THE EVOLUTION OF WAVE CORRELATIONS IN UNIFORMLY TURBULENT, WEAKLY NONLINEAR SYSTEMS

Ronald C. Davidson

Department of Physics, University of California, Berkeley, California

ABSTRACT

A formalism is developed describing the time evolution of wave correlations in a uniformly turbulent ensemble of weakly nonlinear systems. The statistics are built into the formalism a priori. With closure (of the hierarchy of equations for wave correlations) appropriate to the inclusion of resonant three-wave interactions, a (nonlinear) kinetic equation for the two-wave correlations is derived and various properties of this equation are discussed. The effects of both bilinear and trilinear nonlinearities are considered. Modifications of the formalism in situations where there is a weak (linear) instability γ_k, $(|\gamma_k/\omega_k| \ll 1)$ are also considered.

January, 1967

I: INTRODUCTION

In recent years, considerable research[1-8] has been devoted to the study of the time behavior of weakly nonlinear systems evolving according to equations of the form

$$i \frac{\partial}{\partial t} \underset{\sim}{\Psi}(\underset{\sim}{x},t) = \mathcal{L}\, \underset{\sim}{\Psi}(\underset{\sim}{x},t) + \underset{\sim}{B}\{\underset{\sim}{\Psi}(\underset{\sim}{x},t),\underset{\sim}{\Psi}(\underset{\sim}{x},t)\} \quad . \tag{1}$$

Depending on the specific problem, $\underset{\sim}{\Psi}(\underset{\sim}{x},t)$ is a finite dimension column vector whose elements are the physical quantities of interest; $\mathcal{L}$ is a linear differential operator with respect to the position variable $\underset{\sim}{x}$, and $\underset{\sim}{B}$ is a bilinear differential operator with respect to $\underset{\sim}{x}$.

Dynamical equations of this form arise in various areas of research. For example Galeev and Karpman[2] utilize the magnetohydrodynamic equations in the description of a low density plasma in the presence of a strong magnetic field. In this case $\underset{\sim}{\Psi}(\underset{\sim}{x},t)$ consists of seven elements, namely, the perturbations in fluid density, fluid velocity and magnetic field, from a spatially homogeneous, velocity free equilibrium in which there is a uniform magnetic field, $\underset{\sim}{B}_0$. In a model relevant to the description of long wavelength phenomena in anharmonic crystals, or the description of shallow water waves, Zabusky and Kruskal[7] make a numerical study of the Korteweg-deVries equation, which is of a form similar to Eq. (1). Hasselman,[4,5] in investigating the nonlinear interactions of gravity waves in a fluid of constant depth, uses a hydrodynamic description assuming incompressible, irrotational motion and that pressure effects are negligible. In this model $\underset{\sim}{\Psi}(\underset{\sim}{x},t)$ consists of the velocity potential and the displacement of the free surface. As a final example, Litvak[6] makes use of a plasma model consisting of the Maxwell equations and two-fluid equations of continuity and momentum transfer. This system of equations may also be cast into the form of Eq. (1). In this case there are twelve physical quantities of interest and hence as many elements of $\underset{\sim}{\Psi}(\underset{\sim}{x},t)$. The operators $\mathcal{L}$ and $\underset{\sim}{B}$ occurring in the above examples are of varying degrees of complexity depending on the explicit physical problem being investigated.

The purpose of this article is to develop a formalism describing spatially uniform turbulence within the context of the general dynamical equation, Eq. (1). The aforementioned examples are to be kept in mind as specific applications. In order that the problem be mathematically tractable, the turbulence is assumed weak, corresponding to a small amplitude analysis of Eq. (1).

In Section II, a Fourier analysis of Eq. (1) with respect to the position variable is carried out; the resulting equation is then rewritten in a representation in which the basis vectors are solutions to the linear equation ($\underset{\sim}{B} \equiv 0$). For the purpose of introductory remarks, we record at this point the equation for the time evolution of the wave amplitude associated with the αth mode, $A_\alpha(\underset{\sim}{k},t)$, in this representation, i.e.,

$$\frac{\partial A_\alpha}{\partial t}(\underset{\sim}{k},t) = \sum_{\alpha_1\alpha_2}\int d\underset{\sim}{k}_1\, \mathcal{K}^{\alpha\alpha_1\alpha_2}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}-\underset{\sim}{k}_1)A_{\alpha_1}(\underset{\sim}{k}_1,t)A_{\alpha_2}(\underset{\sim}{k}-\underset{\sim}{k}_1)\exp\{i(\omega_\alpha(\underset{\sim}{k})-\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_2}(\underset{\sim}{k}-\underset{\sim}{k}_1))t\} \tag{2}$$

The kernel, $\mathcal{K}$, depends on the details of the bilinear operator, $\underset{\sim}{B}$ and $\{\omega_\alpha(\underset{\sim}{k})\}$ are the eigenvalues of the linear ($\underset{\sim}{B} \equiv 0$) equation in Fourier space. It is assumed in Sections II-V that these eigenvalues are real. It is clear that a small amplitude theory of Eq. (2) yields no change in A_α in lowest order; in higher orders, however, the nonlinear terms act as perturbations which, through the interaction between waves of differing wave number, causes A_α to change in the course of time. The general procedure for treating problems of weak turbulence in relation to equations of the above form has traditionally been to obtain a perturbation solution to some order;[1-6] then, by performing appropriate statistical averages over a spatially homogeneous ensemble a posteriori, a kinetic equation is deduced, describing the time evolution of the correlation between two waves. This may be obtained by considering suitable (average) transition probabilities per unit time as in Ref. 2. Alternatively, a more elegant treatment (of the particular problem in which there are only two modes, $\omega_{\alpha_1}(k) = \omega(k)$ and $\omega_{\alpha_2}(k) = -\omega(k)$) has been given by Benney and Saffman[1] who carry out a multiple-time perturbation analysis of Eq. (2). In all cases, however, Refs. 1-6 perform appropriate statistical averages after obtaining the perturbation solution to Eq. (2) to some order. These approaches, by their very nature are somewhat cumbersome since the perturbation solution to Eq. (2) entails more information than necessary to describe the ensemble. A far more direct approach would be to work within a framework in which the statistical averaging has been carried out a priori.[8]

This simple concept of "a priori statistics" is applied in Section III, with Eq. (2) as the dynamical equation for an individual system. There results a hierarchy of equations describing the evolution of wave correlations characterizing the spatially homogeneous ensemble. The correlation between two waves is driven by the correlation between three waves, and in turn, the three-wave correlations are driven by four-wave correlations, and so on. In Section IV, with closure of this hierarchy appropriate to the description of resonant three-wave interactions, a simple multiple time analysis results in a kinetic equation for the correlation between two waves.

Several properties of this kinetic equation relating to the creation of correlations, invariants of the equation, multiple resonances, etc., are discussed in Section V. In Section VI, the modifications of the techniques and results of Sections III and IV are discussed for situations in which there is a weak instability (or weak damping). The effects of including a trilinear term, $\underset{\sim}{T}\{\underset{\sim}{\Psi}(\underset{\sim}{x},t),\underset{\sim}{\Psi}(\underset{\sim}{x},t),\underset{\sim}{\Psi}(\underset{\sim}{x},t)\}$, in Eq. (1), are discussed in Section VII.

II: THE BASIC EQUATION IN FOURIER SPACE

Fourier analyzing with respect to the position variable according to

$$\underset{\sim}{\Psi}(\underset{\sim}{x},t) = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \exp\{i\underset{\sim}{k}\cdot\underset{\sim}{x}\}\underset{\sim}{\Psi}(\underset{\sim}{k},t) ,$$

and

$$\underset{\sim}{\Psi}(\underset{\sim}{k},t) = \int d\underset{\sim}{x} \exp\{-i\underset{\sim}{k}\cdot\underset{\sim}{x}\}\underset{\sim}{\Psi}(\underset{\sim}{x},t) , \qquad (3)$$

Eq. (1) may be written in the form

$$i \frac{\partial}{\partial t} \underset{\sim}{\Psi}(\underset{\sim}{k},t) = \underset{\approx}{H}_0(\underset{\sim}{k})\cdot\underset{\sim}{\Psi}(\underset{\sim}{k},t)+\iint d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 \delta(\underset{\sim}{k}-\underset{\sim}{k}_1-\underset{\sim}{k}_2)\underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{\Psi}(\underset{\sim}{k}_1,t),\underset{\sim}{\Psi}(\underset{\sim}{k}_2,t)] . \qquad (4)$$

The matrix, $\underset{\approx}{H}_0(\underset{\sim}{k})$, and column vector $\underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{\Psi}(\underset{\sim}{k}_1,t),\underset{\sim}{\Psi}(\underset{\sim}{k}_2,t)]$ depend upon the structure of the operators $\mathcal{L}$ and $\underset{\sim}{B}_1$ respectively. In addition, $\underset{\sim}{H}_1$ is bilinear in its $\underset{\sim}{\Psi}$ arguments. Analogous to the techniques employed in time dependent perturbation theory in quantum mechanics, it is advantageous to rewrite Eq. (4) in a representation in which the basis vectors are solutions to the equation

$$i \frac{\partial}{\partial t} \underset{\sim}{u}(\underset{\sim}{k},t) = \underset{\approx}{H}_0(\underset{\sim}{k})\cdot\underset{\sim}{u}(\underset{\sim}{k},t) . \qquad (5)$$

With $\underset{\sim}{u}(\underset{\sim}{k},t) = \underset{\sim}{u}(\underset{\sim}{k})e^{-i\omega t}$, the eigenfrequencies of Eq. (5) are given by the characteristic equation

$$\mathrm{Det}(\omega\delta_{ij}-[H_0]_{ij}) = 0 . \qquad (6)$$

For purposes of the present discussion we limit ourselves to situations where the solutions to Eq. (6) are real. In situations where there is a weak instability, the analysis is a simple extension of the technique described here and is briefly discussed in Section VI. Denoting the solutions of Eq. (6) by $\{\omega_\alpha(\underset{\sim}{k})\}$, where α labels a particular eigenvalue, the corresponding eigenvectors $\{\underset{\sim}{u}_\alpha(\underset{\sim}{k})\}$, are determined from

$$\omega_\alpha(\underset{\sim}{k})\underset{\sim}{u}_\alpha(\underset{\sim}{k}) = \underset{\approx}{H}_0(\underset{\sim}{k})\cdot\underset{\sim}{u}_\alpha(\underset{\sim}{k}) \ . \tag{7}$$

At this point it is convenient to define a row vector $\underset{\sim}{u}^\beta(\underset{\sim}{k})$, adjoint to the column vector $\underset{\sim}{u}_\beta(\underset{\sim}{k})$, by the relation

$$\omega_\beta(\underset{\sim}{k})\underset{\sim}{u}^\beta(\underset{\sim}{k}) = \underset{\sim}{u}^\beta(\underset{\sim}{k})\cdot\underset{\approx}{H}_0(\underset{\sim}{k}) \ . \tag{8}$$

If $\underset{\approx}{H}_0$ is Hermitean, $\underset{\sim}{u}^\beta(\underset{\sim}{k})$ is simply the Hermitean adjoint of $\underset{\sim}{u}_\beta(\underset{\sim}{k})$.†

It follows from Eqs. (7) and (8) that

$$(\omega_\alpha(\underset{\sim}{k})-\omega_\beta(\underset{\sim}{k}))(\underset{\sim}{u}^\beta(\underset{\sim}{k})\cdot\underset{\sim}{u}_\alpha(\underset{\sim}{k})) = 0 \ , \tag{9}$$

and hence the orthogonality condition

$$\underset{\sim}{u}^\beta(\underset{\sim}{k})\cdot\underset{\sim}{u}_\alpha(\underset{\sim}{k}) = 0, \text{ for } \omega_\alpha(\underset{\sim}{k}) \neq \omega_\beta(\underset{\sim}{k}) \ . \tag{10}$$

In situations where there is a degeneracy associated with certain eigenvalues a simple variation of the Schmidt orthogonalization procedure may be carried out. In any case we assume that $\{\underset{\sim}{u}^\beta(\underset{\sim}{k})\}$ and $\{\underset{\sim}{u}_\alpha(\underset{\sim}{k})\}$ have been chosen orthogonal in such a way that

$$\underset{\sim}{u}^\beta(\underset{\sim}{k})\cdot\underset{\sim}{u}_\alpha(\underset{\sim}{k}) = \delta_{\alpha\beta} \ . \tag{11}$$

Writing

$$\begin{aligned}\underset{\sim}{\Psi}(\underset{\sim}{k},t) &= \sum_\alpha A_\alpha(\underset{\sim}{k},t)\underset{\sim}{u}_\alpha(\underset{\sim}{k},t) \\ &= \sum_\alpha A_\alpha(\underset{\sim}{k},t)\underset{\sim}{u}_\alpha(\underset{\sim}{k})\exp\{-i\omega_\alpha(\underset{\sim}{k})t\} \ ,\end{aligned} \tag{12}$$

Equation (4) becomes

$$i \sum_{\beta} \frac{\partial}{\partial t} A_{\beta}(\underset{\sim}{k},t)\underset{\sim}{u}_{\beta}(\underset{\sim}{k})\exp\{-i\omega_{\beta}(\underset{\sim}{k})t\}$$

$$= \sum_{\alpha_1,\alpha_2} \iint d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 A_{\alpha_1}(\underset{\sim}{k}_1,t) A_{\alpha_2}(\underset{\sim}{k}_2,t) \underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{u}_{\alpha_1}(\underset{\sim}{k}_1),\underset{\sim}{u}_{\alpha_2}(\underset{\sim}{k}_2)]$$

$$\times \exp\{-i(\omega_{\alpha_1}(\underset{\sim}{k}_1)+\omega_{\alpha_2}(\underset{\sim}{k}_2))t\}\delta(\underset{\sim}{k}-\underset{\sim}{k}_1-\underset{\sim}{k}_2) \ . \tag{13}$$

Taking the outer product of Eq. (13) with $\underset{\sim}{u}^{\alpha}(\underset{\sim}{k})$, and using the orthogonality condition given in Eq. (11), readily yields

$$\frac{\partial}{\partial t} A_{\alpha}(\underset{\sim}{k},t) = \sum_{\alpha_1,\alpha_2} \iint d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 \delta(\underset{\sim}{k}-\underset{\sim}{k}_1-\underset{\sim}{k}_2) A_{\alpha_1}(\underset{\sim}{k}_1,t) A_{\alpha_2}(\underset{\sim}{k}_2,t)$$

$$\times K^{\alpha\alpha_1\alpha_2}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2)\exp\{i(\omega_{\alpha}(\underset{\sim}{k})-\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_2}(\underset{\sim}{k}_2))t\} \tag{14}$$

where

$$K^{\alpha\alpha_1\alpha_2}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2) \equiv -i(\underset{\sim}{u}^{\alpha}(\underset{\sim}{k})\cdot\underset{\sim}{H}_1[\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{u}_{\alpha_1}(\underset{\sim}{k}_1),\underset{\sim}{u}_{\alpha_2}(\underset{\sim}{k}_2)]) \ . \tag{15}$$

To ensure the reality of $\underset{\sim}{\psi}(\underset{\sim}{x},t)$, we adopt the conventions

$$\underset{\sim}{u}_{\alpha}(-\underset{\sim}{k}) = \underset{\sim}{u}_{\alpha}(\underset{\sim}{k})^{*} \ , \tag{16}$$

$$A_{\alpha}(-\underset{\sim}{k}) = A_{\alpha}(\underset{\sim}{k})^{*} \ , \tag{17}$$

and

$$\omega_{\alpha}(-\underset{\sim}{k}) = -\omega_{\alpha}(\underset{\sim}{k}) \ . \tag{18}$$

In addition, the symmetry property

$$K^{\alpha\alpha_1\alpha_2}(\underset{\sim}{k},\underset{\sim}{k}_1,\underset{\sim}{k}_2)^{*} + K^{\alpha\alpha_2\alpha_1}(\underset{\sim}{k},\underset{\sim}{k}_2,\underset{\sim}{k}_1)^{*} = K^{\alpha\alpha_1\alpha_2}(-\underset{\sim}{k},-\underset{\sim}{k}_1,-\underset{\sim}{k}_2) + K^{\alpha\alpha_2\alpha_1}(-\underset{\sim}{k},-\underset{\sim}{k}_2,-\underset{\sim}{k}_1) \ , \tag{19}$$

follows directly from relations (14)-(18).

Equation (14), advancing the amplitudes A_α in time, is exact in the context of the original dynamical equation; the form, however, is more amenable to a direct analysis than that of Eq. (4). In a small amplitude theory of Eq. (14), A_α does not change in the lowest approximation, thus giving a $\underset{\sim}{\Psi}(\underset{\sim}{k},t)$ in which waves of different wave number propagate independently. In higher order however, the nonlinear terms act as perturbations causing $A_\alpha(\underset{\sim}{k},t)$ to change in the course of time through interactions between waves of differing wavenumber.

For the purpose of the study of weak turbulence, the general procedure in the literature has been to solve Eq. (14) in a perturbation expansion, i.e.,

$$A_\alpha(\underset{\sim}{k},t) \cong \lambda A_\alpha^{(1)}(\underset{\sim}{k},t) + \lambda^2 A_\alpha^{(2)}(\underset{\sim}{k},t) + \ldots \quad ,$$

where

$$\lambda \ll 1 \ , \quad \lambda \sim A_\alpha(\underset{\sim}{k},0) \ .$$

Once a solution is obtained to order λ^3 say, appropriate statistical averages are <u>then</u> carried out over a spatially homogeneous ensemble.[1-6] In the section which follows, a formalism is developed in which the statistical averaging is done at the outset. Using Eq. (14) as the dynamical equation for an individual system, this method of "a priori statistics" leads to equations of evolution for wave correlations characterizing the spatially homogeneous ensemble.

III: THE HIERARCHY FOR WAVE CORRELATIONS[8]

We rewrite Eq. (14) in the following notation,

$$\frac{\partial}{\partial t} A_{\alpha_1}(\underset{\sim}{k}_1,t) = \sum_{\alpha_2,\alpha_3} \iint d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 \delta(\underset{\sim}{k}_1-\underset{\sim}{k}_2-\underset{\sim}{k}_3) A_{\alpha_2}(\underset{\sim}{k}_2,t) A_{\alpha_3}(\underset{\sim}{k}_3,t)$$

$$K^{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3) \exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_2}(\underset{\sim}{k}_2)-\omega_{\alpha_3}(\underset{\sim}{k}_3))t\} \quad . \qquad (20)$$

Equations advancing the product of two A's in terms of the product of three A's, and three A's in terms of four A's, etc., are simply constructed from Eq. (20), namely,

$$\frac{\partial}{\partial t}\left(A_{\alpha_1}(\underset{\sim}{k}_1,t)A_{\alpha_2}(\underset{\sim}{k}_2,t)\right)$$
$$=\sum_{\alpha_3\alpha_4}\iint d\underset{\sim}{k}_3 d\underset{\sim}{k}_4\Big\{\Big(\delta(\underset{\sim}{k}_1-\underset{\sim}{k}_3-\underset{\sim}{k}_4)A_{\alpha_2}(\underset{\sim}{k}_2,t)A_{\alpha_3}(\underset{\sim}{k}_3,t)A_{\alpha_4}(\underset{\sim}{k}_4,t)$$
$$\times K^{\alpha_1\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_3,\underset{\sim}{k}_4)\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_3}(\underset{\sim}{k}_3)-\omega_{\alpha_4}(\underset{\sim}{k}_4))t\}\Big)+(1\leftrightarrow 2)\Big\} \qquad (21)$$

$$\vdots$$

$$\frac{\partial}{\partial t}\left(A_{\alpha_1}(\underset{\sim}{k}_1,t)\cdots A_{\alpha_s}(\underset{\sim}{k}_s,t)\right)$$
$$=\sum_{s'=2}^{s}\sum_{\alpha_{s+1}\alpha_{s+2}}\iint d\underset{\sim}{k}_{s+1}d\underset{\sim}{k}_{s+2}\Big\{\Big(\delta(\underset{\sim}{k}_1-\underset{\sim}{k}_{s+1}-\underset{\sim}{k}_{s+2})A_{\alpha_s}(\underset{\sim}{k}_s,t)A_{\alpha_{s-1}}(\underset{\sim}{k}_{s-1},t)\cdots A_{\alpha_2}(\underset{\sim}{k}_2,t)$$
$$\times A_{\alpha_{s+1}}(\underset{\sim}{k}_{s+1},t)A_{\alpha_{s+2}}(\underset{\sim}{k}_{s+2},t)K^{\alpha_1\alpha_{s+1}\alpha_{s+2}}(\underset{\sim}{k}_1,\underset{\sim}{k}_{s+1},\underset{\sim}{k}_{s+2})\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_{s+1}}(\underset{\sim}{k}_{s+1})-\omega_{\alpha_{s+2}}(\underset{\sim}{k}_{s+2}))t\}\Big)$$
$$+(s'\leftrightarrow 1)\Big\} \qquad (22)$$

$$\vdots$$

The notation, $(1\leftrightarrow 2)$, denotes the interchange $\begin{pmatrix}\underset{\sim}{k}_1\leftrightarrow\underset{\sim}{k}_2\\ \alpha_1\leftrightarrow\alpha_2\end{pmatrix}$.

The point of view we adopt is the following. Equation (20) is to be interpreted as the dynamical equation for an individual system. We now imagine a collection of such systems and average Eqs. (20)-(22) over this ensemble. The ensemble average, $\langle A_{\alpha_1}\cdots A_{\alpha_n}\rangle$, may be viewed as the arithmetic mean of $A_{\alpha_1}\cdots A_{\alpha_n}$ taken over a large number of systems or as an average over a (probability) distribution of systems. The net result is an interconnected chain of equations whereby the average of the product of s amplitudes, $\langle A_{\alpha_1}\cdots A_{\alpha_s}\rangle$ is driven by an integral operator acting on the average of the product of s+1 amplitudes, namely

$$\frac{\partial}{\partial t}\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\ldots A_{\alpha_s}(\underset{\sim}{k}_s,t)\rangle$$

$$=\sum_{s'=2}^{s}\sum_{\alpha_{s+1}\alpha_{s+2}}\iint d\underset{\sim}{k}_{s+1}d\underset{\sim}{k}_{s+2}\Big\{\Big(\delta(\underset{\sim}{k}_1-\underset{\sim}{k}_{s+1}-\underset{\sim}{k}_{s+2})$$

$$\langle A_{\alpha_s}(\underset{\sim}{k}_s,t)\ldots A_{\alpha_2}(\underset{\sim}{k}_2,t)A_{\alpha_{s+1}}(\underset{\sim}{k}_{s+1},t)A_{\alpha_{s+2}}(\underset{\sim}{k}_{s+2},t)\rangle$$

$$\times K^{\alpha_1\alpha_{s+1}\alpha_{s+2}}(\underset{\sim}{k}_1,\underset{\sim}{k}_{s+1},\underset{\sim}{k}_{s+2})\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_{s+1}}(\underset{\sim}{k}_{s+1})-\omega_{\alpha_{s+2}}(\underset{\sim}{k}_{s+2}))t\}\Big)+(1\leftrightarrow s')\Big\}\ . \tag{23}$$

With the problem of uniform turbulence in mind, our basic assumption regarding the ensemble is that the averages, $\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\ldots A_{\alpha_s}(\underset{\sim}{k}_s,t)\rangle$, correspond to a spatially homogeneous situation. In addition, it is convenient to construct from the above chain for $\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\rangle$, $\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)A_{\alpha_2}(\underset{\sim}{k}_s,t)\rangle,\ldots$, equations for irreducible correlations. The irreducible s-correlation is <u>defined</u> by subtracting from $\langle A_{\alpha_1}\ldots A_{\alpha_s}\rangle$ all irreducible correlations of lower order. We first record these definitions within the framework of spatial homogeneity and the assumption that $\langle A_{\alpha_i}\rangle = 0$,[‡] and then comment on their significance:

$$0=\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\rangle\ , \tag{24}$$

$$_2G_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2)\equiv\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)A_{\alpha_2}(\underset{\sim}{k}_2,t)\rangle\ , \tag{25}$$

$$_3G_{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\equiv\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)A_{\alpha_2}(\underset{\sim}{k}_2,t)A_{\alpha_3}(\underset{\sim}{k}_3,t)\rangle\ , \tag{26}$$

$$\begin{aligned}_4G_{\alpha_1\alpha_2\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3+\underset{\sim}{k}_4)&\equiv\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\ldots A_{\alpha_4}(\underset{\sim}{k}_4,t)\rangle\\
&-\ {_2G_{\alpha_1\alpha_2}}(\underset{\sim}{k}_1,t)\,{_2G_{\alpha_3\alpha_4}}(\underset{\sim}{k}_3,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2)\delta(\underset{\sim}{k}_3+\underset{\sim}{k}_4)\\
&-\ {_2G_{\alpha_1\alpha_3}}(\underset{\sim}{k}_1,t)\,{_2G_{\alpha_2\alpha_4}}(\underset{\sim}{k}_2,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_3)\delta(\underset{\sim}{k}_2+\underset{\sim}{k}_4)\\
&-\ {_2G_{\alpha_1\alpha_4}}(\underset{\sim}{k}_1,t)\,{_2G_{\alpha_2\alpha_3}}(\underset{\sim}{k}_2,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_4)\delta(\underset{\sim}{k}_2+\underset{\sim}{k}_3)\ ,\end{aligned} \tag{27}$$

$$\vdots$$

$$
{}_sG_{\alpha_1\cdots\alpha_s}(\underset{\sim}{k}_1,\ldots,t)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\cdots+\underset{\sim}{k}_s) \equiv \langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\cdots A_{\alpha_s}(\underset{\sim}{k}_s,t)\rangle
$$

$$
-\sum_{s'}\sum_{\{1,2,\ldots,s\}} {}_{s'}G_{\alpha_1\cdots\alpha_{s'}}\ {}_{s-s'}G_{\alpha_{s'+1}\cdots\alpha_s}\delta(\underset{\sim}{k}_1+\cdots+\underset{\sim}{k}_{s'})\delta(\underset{\sim}{k}_{s'+1}+\cdots+\underset{\sim}{k}_s)
$$

$$
-\sum_{s'}\sum_{s''}\sum_{\{1,2,\ldots,s\}} {}_{s'}G_{\alpha_1\cdots\alpha_{s'}}\ {}_{s''}G_{\alpha_{s'+1}\cdots\alpha_{s'+s''}}\ {}_{s-s'-s''}G_{\alpha_{s'+s''+1}\cdots\alpha_s}
$$

$$
\times\,\delta(\underset{\sim}{k}_1+\cdots+\underset{\sim}{k}_{s'})\delta(\underset{\sim}{k}_{s'+1}\cdots+\underset{\sim}{k}_{s'+s''})\delta(\underset{\sim}{k}_{s'+s''+1}+\cdots+\underset{\sim}{k}_s) \qquad (28)
$$

$$
-\ \cdots\ .
$$

The sum, $\sum_{\{1,2,\ldots,s\}}$, is over permutations of 1,2,...,s; the summation over s' in the second term on the right hand side of Eq. (28) runs from s'=2 to $s'=s_M'$, where $s_M'=s/2$ if s is even and $(s-1)/2$ if s is odd, etc.

Equation (24) corresponds to the assumption that $\underset{\sim}{\Psi}(\underset{\sim}{x},t)$ averaged over the ensemble is zero. A sufficient condition for this to be true, if true initially, is

$$
\dot{K}^{\alpha_1\alpha_2\alpha_3}(0,\underset{\sim}{k}_2,-\underset{\sim}{k}_2) = 0\ . \qquad (29)
$$

This follows directly from averaging Eq. (20) and utilizing relation (29), whereupon

$$
\frac{\partial}{\partial t}\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\rangle = 0\ . \qquad (30)
$$

Whether or not relation (29) is true depends on the detailed structure of $\dot{K}$ and hence on the explicit physical problem being examined; in any case, for purposes of the present analysis it is assumed throughout the remainder of this article that $\dot{K}^{\alpha_1\alpha_2\alpha_3}(0,\underset{\sim}{k}_2,-\underset{\sim}{k}_2) = 0$, and consequently

$$
\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\rangle = \langle A_{\alpha_1}(\underset{\sim}{k}_1,0)\rangle = 0\ . \qquad (31)
$$

The delta functions occurring in the definitions (25)-(28) are manifestations of the spatial homogeneity of the ensemble. Equation (25) ensures that the average, $\langle \Psi_i(\underset{\sim}{x}_1,t)\Psi_j(\underset{\sim}{x}_2,t)\rangle$, where Ψ_i is the ith element of the column vector $\underset{\sim}{\Psi}$, is invariant under translation as it depends on the difference $\underset{\sim}{x}_1-\underset{\sim}{x}_2$ (see also Ref. 9). Similarly, by virtue of Eq. (26), $\langle \Psi_i(\underset{\sim}{x}_1,t)\Psi_j(\underset{\sim}{x}_2,t)\Psi_\ell(\underset{\sim}{x}_3,t)\rangle$ is invariant under translation, and so on. We remind the reader that the definitions of the irreducible correlations, ${}_2G$ and ${}_3G$, in Eqs. (25) and (26), are of a particularly simple form because $\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\rangle = 0$.

Upon using the definitions of the irreducible correlations in Eq. (22), the following hierarchy of equations for ${}_2G, {}_3G, \ldots$, results:

$$\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2)\frac{\partial}{\partial t}\,{}_2G_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t)$$
$$= \Big\{ \Big(\sum_{\alpha_3\alpha_4} \int d\underset{\sim}{k}' K^{\alpha_1\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}')\,{}_3G_{\alpha_3\alpha_4\alpha_2}(\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}',t)$$
$$\times \exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_3}(\underset{\sim}{k}')-\omega_{\alpha_4}(\underset{\sim}{k}_1-\underset{\sim}{k}'))t\}\Big) + (1 \leftrightarrow 2)\Big\}\,\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2) \quad , \qquad (32)$$

$$\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\frac{\partial}{\partial t}\,{}_3G_{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t)$$
$$= \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\Big\{ \Big(\sum_{\alpha_4\alpha_5} \int d\underset{\sim}{k}' K^{\alpha_1\alpha_4\alpha_5}(\underset{\sim}{k}_1,\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}')\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_4}(\underset{\sim}{k}')-\omega_{\alpha_5}(\underset{\sim}{k}_1-\underset{\sim}{k}'))t\}$$
$$\times\,{}_4G_{\alpha_4\alpha_5\alpha_2\alpha_3}(\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}',\underset{\sim}{k}_2,t)\Big) + (2 \leftrightarrow 1) + (3 \leftrightarrow 1)\Big\}$$
$$+ \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\Big\{ \Big(\Big[\sum_{\alpha_4\alpha_5} K^{\alpha_1\alpha_4\alpha_5}(\underset{\sim}{k}_1,\underset{\sim}{k}_3,\underset{\sim}{k}_1+\underset{\sim}{k}_3)\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_4}(\underset{\sim}{k}_3)-\omega_{\alpha_5}(\underset{\sim}{k}_1+\underset{\sim}{k}_3))t\}$$
$$\times\,{}_2G_{\alpha_2\alpha_5}(\underset{\sim}{k}_2,t)\,{}_2G_{\alpha_3\alpha_4}(\underset{\sim}{k}_3,t)\Big] + [2 \leftrightarrow 3]\Big) + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\Big\} \quad , \qquad (33)$$

$$\vdots$$

As one continues to construct the equations for ${}_4G, {}_5G, \ldots$ it is found that ${}_4G$ is driven by ${}_5G$ and ${}_2G\,{}_3G$ terms and in general ${}_sG$ is driven by ${}_{s+1}G$ and

${}_pG_{s+1-p}G$ terms, $p=2,\ldots,s-1$. The procedure and resulting structure of equations is analogous to the derivation of equations for irreducible correlations within the BBGKY formalism.[10]

The hierarchy (32),(33),..., represents the dynamical system of equations to be used in describing the evolution of wave correlations characterizing the spatially homogeneous ensemble. Such a description is clearly practical only if the correlation between s waves, ${}_sG$, becomes small as the number of waves is increased, and closure can be obtained at some level of description. That this is in fact the case is most easily demonstrated in the following manner. Consistent with assumption of weak nonlinearity in relation to Eq. (14), we assume to leading order that

$$ {}_2G \sim \epsilon \; , \quad \epsilon \ll 1 \; . \tag{34} $$

Since ${}_3G$ is driven by ${}_2G_2G$ terms, we assume

$$ {}_3G \sim \epsilon^2 \text{ to leading order.} $$

Similarly

$$ {}_4G \sim \epsilon^3 $$

and in general

$$ {}_sG \sim \epsilon^{s-1} \; . \tag{35} $$

It is clear from Eq. (32) that the level of sophistication with which we describe the evolution of the correlation between two waves, ${}_2G$, depends vitally on the accuracy with which we describe ${}_3G$. For example, in order to calculate ${}_2G$ to order ϵ^2 and describe the leading order ${}_2G$ for times $t \sim 1/\epsilon$, we need ${}_3G$ to order ϵ^2; consequently, ${}_4G$ and higher correlations can be neglected. Similarly, to calculate ${}_2G$ to order ϵ^n and describe the leading order ${}_2G$ for times $t \sim 1/\epsilon^{n-1}$, closure may be obtained by neglecting ${}_{n+2}G$ and higher correlations.

IV: THE LOWEST ORDER (NON-TRIVIAL) KINETIC EQUATION FOR ${}_2G_{\alpha_1\alpha_2}$

In the context of the estimates in the previous section, we use the multiple time perturbation techniques of Frieman[11] and Sandri[12] in reference to the hierarchy (32),(33),..., with

$$ {}_2G_{\alpha_1\alpha_2} \cong \epsilon\, {}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\dots)+\epsilon^2\, {}_2G^{(2)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\dots)+\dots\ , \qquad (36) $$

$$ {}_3G_{\alpha_1\alpha_2\alpha_3} \cong \epsilon^2\, {}_3G^{(2)}_{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t,\epsilon t,\dots)+\dots\ , \qquad (37) $$

$$ {}_4G_{\alpha_1\alpha_2\alpha_3\alpha_4} \cong \epsilon^3\, {}_4G^{(3)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3,t,\epsilon t,\dots)+\dots\ , \qquad (38) $$

$$ \vdots $$

Evidently,

$$ \frac{\partial}{\partial t}\, {}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\dots) = 0\ , \qquad (39) $$

$$ \frac{\partial}{\partial t}\, {}_2G^{(2)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\dots) + \frac{\partial}{\partial \epsilon t}\, {}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\dots) $$

$$ = \left\{ \left(\sum_{\alpha_3\alpha_4} \int d\underset{\sim}{k}'\, K^{\alpha_1\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}')\, {}_3G^{(2)}_{\alpha_3\alpha_4\alpha_2}(\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}',t,\epsilon t,\dots) \right.\right. $$

$$ \left.\left. \times \exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_3}(\underset{\sim}{k}')-\omega_{\alpha_4}(\underset{\sim}{k}_1-\underset{\sim}{k}'))t\} \right) + \begin{pmatrix} \underset{\sim}{k}_1 \leftrightarrow -\underset{\sim}{k}_1 \\ \alpha_1 \leftrightarrow \alpha_2 \end{pmatrix} \right\}\ , \qquad (40) $$

$$ \vdots $$

and

$$ \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\, \frac{\partial}{\partial t}\, {}_3G^{(2)}_{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t,\epsilon t,\dots) $$

$$ = \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3) \left\{ \left(\left[\sum_{\alpha_4\alpha_5} K^{\alpha_1\alpha_4\alpha_5}(\underset{\sim}{k}_1,-\underset{\sim}{k}_3,\underset{\sim}{k}_1+\underset{\sim}{k}_3)\, {}_2G^{(1)}_{\alpha_2\alpha_5}(\underset{\sim}{k}_2,t,\epsilon t,\dots) \right.\right.\right. $$

$$ \left.\left. \times\, {}_2G^{(1)}_{\alpha_3\alpha_4}(\underset{\sim}{k}_3,t,\epsilon t,\dots) \exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)+\omega_{\alpha_4}(\underset{\sim}{k}_3)-\omega_{\alpha_5}(\underset{\sim}{k}_1+\underset{\sim}{k}_3))t\} \right] + \begin{bmatrix} \underset{\sim}{k}_2 \leftrightarrow \underset{\sim}{k}_3 \\ \alpha_2 \leftrightarrow \alpha_3 \end{bmatrix} \right) $$

$$+\begin{pmatrix} \underset{\sim}{k}_1 \leftrightarrow \underset{\sim}{k}_2 \\ \alpha_1 \leftrightarrow \alpha_2 \end{pmatrix} + \begin{pmatrix} \underset{\sim}{k}_1 \leftrightarrow \underset{\sim}{k}_3 \\ \alpha_1 \leftrightarrow \alpha_3 \end{pmatrix} \Bigg\} \ , \tag{41}$$

$$\vdots$$

From Eq. (39) it is clear that

$$_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\ldots) = {_2G^{(1)}_{\alpha_1\alpha_2}}(\underset{\sim}{k}_1,0,\epsilon t,\ldots) \ .$$

Since $_2G^{(1)}_{\alpha_1\alpha_2}$ does not vary on the short time scale, t, Eq. (41) may be integrated directly on the t scale to give

$$\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\,{_3G^{(2)}_{\alpha_1\alpha_2\alpha_3}}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,t,\epsilon t,\ldots) = \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\,{_3G^{(2)}_{\alpha_1\alpha_2\alpha_3}}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,0,\epsilon t,\ldots)$$

$$+ \delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)\Bigg\{\Bigg(\Bigg[\sum_{\alpha_4\alpha_5} K^{\alpha_1\alpha_4\alpha_5}_{(\underset{\sim}{k}_1,-\underset{\sim}{k}_3,\underset{\sim}{k}_1+\underset{\sim}{k}_3)}\,{_2G^{(1)}_{\alpha_2\alpha_5}}(\underset{\sim}{k}_2,\epsilon t,\ldots)$$

$$\times\, {_2G^{(1)}_{\alpha_3\alpha_4}}(\underset{\sim}{k}_3,\epsilon t,\ldots)\,\frac{\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)+\omega_{\alpha_4}(\underset{\sim}{k}_3)-\omega_{\alpha_5}(\underset{\sim}{k}_1+\underset{\sim}{k}_3))t\}-1}{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)+\omega_{\alpha_4}(\underset{\sim}{k}_3)-\omega_{\alpha_5}(\underset{\sim}{k}_1+\underset{\sim}{k}_3))}\Bigg]$$

$$+\begin{bmatrix} \underset{\sim}{k}_2 \leftrightarrow \underset{\sim}{k}_3 \\ \alpha_2 \leftrightarrow \alpha_3 \end{bmatrix}\Bigg) + \begin{pmatrix} \underset{\sim}{k}_1 \leftrightarrow \underset{\sim}{k}_2 \\ \alpha_1 \leftrightarrow \alpha_2 \end{pmatrix} + \begin{pmatrix} \underset{\sim}{k}_1 \leftrightarrow \underset{\sim}{k}_3 \\ \alpha_1 \leftrightarrow \alpha_3 \end{pmatrix}\Bigg\} \ . \tag{42}$$

This equation, when integrated over $\underset{\sim}{k}_3$, gives an explicit expression for $_3G^{(2)}_{\alpha_1\alpha_2\alpha_3}$ which is to be substituted in the integrand on the right hand side of Eq. (40) to determine the time behavior of $_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt scale and $_2G^{(2)}_{\alpha_1\alpha_2}$ on the t scale. Of particular interest is the evolution of the dominant order $_2G^{(1)}_{\alpha_1\alpha_2}$ on the long time scale, ϵt. This is obtained by asking that Eq. (40) not yield a secular solution for $_2G^{(2)}_{\alpha_1\alpha_2}$ on the time scale, t, as $t \to \infty$. Clearly $\frac{\partial}{\partial \epsilon t}\,{_2G^{(1)}_{\alpha_1\alpha_2}}$ yields a term proportional to t in $_2G^{(2)}_{\alpha_1\alpha_2}$; similarly, any portions of the right hand side of Eq. (40) that tend to a constant (on the t scale) for large t, will give contributions to $_2G^{(2)}_{\alpha_1\alpha_2}$ secular as t. Removing the secular behavior in $_2G^{(2)}_{\alpha_1\alpha_2}$ by equating the net coefficient of t to zero, then yields an equation describing the evolution of ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt scale. With this in mind, we turn to an examination of the right hand side of Eq. (40).

Upon using expression (42), the integrand in Eq. (40) contains three distinct types of time behavior on the t scale, which can be written schematically as

$$\exp\{if(\underset{\sim}{k}_1,\underset{\sim}{k}')t\} \qquad (i)$$

$$\frac{\exp\{if(\underset{\sim}{k}_1,\underset{\sim}{k}')t\}-1}{if(\underset{\sim}{k}_1,\underset{\sim}{k}')} \qquad (ii)$$

$$\exp\{if(\underset{\sim}{k}_1,\underset{\sim}{k}')t\}\frac{\exp\{-ig(\underset{\sim}{k}_1,\underset{\sim}{k}')t\}-1}{-ig(\underset{\sim}{k}_1,\underset{\sim}{k}')}\,, \quad g \neq f \qquad (iii)\,.$$

The abbreviations f and g stand for appropriate sums or differences of three frequencies. It is assumed that K, ${}_2G^{(1)}$, and the initial value of ${}_3G^{(2)}$ on the t scale are relatively smooth functions of their Fourier space arguments in carrying out the $\underset{\sim}{k}'$-integration in Eq. (40). Without belaboring algebraic details which have been discussed elsewhere,[8,13] the following statements can be made regarding the time behavior, for large t, of the right hand side of Eq. (40).

Integration over $\underset{\sim}{k}'$ of terms involving expression (i), decay as $t \to \infty$. In situations where a stationary phase analysis is applicable, the behavior for large t is to leading order

$$\frac{(\text{oscillation})}{t^{3/2}}\,, \text{ in a three dimensional problem} \qquad (43)$$

and

$$\frac{(\text{oscillation})}{t^{1/2}}\,, \text{ in a one dimensional problem}\,. \qquad (44)$$

On the other hand, expression (ii) behaves to leading effectively as if

$$\left(\frac{\exp\{if(\underset{\sim}{k},\underset{\sim}{k}')t\}-1}{if(\underset{\sim}{k},\underset{\sim}{k}')}\right)_{t\to\infty}\to\left(\frac{i}{f(\underset{\sim}{k},\underset{\sim}{k}')+i\Delta}\right)_{\Delta\to 0_+} \tag{45}$$

insofar as integrations over $\underset{\sim}{k}'$ are concerned. Corrections to this are of the form (43) or (44). Integrations over $\underset{\sim}{k}'$ involving (iii) lead to terms which are purely oscillatory for large t as well as terms decaying like expressions (43) or (44).

It is thus evident that the only terms on the right hand side of Eq. (40) that yield secular behavior in ${}_2G^{(2)}$ are terms involving expression (ii). Removal of the net secular behavior ($\propto t$) in ${}_2G^{(2)}$ for large t readily gives the equation of evolution for ${}_2G^{(1)}$ on the ϵt scale, namely

$$\frac{\partial}{\partial\epsilon t}\,{}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,\epsilon t,\ldots)=\frac{1}{2}\sum_{\alpha_3\alpha_4}\int d\underset{\sim}{k}'\,\frac{i\,\mathcal{M}^{\alpha_1\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}',\underset{\sim}{k}_1-\underset{\sim}{k}')}{(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_3}(\underset{\sim}{k}')-\omega_{\alpha_4}(\underset{\sim}{k}_1-\underset{\sim}{k}')+i\Delta)}$$

$$\times\Big\{\mathcal{M}^{\alpha_3\alpha_4\alpha_1}(\underset{\sim}{k}',\underset{\sim}{k}'-\underset{\sim}{k}_1,\underset{\sim}{k}_1)\,{}_2G^{(1)}_{\alpha_4\alpha_4}(\underset{\sim}{k}_1-\underset{\sim}{k}',\epsilon t,\ldots)\,{}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,\epsilon t,\ldots)$$

$$+\mathcal{M}^{\alpha_4\alpha_3\alpha_1}(\underset{\sim}{k}_1-\underset{\sim}{k}',-\underset{\sim}{k}',\underset{\sim}{k}_1)\,{}_2G^{(1)}_{\alpha_3\alpha_3}(\underset{\sim}{k}',\epsilon t,\ldots)\,{}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,\epsilon t,\ldots)$$

$$+\,\delta_{\alpha_1\alpha_2}\,\mathcal{M}^{\alpha_1\alpha_3\alpha_4}(-\underset{\sim}{k}_1,-\underset{\sim}{k}',\underset{\sim}{k}'-\underset{\sim}{k}_1)\,{}_2G^{(1)}_{\alpha_3\alpha_3}(\underset{\sim}{k}',\epsilon t,\ldots)\,{}_2G^{(1)}_{\alpha_4\alpha_4}(\underset{\sim}{k}_1-\underset{\sim}{k}',\epsilon t,\ldots)\Big\}$$

$$+\begin{pmatrix}\alpha_1\leftrightarrow\alpha_2\\ \underset{\sim}{k}_1\leftrightarrow-\underset{\sim}{k}_1\end{pmatrix};\quad\Delta\to 0_+\quad. \tag{46}$$

In writing Eq. (46), we have utilized relation (45), and the definition

$$\mathcal{M}^{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3)\equiv\mathcal{K}^{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3)+\mathcal{K}^{\alpha_1\alpha_3\alpha_2}(\underset{\sim}{k}_1,\underset{\sim}{k}_3,\underset{\sim}{k}_2)\;. \tag{47}$$

V: COMMENTS ON THE KINETIC EQUATION FOR ${}_2G^{(1)}_{\alpha_1\alpha_2}$

Equation (46) is a nonlinear integral differential equation describing the kinetic behavior of the correlation between two waves, ${}_2G^{(1)}_{\alpha_1\alpha_2}$, on the ϵt time scale. In practice one is usually content with the description afforded by Eq. (46); in principle, however, armed with a solution to this (nonlinear) equation one could proceed to a higher order description of ${}_2G_{\alpha_1\alpha_2}$ on longer time scales. A notable exception to the complexity of Eq. (46) occurs in problems where the principal value integrations associated with

$$\frac{P}{\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_\beta(\underset{\sim}{k}')-\omega_\delta(\underset{\sim}{k}_1-\underset{\sim}{k}')} ,$$

give zero contribution, and the condition for resonant three wave interactions,

$$\omega_{\alpha_1}(\underset{\sim}{k}_1) = \omega_\beta(\underset{\sim}{k}')+\omega_\delta(\underset{\sim}{k}_1-\underset{\sim}{k}') \tag{48}$$

cannot be satisfied. In this situation

$$\frac{\partial}{\partial\epsilon t}\, {}_2G^{(1)}_{\alpha_1\alpha_2} = 0 , \tag{49}$$

and one must continue to higher order with the inclusion of four-wave correlations, ${}_4G^{(3)}$, in order to obtain a nontrivial kinetic equation for ${}_2G^{(1)}_{\alpha_1\alpha_2}$. Such, apparently are the circumstances in the work of Hasselman.[4,5] It will be assumed however, throughout the remainder of this article, that we are dealing with physical problems where the condition for resonant three-wave interactions, Eq. (48), can be satisfied at least for certain modes.

It is evident that the derivation of the preceding section has been very direct and simple compared to the usual formalisms which solve Eq. (14) to a certain order in a perturbation expansion, and then perform appropriate statistical averages a posteriori. Another notable difference is that there is no apparent reason to use the time scales, $t,\lambda t,\lambda^2 t,\ldots,(\lambda^2 \sim \epsilon)$ as is the case in the techniques of Benney and Saffman[1] which involve a perturbation solution to Eq. (14). The time scales t, ϵt,... are quite natural choices in a formalism which describes wave correlations characterizing the ensemble.

The kinetic equation, Eq. (46), for ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt time scale is similar to results elsewhere in the literature. One point of difference is that we have not (a priori) assumed that $G^{(1)}_{\alpha_1\alpha_2}$ is zero for α_1 distinct from α_2. In addition, Eq. (46) is quite general; since no restriction has been made to a particular physical problem, the number of modes, the dimension of $\underset{\sim}{\Psi}$, and the structure of the kernel $\mathcal{K}$ are quite arbitrary. Although Eq. (46) is impossible to solve without specialization to a particular problem, several general comments can be made regarding the creation of correlations, conservation laws, etc.

(a) The Creation of Correlations

We note from Eqs. (17) and (25) that

$$ {}_2G_{\alpha_1\alpha_2}(\underset{\sim}{k}_1) = {}_2G_{\alpha_2\alpha_1}(-\underset{\sim}{k}_1) = {}_2G_{\alpha_1\alpha_2}(-\underset{\sim}{k}_1)^* \; . \tag{50} $$

That is to say, ${}_2G_{\alpha\alpha}(\underset{\sim}{k})$ is real and an even function of its $\underset{\sim}{k}$ argument; however, for α_1 distinct from α_2, ${}_2G_{\alpha_1\alpha_2}$ is in general complex. It is clear from the structure of Eq. (46) and using relations (50), that if ${}_2G^{(1)}_{\alpha_1\alpha_2}$ is initially zero (for α_1 distinct from α_2), it remains so on the t and ϵt scales. That is to say, correlations between two different modes do not develop in the course of time if they are initially uncorrelated.

The situation is considerably different for the correlation, ${}_2G^{(1)}_{\alpha\alpha}$, between two waves belonging to the same mode α. In this case, provided there is at least one other mode β for which ${}_2G^{(1)}_{\beta\beta}$ is initially non zero, the correlations, ${}_2G^{(1)}_{\alpha\alpha}$, may be driven into the system even if initially zero. This is simply demonstrated directly from Eq. (46). By hypothesis

$$ {}_2G^{(1)}_{\alpha\alpha} = 0 \text{ for } \epsilon t = 0 \, , $$

$$ {}_2G^{(1)}_{\beta\beta} \neq 0 \text{ for } \epsilon t = 0, \text{ for one mode, } \beta \text{ say.} $$

We then have

$$ \frac{\partial}{\partial \epsilon t} \, {}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_1, \epsilon t)\Big|_{\epsilon t=0} $$

$$ = \pi \int \underset{\approx}{dk}' \, |\mathcal{M}^{\alpha\beta\beta}(\underset{\sim}{k}_1, \underset{\sim}{k}', \underset{\sim}{k}_1 - \underset{\sim}{k}')|^2 \, {}_2G^{(1)}_{\beta\beta}(\underset{\sim}{k}', \epsilon t=0) \, {}_2G^{(1)}_{\beta\beta}(\underset{\sim}{k}_1 - \underset{\sim}{k}', \epsilon t=0) $$

$$ \times \, \delta(\omega_\alpha(\underset{\sim}{k}_1) - \omega_\beta(\underset{\sim}{k}') - \omega_\beta(\underset{\sim}{k}_1 - \underset{\sim}{k}')) \, , \qquad (51) $$

where we have utilized Eq. (19). Clearly if the equation

$$ \omega_\alpha(\underset{\sim}{k}_1) = \omega_\beta(\underset{\sim}{k}') + \omega_\beta(\underset{\sim}{k}_1 - \underset{\sim}{k}') $$

has a solution, $\underset{\sim}{k}' = \underset{\sim}{k}'(\underset{\sim}{k}_1)$, the quantity

$$ \frac{\partial}{\partial \epsilon t} \, {}_2G^{(1)}_{\alpha\alpha}\Big|_{\epsilon t=0} $$

is non zero; consequently, the correlation ${}_2G^{(1)}_{\alpha\alpha}$ is driven into the system even though initially zero. The mechanism for this has been a coupling between the α and β modes as manifested by the delta-function in Eq. (51).

(b) Sign of the Correlations, ${}_2G^{(1)}_{\alpha\alpha}$

Let us assume in the remainder of Section V that the correlation between unlike modes is initially zero. As previously demonstrated this remains true on the t and ϵt scales. We now turn to an examination of the correlations, ${}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_1)$, between any two like modes. We note that ${}_2G_{\alpha\alpha}(\underset{\sim}{k}_1)$, which is the energy spectral density associated with the mode α, is real and an even function of $\underset{\sim}{k}_1$ by virtue of Eq. (50); in addition it is non-negative as is simply demonstrated by means of a "filter" theorem (Appendix A), i.e.,

$$ {}_2G_{\alpha\alpha}(\underset{\sim}{k}_1) \geqq 0, \text{ and } {}_2G_{\alpha\alpha}(\underset{\sim}{k}_1) = {}_2G_{\alpha\alpha}(-\underset{\sim}{k}_1), \; {}_2G_{\alpha\alpha}(\underset{\sim}{k}_1) \text{ real,} $$

The above statements can be made a priori within the general framework of stationary random processes. Consequently, in order that Eq. (46) be an acceptable kinetic equation for ${}_2G^{(1)}_{\alpha\alpha}$, it is necessary that this equation preserve the non-negative nature of ${}_2G^{(1)}_{\alpha\alpha}$. The following reductio ad absurdum argument in fact demonstrates that this is the case.

Take all correlations, ${}_2G^{(1)}_{\alpha_1\alpha_1}$, as initially positive definite. Assume that the correlation to first turn negative is associated with the mode α and that this occurs for $\underset{\sim}{k}_1 = \underset{\sim}{k}_0$. It follows, that at the instant ${}_2G^{(1)}_{\alpha\alpha}$ is passing through zero,

$$ {}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_0) = 0 $$

$$ {}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_1 \neq \underset{\sim}{k}_0) \geqq 0 $$

and

$$ {}_2G^{(1)}_{\beta\beta}(\underset{\sim}{k}_1) \geqq 0, \; \beta \text{ distinct from } \alpha. $$

Also, at the instant ${}_2G^{(1)}_{\alpha\alpha}$ is passing through zero, Eq. (46) gives

$$ \frac{\partial}{\partial\epsilon t}\, {}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_0) = \sum_{\beta,\delta} \pi \int \underset{\sim}{dk'} \, |\mathcal{M}^{\alpha\beta\delta}(\underset{\sim}{k}_0, \underset{\sim}{k}', \underset{\sim}{k}_0 - \underset{\sim}{k}')|^2 \delta(\omega_\alpha(\underset{\sim}{k}_0) - \omega_\beta(\underset{\sim}{k}') - \omega_\delta(\underset{\sim}{k}_0 - \underset{\sim}{k}')) $$
$$ \times \, {}_2G^{(1)}_{\beta\beta}(\underset{\sim}{k}')\, {}_2G^{(1)}_{\delta\delta}(\underset{\sim}{k}_0 - \underset{\sim}{k}') \quad . \tag{52} $$

The right hand side of Eq. (52) is manifestly positive definite in nature, i.e.,

$$ \frac{\partial}{\partial\epsilon t}\, {}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_0) \geqq 0 \; . \tag{53} $$

This contradicts our original hypothesis of ${}_2G^{(1)}_{\alpha\alpha}$ turning negative. The argument may be continued to show that no correlations, ${}_2G^{(1)}_{\alpha_1\alpha_1}$, turn negative if all correlations are originally positive definite.

(c) Multiple Resonances

Inherent in the derivation of the kinetic equation for ${}_2G^{(1)}_{\alpha_1\alpha_2}$ is the assumption that

$$\frac{d}{d\underset{\sim}{k}'}\left(\omega_\beta(\underset{\sim}{k}')+\omega_\delta(\underset{\sim}{k}-\underset{\sim}{k}')\right) = 0 \ , \tag{54}$$

and

$$\omega_\beta(\underset{\sim}{k}')+\omega_\delta(\underset{\sim}{k}-\underset{\sim}{k}') = \omega_\alpha(\underset{\sim}{k}) \ , \tag{55}$$

are not satisfied simultaneously for some $\underset{\sim}{k}'$ (given $\underset{\sim}{k}$). This does not appear to be a serious limitation on the theory, however it is of some interest to understand how this restriction arises mathematically, and its physical implications.

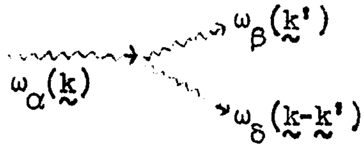
Mathematically, the delta-function portion of the integrand in Eq. (46), i.e., $\delta(\omega_\alpha(\underset{\sim}{k})-\omega_\beta(\underset{\sim}{k}')-\omega_\delta(\underset{\sim}{k}-\underset{\sim}{k}'))$, leads to divergent results in the $\underset{\sim}{k}'$-integration in situations where both Eq. (54) and Eq. (55) can be satisfied for the same $\underset{\sim}{k}'$. This difficulty may be traced back to the fact that replacing

$$\frac{\exp\{i(\omega_\alpha(\underset{\sim}{k})-\omega_\beta(\underset{\sim}{k}')-\omega_\delta(\underset{\sim}{k}-\underset{\sim}{k}'))t\}-1}{i(\omega_\alpha(\underset{\sim}{k})-\omega_\beta(\underset{\sim}{k}')-\omega_\delta(\underset{\sim}{k}-\underset{\sim}{k}'))} \text{ by } \frac{i}{(\omega_\alpha(\underset{\sim}{k})-\omega_\beta(\underset{\sim}{k}')-\omega_\delta(\underset{\sim}{k}-\underset{\sim}{k}')+i\Delta)} \quad \Delta\to 0_+ \ ,$$

in calculating the $t\to\infty$ behavior of the various integrals over $\underset{\sim}{k}'$, is incorrect under these conditions. It may be shown, for example, that in a one-dimensional problem these integrals diverge as $t^{1/2}$.[8] In such situations, more general multiple time scales than $t,\epsilon t,\ldots$ must be used,[14] since the wave-wave interactions take place rapidly and are of a more complicated nature.

Physically, by assuming that Eq. (54) is not satisfied for those $\underset{\sim}{k}'$ determined from Eq. (55), we are excluding problems in which the wave-wave interactions are "multiple" in the following sense. The resonance condition, Eq. (55), may be interpreted as the decay of the mode $\omega_\alpha(\underset{\sim}{k})$ into two components,

$\omega_\beta(\mathbf{k}')$ and $\omega_\delta(\mathbf{k}-\mathbf{k}')$ (or as the inverse process).[3] We represent this schematically as



The group velocities of the decay modes $\omega_\beta(\mathbf{k}')$ and $\omega_\delta(\mathbf{k}-\mathbf{k}')$ are $d\omega_\beta(\mathbf{k}')/d\mathbf{k}'$ and $d\omega_\delta(k'-\mathbf{k})/d\mathbf{k}'$ respectively. These group velocities are clearly equal if condition (54) is satisfied. Under such circumstances the associated wave disturbances move away with the same velocity and are thus capable of further multiple interactions with one another. However, under the assumption that

$$\frac{d}{d\mathbf{k}'}\left(\omega_\beta(\mathbf{k}') + \omega_\delta(\mathbf{k}-\mathbf{k}')\right) \neq 0 \ ,$$

for those $\mathbf{k}'$ such that

$$\omega_\beta(\mathbf{k}') + \omega_\delta(\mathbf{k}-\mathbf{k}') = \omega_\alpha(\mathbf{k}) \ ,$$

the wave disturbances move away from one another and do not further interact effectively.

(d) Quasi-Particle Conservation Laws

Let us now examine certain invariants of Eq. (46), and introduce the notation

$$\mathcal{N}^{\alpha_1\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \equiv \sqrt{\pi}\ \omega_{\alpha_2}(\mathbf{k}_2)\omega_{\alpha_3}(\mathbf{k}_3)\mathcal{M}^{\alpha_1\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \ . \tag{56}$$

We define the "number of quasi-particles" associated with the mode α and the wave vector $\mathbf{k}_1$ as

$$n_\alpha(\mathbf{k}_1,\epsilon t) \equiv \frac{2G^{(1)}_{\alpha\alpha}(\mathbf{k}_1,\epsilon t)}{\omega_\alpha(\mathbf{k}_1)} \ . \tag{57}$$

The quantity $n_\alpha(\mathbf{k}_1)$ is essentially the action associated with the mode α.

By virtues of the definitions (47) and (56), and Eq. (19), $\mathcal{N}$ possesses the following symmetry properties:

$$\mathcal{N}^{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3) = \mathcal{N}^{\alpha_1\alpha_3\alpha_2}(\underset{\sim}{k}_1,\underset{\sim}{k}_3,\underset{\sim}{k}_2) \;, \tag{58}$$

and

$$\mathcal{N}^{\alpha_1\alpha_2\alpha_3}(-\underset{\sim}{k}_1,-\underset{\sim}{k}_2,-\underset{\sim}{k}_3) = \mathcal{N}^{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3)^* \;. \tag{59}$$

In asking that the original dynamical equation, Eq. (14), be derivable from a Hamiltonian, Litvak[6] has deduced certain symmetry properties of the kernel K equivalent to the following statements.

$$\mathcal{N}^{\alpha_1\alpha_2\alpha_3}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3) = \mathcal{N}^{\alpha_2\alpha_1\alpha_3}(-\underset{\sim}{k}_2,-\underset{\sim}{k}_1,\underset{\sim}{k}_3) = \mathcal{N}^{\alpha_3\alpha_1\alpha_2}(-\underset{\sim}{k}_3,-\underset{\sim}{k}_1,\underset{\sim}{k}_2), \quad \text{for } \underset{\sim}{k}_1 = \underset{\sim}{k}_2+\underset{\sim}{k}_3 \;. \tag{60}$$

Assuming the validity of these relations, the kinetic equation for ${}_2G^{(1)}_{\alpha\alpha}(\underset{\sim}{k}_1)$ may be rewritten as

$$\frac{\partial}{\partial\epsilon t}\, n_\alpha(\underset{\sim}{k}_1,\epsilon t) = \sum_{\beta,\delta}\iint d\underset{\sim}{k}_2 d\underset{\sim}{k}_3\, \delta(\underset{\sim}{k}_1-\underset{\sim}{k}_2-\underset{\sim}{k}_3)\,\delta(\omega_\alpha(\underset{\sim}{k}_1)-\omega_\beta(\underset{\sim}{k}_2)-\omega_\delta(\underset{\sim}{k}_3))$$
$$\times \frac{|\mathcal{N}^{\alpha\beta\delta}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3)|^2}{\omega_\alpha(\underset{\sim}{k}_1)\omega_\beta(\underset{\sim}{k}_2)\omega_\delta(\underset{\sim}{k}_3)}\,(n_\beta(\underset{\sim}{k}_2,\epsilon t)n_\delta(\underset{\sim}{k}_3,\epsilon t)-n_\alpha(\underset{\sim}{k}_1,\epsilon t)n_\beta(\underset{\sim}{k}_2,\epsilon t)-n_\alpha(\underset{\sim}{k}_1,\epsilon t)n_\delta(\underset{\sim}{k}_3,\epsilon t)), \tag{61}$$

where use has been made of Eqs. (56)-(60) and (46). Equation (61) describes the evolution of the number of quasi-particles, $n_\alpha(\underset{\sim}{k}_1,\epsilon t)$, due to resonant three-wave interactions. Since energy and momentum are conserved on the microscopic scale, i.e.,

$$\omega_\alpha(\underset{\sim}{k}_1) = \omega_\beta(\underset{\sim}{k}_2) + \omega_\delta(\underset{\sim}{k}_3) \;,$$

and

$$\underset{\sim}{k}_1 = \underset{\sim}{k}_2 + \underset{\sim}{k}_3 \;,$$

it may be expected that the total quasi-momentum, $\sum_\alpha \int d\underset{\sim}{k}_1 n_\alpha(\underset{\sim}{k}_1,\epsilon t)\underset{\sim}{k}_1$ and total quasi-energy $\sum_\alpha \int d\underset{\sim}{k}_1 n_\alpha(\underset{\sim}{k}_1,\epsilon t)\omega_\alpha(\underset{\sim}{k}_1)$, are invariants of Eq. (61). We check that this is the case. Making use of the oddness of n_α and ω_α as functions of their Fourier arguments, as well as the symmetry properties, (60), it follows that

$$\frac{\partial}{\partial \epsilon t}\left(\sum_\alpha \int n_\alpha(\underset{\sim}{k}_1,\epsilon t)\underset{\sim}{k}_1 d\underset{\sim}{k}_1\right)$$

$$= \sum_{\alpha,\beta,\delta} \iiint d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 \delta(\underset{\sim}{k}_1-\underset{\sim}{k}_2-\underset{\sim}{k}_3)\delta(\omega_\alpha(\underset{\sim}{k}_1)-\omega_\beta(\underset{\sim}{k}_2)-\omega_\delta(\underset{\sim}{k}_3))$$

$$\times \frac{|\overset{\alpha\beta\delta}{\mathcal{N}}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3)|^2}{\omega_\alpha(\underset{\sim}{k}_1)\omega_\beta(\underset{\sim}{k}_2)\omega_\delta(\underset{\sim}{k}_3)}(\underset{\sim}{k}_1-\underset{\sim}{k}_2-\underset{\sim}{k}_3)n_\beta(\underset{\sim}{k}_2,\epsilon t)n_\delta(\underset{\sim}{k}_3,\epsilon t)$$

$$\equiv 0 \; . \tag{62}$$

Similarly,

$$\frac{\partial}{\partial \epsilon t}\left(\sum_\alpha \int n_\alpha(\underset{\sim}{k}_1,\epsilon t)\omega_\alpha(\underset{\sim}{k}_1)d\underset{\sim}{k}_1\right) \equiv 0 \; . \tag{63}$$

VI: MODIFICATIONS DUE TO A WEAK INSTABILITY

We now consider modifications of the results of Sections III and IV in situations where there is a (weak) instability, that is

$$\omega_\alpha(\underset{\sim}{k}) = \omega_\alpha{}^R(\underset{\sim}{k}) + i\gamma_\alpha(\underset{\sim}{k}) \; , \tag{64}$$

where $\omega_\alpha{}^R$ and γ_α are real ($\gamma_\alpha > 0$) and

$$\omega_\alpha{}^R(\underset{\sim}{k}) = -\omega_\alpha{}^R(-\underset{\sim}{k}) \; , \quad \gamma_\alpha(\underset{\sim}{k}) = \gamma_\alpha(-\underset{\sim}{k}) \; .$$

In place of Eq. (12), we now write

$$\underset{\sim}{\Psi}(\underset{\sim}{k},t) = \sum_\alpha A_\alpha(\underset{\sim}{k},t)\underset{\sim}{u}_\alpha(\underset{\sim}{k})\exp\{-i\omega_\alpha{}^R(\underset{\sim}{k})t\} \tag{65}$$

An analysis, completely analogous to that given in Section II, then yields the following equation advancing $A_{\alpha_1}(\underset{\sim}{k}_1,t)$ in time, i.e.,

$$\frac{\partial}{\partial t} A_{\alpha_1}(\underset{\sim}{k}_1,t) = \gamma_{\alpha_1}(\underset{\sim}{k}_1)A_{\alpha_1}(\underset{\sim}{k}_1,t)$$

$$+ \sum_{\alpha_2,\alpha_3} \iint d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 \delta(\underset{\sim}{k}_1-\underset{\sim}{k}_2-\underset{\sim}{k}_3)\, K^{\alpha_1\alpha_2\alpha_3}_{(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3)}$$

$$\times A_{\alpha_2}(\underset{\sim}{k}_2,t)A_{\alpha_3}(\underset{\sim}{k}_3,t)\exp\{i(\omega^R_{\alpha_1}(\underset{\sim}{k}_1)-\omega^R_{\alpha_2}(\underset{\sim}{k}_2)-\omega^R_{\alpha_3}(\underset{\sim}{k}_3))t\}. \quad (66)$$

Equation (66) is of the same form as Eq. (14) with the exception of the appearance of the term, $\gamma_{\alpha_1}(\underset{\sim}{k}_1)A_{\alpha_1}(\underset{\sim}{k}_1,t)$, on the right hand side. The construction of a hierarchy using Eq. (66) as the dynamical equation may be carried out in a manner identical to the analysis of Section III. The only modification of the equation for ${}_sG_{\alpha_1\alpha_2\ldots\alpha_s}$ (s = 2,...) is the inclusion of a term

$$(\gamma_{\alpha_1}(\underset{\sim}{k}_1)+\ldots+\gamma_{\alpha_s}(\underset{\sim}{k}_s))\,{}_sG_{\alpha_1\ldots\alpha_s}(\underset{\sim}{k}_1,\ldots,\underset{\sim}{k}_{s-1})\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\ldots+\underset{\sim}{k}_s)$$

on the right hand side.

A theory describing the evolution of ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt time scale may still be obtained as in Section IV, if the instability index, $|\gamma_\alpha(\underset{\sim}{k})/\omega_\alpha(\underset{\sim}{k})|$, is of order ϵ. The situation in which the instability index is of order unity is mathematically intractable since the correlations increase by an order of magnitude on the short time scale t (for non zero initial values). Writing

$$\gamma_{\alpha_1}(\underset{\sim}{k}_1) \cong \epsilon\gamma^{(1)}_{\alpha_1}(\underset{\sim}{k}_1)+\ldots ,$$

the only modification to Eqs. (39), (40) and (41), occurs in Eq. (40) which becomes

$$\frac{\partial}{\partial t}\,{}_2G^{(2)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\ldots) + \frac{\partial}{\partial \epsilon t}\,{}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\ldots)$$

$$= (\gamma^{(1)}_{\alpha_1}(\underset{\sim}{k}_1)+\gamma^{(1)}_{\alpha_2}(-\underset{\sim}{k}_1))\,{}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1,t,\epsilon t,\ldots) + (\text{r.h.s. of Eq. (40)}) . \quad (67)$$

The equation describing ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt time scale is then

$$\frac{\partial}{\partial \epsilon t}\, {}_2G^{(1)}_{\alpha_1\alpha_2} = (\gamma^{(1)}_{\alpha_1}(\underset{\sim}{k}_1)+\gamma^{(1)}_{\alpha_2}(-\underset{\sim}{k}_1))\,{}_2G^{(1)}_{\alpha_1\alpha_2}$$

$$+ \text{(r.h.s. of Eq. (46))} \ . \qquad (68)$$

It is possible of course that the solution to Eq. (68) may be such that ${}_2G^{(1)}$ increases by an order of magnitude on the ϵt time scale and consequently destroys the ordering. However, since the driving terms nonlinear in ${}_2G^{(1)}$ are of the same order (initially on the ϵt time scale) as the (unstable) linear driving term, the description by Eq. (68) may be valid for a considerable length of time on the ϵt scale.

It is often the situation in practice that $\gamma_\alpha(\underset{\sim}{k})$ varies slowly in time, say on the ϵt time scale. Evidently the analysis in arriving at Eq. (68) from Eq. (66) is the same, the only exception being that $\gamma_{\alpha_i}(\underset{\sim}{k}_i)$ is replaced by $\gamma_{\alpha_i}(\underset{\sim}{k}_i,\epsilon t)$. It should also be pointed out that nowhere in the derivation of Eq. (68) is the assumption that $\gamma_\alpha > 0$ utilized; consequently, the resulting equation for ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt scale is equally valid if the growth rate, γ_α, is negative corresponding to damping.

VII: MODIFICATIONS DUE TO TRILINEAR TERMS

Many problems of physical interest contain in addition to the bilinear term, $\underset{\sim}{B}(\underset{\sim}{\Psi},\underset{\sim}{\Psi})$, in Eq. (1), a trilinear term, $\underset{\sim}{T}(\underset{\sim}{\Psi},\underset{\sim}{\Psi},\underset{\sim}{\Psi})$, and perhaps higher multilinear terms. Such situations usually arise when the physical model becomes more sophisticated. For example, a plasma description by means of the Maxwell equations and the zero pressure two-fluid equations of continuity and momentum transfer can be put into the form of Eq. (1), with the only nonlinearity being bilinear. However, if the zero pressure assumption is relaxed, additional trilinear and higher multilinear terms appear.

One is still able to carry out an analysis similar to that of Section II. With the problem in mind of describing the evolution of ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt time scale, it turns out to be necessary to include only the modifications presented by trilinear effects. This gives an additional term to the right hand side of Eq. (14) of the form.

$$\sum_{\alpha_2\alpha_3\alpha_4} \iiint d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 d\underset{\sim}{k}_4 \delta(\underset{\sim}{k}_1-\underset{\sim}{k}_2-\underset{\sim}{k}_3-\underset{\sim}{k}_4) \mathcal{K}^{\alpha_1\alpha_2\alpha_3\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3,\underset{\sim}{k}_4)$$

$$\times A_{\alpha_2}(\underset{\sim}{k}_2,t)A_{\alpha_3}(\underset{\sim}{k}_3,t)A_{\alpha_4}(\underset{\sim}{k}_4,t)\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_2}(\underset{\sim}{k}_2)-\omega_{\alpha_3}(\underset{\sim}{k}_3)-\omega_{\alpha_4}(\underset{\sim}{k}_4))t\} \; , \tag{69}$$

where the kernel $\mathcal{K}^{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1,k_2,k_3,k_4)$ is constructed in a similar manner to the kernel $\mathcal{K}^{\alpha_1\alpha_2\alpha_3}(k_1,k_2,k_3)$. Paralleling the analysis of Section III, we ask that

$$\mathcal{K}^{\alpha_1\alpha_2\alpha_3\alpha_4}(0,k_2,k_3,-k_2-k_3) = 0 \; , \tag{70}$$

in order that $\langle A_{\alpha_1}(\underset{\sim}{k}_1,t)\rangle$ remain zero if initially so. With the inclusion of expression (69) in the formalism, it is evident that Eq. (32) for ${}_2G$ is modified by ${}_4G$ and ${}_2G_2G$ driving terms. Similarly Eq. (33) for ${}_3G$ is modified by ${}_5G$ and ${}_2G_3G$ driving terms, and so on. To the order necessary to describe the evolution ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt time scale, the only changes in Eqs. (39), (40) and (41), occur in Eq. (40) which now has an additional term on the right hand side, namely

$$\Bigg[\sum_{\alpha_3\alpha_4\alpha_5} \int d\underset{\sim}{k}_4 \, {}_2G^{(1)}_{\alpha_3\alpha_2}(\underset{\sim}{k}_1) \, {}_2G^{(1)}_{\alpha_4\alpha_5}(\underset{\sim}{k}_4)\exp\{i(\omega_{\alpha_1}(\underset{\sim}{k}_1)-\omega_{\alpha_3}(\underset{\sim}{k}_1)+\omega_{\alpha_5}(\underset{\sim}{k}_4)-\omega_{\alpha_4}(\underset{\sim}{k}_4))t\}$$

$$\times\left(\mathcal{K}^{\alpha_1\alpha_3\alpha_4\alpha_5}(\underset{\sim}{k}_1,\underset{\sim}{k}_1,\underset{\sim}{k}_4,-\underset{\sim}{k}_4)+\mathcal{K}^{\alpha_1\alpha_4\alpha_3\alpha_5}(\underset{\sim}{k}_1,\underset{\sim}{k}_4,\underset{\sim}{k}_1,-\underset{\sim}{k}_4)+\mathcal{K}^{\alpha_1\alpha_5\alpha_4\alpha_3}(\underset{\sim}{k}_1,-\underset{\sim}{k}_4,\underset{\sim}{k}_4,\underset{\sim}{k}_1)\right)\Bigg] + \begin{bmatrix} k_1 \to -k_1 \\ \alpha_1 \leftrightarrow \alpha_2 \end{bmatrix} . \tag{71}$$

Carrying out an analysis similar to that in Section IV with the inclusion of expression (71), yields an additional term in the equation of evolution for the correlation ${}_2G^{(1)}_{\alpha_1\alpha_2}$ on the ϵt time scale. In particular, the expression

$$\sum_{\alpha_4} \int d\underset{\sim}{k}_4\, {}_2G^{(1)}_{\alpha_4\alpha_4}(\underset{\sim}{k}_4)\, {}_2G^{(1)}_{\alpha_1\alpha_2}(\underset{\sim}{k}_1)\, \mathcal{H}^{\alpha_1\alpha_2\alpha_4\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_1,\underset{\sim}{k}_4,-\underset{\sim}{k}_4) + \begin{pmatrix} k_1 \to -k_1 \\ \alpha_1 \leftrightarrow \alpha_2 \end{pmatrix}, \tag{72}$$

where

$$\mathcal{H}^{\alpha_1\alpha_2\alpha_4\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_1,\underset{\sim}{k}_4,-\underset{\sim}{k}_4) \equiv \mathcal{K}^{\alpha_1\alpha_2\alpha_4\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_1,\underset{\sim}{k}_4,-\underset{\sim}{k}_4) + \mathcal{K}^{\alpha_1\alpha_4\alpha_2\alpha_4}(\underset{\sim}{k}_1,\underset{\sim}{k}_4,\underset{\sim}{k}_1,-\underset{\sim}{k}_4) + \mathcal{K}^{\alpha_1\alpha_4\alpha_4\alpha_2}(\underset{\sim}{k}_1,-\underset{\sim}{k}_4,\underset{\sim}{k}_4,\underset{\sim}{k}_1), \tag{73}$$

is added to the right hand side of Eq. (46).

CONCLUDING REMARKS:

Working within a framework in which the statistical averaging has been carried out a priori is conceptually, a natural and direct approach. The simplicity of mathematical analysis (in Section IV) bears witness to the practicality of such a formalism.

It is also evident that few difficulties have been encountered by staying within the context of a quite general nonlinear equation. Since the dimension of $\underset{\sim}{\Psi}$ is arbitrary, and since nowhere has an explicit form for the kernel $\mathcal{K}$, or $\{\omega_\alpha\}$ been assumed, the formalism and resulting kinetic equation for ${}_2G^{(1)}_{\alpha_1\alpha_2}$ are applicable to a wide variety of physical problems. In addition it has been demonstrated that a generalization of the technique for situations in which there is a weak instability or trilinear nonlinearities, is a relatively simple process.

ACKNOWLEDGEMENTS

The author is most grateful to Professor Allan Kaufman for many stimulating discussions and comments regarding various aspects of this article.

This research was supported by the National Aeronautics and Space Administration, Grant No. NGR 05-003-220.

APPENDIX A:

The following "filter" theorem due to Kaufman[15] represents a simple technique of demonstrating the non-negative nature of ${}_2G_{\alpha\alpha}(\underset{\sim}{k})$, and is essentially a mathematical formulation of a physical argument given by Papoulis.[16] For simplicity of notation we omit the subscript α and variable t in the discussion that follows.

Let us define a function $\mathcal{A}(\underset{\sim}{k})$ by the relation

$$\mathcal{A}(\underset{\sim}{k}) = F(\underset{\sim}{k})A(\underset{\sim}{k}) \ , \tag{A-1}$$

where $F(\underset{\sim}{k})$, the "filter" function, is the Fourier transform of a real valued (but otherwise arbitrary) function of $\underset{\sim}{x}$. By virtue of the relation

$$F(-\underset{\sim}{k}) = F(\underset{\sim}{k})^* \ , \tag{A-2}$$

and Eq. (17), it follows that

$$\mathcal{A}(-\underset{\sim}{k}) = \mathcal{A}(\underset{\sim}{k})^* \ . \tag{A-3}$$

We thus have from Eqs. (25), (A-1) and (A-2) that

$$\langle \mathcal{A}(\underset{\sim}{k}_1)\mathcal{A}(\underset{\sim}{k}_2)\rangle = \mathcal{G}(\underset{\sim}{k}_1)\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2) \ , \tag{A-4}$$

where

$$\mathcal{G}(\underset{\sim}{k}_1) = |F(\underset{\sim}{k}_1)|^2 G(\underset{\sim}{k}_1) \ . \tag{A-5}$$

It is readily shown from Eqs. (17), (25), (A-3), and (A-4) that

$$\int \frac{d\underset{\sim}{k}}{(2\pi)^6} G(\underset{\sim}{k}) = \langle (A(\underset{\sim}{x}))^2 \rangle \ , \tag{A-6}$$

and

$$\int \frac{d\underset{\sim}{k}}{(2\pi)^6} \mathcal{G}(\underset{\sim}{k}) = \langle (\mathcal{A}(\underset{\sim}{x}))^2 \rangle \ , \tag{A-7}$$

where the quantities $A(\underset{\sim}{x})$ and $\mathcal{A}(\underset{\sim}{x})$ are the (real-valued) inverse Fourier transforms of $A(\underset{\sim}{k})$ and $\mathcal{A}(\underset{\sim}{k})$ respectively. Consequently we have the inequalities:

$$\int d\underset{\sim}{k}\; G(\underset{\sim}{k}) \geqq 0 \;, \tag{A-8}$$

and

$$\int d\underset{\sim}{k} |F(\underset{\sim}{k})|^2 G(\underset{\sim}{k}) \geqq 0 \;. \tag{A-9}$$

Relations (A-8) and (A-9) together with the arbitrariness of the filter function $F(\underset{\sim}{k})$ are sufficient to prove the fact that $G(\underset{\sim}{k})$ is non-negative. This is most simply seen by the following reductio ad absurdum argument: Let us assume that $G(\underset{\sim}{k}_1)$ is negative in a region R of $\underset{\sim}{k}$-space, and non-negative elsewhere. We now choose $F(\underset{\sim}{k}_1)$ such that

$$F(\underset{\sim}{k}) = 0, \quad \underset{\sim}{k} \notin R \;,$$

and $F(\underset{\sim}{k})$ non zero for some region $R_1 \subseteq R$. It then follows that

$$\int d\underset{\sim}{k} |F(\underset{\sim}{k})|^2 G(\underset{\sim}{k}) = \int_{\underset{\sim}{k} \in R} d\underset{\sim}{k} |F(\underset{\sim}{k})|^2 G(\underset{\sim}{k}) < 0 \;,$$

which contradicts statement (A-9). Consequently $G(\underset{\sim}{k})$ is non-negative.

REFERENCES:

1. D. J. Benney and P. G. Saffman, Proc. Roy. Soc. 289, 301 (1966).

2. A. A. Galeev and V. I. Karpman, Soviet Phys. J.E.T.P. 17, 403 (1963).

3. B. B. Kadomstev, Plasma Turbulence (Academic Press, New York, 1965).

4. K. Hasselmann, J. Fl. Mech. 12, 481 (1962).

5. K. Hasselmann, J. Fl. Mech. 15, 273 and 385 (1963).

6. M. M. Litvak, Avco Everett Research Report Number 92 (1960).

7. N. J. Zabusky and M. D. Kruskal, Phys. Rev. Letters 15, 240 (1965).

8. R. C. Davidson, Princeton University Plasma Physics Laboratory Report, Matt - 496, 56 and 109 (1966).

9. G. K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, 1956).

10. S. V. Iordanskii and A. G. Kulikovskii, Soviet Phys. J.E.T.P. 19, 499(1964)

11. E. Frieman, J. Math. Phys. 4, 410 (1963).

12. G. Sandri, Ann. Phys. 24, 332 and 380 (1963).

13. M. J. Lighthill, J. Inst. Math. Applics 1, 1 (1965).

14. G. Sandri, Il Nuovo Cimento, Series X, 36, 67 (1965).

15. Allan N. Kaufman, private communication.

16. A. Papoulis, Probability, Random Variables and Stochastic Processes, McGraw-Hill Publishing Co., (1965).

† In this analysis it is not assumed that $\underset{\approx}{H}_0$ is Hermitean since this is only a sufficient condition for the reality of $\{\omega_\alpha\}$.

‡ In general, for a spatially homogeneous ensemble, $\langle A_\alpha(\underset{\sim}{k},t)\rangle \equiv {}_1G_\alpha(t)\delta(\underset{\sim}{k})$; we make the additional assumption that ${}_1G_\alpha(t) = 0$.