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ROTATING EMISSION RING IN BINARY SYSTEMS

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ABSTRACT

The periodic orbits around the more massive component in the restricted three-body problem have been taken as representing quantitatively the rotating gaseous rings discovered through their emission lines in many eclipsing binaries. The periodic orbits have been obtained first through a series expansion and further improved by the successive approximation. Thus, a theoretical relation, in a tabulated form, has been obtained between the velocity of the emission ring and its distance from the more massive component for each of six values of the mass ratio of the two component stars.

By using this theoretical relation, the masses of two components in the binary system may be determined by observing the emission ring during eclipse. As an illustration the masses of the two components of RY Gem have been found.

Applying the same idea of the periodic orbits, we have also examined the nature of non-synchronous rotation in U Cep and RZ Sct.

I. INTERPRETATION OF THE ROTATING RINGS IN TERMS OF PERIODIC ORBITS

Since the discovery by Joy (1942, 1947) of the gaseous emission ring around the primary component in RW Tau, it has been found that ring occurrence is a common phenomenon in binaries of the Algol type (e.g., Sahade 1960). While its formation in such binaries can only be qualitatively understood (Huang 1957) because of the mathematical difficulty intrinsic to such a problem, its behavior, once it has been already formed, can be represented in a quantitative manner.

The simplest representation of the gaseous ring in the binary system is by the classical two-body problem. The particles in the ring may be treated as moving in the gravitational field of the primary component alone. This is obviously an over-simplification because of the neglect of the secondary component. On the other hand, a realistic representation of this problem from the point of view of hydrodynamic flow turns out to be very difficult. The actual attempts to solve the problem in this way have been confined only to the case where the pressure is neglected. In other words the hydrodynamic approach so far has taken into consideration only collisions. Even with this simplification, the significance of the result remains doubtful. (Prendergast 1960, Huang 1965). Hereby we suggest a compromise method of mathematical representation which, though not as general as a hydrodynamic solution, takes the effect of both components into consideration. As we shall see, this method will not be invalidated by the collisions of particles as long as the pressure term can be neglected. Therefore it is equivalent to the hydrodynamic approach without the pressure term.

In order to see this point let us consider the restricted three-body problem. The calculation of the paths of a test particle in a binary system under different initial conditions by the aid of the restricted three-body problem has been performed previously by several investigators (Gould 1959, Kopal 1959). There are indications that particles ejected from the less massive component have the tendency to circle the more massive component if the mass ratio of the two component stars and the initial conditions are favorable. However the present method differs from the previous calculations by pointing out the function of periodic orbits that exist for the third body around each component of the system. These periodic orbits are very nearly circular and we may take the gaseous rings as following their paths. Such a view is, of course, not new. It has been pointed out before that the restricted three-body problem bears a formal analogy to the steady flow of an incompressible fluid in which certain streamlines correspond to periodic orbits (e.g. Szebehely 1961). According to this point of view the necessary condition for the existence of gaseous rings is that the periodic orbits are stable. If they are stable, a slight change in its velocity at any point will not disturb the particle far from the periodic one. In other words, in the neighborhood of each stable orbit there exist an infinite number of nearly periodic orbits, none of which deviates drastically from the periodic one. We may then visualize the motion of individual particles in the rotating gaseous ring as each following one of these nearly periodic orbits. As a result of collisions among themselves, each particle shifts from one nearly periodic orbit to another. Taken as a whole the rotating ring keeps its stationary appearance at least for a short time scale, say for a period. Here we can see how the present method includes the collisions of particles into consideration. On the other hand, if the periodic orbit is unstable, the ring can never be maintained because particles will depart from the periodic solution rapidly as soon as they are slightly deviated from the exact periodic solution.

Such a view of the rotating gaseous ring in terms of the stability of periodic orbits in the restricted three-body problem has its advantage in two respects. In the first place, the necessary condition for the ring formation can be mathematically treated because the problem of stability of the periodic orbit in the restricted three-body problem has been formulated. The early method introduced by Hill (1886) and Darwin (1897) for the stability criterion is tedious because it involves the calculation of an infinite matrix. However, recently simpler numerical methods (Huang 1963, Deprit and Price 1965, Hénon 1965) have been devised. The method suggested by Huang and that by Deprit and Price are identical. For periodic orbits in the plane of symmetry the stability criterion is reduced to the investigation of a fourth order matrix. The second advantage which has an immediate application to empirical data is that the rotational velocities of the gaseous ring may now be realistically evaluated, since the periodic orbits in the restricted three-body problem can be determined either analytically as shown in the following section or by the method of successive approximations.

II. A SERIES SOLUTION FOR PERIODIC ORBITS

A series solution for the periodic orbits may be obtained by the usual technique. Let us take the total mass of the two finite bodies as the unit of mass, their separation the unit of length and $P/(2\pi)$ the unit of time where P represents the orbital period of the two finite bodies. If we choose the pole to be located at the more massive body, the equation of motion of the third (infinitesimal) body in the restricted three-body problem in the orbital plane may be written in the rotating polar coordinate system as

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - 2r\frac{d\theta}{dt} = r - \frac{1-\mu}{\rho_1^2} - \frac{\mu r}{\rho_2^3} + \mu\left(\frac{1}{\rho_2^3} - 1\right)\cos\theta, \tag{1}$$

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} + 2\frac{dr}{dt} = -\mu\left(\frac{1}{\rho_2^3} - 1\right)\sin\theta, \tag{2}$$

where (r , θ) denotes the polar coordinates of the third body, and ρ_1 and ρ_2 respectively the distances of the infinitesimal body from the two finite bodies. (Obviously $r \equiv \rho_1$.) The polar axis which rotates with the two finite bodies is directed from the more massive body (of mass $1-\mu$) to the less massive body (of mass μ).

When μ equals zero, it becomes a two-body problem. Thus, all circles with their center at the origin are the periodic solutions of the problem. We can now argue that if μ is small, the periodic orbit deviates only slightly from the circular one. The deviation obviously depends upon μ . Therefore, if the periodic solutions exist, they may be represented by

$$r = r_0 + \mu r_1(t) + \mu^2 r_2(t) + \cdots , \qquad (3)$$

$$\theta = \lambda t + \mu \theta_1(t) + \mu^2 \theta_2(t) + \cdots , \qquad (4)$$

where r_0 and λ are to be determined and are independent of time (Huang 1964). Substitution of equations (3) and (4) into equations (1) and (2) gives equations of perturbation in different orders of approximation.

The zeroth order (μ^0) approximation gives

$$(\lambda+1)^2 = \frac{1-\mu}{r_0^3} , \qquad (5)$$

which is simply Kepler's third law in the problem of two bodies, namely the $1 - \mu$ component and the third, infinitesimal body. The first term includes $\lambda+1$ instead of λ because the equations are expressed in the rotating coordinate system. Thus, λ may have positive or negative values, corresponding respectively to the direct and retrograde motion of the third body.

The first order (μ^1) approximation yields

$$\frac{d^2r_1}{dt^2} - 2(\lambda+1)r_0\frac{d\theta_1}{dt} - 3(\lambda+1)^2 r_1 = \frac{1}{2}r_0 + \frac{3}{2}r_0\cos 2\lambda t + \frac{1}{8}r_0^2(9\cos\lambda t + 15\cos 3\lambda t), \tag{6}$$

$$r_0\frac{d^2\theta_1}{dt^2} + 2(\lambda+1)\frac{dr_1}{dt} = -\frac{3}{2}r_0\sin 2\lambda t - \frac{3}{8}r_0^2(\sin\lambda t + 5\sin 3\lambda t), \tag{7}$$

if the terms involving third and higher orders of r_0 are neglected.

With the same degree of approximation as regards to the series in r_0 , the second order (μ^2) equations are:

$$\begin{aligned}\frac{d^2r_2}{dt^2} &- 2(\lambda+1)r_0\frac{d\theta_2}{dt} - 3(\lambda+1)^2 r_2 \\ &= r_0\left(\frac{d\theta_1}{dt}\right)^2 + 2(\lambda+1)r_1\frac{d\theta_1}{dt} - 3(\lambda+1)^2\frac{r_1^2}{r_0} \\ &+ \frac{1}{2}r_1(1+3\cos 2\lambda t) + \frac{1}{4}r_0 r_1(9\cos\lambda t + 15\cos 3\lambda t) \\ &- 3r_0\theta_1\sin 2\lambda t - \frac{r_0^2\theta_1}{8}(45\sin 3\lambda t + 9\sin\lambda t),\end{aligned} \tag{8}$$

and

$$r_0 \frac{d^2\theta_2}{dt^2} + 2(\lambda+1)\frac{dr_2}{dt}$$

$$= -2\frac{dr_1}{dt}\frac{d\theta_1}{dt} - r_1\frac{d^2\theta_1}{dt^2} - \frac{3}{2}r_1 \sin 2\lambda t - 3r_0\theta_1 \cos 2\lambda t$$

$$- \frac{1}{4} r_0 r_1 (3 \sin \lambda t + 15 \sin 3\lambda t) \tag{9}$$

$$- \frac{1}{8} r_0^2 \theta_1 (3 \cos \lambda t + 45 \cos 3\lambda t).$$

In a similar way, equations of perturbation in higher orders can be derived in terms of the solutions of the equations of lower orders.

The solutions of equations (6) and (7) can be easily found:

$$r_1 = -\frac{r_0}{6(\lambda+1)^2} + A_1' \cos(\lambda+1)t + A_2' \sin(\lambda+1)t$$

$$- \frac{2B_1'}{3(\lambda+1)} + \sum_{n=1}^{3} k_n \cos n\lambda t, \tag{10}$$

$$r_0\theta_1 = B_1' t + B_2' - 2A_1' \sin(\lambda+1)t + 2A_2' \cos(\lambda+1)t$$

$$+ \sum_{n=1}^{3} l_n \sin n\lambda t, \tag{11}$$

where k_n and l_n are defined by

$$k_1 = \frac{15\lambda+6}{8\lambda(2\lambda+1)} r_0^2 , \qquad l_1 = - \frac{3(10\lambda^2+12\lambda+3)}{8\lambda^2(2\lambda+1)} ,$$

$$k_2 = - \frac{3(2\lambda+1)}{2\lambda(3\lambda+1)(\lambda-1)} r_0 , \qquad l_2 = \frac{3(11\lambda^2+10\lambda+3)}{8\lambda^2(3\lambda+1)(\lambda-1)} r_0 , \tag{12}$$

$$k_3 = - \frac{5(5\lambda+2)}{8\lambda(4\lambda+1)(2\lambda-1)} r_0^2 , \qquad l_3 = \frac{5(6\lambda^2+4\lambda+1)}{8\lambda^2(4\lambda+1)(2\lambda-1)} r_0^2 .$$

It should be noted that λ (or equivalently, r_0) is an integration constant in the unperturbed case. It defines as well as labels the circular orbits, each of which is used as the reference orbit for the perturbed calculation. In other words, the solutions for different orders of perturbation represent small variations from the circular orbit which is fixed by assigning a definite value to λ in each case. This is the basic assumption involved in the present method. It then becomes obvious that the presence of the B_1' term in the solution of the perturbed equation is incompatible with this assumption and consequently B_1' must vanish, because $B_1' t$, which increases with t , cannot be kept small indefinitely. Hence, although the four arbitrary constants in the solution of the differential equations (6)-(7) are A_1' , A_2' , B_1' , and B_2' , the four arbitrary constants in the solution of equations (1)-(2) are λ , A_1' , A_2' , and B_2' .

The previous argument may be stated in another way. Let us introduce λ' defined by

$$\lambda' = \lambda + \frac{B_1'\mu}{n_0} \tag{13}$$

Clearly λ' denotes the mean angular velocity in the perturbed case and therefore is an integration constant for the solution of equations (1)-(2). But we can combine λ' and $B_1'\mu/n_0$ to form a single constant λ according to equations (13) leaving in total four integration constants λ, A_1', A_2', and B_2' in the final solution.

In general, if higher orders of μ are considered, the integration constants will be λ, A_1, A_2, and B_2 defined by

$$A_1 = A'\mu + A''\mu^2 + \cdots \quad , \text{etc.,}$$

because, as we can see easily, the complementary functions in different orders of approximation are of the same form. However, the problem of convergence of the series has not been investigated.

The general solution does not give the periodic orbits because there are two fundamental periods, $2\pi/(\lambda + 1)$ and $2\pi/\lambda$, with several harmonics of the latter. However, since the existence of some periodic solutions has been proved, these periodic orbits must correspond to the particular integral obtained from equations (10) and (11) by setting $A_1' = A_2' = B_1' = B_2' = 0$. Then the solutions contain only terms with period $2\pi/\lambda$ of the fundamental oscillation, and shorter periods corresponding to its harmonics. Thus, the periodic orbits around the $1 - \mu$ component may be given, to the first order of μ and the second order of r_0, by

$$r = r_0 + \mu r_1 \quad \text{and} \quad \theta = \lambda t + \mu \theta_1 \tag{14}$$

where

$$r_1 = -\frac{r_0}{6(\lambda+1)^2} + \sum_{n=1}^{3} k_n \cos n\lambda t, \tag{15}$$

$$r_0 \theta_1 = \sum_{n=1}^{3} l_n \sin n\lambda t. \tag{16}$$

It follows from the solution given by equations (10) and (11) that, in general, a periodic solution can be obtained for any given value of λ only by setting A_1, A_2, B_1, and B_2 equal to zero. Thus, one periodic orbit is associated with one value of the period. However, if λ, and consequently $\lambda/(\lambda + 1)$ are ratios of two integers, arbitrary (small) values for the other three integration constants may all lead to periodic orbits. These commeasurable orbits are perhaps unstable. For example, the coefficients defined by equations (12) diverge when $\lambda = 1$, $\lambda = \pm \frac{1}{2}$, etc. This is credible because both the asteroids and Saturn's rings show gaps due to commeasurability. However this divergence could be introduced by the particular way of expansion of the solution. If so, this would indicate that the present method cannot be applied to the commeasurable periodic orbits.

When the solutions of r_1 and $r_0 \theta_1$ given by equations (15) and (16) are substituted into the second order equations (8) and (9) and when the resulting equations are simplified, we obtain:

$$\frac{d^2 r_2}{dt^2} - 2(\lambda+1) r_0 \frac{d\theta_2}{dt} - 3(\lambda+1)^2 r_2 = \beta_0 + \sum_{n=1} \beta_n \cos n\lambda t, \qquad (17)$$

$$r_0 \frac{d^2\theta_2}{dt^2} + 2(\lambda+1)\frac{dr_2}{dt} = \sum_{n=1} \gamma_n \sin n\lambda t. \qquad (18)$$

These equations have the same form as equations (6) and (7), except for more terms on the righthand side. Therefore, the solution can be derived in the same manner as in the case of the first order equations, although finding the explicit expressions of the solutions is much more tedious because of the lengthy equations that define β_n and γ_n.

III. RADIUS AND VELOCITY RELATION

Under the approximation by the two-body problem, the velocity of gaseous particles revolving around the more massive component is related to the radius of the circular ring by the simple relation

$$v = \left(\frac{1-\mu}{\rho}\right)^{1/2} \tag{19}$$

in the adopted unit system. When the effect of the μ component is considered, the periodic orbit is not exactly circular (although it is nearly so) and the speed of the particle moving in any periodic orbit is no longer constant. However, observationally we observe the radius and the velocity of the rotating gaseous ring only at the time of principal eclipse. Therefore, we can find a relation between the distance from the central star (i.e., ρ) and the velocity v at the time of principal eclipse. Obviously the relation involves μ as a parameter.

We have first applied the result of the previous section to obtain the periodic orbits around the $1-\mu$ component. For large orbits, a method of successive approximation

(Huang and Wade 1963) has to be used in order to improve the analytical result which is only approximate. Thus six series of periodic orbits around the $1-\mu$ component have been derived in the plane of symmetry. In Table 1 we have expressed the initial conditions in the rectangular coordinate (x , y) system defined by $x = r\cos\theta - \mu$, $y = r\sin\theta$, i.e., the origin of the (x , y) system is at the center of mass of the two finite bodies. In all cases, the initial values of y and $\dot{x}$ are zero. Hence, only the initial values of x and $\dot{y}$ are given.

Once the periodic orbits have been found, we can proceed to determine the values of y and of $\dot{x}$ at the points where $\dot{y}$ = 0. These values are given respectively in the first and second columns of Table 2 again for six series corresponding to six values of μ . They should represent the radius and the velocity of the gaseous ring observed during principal eclipse. It may be noted that there are two points in each periodic orbit where $\dot{y}$ = 0 with both y and $\dot{x}$ in opposite signs. But the magnitudes of y and $\dot{x}$ calculated at these two points are identical within the accuracy of the figures given in the table. As a comparison we list in the last column of Table 2, the circular velocity obtained by setting $r = y$ in equation (19).

As we have pointed out before, gaseous rings are most likely formed in binaries with $\mu \leq 0.3$. At the same time binaries with $\mu < 0.1$ have little chance to have the mass ratio accurately determined by observation. Therefore, the table covers the range of μ that has a practical interest.

Periodic orbits exist also around the less massive component of a binary system. Therefore, based on this fact alone one would expect to find some gaseous ring revolving around the less massive component. Indeed, such is the case of Saturn's ring if we take the Sun and Saturn as a binary system. However, no emission rings have yet been found around the less massive component in eclipsing binaries. This could result from the observational selection. On the other hand, there are some intrinsic reasons that induce us to suggest that such a situation does not happen easily in close binaries (Huang 1965).

As can be seen from Table 2, a difference exists between the velocity of the gaseous ring calculated from periodic orbits in the three-body problem and that from equation (19) in the sense that the latter is systematically larger than the former. This difference measures the effect of the presence of the secondary component. Thus, it increases with r obviously because the farther away the ring from the primary component, the greater is the influence of the secondary component. It is equally obvious that the difference increases with μ for the same value of r .

The velocity, v , in Table 2 from the three-body calculation represents a function of two variables r and μ , i.e., $v = v(r, \mu)$. Although it cannot be written down analytically, an interpolation formula may be easily devised from the table. One interpolation formula may be of the form, say

$$v = \left(\frac{1-\mu}{r}\right)^{1/2} - (c_0 + c_1\mu)r - (c_2 + c_3\mu)r^2$$

with constants c_0 , c_1 , c_2 , and c_3 to be determined by the tabulated values.

IV. DETERMINATION OF THE MASS RATIO FROM THE RADIUS-VELOCITY RELATION

Since the radius-velocity relations of the gaseous emission ring depends upon the mass ratio, μ, of the binary system, we may determine the mass-ratio from the observed values of the radius and the velocity of the ring. From a consideration of symmetry, the ring must lie in the orbital plane of the binary system. Its inclination being known, its velocity at the time of principal eclipse can then be obtained simply from the Doppler shifts of the emission line. The radius of the ring may be obtained from the time of eclipse of the emission line in the same way that the radii of the two component stars are determined from the duration of the principal and secondary eclipses. Unfortunately all the previous works on the emission line depend upon the spectrogram which takes a long time to expose. Consequently, in no case the radius of the gaseous ring has been determined accurately. What can be suggested for the future study is to derive a light curve for each one of the emission lines. Perhaps with the use of an image intensifying device, this can now be done.

The major problem in designing this kind of instruments is how to take into account the Doppler shift of the line such that the measured light intensity covers no more and no less than the entire emission line at any phase. Such an instrument will be useful not only in the present case of the gaseous emission ring but also for studying the surface activity of the star in the eclipsing binary system in more or less the same way that the spectroheliograph is useful for the study of the solar surface.

To return to the problem of mass determination we may recall that the velocities used in the previous calculation and given in Tables 1 and 2 are dimensionless. Thus, the actually observed velocity of the emission ring, V_{em}, at the time of principal eclipse is related to v of Table 2 by

$$V_{em} = \left[\frac{G(M_1 + M_2)}{a} \right]^{1/2} v \sin i \,, \tag{20}$$

where i is the inclination of the orbital plane, a is the semi-major axis of the relative orbit of the two components with masses M_1 and M_2, all expressed in the c.g.s. units. It can be easily obtained from equation (20) that

$$\frac{V_{em}}{K_1} = \frac{1}{\mu}(1-e^2)^{1/2} v(r,\mu) \,, \tag{21}$$

where K_1 denotes the semi-amplitude of the velocity curve of the primary component and e the eccentricity of the binary orbit. Equation (21) determines the mass ratio μ from the observable quantities V_{em} and n. As a suggestion for the practical procedure we may plot the curves V_{em}/K_1 versus n for different values of μ given in Table 2. Then the value of μ for any particular case may simply be read off from the diagram by finding the curve of μ on which the observed values, V_{em}/K_1 and n, fall on.

Once μ is known, the masses of both components can be easily derived from the mass function. Thus, as far as the determination of stellar masses is concerned, the observation of the gaseous ring is equivalent to the observation spectroscopically of the less massive component which is usually not seen outside the principal eclipse. However, since v is calculated from the restricted three-body problem, we can apply this method for determining stellar masses only for systems with small e. This does not mean a drawback of the present method because rotating rings are unlikely to be associated with binaries having large eccentricities where motion of the secondary component exerts a large disturbing effect on the stability of the gaseous ring.

As an example, we may apply the present method to determine the masses of RY Gem which has been studied by Wyse (1934) Gaposchkin (1946) and McKellar (1949). According to McKellar, $V_{em} = 200$ km/sec, $K_1 = 27.1$ km/sec, the mass function $= 0.018 M_{\odot}$, and the linear size of the ring is three times the radius of the primary component. The radius of the primary component has not been exactly determined from photometric data. Wyse gave a solution (that considers limb darkening) of 0.108 for the primary but Gaposchkin gave the same quantity two values --0.086 from photographic measures and 0.079 -- from visual measures. As a rough estimate we take the mean of these three values. This would give r = .273 for the gaseous ring. With V_{em}/K_1 = 7.38, we can easily find from the family of V_{em}/K_1 versus r curves that μ is about equal to 0.195. It then follows from the mass function that $M_1 = 2 M_\odot$ and $M_2 = 0.5 M_\odot$ if $i = 85^0$ as determined from the light curve used. As a comparison McKellar gives an estimate of 1.7 $M_\odot$ for M_1 . The results derived here are tentative since r is not accurately measured from the eclipse of the emission ring. Also, the stellar absorption line may be affected to a sufficient extent by the gaseous stream. This makes the semi-amplitude K_1 of the velocity curve somewhat uncertain. However, these are all observational uncertainties and do not reflect any flaw on the part of the basic principle of such a determination of stellar mass.

V. NON-SYNCHRONIZATION OF U CEP AND RZ SCT

It has been found observationally that axial rotation of close binary components are in general synchronized to their orbital motion. This result is indeed expected from theoretical considerations. However, there are a few close binaries where synchronization is not observed. In some cases the star rotates slower, and in other cases it rotates faster than expected from synchronism. Non-synchronization of the latter cases has been found in U Cep and RZ Sct (Struve 1944, 1963).

We have suggested that non-synchronization of U Cep and RZ Sct is due to the transport of angular momentum to the primary component by the ejected particles from the secondary component (Huang 1966). Two possibilities arise from this suggestion. First, the ejected particles may form a rotating ring that is so close to the primary such that we cannot observationally distinguish it from the reversing layer. Or, instead of developing a rotating ring, the gaseous particles fall into the atmosphere of the primary. The axial rotation of the atmosphere will then be accelerated by the falling particles of large angular momentum per unit mass. In either case we will see large rotational disturbances in the velocity curve and obtain the result that the primary rotates faster than the synchronized velocity.

In order to determine which one of the two possibilities corresponds to the true nature of the event we may first obtain the periodic orbits very close to the primary component. The initial conditions that lead to these periodic solutions are:
$x = -0.49$, $\dot{y} = -1.6985$ for U Cep with $\mu = 0.29$
and $x = -0.349$, $\dot{y} = -1.7345$ for RZ Sct with $\mu = 0.119$.
(In both cases $y = \dot{x} = 0$.) The periodic orbits corresponding to these initial conditions have the following properties. In the case of U Cep, the velocity magnitude, v, varies from 1.65 to 1.70 and its distance, r, to the center of the primary varies from 0.200 to 0.204 as compared with 0.20 for the photometric radius of the primary. In the case of RZ Sct, v and r range respectively between 1.71 and 1.73, and

between 0.230 and 0.232, while the photometric radius of the primary is 0.23. In either case, the orbit is very nearly circular and close to the primary's surface. Therefore, if a gaseous ring is indeed formed close to the primary, its rotational velocity must be given by that of the periodic orbit. Since the computed range of rotational velocities at different points in the orbit is narrow, and since the observational value is not accurate, we may simply take the mean of the computed range for the purpose of comparing it with observation. Converted to the conventional units the rotational velocity of the gaseous ring close to the primary of U Cep should be about 470 km/sec, while the actual value determined from the disturbance of the velocity curve amounts to 300 km/sec. For RZ Sct the rotational velocity of a gaseous ring close to the primary would be 340 km/sec, while the observed value is 200 km/sec at most. Thus it appears that what produces rotational disturbance in the velocity curve could not be due to gaseous ring located close to the stellar surface. In other words the other possibility that the falling particles drive the atmosphere of the primary to rapid rotation may be what actually happens to these two binaries.

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TABLE 1

INITIAL CONDITIONS THAT LEAD TO PERIODIC ORBITS AROUND THE MORE MASSIVE COMPONENT

($y = \dot{x} = 0$ For All Cases)

$x+\mu$	$\dot{y}$					
	$\mu = 0.1$	$\mu = 1/7$	$\mu = 1/6$	$\mu = 0.2$	$\mu = 0.25$	$\mu = 0.3$
-.10	-2.9009	-2.8289	-2.7882	-2.7315	-2.6408	-2.5484
-.15	-2.3016	-2.2435	-2.2106	-2.1638	-2.0917	-2.0173
-.20	-1.9255	-1.8764	-1.8486	-1.8091	-1.7484	-1.6858
-.25	-1.6549	-1.6129	-1.5893	-1.5557	-1.5046	-1.4527
-.30	-1.4449	-1.4099	-1.3905	-1.3634	-1.3237	-1.2870
-.35	-1.2752	-1.2490	-1.2355	-1.2183	-1.1957	-1.1612
-.40	-1.1376	-1.1257	-1.1189	-1.1019	-1.0585	-1.0020
-.45	-1.0350	-1.0226	-1.0049	-0.9733	-0.9180	-0.8571

TABLE 2

VELOCITIES, v, OF GASEOUS RINGS AT TIME OF ECLIPSE

$r = y$	From three-body problem	From equation (19)	$r = y$	From three-body problem	From equation (19)
	μ = 0.1			μ = 0.2	
.10006	2.898	2.999	.10022	2.724	2.825
.15028	2.295	2.447	.15063	2.151	2.305
.20092	1.9122	2.116	.2021	1.7799	1.9895
.2524	1.6292	1.8882	.2559	1.4976	1.7681
.3058	1.3979	1.7155	.3152	1.2504	1.5932
.3634	1.1897	1.5738	.3901	.9876	1.4321
.4316	.9745	1.4440	.4876	.6951	1.2809
.5329	.6899	1.2996	.5579	.5121	1.1975
	μ = 1/7			μ = 0.25	
.10009	2.826	2.926	.10017	2.635	2.736
.15042	2.234	2.387	.15085	2.075	2.230
.2014	1.8566	2.063	.2029	1.7101	1.9227
.2538	1.5744	1.8379	.2582	1.4264	1.7042
.3092	1.3378	1.6649	.3226	1.1638	1.5248
.3722	1.1125	1.5175	.4139	.8513	1.3462
.4564	.8501	1.3704	.4903	.6341	1.2369
.5646	.5582	1.2321	.5350	.5127	1.1841

Legends

Fig. 1 - Functions $\Phi_1(y)$ and $\Phi_3(y)$. The function $\Phi_1(y)$, which represents the distribution of projected rotational velocities of stars if their equatorial rotational velocities follow the Maxwellian distribution, is shown by the curve labeled by numeral 1. The rest of the curves are cases of $\Phi_3(y)$ which shows the effect of the time and other factors on the distribution discussed in Section II. The numerals accompanying the respective curves denote λ_1. Because the scale of rotational velocities in the statistical model is not fixed, the actual curves may be compressed or stretched along the abscissa with a consequent change in the ordinate in order to maintain the normalization. In consequence of this degree of uncertainty, the intrinsic difference among these curves is too slight and is not expected to be detected by observation.

Fig. 2 - Function $\Phi_2(y)$. The curves represent different degrees of braking for the case that the rate of decrease in the spin angular momentum of a star at any time is proportional to the square of the spin angular momentum. The labeled value, y_2, measures the inverse of the braking strength.