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AN ANALYSIS OF A FIRST-ORDER POLYNOMIAL PREDICTOR FOR DATA COMPRESSION

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SUMMARY

A theoretical analysis is presented of a first-order polynomial predictor for use in data compression. The investigation is concerned with compressing data which existed in the form of samples taken at a constant rate. This analysis is also limited to data having a Gaussian distribution of amplitude. Four pseudo-random time histories with different amplitude spectra are compared by using this method. A digital computer was used to generate data approximating the four time histories and to measure the compression possible in the generated data for comparison with the theoretical calculations.

The results of this investigation show that, under known conditions, it is possible to calculate the amount of data compression which can be obtained through the use of a first-order polynomial predictor. This investigation also shows that high frequencies in the data reduce the amount of compression and that there is a trade-off between the amount of error introduced by the compression system and the amount of compression obtained. This method of analysis is limited to data with known statistics.

INTRODUCTION

Compression of sampled data is defined as the reduction in the total number of samples required for the reconstruction of the original data within a given accuracy. One of the techniques for compressing data is the application of polynomial prediction. The amplitude of a given sample is predicted as a function of the previous samples. The order of the prediction scheme indicates the order of the polynomial equation and the number of previous samples used for prediction. A zero-order polynomial predictor uses only one previous sample and predicts that the succeeding sample will have the same amplitude as the previous sample. A first-order polynomial predictor uses two previous samples to generate amplitude and slope information for its prediction.

[1]The information presented herein was offered as a thesis in partial fulfillment of the requirements for the degree of Master of Electrical Engineering, University of Virginia, Charlottesville, Virginia, August 1965.

Among those actively conducting research in this area are Gardenhire (ref. 1) and Medlin (ref. 2). Their work has been essentially experimental and has consisted in making compression measurements on data from actual telemetry systems or on artificially generated data. There has been a scarcity of work directed toward calculating the amount of compression possible by polynomial prediction from the statistics of the data. Hochman in reference 3 analyzed theoretically the compression by a zero-order polynomial predictor. The purpose of the present investigation is to analyze theoretically the first-order polynomial predictor for data compression, using Hockman's techniques as an outline, and to compare these results with actual measurements on data simulated by a computer. The first-order polynomial predictor was analyzed by using assumed statistics for the data and by solving a random walk problem.

The results of this investigation show that the amount of data compression for a first-order polynomial predictor can be calculated by using the statistics of the data. This method should be a useful tool to a designer contemplating the use of first-order polynomial prediction for data compression.

SYMBOLS

C_R compression ratio

$E(x)$ expected value

$F(\omega)$ amplitude spectrum

f_m maximum frequency component

f_s sampling frequency

K tolerance limit as a function of σ

k_i amplitude of frequency component

$L_{11}, L_{12}; L_{21}, L_{22} \ldots$ plus and minus tolerance limits

$\|M\|$ matrix

$|M|$ determinant of matrix $\|M\|$

M_{ij} cofactor of determinant $|M|$

n number of samples since a change in the reference samples

$p(x)$ probability density function

t	time
U	difference between Δ's
x_n	amplitude of nth sample
Z_n	difference between actual amplitude and predicted amplitude
Δ	difference in amplitude between adjacent samples
ρ	correlation coefficient
σ	standard deviation
σ^2	variance
τ	time displacement
$\psi(t)$	moment generating function
ω	angular rate in radians per second
ω_m	maximum angular rate

THEORETICAL ANALYSIS OF A FIRST-ORDER POLYNOMIAL PREDICTOR

In general, a first-order polynomial predictor performs a prediction of the amplitude of a given sample by using information on both amplitude and slope that has been obtained from previous samples. The particular first-order predictor to be considered herein uses two adjacent samples to predict the amplitude of succeeding samples. The general equation for this type of prediction is:

$$x_n = x_1 + (n - 1)(x_2 - x_1)$$

In this analysis the samples are assumed to be evenly spaced in time.

Figure 1 shows the operation of the first-order polynomial predictor in data compression. The curve in figure 1 is an amplitude-time plot of the input data. The data have been sampled at a rate f_s and the samples have an amplitude x_n. Two adjacent samples, x_1 and x_2, are chosen as the first reference samples. A reference line is drawn between the two points; predicted amplitudes of future samples will fall along this line. Allowable tolerance limits, L_{11} and L_{12}, are placed on either side of the reference line. All succeeding samples are discarded until a sample falls outside the tolerance band. In figure 1 the first sample to fall outside the tolerance band is the fifth sample. When this phenomenon occurs, the previous sample, x_4, is used in conjunction with x_5 to generate a new reference line. The previous sample is used so that the data may be reconstructed by joined straight line segments. The maximum error in the reconstructed data due to the compression is one-half the width of the tolerance band.

In order to analyze the first-order predictor theoretically, the following assumptions are made:

(1) The power density spectrum of the data is known.

(2) The probability density of the amplitude of the data is Gaussian with a mean of zero and a standard deviation of σ.

(3) The sampling rate is greater than twice the highest frequency present in the data.

As assumption (2) shows, this analysis is concerned only with the Gaussian distribution of amplitudes. However, the method could be applied to other probability distributions.

The compression ratio is defined as the total number of samples handled divided by the number of samples retained. The retained samples are the two adjacent samples which are used to generate the reference line. The method of analysis is similar to that of Hochman (ref. 3). The problem is approached by solving a random walk problem in order to find the average number of samples required to reach one of the tolerance limits. This number is the average number of samples handled, and the number of samples retained is two. Therefore, the theoretical compression ratio is one-half the average number of steps to the boundary.

The difference in amplitude between the actual amplitude of the nth sample and the predicted value of the nth sample is Z_n. In appendix A, Z_n is found to be

$$Z_n = \sum_{i=1}^{n} U_i$$

where

$$U_i = \left(x_{i+2} - x_{i+1}\right) - \left(x_2 - x_1\right)$$

$$U_i = \Delta_{i+1} - \Delta_1$$

and Δ_{i+1} is the difference in amplitude between samples x_{i+2} and x_{i+1}.

The probability density for U_n can be found by using a joint probability density function for four variables, $p\left(x_1, x_2, x_{n+1}, x_{n+2}\right)$. If

$$x_2 - \Delta_1 = x_1$$

$$p\left(x_2 - \Delta_1, x_2, x_{n+1}, x_{n+2}\right) = p\left(x_1, x_2, x_{n+1}, x_{n+2}\right)$$

According to reference 4 (p. 36),

$$p\left(\Delta_1, x_{n+1}, x_{n+2}\right) = \int_{-\infty}^{\infty} p\left(x_2 - \Delta_1, x_2, x_{n+1}, x_{n+2}\right) dx_2$$

Then, let

$$x_{n+2} - \Delta_{n+1} = x_{n+1}$$

$$p\left(\Delta_1, \Delta_{n+1}\right) = \int_{-\infty}^{\infty} p\left(\Delta_1, x_{n+2} - \Delta_{n+1}, x_{n+2}\right) dx_{n+2}$$

Since

$$U_n = \Delta_{n+1} - \Delta_1$$

$$p\left(U_n\right) = \int_{-\infty}^{\infty} p\left(\Delta_{n+1} - U_n, \Delta_{n+1}\right) d\Delta_{n+1}$$

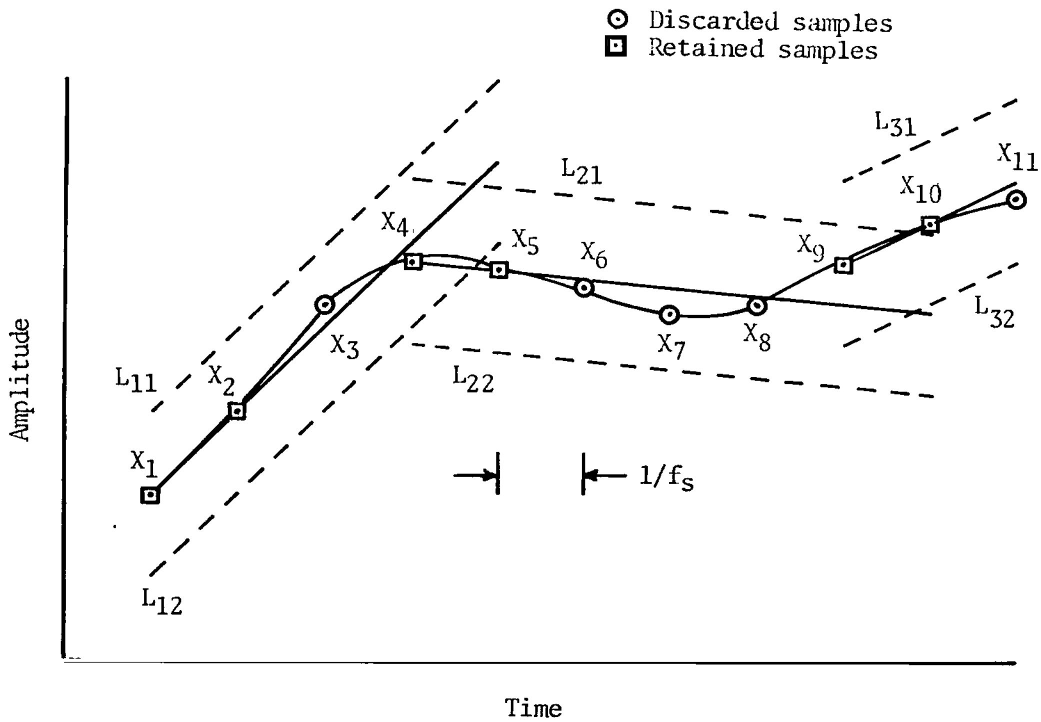

Figure 1.- A first-order polynomial predictor.

The probability density function for U_n is derived in appendix B and is found to be a Gaussian density function with mean equal to zero.

In reference 5, Blackwell and Girshick solve a problem in sequential analysis, which is in essence a random-walk problem. Appendix C presents this solution, in the notation used herein, to obtain the expected value of Z_n^2:

$$E\left(Z_n^2\right) = L_{11}L_{12} = K_1K_2\sigma^2 \tag{1}$$

where σ^2 is the variance of the amplitude of the data and

$$L_{11} = K_1\sigma$$

$$L_{12} = K_2\sigma$$

the quantities L_{11} and L_{12} are the plus and minus tolerance limits. By the definition of Z_n, together with the definition for the second moment (ref. 6),

$$E\left(Z_n^2\right) = E\left\{\left(\sum_{i=1}^{n} U_i\right)^2\right\}$$

$$E\left(Z_n^2\right) = \sum_{i=1}^{n} \sigma_{U_i}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} E\left(U_iU_j\right) \tag{2}$$

where $\sigma_{U_i}^2$ is the variance of U_i and where i is never equal to j. In this analysis it is assumed that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(U_iU_j\right) = 0 \qquad (i \neq j)$$

This assumption is considered reasonable since the expected value of U_iU_j may be either positive or negative. This relationship, in effect, assumes that there is no correlation between U_i and U_j for $i \neq j$.

If the plus and minus tolerance limits are assumed to be equal and equations (1) and (2) for $E\left(Z_n^2\right)$ are equated,

$$K = K_1 = K_2$$

$$K^2\sigma^2 = \sum_{i=1}^{n} \sigma_{U_i}^2 \tag{3}$$

The n in equation (3) is the average number of steps to one of the tolerance limits. It has already been determined that the compression ratio C_R is

$$C_R = \frac{n}{2}$$

However, it is not possible to solve for the compression ratio explicitly as a function of K, but it is possible to solve for K as a function of C_R.

$$K = \frac{1}{\sigma}\sqrt{\sum_{i=1}^{2C_R} \sigma_{U_i}^2}$$

The values of $\sigma_{U_i}^2$ were calculated from the equation obtained in appendix B for various values of i. These values were then used to compute K as a function of C_R.

Figure 2 is a plot of the theoretical compression ratio for a first-order predictor operating on four types of input data. In this figure the sampling rate was 5 times the cutoff frequency of the data. The abscissa is K, the allowable tolerance limit expressed in terms of σ, which is the standard deviation of the amplitude of the data.

The amplitude spectra for the four types of input shown in figure 2 are:

L spectrum:

$$F(\omega) = k_1 \qquad \left(\text{For } 0 \leqq |\omega| \leqq 0.1\omega_m\right)$$

$$F(\omega) = 0.01k_1 \qquad \left(\text{For } 0.1\omega_m \leqq |\omega| \leqq \omega_m\right)$$

$$F(\omega) = 0 \qquad (\text{For other values of } \omega)$$

Exponential spectrum:

$$F(\omega) = k_2 \exp\left(-\frac{5|\omega|}{\omega_m}\right) \qquad \left(\text{For} \quad 0 \leqq |\omega| \leqq \omega_m\right)$$

$$F(\omega) = 0 \qquad (\text{For other values of} \quad \omega)$$

Triangular spectrum:

$$F(\omega) = k_3\left(1 - \frac{|\omega|}{\omega_m}\right) \qquad \left(\text{For} \quad 0 \leqq |\omega| \leqq \omega_m\right)$$

$$F(\omega) = 0 \qquad (\text{For other values of} \quad \omega)$$

Rectangular spectrum:

$$F(\omega) = k_4 \qquad \left(\text{For} \quad 0 \leqq |\omega| \leqq \omega_m\right)$$

$$F(\omega) = 0 \qquad (\text{For other values of} \quad \omega)$$

Figures 3, 4, 5, and 6 show the amplitude spectra and the correlation coefficients for the four types of input. The abscissa for the correlation coefficient is the displacement n expressed as a multiple of the number of samples away from the first reference sample with the sampling rate being five times the maximum frequency in the data. For example, the correlation between adjacent samples, taken at the given sampling rate, is found from the curves at a displacement equal to one.

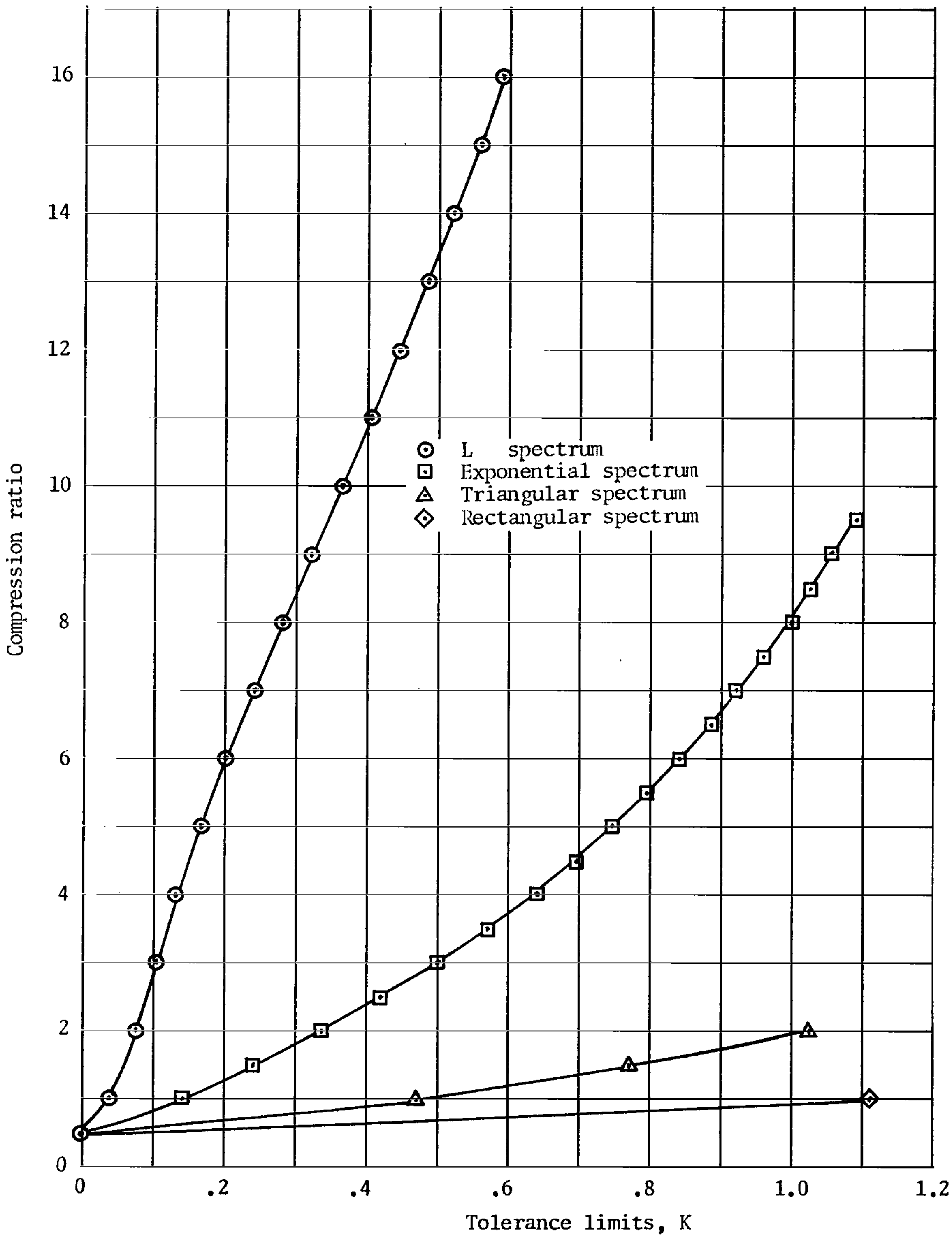

Figure 2.- Theoretical compression ratio for a first-order predictor.

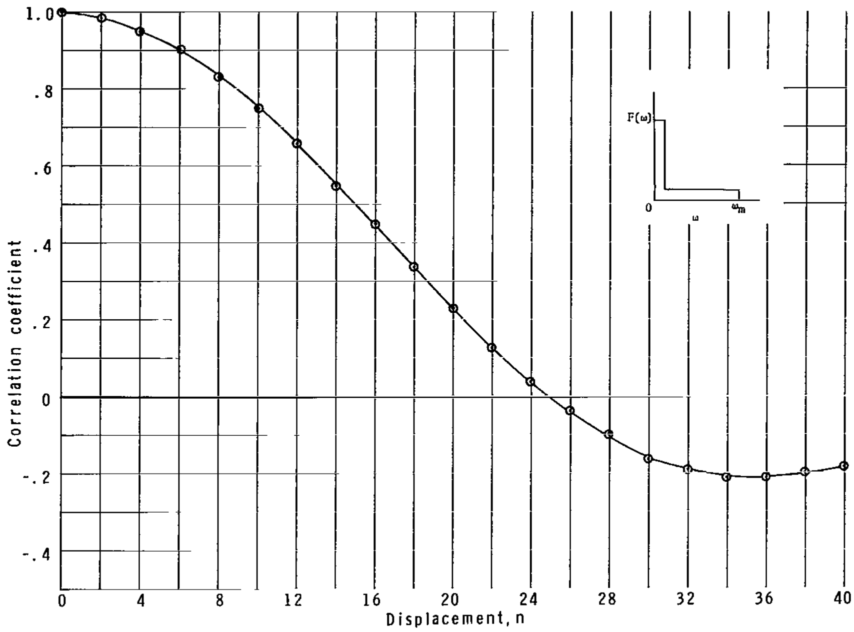

Figure 3.- Correlation coefficient for L spectrum.

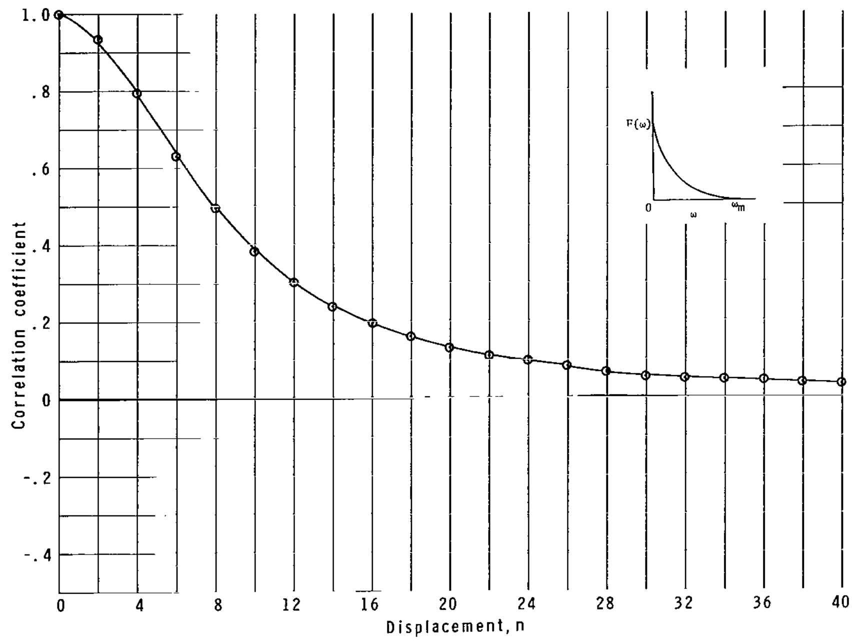

Figure 4.- Correlation coefficient for exponential spectrum.

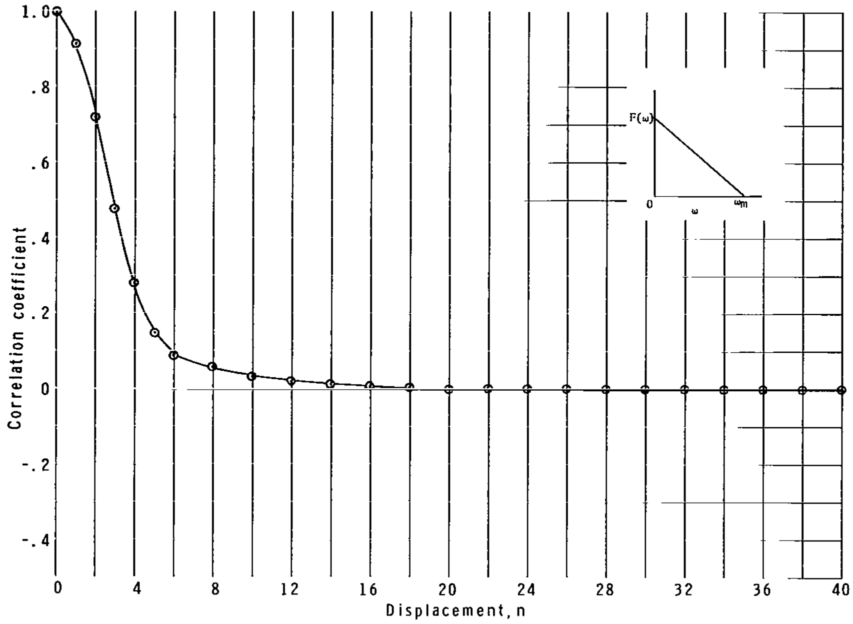

Figure 5.- Correlation coefficient for triangular spectrum.

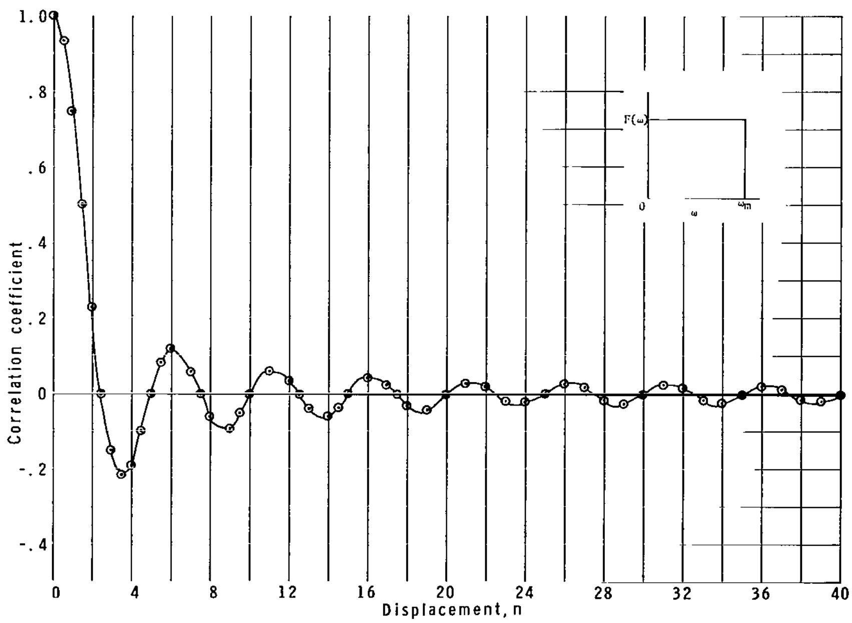

Figure 6.- Correlation coefficient for rectangular spectrum.

EXPERIMENTAL PROCEDURE

Since all experimental work involved sampled data, a breadboard model of the first-order predictor for data compression would have been a special purpose digital computer. For this reason no breadboard model of the predictor was constructed, but all experimental work was performed on a digital computer located at the Langley Research Center.

Data were generated by computer approximations of the characteristics used in the theoretical analysis. These data were then examined by the computer to determine the amount of compression obtainable by a first-order predictor as a function of the width of the tolerance band K.

The procedure used for generating the data with the desired characteristics was as follows:

(1) Pseudo-random numbers were generated by approximating a Gaussian probability density with mean of zero, variance of one (refs. 7 and 8), and a flat amplitude spectrum.

(2) Numerical filtering (ref. 9) by use of weighting functions and a convolution transformed the flat amplitude spectrum into that of the four spectra used in the theoretical analysis.

These two steps gave a sequence of numbers for each of the four spectra. If this sequence of numbers is taken to be a sequence of samples of a continuous function, the continuous function would have a probability density approximating a Gaussian probability density function and amplitude spectra approximating that of one of the four spectra.

The procedure used for measuring the compression possible on the data generated by a first-order predictor was as follows:

(1) The first two samples, x_1 and x_2, of the sequence were used to find the slope of the reference line, Δ_1:

$$\Delta_1 = x_2 - x_1$$

(2) The predicted value of x_3 is $x_2 + \Delta_1$.

(3) The difference in the actual value of x_3 and the predicted value was calculated. If this difference was less than the width of the tolerance band, the sample was considered redundant. The amplitude of x_4 was then predicted by $x_2 + 2\Delta_1$. This process was continued until the difference in actual and predicted value of a sample was greater than the width of the tolerance band.

(4) When a nonredundant sample was located, the computer returned to step (1) with the nonredundant sample as x_2 and the previous sample as x_1.

(5) A record was kept of the number of samples examined and the number found to be nonredundant. The compression ratio is the total number of samples divided by twice the number of retained samples.

RESULTS

The simulated data generated by the computer was a sequence of 5000 numbers. The autocorrelation of the pseudo-random numbers was calculated to check the randomness of the numbers. At all nonzero displacements, the autocorrelation was found to be less than 3 percent of the value at zero displacement. Figure 7 is a plot of the first one hundred points of the pseudo-random data before being filtered. Figures 8, 9, 10, and 11 show the data after being filtered to conform to the spectral requirements of the four different inputs. These figures, as expected, show that the data having the rectangular spectrum has the greatest content of high frequencies.

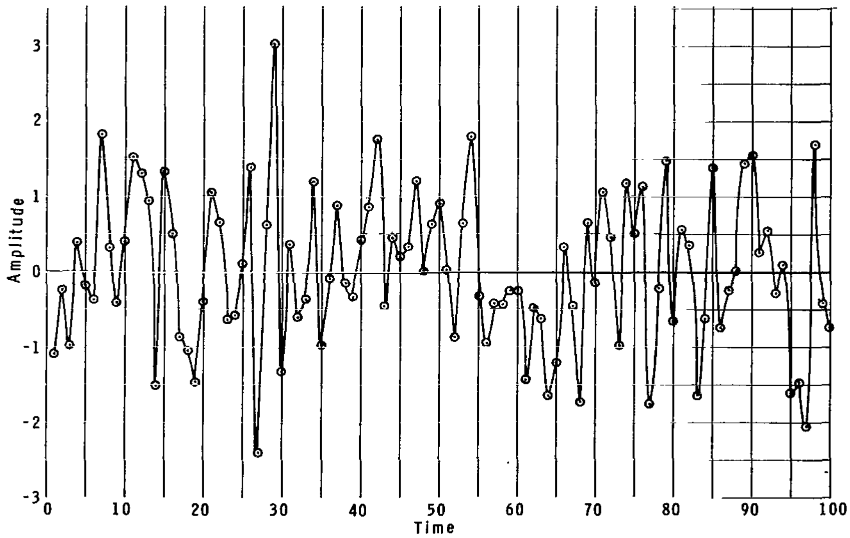

Figure 7.- Random data. Gaussian density before filtering. Mean, 0.000849; standard deviation, 1.003534.

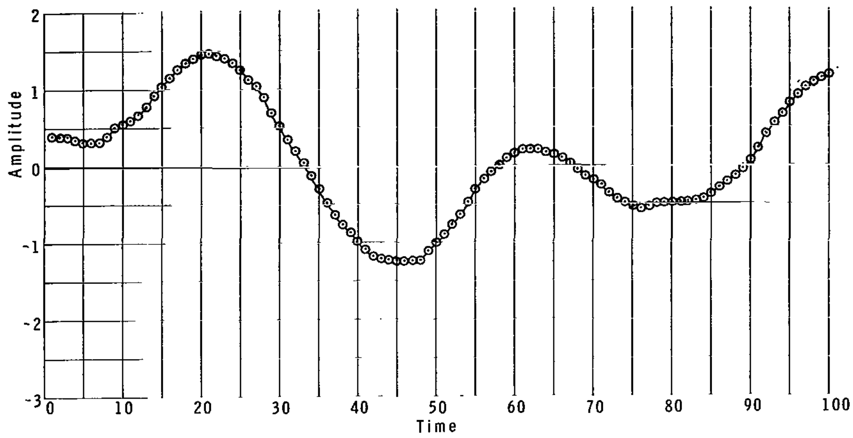

Figure 8.- Random data. Gaussian density with L-spectrum filter. Mean, 0.004785; standard deviation, 0.943357.

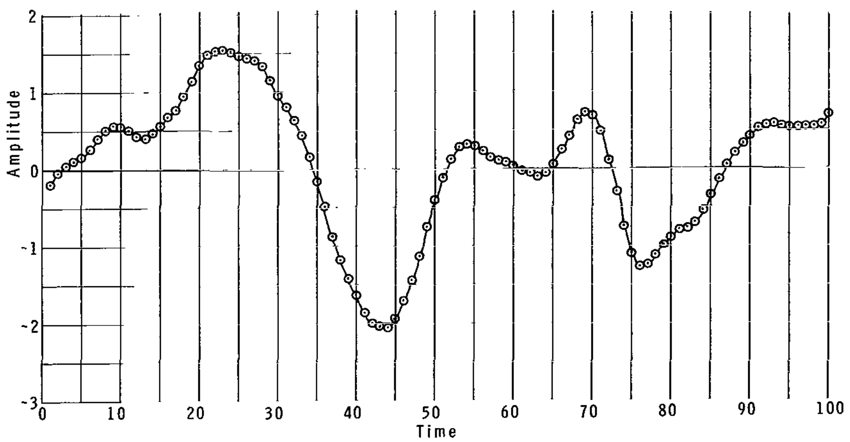

Figure 9.- Random data. Gaussian density with exponential-spectrum filter. Mean, 0.003667; standard deviation, 0.966783.

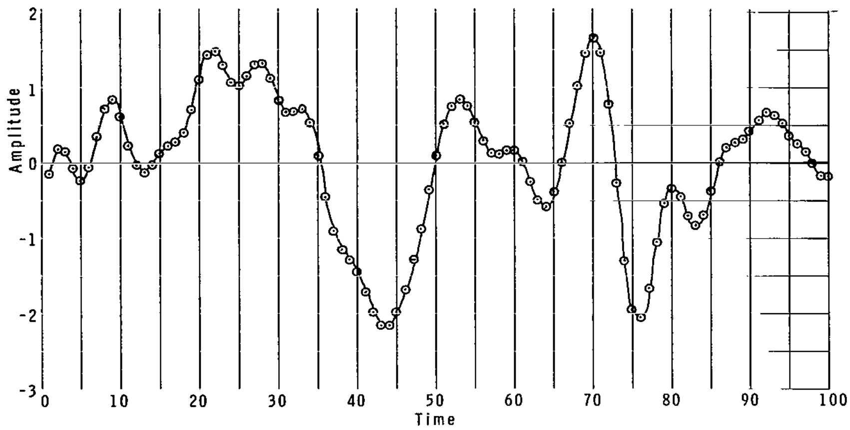

Figure 10.- Random data. Gaussian density with triangular-spectrum filter. Mean, 0.002235; standard deviation, 0.987779.

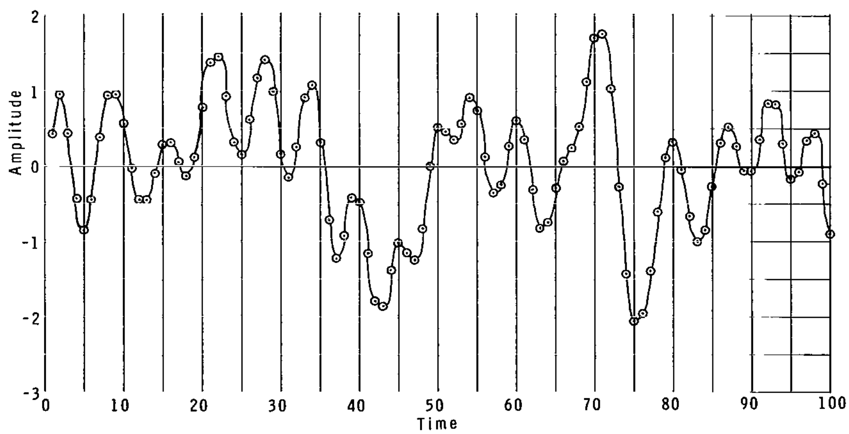

Figure 11.- Random data. Gaussian density with rectangular-spectrum filter. Mean, 0.001498; standard deviation, 0.989968.

The amplitude spectrum was measured from the actual data in order to make a comparison with the theoretical spectra. Figures 12, 13, 14, and 15 show both theoretical and measured amplitude spectra for the four inputs. It can be observed from these figures that the actual amplitude spectra have the general shape of the theoretical spectra. The difference in the two curves is due to the fact that numerical integration was used in filtering the data with only forty-one points used to approximate the amplitude spectra.

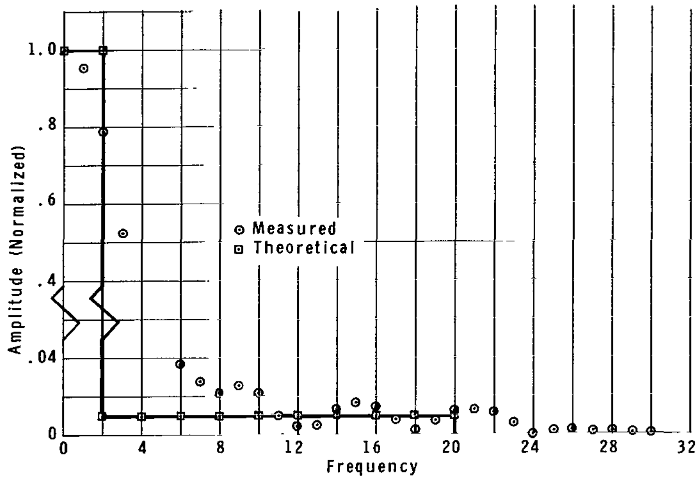

Figure 12.- Amplitude spectrum for L-spectrum filter.

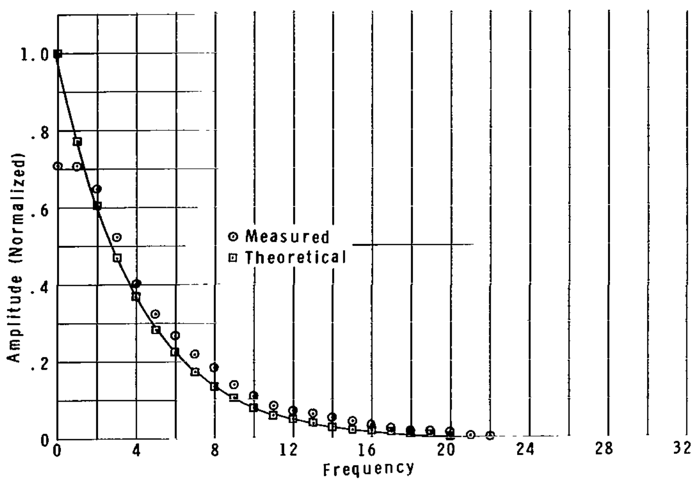

Figure 13.- Amplitude spectrum for exponential-spectrum filter.

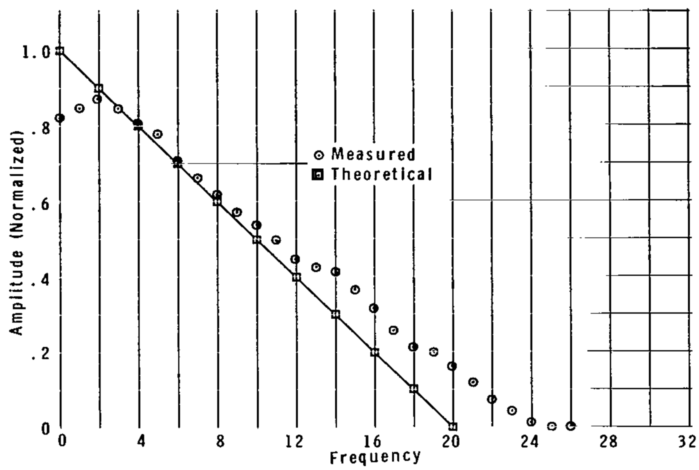

Figure 14.- Amplitude spectrum for triangular-spectrum filter.

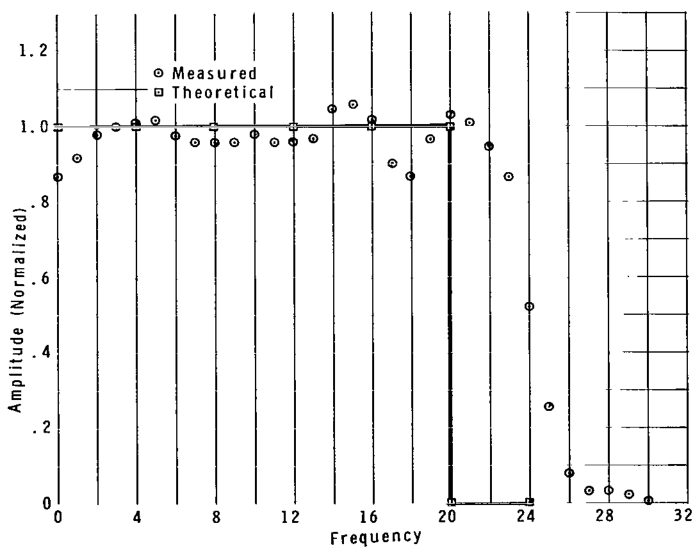

Figure 15.- Amplitude spectrum for rectangular-spectrum filter.

The data were also checked to insure that the amplitude distribution followed a Gaussian probability curve. Figure 16 is a plot of the cumulative relative frequency of the random data before filtering, which has been plotted on Gaussian probability paper. If the data do have a Gaussian distribution, the points will form a straight line. The random data before and after filtering were found to approximate Gaussian distributed data out to approximately $\pm 2.5\sigma$. Figures 17, 18, 19, and 20 are the plots of the cumulative frequency for the data after it had been filtered according to the requirements of the four spectra.

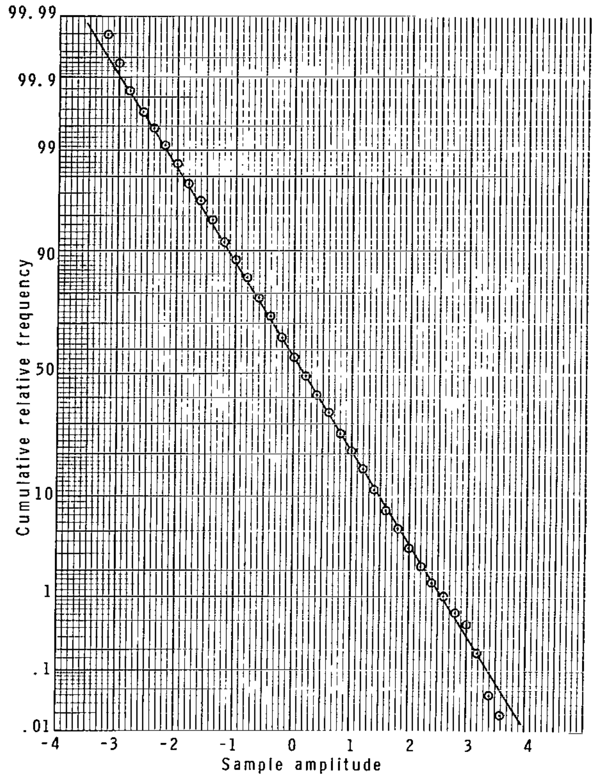

Figure 16.- Random data before filtering.

Figure 21 shows the measured compression ratio as a function of the width of the tolerance band for the first-order predictor. The measured values of the compression do not agree exactly with the theoretical values. However, the theoretical and actual compression ratios have the same shape, general range of values, and response to the four types of input data. Both the theoretical and measured curves show that the presence of high frequencies in the data reduces the amount of data compression which can be obtained. It is anticipated that the differences between the measured and theoretical compression ratios might be reduced by using data which more closely approximate the theoretical data. However, the results obtained herein are sufficient for comparison with theoretical results.

It can be seen from the curves in figures 2 and 21 that there is a tradeoff between the amount of compression and the width of the tolerance band. As the compression system is allowed to introduce more error with sample rate held constant, the amount of compression is increased.

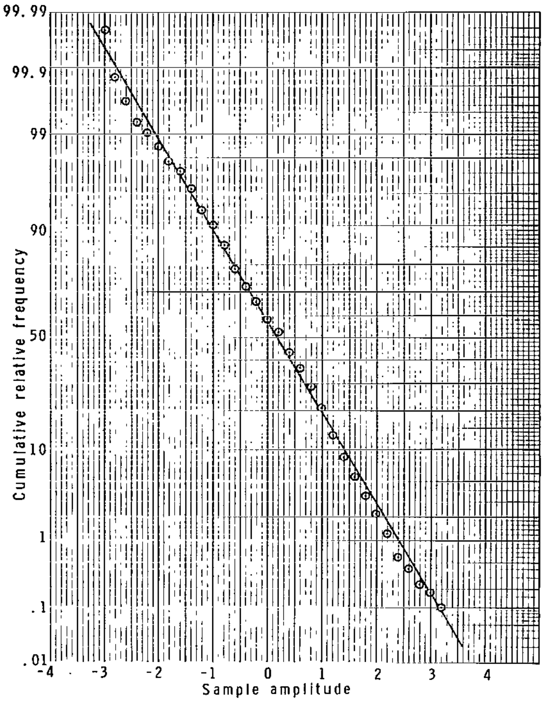

Figure 17.- Random data, L-spectrum filter.

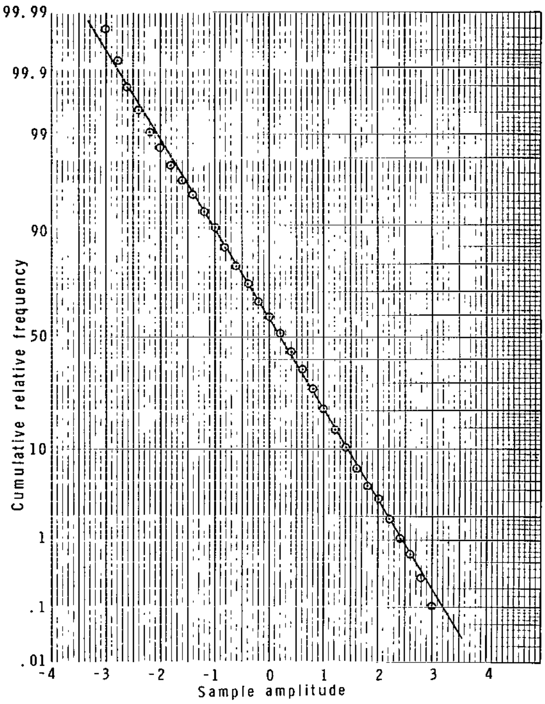

Figure 18.- Random data, exponential-spectrum filter.

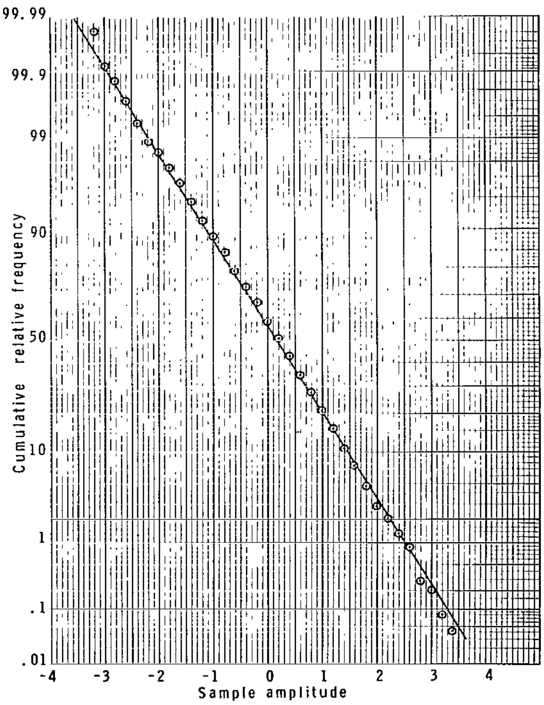

Figure 19.- Random data, triangular-spectrum filter.

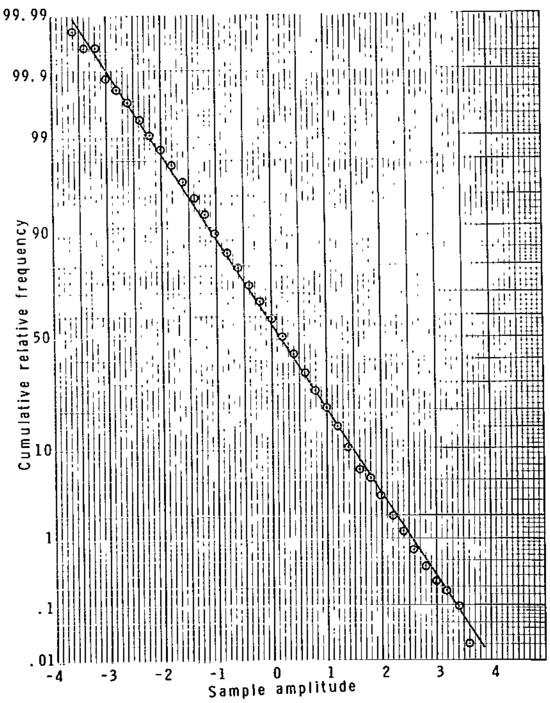

Figure 20.- Random data, rectangular-spectrum filter.

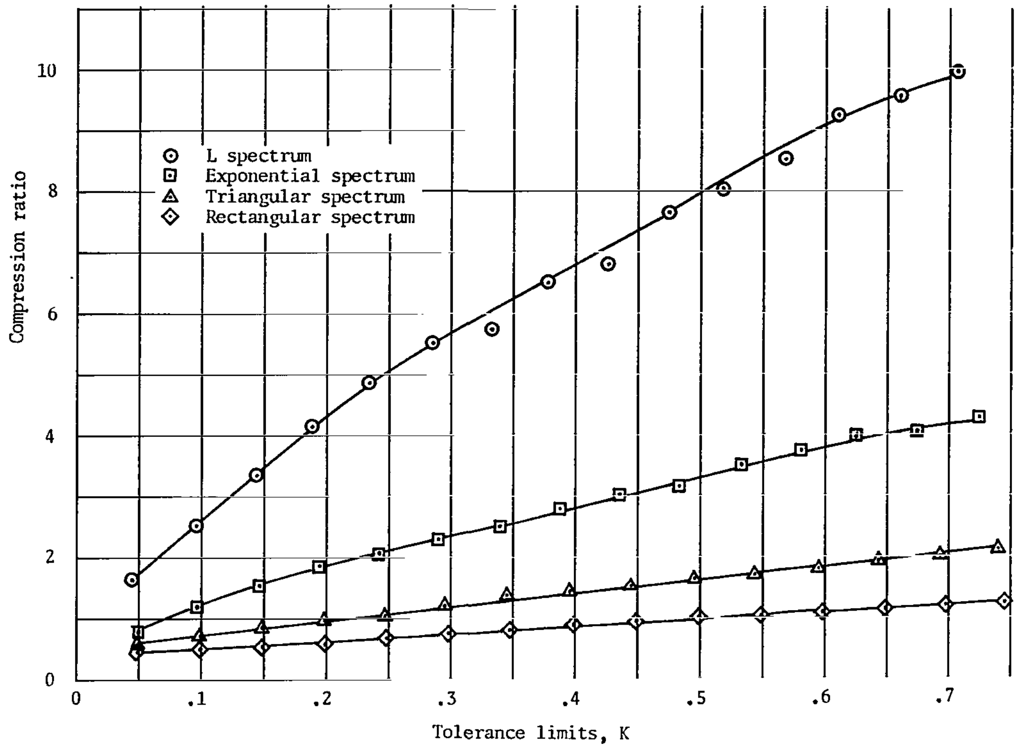

Figure 21.- Actual compression ratio.

CONCLUDING REMARKS

The results of this investigation have shown that it is possible under some conditions to calculate the amount of data compression which can be obtained by using a first-order polynomial predictor. This method is limited to the cases in which the statistics of the data are known; that is, the amplitude spectrum and the probability distribution of amplitudes are known. This investigation has been concerned only with data having a Gaussian probability distribution simply for mathematical convenience but could be extended to other probability distributions. This investigation has also shown that there is a tradeoff between the amount of error introduced by the compression system and the amount of compression obtained.

The methods of analysis presented should be useful as a basis for the extension of theoretical analysis to other types of first-order predictors and to data with other types of probability distributions.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., October 27, 1966,
125-21-02-04-23.

APPENDIX A

DERIVATION OF THE EQUATION FOR THE DISTANCE FROM THE PREDICTED VALUE IN A FIRST-ORDER SYSTEM

Definitions of quantities required for derivation of an equation for the distance from the predicted value in a first-order system are as follows:

x_i — amplitude of the ith sample

Δ_i — difference in amplitude of the (i + 1) sample and the ith sample; that is, $\Delta_i = x_{i+1} - x_i$

Δ_1 — reference difference; $\Delta_1 = x_2 - x_1$

U_i — difference in amplitude of the (i + 1) difference and the reference difference; $U_i = \Delta_{i+1} - \Delta_1$

n — number of samples beyond the second reference sample; that is, $n = i - 2$

Z_n — difference in amplitude between the actual sample and the predicted value of the nth sample; that is, $Z_n = x_{n+2} - \left[x_1 + (n + 1)\Delta_1\right]$

From these definitions,

$$Z_n = x_{n+2} - \left[x_1 + (n + 1)\Delta_1\right]$$

$$Z_n = x_{n+2} - x_1 - (n + 1)\Delta_1$$

$$Z_n = \left(x_{n+2} - x_{n+1}\right) + \left(x_{n+1} - x_n\right) + \ . \ . \ . \ + \left(x_2 - x_1\right) - (n - 1)\Delta_1$$

$$Z_n = \Delta_{n+1} + \Delta_n + \Delta_{n-1} + \ . \ . \ . \ + \Delta_1 - (n + 1)\Delta_1$$

$$Z_n = \left(\Delta_{n+1} - \Delta_1\right) + \left(\Delta_n - \Delta_1\right) + \ . \ . \ . \ + \left(\Delta_2 - \Delta_1\right)$$

$$Z_n = U_n + U_{n-1} + \ . \ . \ . \ + U_1$$

$$Z_n = \sum_{j=1}^{n} U_j$$

APPENDIX B

DERIVATION OF THE PROBABILITY DENSITY OF U_n

If x_1, x_2, x_{n+1}, and x_{n+2} are variables with Gaussian probability densities and each has a mean of zero and a standard deviation of σ,

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

and the joint probability density function for four variables is (ref. 10)

$$p(x_1, x_2, x_{n+1}, x_{n+2}) = \frac{1}{(2\pi)^2\sqrt{|M|}} \exp\left[\frac{-1}{2|M|} \sum_{i,j=1}^{4} M_{ij}x_i x_j\right]$$

In the preceding equation, $|M|$ is the determinant of the matrix $\|M\|$.

$$\|M\| = \text{Matrix}\begin{Vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{Vmatrix}$$

in which

$$d_{ij} = E\left(x_i x_j\right)$$

and the M_{ij}'s are the cofactors of the determinant. The coefficient $\rho(1)$ is the correlation coefficient between samples which are separated by one sample time (x_1 and x_2, x_{n+1} and x_{n+2}); $\rho(n - 1)$ is the correlation coefficient between samples which are separated by (n - 1) sample times (x_2 and x_{n+1}); $\rho(n)$ is the correlation coefficient between samples which are separated by n sample times (x_1 and x_{n+1}, x_2 and x_{n+2}); and $\rho(n + 1)$ is the correlation coefficient between samples which are separated by (n + 1) sample times (x_1 and x_{n+2}). Therefore,

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$$\|M\| = \left\| \begin{array}{cccc} \sigma^2 & \rho(1)\sigma^2 & \rho(n)\sigma^2 & \rho(n+1)\sigma^2 \\ \rho(1)\sigma^2 & \sigma^2 & \rho(n-1)\sigma^2 & \rho(n)\sigma^2 \\ \rho(n)\sigma^2 & \rho(n-1)\sigma^2 & \sigma^2 & \rho(1)\sigma^2 \\ \rho(n+1)\sigma^2 & \rho(n)\sigma^2 & \rho(1)\sigma^2 & \sigma^2 \end{array} \right\|$$

$$|M| = \sigma^8\Big\{\left[\rho(1)^2 - \rho(n)^2\right]^2 + \left[\rho(n-1)^2 - 1\right]\left[\rho(n+1)^2 - 1\right]$$
$$- 2\left[\rho(1)^2 + \rho(n)^2\right]\left[1 + \rho(n-1)\rho(n+1)\right] + 4\rho(1)\rho(n)\left[\rho(n-1) + \rho(n+1)\right]\Big\}$$

$$M_{11} = M_{44} = \sigma^6\left[1 + 2\rho(1)\rho(n-1)\rho(n) - \rho(1)^2 - \rho(n-1)^2 - \rho(n)^2\right]$$

$$M_{12} = M_{21} = M_{34} = M_{43} = \sigma^6\Big[\rho(n)\rho(n+1) + \rho(1)^3 + \rho(n-1)\rho(n) - \rho(1) - \rho(1)\rho(n)^2$$
$$- \rho(1)\rho(n-1)\rho(n+1)\Big]$$

$$M_{13} = M_{31} = M_{24} = M_{42} = \sigma^6\Big[\rho(1)\rho(n-1) + \rho(1)\rho(n+1) + \rho(n)^3 - \rho(n-1)\rho(n)\rho(n+1)$$
$$- \rho(1)^2\rho(n) - \rho(n)\Big]$$

$$M_{14} = M_{41} = \sigma^6\left[\rho(n-1)^2\rho(n+1) + 2\rho(1)\rho(n) - \rho(1)^2\rho(n-1) - \rho(n+1) - \rho(n-1)\rho(n)^2\right]$$

$$M_{23} = M_{32} = \sigma^6\left[\rho(n-1)\rho(n+1)^2 + 2\rho(1)\rho(n) - \rho(n-1) - \rho(n)^2\rho(n+1) - \rho(1)^2\rho(n+1)\right]$$

$$M_{22} = M_{33} = \sigma^6\left[1 + 2\rho(1)\rho(n)\rho(n+1) - \rho(n+1)^2 - \rho(1)^2 - \rho(n)^2\right]$$

$$p\left(x_1, x_2, x_{n+1}, x_{n+2}\right) = \frac{1}{(2\pi)^2\sqrt{|M|}} \exp\Big\{\frac{-1}{2|M|}\Big[M_{11}x_1^2 + M_{22}x_2^2 + M_{33}x_{n+1}^2 + M_{44}x_{n+2}^2$$
$$+ \left(M_{12} + M_{21}\right)x_1x_2 + \left(M_{13} + M_{31}\right)x_1x_{n+1} + \left(M_{14} + M_{41}\right)x_1x_{n+2}$$
$$+ \left(M_{23} + M_{32}\right)x_2x_{n+1} + \left(M_{24} + M_{42}\right)x_2x_{n+2} + \left(M_{34} + M_{43}\right)x_{n+1}x_{n+2}\Big]\Big\}$$

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Let $x_1 = x_2 - \Delta_1$. Then

$$p\left(\Delta_1, x_{n+1}, x_{n+2}\right) = \int_{-\infty}^{\infty} p\left(x_2 - \Delta_1, x_2, x_{n+1}, x_{n+2}\right) dx_2$$

Let $x_{n+1} = x_{n+2} - \Delta_{n+1}$. Then

$$p\left(\Delta_1, \Delta_{n+1}\right) = \int_{-\infty}^{\infty} p\left(\Delta_1, x_{n+2} - \Delta_{n+1}, x_{n+2}\right) dx_{n+2}$$

$$p\left(\Delta_1, \Delta_{n+1}\right) = \frac{\sqrt{|M|}}{\pi\sqrt{4H_1^2 - H_3^2}} \exp\left\{\frac{-1}{2|M|\left(4H_1^2 - H_3^2\right)}\left[H_5\left(4H_1^2 - H_3^2\right) - H_2^2 H_1 - H_1 H_4^2 + H_2 H_3 H_4\right]\right\}$$

where

$$H_1 = M_{11} + 2M_{12} + M_{22}$$

$$H_2 = 2\Delta_1\left[M_{11} + M_{12}\right] + 2\Delta_{n+1}\left[M_{13} + M_{23}\right]$$

$$H_3 = 2\left[2M_{13} + M_{14} + M_{23}\right]$$

$$H_4 = 2\Delta_1\left[M_{13} + M_{14}\right] + 2\Delta_{n+1}\left[M_{33} + M_{34}\right]$$

$$H_5 = M_{11}\Delta_1^2 + 2M_{13}\Delta_1\Delta_{n+1} + M_{33}\Delta_{n+1}^2$$

Let $\Delta_1 = \Delta_{n+1} - U_n$. Then,

$$p\left(U_n\right) = \int_{-\infty}^{\infty} p\left(\Delta_{n+1} - U_n, \Delta_{n+1}\right) d\Delta_{n+1}$$

$$p\left(U_n\right) = \frac{\sqrt{2}|M|}{\sqrt{\pi}\sqrt{C_1 + C_2 + C_3}} \exp\left\{\frac{-1}{8|M|\left(4H_1^2 - H_3^2\right)}\left(\frac{4C_1C_3 - C_2^2}{C_1 + C_2 + C_3}\right)U_n^2\right\}$$

where

$$C_1 = M_{11}\left(4H_1^{\,2} - H_3^{\,2}\right) - 4H_1\left[\left(M_{11} + M_{12}\right)^2 + \left(M_{13} + M_{14}\right)^2\right] + 4H_3\left(M_{11} + M_{12}\right)\left(M_{13} + M_{14}\right)$$

$$C_2 = 2M_{13}\left(4H_1^{\,2} - H_3^{\,2}\right) - 8H_1\left[\left(M_{11} + M_{12}\right)\left(M_{13} + M_{23}\right) + \left(M_{13} + M_{14}\right)\left(M_{12} + M_{22}\right)\right]$$
$$+ 4H_3\left[\left(M_{11} + M_{12}\right)\left(M_{12} + M_{22}\right) + \left(M_{13} + M_{23}\right)\left(M_{13} + M_{14}\right)\right]$$

$$C_3 = M_{22}\left(4H_1^{\,2} - H_3^{\,2}\right) - 4H_1\left[\left(M_{13} + M_{23}\right)^2 + \left(M_{12} + M_{22}\right)^2\right] + 4H_3\left(M_{13} + M_{23}\right)\left(M_{12} + M_{22}\right)$$

From the previous equation for $p(U_n)$, U_n can be observed to have a Gaussian probability density with a mean of zero and a variance of

$$\sigma_{U_n}^2 = \frac{4|M|\left(C_1 + C_2 + C_3\right)\left(4H_1^{\,2} - H_3^{\,2}\right)}{4C_1C_3 - C_2^{\,2}}$$

$$= \frac{C_1 + C_2 + C_3}{4|M|^2}$$

The two expressions for the variance result from the fact that the variance appears in two places in the equation for the normal probability density function.

APPENDIX C

A RANDOM WALK PROBLEM WITH ABSORBING BOUNDARIES

In order to find the average number of steps required to reach either the plus or minus boundary in the first-order predictor, a random walk problem with absorbing boundaries (ref. 5) is solved. The expected value of Z_n, the difference between the actual value of the sample and the predicted value, is

$$E\left(Z_n\right) = nE(U)$$

if

$$Z_n = \sum_{i=1}^{n} U_i$$

At the time when the data reach one of the boundaries, $Z_n = -L_{12}$ or $Z_n = L_{11}$, let q equal the probability that $Z_n \leqq -L_{12}$; therefore, $(1 - q)$ is the probability that $Z_n \geqq L_{11}$.

Let $\psi_Z(t)$ be the moment-generating function of Z_n (ref. 6).

$$\psi_Z(t) = E\left(e^{tZ_n}\right) = \sum_j p_j e^{tZ_j}$$

Assume that the length of the step is small compared with the distance to the boundary. This assumption, in effect, means that the first sample to occur outside the boundary will occur exactly on the boundary and that when the boundary is crossed, Z_n can have one of two values, L_{11} or $-L_{12}$. Since by definition (ref. 6),

$$E\left(x^2\right) = \sum_{k=1}^{n} p_k x_k^{\,2}$$

$$E\left(Z_n^{\,2}\right) = L_{12}^2 q + L_{11}^2 [1 - q] \tag{C1}$$

Select $t = t_\omega$ so that

$$E\left(e^{t_\omega Z}\right) = 1$$

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Because of the assumption that Z_n can be either L_{11} or $-L_{12}$ at the boundaries,

$$qe^{-L_{12}t_\omega} + (1 - q)e^{L_{11}t_\omega} = 1$$

$$q = \frac{e^{(L_{11}+L_{12})t_\omega} - e^{L_{12}t_\omega}}{e^{(L_{11}+L_{12})t_\omega} - 1}$$

For $t_\omega = 0$, this equation is indeterminate and must be solved by using L'Hospital's rule.

$$q = \lim_{t_\omega \to 0} \frac{(L_{11} + L_{12})e^{(L_{11}+L_{12})t_\omega} - L_{12}e^{L_{12}t_\omega}}{(L_{11} + L_{12})e^{(L_{11}+L_{12})t_\omega}}$$

$$q = \frac{L_{11} + L_{12} - L_{12}}{L_{11} + L_{12}} = \frac{L_{11}}{L_{11} + L_{12}} \tag{C2}$$

Inserting this expression for q (eq. (C2)) into the equation for $E\left(Z_n^{\;2}\right)$ (eq. (C1)) yields

$$E\left(Z_n^{\;2}\right) = L_{11}L_{12}$$

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