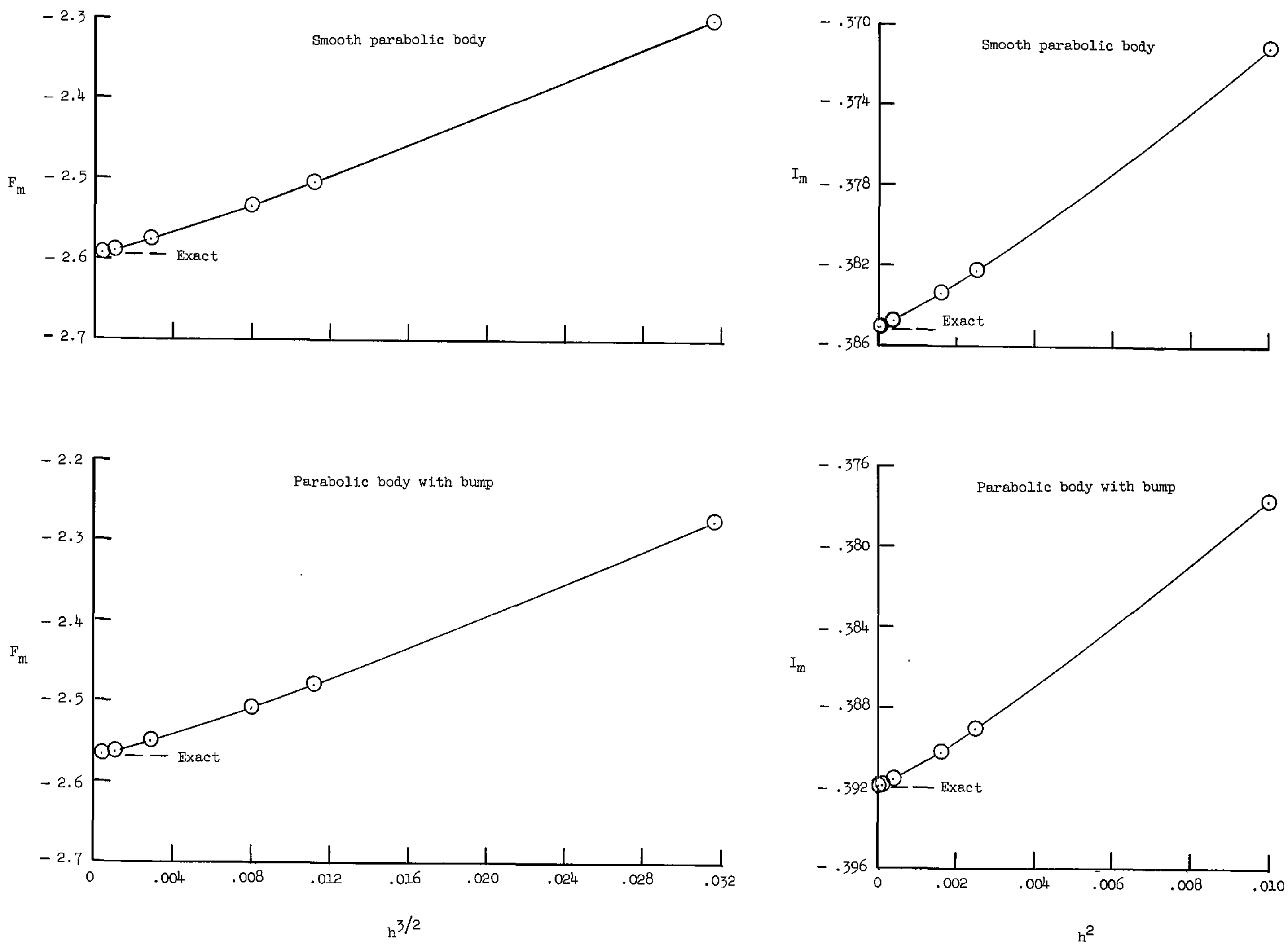


Figure 1.- Comparison of numerical values of A_{check}, F_m, and I_m with exact values for the parabolic bodies.



(a) x = 0.4.

Figure 2.- Extrapolation for F and I functions at body stations x = 0.4 and x = 0.8 for the parabolic bodies.



(b) x = 0.8.

Figure 2.- Concluded.

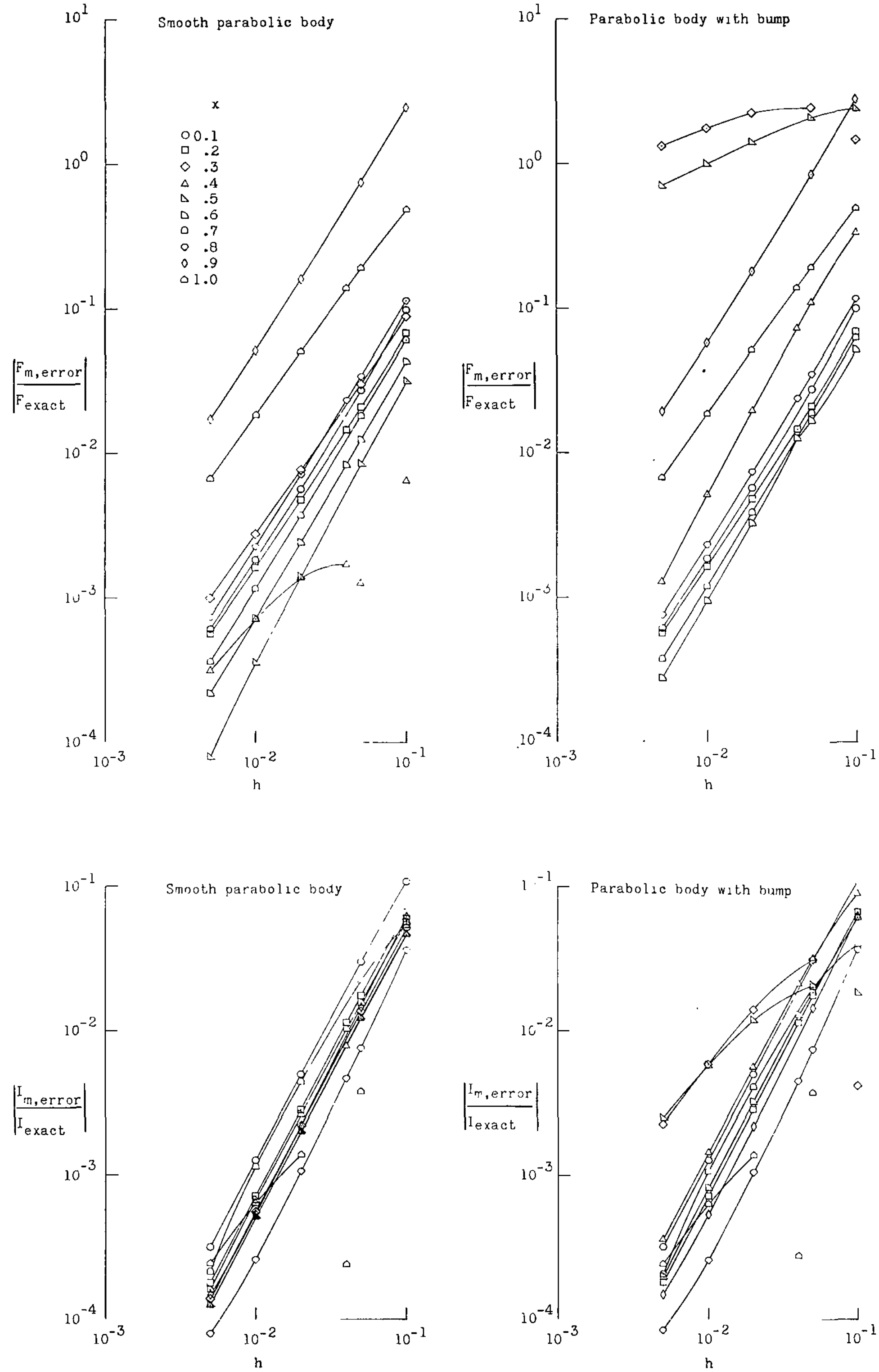


Figure 3.- Error fractions for even station results as a function of h at selected body station points for the parabolic bodies.

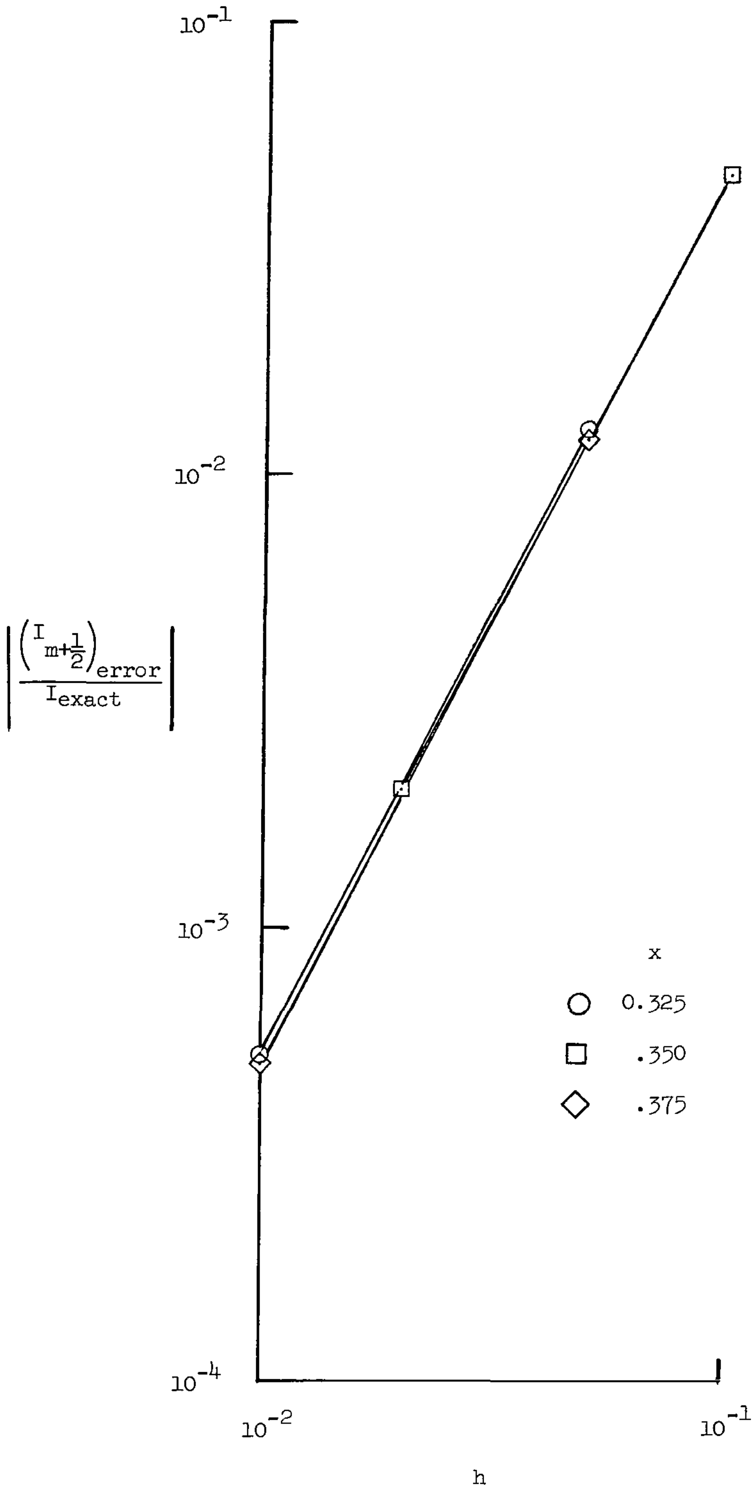


Figure 4.- Error fractions for half-station results as a function of h at selected body stations for the smooth parabolic body.

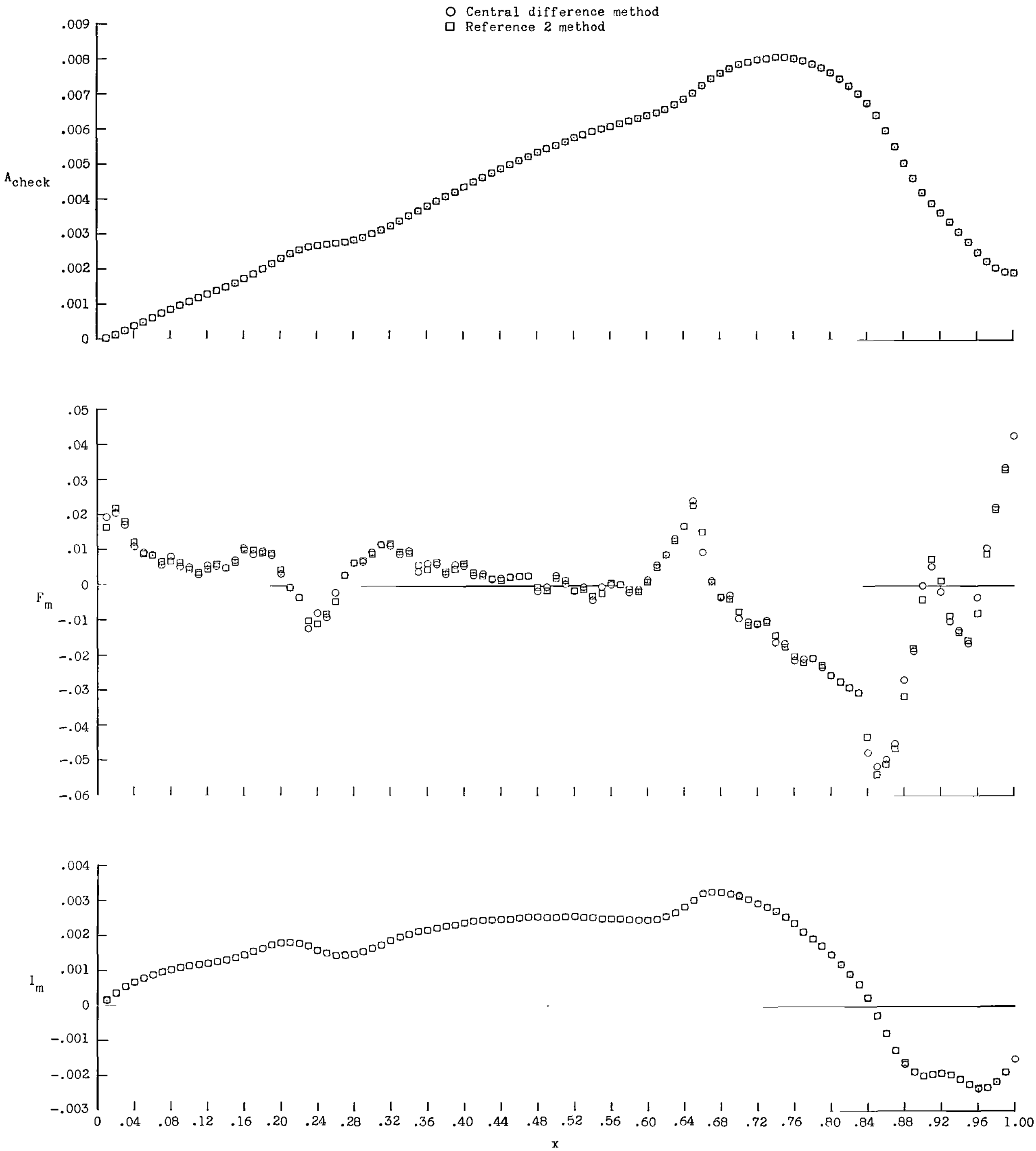


Figure 5.- Comparison of numerical values of A_{check}, F_m, and I_m for a practical airplane shape computed by the central difference method and by the method of reference 2 for $h = 0.01$.

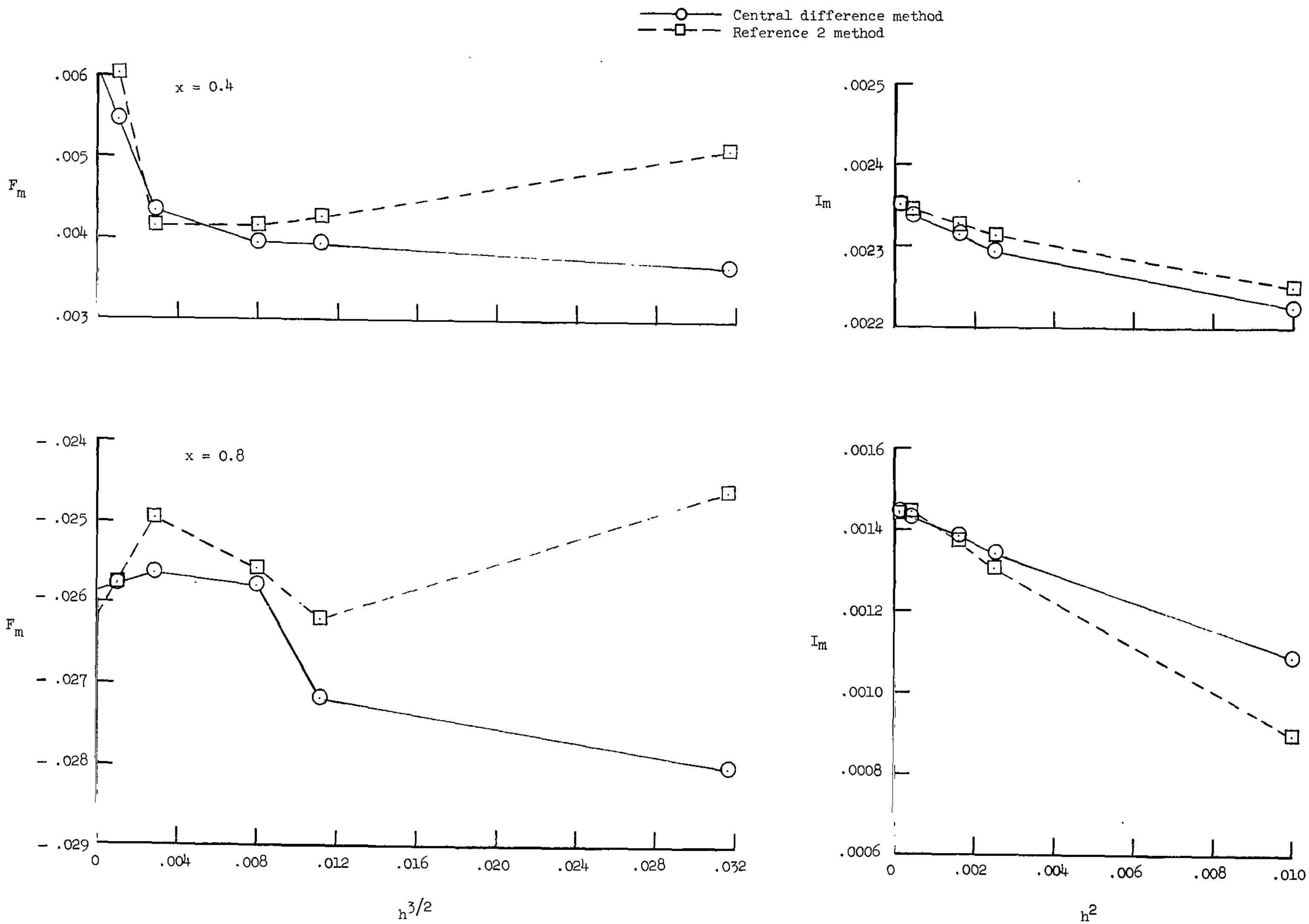


Figure 6.- Extrapolation for F and I functions at body stations x = 0.4 and x = 0.8 for a practical airplane shape.

NASA-Langley, 1967 L-4965

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