DESIGN OF LINEAR AND NONLINEAR CONTROL SYSTEMS VIA STATE VARIABLE FEEDBACK, WITH APPLICATIONS IN NUCLEAR REACTOR CONTROL

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Prepared Under Grant NsG-490
National Aeronautics and Space Administration

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PREFACE

This report represents the completion of one phase of the study of control system design, a study sponsored by the National Aeronautics and Space Administration under Grant NsG-490 on research in and application of modern automatic control theory to nuclear rocket dynamics and control. The report is intended to be a self-contained unit and therefore repeats some of the work presented in previous status reports.

Portions of the work were submitted to the Department of Electrical Engineering at the University of Arizona in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

A method for designing linear closed loop control systems, and nonlinear closed loop control systems having a single, memoryless, nonlinear and/or time-varying gain with an input-output characteristic confined to the first and third quadrants is developed. Nonlinear systems designed by the proposed method have the properties of absolute stability and bounded outputs for bounded inputs for any nonlinear gain of the type considered. Systems in which the gain is time-varying also have these properties if the time-varying gain is constrained to a finite sector in the first and third quadrants. The linear part of the open loop system can have no more than one integration, and all other poles must have negative real parts. In its basic form, the design method requires that the nonlinear and/or time-varying gain be located at the input end of the plant to be controlled.

The design procedure is based on feeding back all the state variables through constant linear gain elements. An equivalent feedback transfer function, $H_{eq}(s)$, from the output is used to determine n (n is the order of the linear part of the open loop system) linear algebraic equations which can be solved for the feedback coefficients. $H_{eq}(s)$ has n-1 zeros which are forced to be equal to n-1 of the poles of the linear part of the forward transfer function, G(s). It is shown that $G(s)H_{eq}(s)$ has one pole and not more than one zero. Thus the Popov frequency criterion for absolute stability is satisfied for all gains of the type considered.

An equivalent system for input-output relations which consist of a first order nonlinear and/or time-varying part in series with an n-1 order stable, linear, time-invariant part is derived. This equivalent system is used to determine the closed loop response for a known input to a system designed by the proposed method. It is also used to show that the closed loop system has n-1 poles equal to n-1 poles of the linear part of the open loop system. This suggests another key step in the design procedure, that of forcing n-1 of the open loop poles to be equal to n-1 desired closed loop poles. Closed loop poles on or near the imaginary axis are not permitted due to structural stability problems.

Problems which might be encountered in applying the design procedure to physical systems are discussed, including applications to multiregion nuclear reactor control and the structural stability problem mentioned above. Modifications in the basic design procedure which might be used when all the state variables cannot be fed back, or when the nonlinearity is not in the proper location for the basic procedure to apply, are discussed.

The Popov theorem for absolute stability and the design of linear systems for a desired closed loop response by feeding back all the state variables are included as background material. The matrix formulation is developed for designing linear systems, and the procedure is extended to the design of linear gain insensitive systems. The procedure for designing linear gain insensitive systems is then extended to nonlinear and/or time-varying systems.

The proposed design procedure for nonlinear and/or time-varying systems is applicable to systems of any order. Examples are given to illustrate the design and analysis procedures. The results of an analog computer simulation of a system designed by the proposed method are given.

CHAPTER I

INTRODUCTION TO THE PROBLEM

Introduction

The performance required of control systems and devices by modern technology has resulted in the development of complicated systems and devices which are not amenable to analysis and synthesis by the classical linear techniques. In addition to the undesired nonlinearities which arise in these systems, nonlinearities are often introduced purposely in order to realize the desired performance better or more economically. No general methods for the analysis and synthesis of such nonlinear systems exist. Rather, the methods in use today can be applied only to particular classes of systems or to particular applications. This is not surprising because of the wide variety of nonlinear systems.

Increased interest is also being manifested in systems with time-varying parameters. One obvious reason for this interest stems from the space program where certain parameters of the control system may vary over very wide ranges as the air density and temperature through which a vehicle is moving changes rapidly. If a parameter variation is dependent on the input to the system, the system is nonlinear. If the variation is caused by some effect other than the input, and if this effect can be expressed independently of the input, the system is time-varying. The comment on the lack of general analysis and synthesis techniques for nonlinear systems applies to time-varying systems as well.

A disproportionate amount of control theory has always been directed towards stability analysis. This is particularly true of nonlinear control theory. The control engineer is handicapped by the lack of synthesis procedures which assure not only the stability of the resulting system, but other desirable operating characteristics as well. The development of a proposed synthesis procedure which can be used to design certain nonlinear and/or time-varying systems is the subject of this dissertation.

Historical Background

The practices and techniques of classical control theory involve primarily frequency domain methods. The trend in modern control theory has been away from these frequency domain methods to analysis and design procedures based on time domain methods. State space concepts and techniques involving matrix equations are becoming increasingly important. Kalman (1964) has used such techniques to show that a linear system subject to a quadratic performance index can be made optimum by feeding back all the state variables through constant coefficients. This result has been a key influence in the proposals by Morgan (1963, 1966) and Schultz (1966) that linear systems be designed for a desired closed loop response by feeding back all the state variables in the proper linear combination.

Other recent developments in control theory involve analysis and design procedures for nonlinear systems. Prior to the past fifteen to twenty years, practically all the literature on the analysis and synthesis of feedback control systems was confined to the consideration of linear systems. Valuable techniques have been developed and made available for the study of linear system characteristics. In recent years, an increasing awareness on the part of control engineers that restriction of thinking to linear systems imposes unnecessary limitations on the design of control systems has resulted in a concerted effort to develop corresponding analytical techniques for nonlinear systems. A completely general theory for nonlinear systems, which behave differently for different inputs, appears to be virtually impossible at the present time because of the mathematical difficulties involved. Although some progress has been made in the theoretical aspects of the problem, most of the procedures which look good in theory become unwieldy when applied to practical systems of order higher than first or second. Also, most of the techniques that have been developed apply only to certain restricted situations. Some of the more widely known methods in use today are: (1) The linearization of nonlinearities about some operating point and the application of linear theory to the resulting system. (2) Graphical methods. (3) Numerical methods. (4) Describing function analysis. (5) Computer methods, both analog and digital. (6) Second method of Liapunov. (7) Popov theory. (8) Optimization techniques based on the maximum principle of Pontryagin and dynamic programming.

Evidence of the increased interest in time-varying systems is apparent in the published results of Rozenvasser (1963), Bongiorno (1963), Sandberg (1964), and Narendra and Goldwyn (1964). Higgins (1966) shows how the work of the last three are related to the Popov criterion.

The Popov theory is of primary interest here, with the isocline method and Liapunov's second method also being used. The Liapunov method illustrates the trend of modern control theory to time domain analysis and design. The Popov stability criterion is a frequency domain criterion, but it is derived from time domain equations. The relationship between Popov's results and the second method of Liapunov has been established by Yakubovich (1962) and Kalman (1963a).

The isocline method is a basic graphical method which applies directly to a first order equation of the form $\dot{x} = f(x,t)$. It is applicable to second order equations of the form $\ddot{x} + f(\dot{x},x) = 0$. It is known as the phase plane method when used with second order systems and is discussed in most books dealing with nonlinear systems, including Truxal (1955) and Gibson (1963).

Liapunov developed his Second Method in the late nineteenth century, but it was not until the 1940's in Russia and the early 1960's in this country that control engineers became interested in the theory. Standard English language references are the books by Hahn (1963) and LaSalle and Lefshetz (1961), and the paper by Kalman and Bertram (1960). The method has evoked widespread interest because of its general nature. However, because of the difficulty in finding Liapunov functions (especially the best one), it has not been possible to apply it generally to systems of order higher than two or three. Some of the better known methods for generating Liapunov functions are found in Letov (1961), Schultz and Gibson (1962), Margolis and Vogt (1963), and Gibson (1963).

The theory of Popov (1961) involves a frequency domain stability criterion which has a simple graphical interpretation. Its usefulness is not limited by the order of the system. It provides absolute stability information and therefore cannot be used to determine regions of stability as in the Second Method. Other references on the Popov theory are Aizerman and Gantmacher (1964) and Lefshetz (1965). Extensions to the Popov theory involving conditions on the slope of the nonlinearity have been made by Yakubovich (1965), Brockett and Willems (1965a, 1965b), Dewey and Jury (1966), and Dewey (1966). The interpretation of the criteria reported in these papers is much more difficult than that for nonlinearities with no restriction on the slope. The results of Dewey appear to be in the most useful form, while those of Brockett and Willems are probably the most general.

Approach to the Problem

In this dissertation, some of the recently developed methods of stability analysis for nonlinear and/or time-varying systems are combined with the modern control concept of feeding back all the state variables to develop a proposed design procedure for a certain class of nonlinear and/or time-varying systems. A method for determining the closed loop response of the resulting system is also developed. The design procedure is applicable to single input, single output, systems containing a single memoryless nonlinear and/or time-varying gain whose input-output characteristic is confined to the first and third quadrants. A significant factor of both the design and analysis procedure is that they are applicable, in a practical sense, to systems of any order.

The design procedure stems from the method of designing linear systems for a desired closed loop response as developed by Schultz (1966). It makes use of both the idea of feeding back all the state variables as suggested by modern control theory and of series compensation as used in classical design procedures. The method of designing linear systems for a desired closed loop response is first extended to a procedure for designing linear gain insensitive systems. This procedure is then used to develop a design procedure for the class of nonlinear and/or time-varying systems given above. The Popov stability criterion is used as a synthesis tool in that the linear part of the system is compensated by feeding back all the state variables in a linear combination such that the Popov stability conditions are always satisfied with no restrictions on the nonlinear and/or time-varying gain. Thus the absolute stability sector of the resulting system includes the entire first and third quadrants. An equivalent system is developed which makes it possible to 1) show that the closed loop system has a bounded output for bounded inputs and 2) determine the output of a particular closed loop system for a known input, regardless of the order of the system. Although the output of the closed loop system depends upon the particular nonlinear and/or time-varying gain characteristics, the system is linearized to the extent that (n-1) of the closed loop poles remain fixed, independent of the gain. This results in considerably more control over the nature of the closed loop response than is possible in the usual nonlinear and/or time-varying system. n refers to the order of the linear part of the system, as discussed in Chapter 2.

The equivalent system mentioned above consists of a first order nonlinear and/or time-varying part and an (n-1)st order time-invariant linear part. The Second Method of Liapunov is used in conjunction with this equivalent system to show that a system designed by the proposed method has bounded solutions, or is Lagrange stable, for bounded inputs.

The isocline method is used to determine the output of the nonlinear and/or time-varying gain from the equivalent system for a given input. With this information, the expression for any other variable in the system can be found by linear transform methods.

Organization

The second chapter discusses the type of nonlinear system to be considered. The stability of such systems is discussed and Popov's stability criterion is given. Also, definitions of terms used in later chapters are given.

Chapter 3 is devoted to linear systems, and in particular to the development of a design procedure for gain insensitive systems. This chapter provides the background for Chapter 4 where the design procedure for gain insensitive systems is extended to nonlinear and/or time-varying systems. A matrix formulation for the state variable feedback method of design is developed and examples are included to illustrate the design procedures.

The basic design procedure of this dissertation is presented in Chapter 4. The procedure for designing gain insensitive systems is extended to systems containing nonlinear and/or time-varying gains. The stability properties of the resulting system are discussed, and an equivalent system is developed which makes it possible to determine input-output relations of the closed loop system. Examples are included to illustrate the design procedure and the determination of the closed loop response.

Chapter 5 discusses the structural stability of the systems designed by the procedure of Chapter 4. That is, it considers the effect on the system's qualitative behavior of small changes in the parameters. The possibility of obtaining more restrictive results when those of Chapter 4 cannot be achieved is discussed. Examples are included to illustrate the proposed modified design procedures.

Chapter 6 contains conclusions and suggestions for further research.

CHAPTER II

DEFINITIONS AND STABILITY CRITERIA

Introduction

The purpose of this chapter is to present certain stability criteria and definitions of terms which are used in the following chapters. The Popov stability criterion is of particular interest as it is used both in the design procedure and in the analysis of the resulting systems. The class of systems to which the criterion applies and which is considered in this dissertation is discussed. The extension of the Popov theory dealing with time-varying systems and with constraints on the slope of the nonlinearity are also given.

System Representation

The system configuration has the form shown in Figure 2-1. $N = \frac{f(\sigma)}{\sigma}$ is a nonlinear gain, $G(s)$ is the transfer function of a linear system whose poles are on the imaginary axis or in the left half plane, σ is the input to the nonlinear gain and $u = f(\sigma)$ is the output. u is also the input control for the linear system. Using the terminology of Aizerman and Gantmacher (1964), the principal case is that case in which all the poles of $G(s)$ are in the left half plane. The particular cases are those cases in which some of the poles of $G(s)$ are on the imaginary axis and the others are in the left half plane. The simplest particular case is that case in which $G(s)$ has a single pole at the origin and the

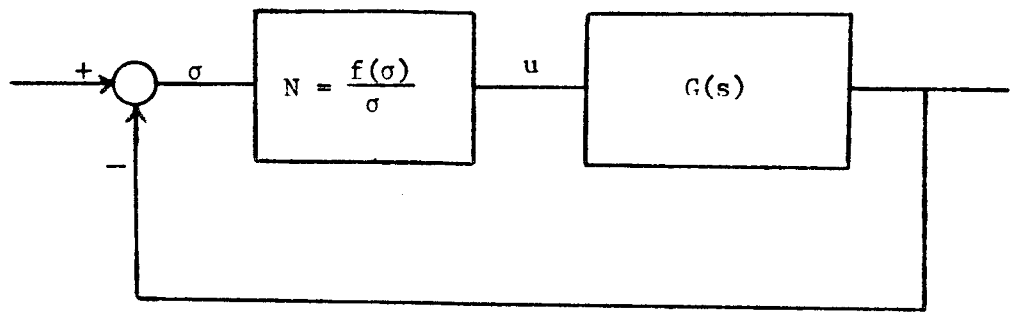


Figure 2-1. n^{th} Order System with One Nonlinearity.

other poles are in the left half plane. The order of G(s) is designated by the letter n.

The system equations are written in the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{b}u \qquad (2\text{-}1)$$

$$u = f(\sigma) \qquad (2\text{-}2)$$

$$\sigma = -\underset{\sim}{k}^T\underset{\sim}{x} \qquad (2\text{-}3)$$

where $\underset{\sim}{x}$ is an n-dimentional state vector, $\underset{\sim}{b}$ is an n-dimensional control vector, $\underset{\sim}{A}$ is the n x n matrix of the linear system G(s), σ and u are the scalar input and output of the nonlinearity, and $\underset{\sim}{k}$ is an n-dimensional feedback vector.

The nonlinearities under consideration satisfy the conditions

$$0 \leq \frac{f(\sigma)}{\sigma} \leq K \qquad (2\text{-}4)$$

for the principal case, or

$$0 < \frac{f(\sigma)}{\sigma} \leq K \qquad (2\text{-}5)$$

for the particular cases, and

$$f(0) = 0 \,. \qquad (2\text{-}6)$$

Definitions

Several definitions of stability and other terms used in this dissertation are given in this section. The origin is assumed to be the equilibrium point in the definitions of asymptotic and absolute stability.

Definition 2-1: The origin is globally asymptotically stable if, for any initial conditions, the system state always returns to the origin as $t \to \infty$.

Definition 2-2: For a given K, the class of systems defined by Equations 2-1 to 2-3 is said to be absolutely stable if for any system in this class, that is, for any $f(\sigma)$ which satisfies Equation 2-4 for the principal case or Equation 2-5 for the particular cases, the origin is globally asymptotically stable.

Definition 2-3: The boundedness of all solutions is described as stability in the sense of Lagrange, or Lagrange stability.

The following criterion for Lagrange stability based on the Second Method of Liapunov is given by Lasalle and Lefschetz (1961):

> <u>Theorem 2-1</u>: Let Ω be a bounded set containing the origin and let $V(\underset{\sim}{x})$ be defined throughout the complement of Ω. If $V \to +\infty$ as $||\underset{\sim}{x}|| \to +\infty$ and if $\dot{V} < 0$ throughout the complement of Ω, then the system $\dot{\underset{\sim}{x}} = \underset{\sim}{X}(\underset{\sim}{x})$ is Lagrange stable.

> <u>Definition 2-4</u>: Structural stability is the property of a physical system such that the qualitative nature of its operation remains unchanged if parameters of the system are subject to small variations.

> <u>Definition 2-5</u>: A system with open loop poles on the imaginary axis is stable-in-the-limit if it is stable for the linear gain $f(\sigma) = \epsilon\sigma$, where ϵ is arbitrarily small. On the s-plane, this means that the imaginary axis poles move into the left half plane for arbitrarily small linear gains in the closed loop system.

> <u>Definition 2-6</u>: A plant is said to be completely controllable if for any t_o each initial state $\underset{\sim}{x}(t_o)$ can be transferred to any final state $\underset{\sim}{x}(t_f)$ in a finite time.

> <u>Definition 2-7</u>: An unforced plant is said to be completely observable on $[t_o, t_f]$ if for given t_o and t_f every state $\underset{\sim}{x}(t_o)$ can be determined from the knowledge of $y(t)$ on $[t_o, t_f]$, where $y(t) = \underset{\sim}{c}^T\underset{\sim}{x}(t)$ is the output.

Definition 2-4 is given by Cunningham (1958) and Definitions 2-6 and 2-7 by Kreindler and Sarachik (1964). It is desirable to express the conditions for a system to be controllable and observable in terms of the coefficients in the system equations. It can be shown that an n^{th} order process characterized by $\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{b}u$ is completely controllable if and only if the vectors $\underset{\sim}{b}$, $\underset{\sim}{A}\underset{\sim}{b}$, ----, $\underset{\sim}{A}^{n-1}\underset{\sim}{b}$ are linearly independent, and completely observable if and only if the vectors $\underset{\sim}{c}^T$, $\underset{\sim}{A}^T\underset{\sim}{c}^T$, ----, $\underset{\sim}{A}^{Tn-1}\underset{\sim}{c}^T$ are linearly independent (Kalman, 1963b).

<u>The Stability Criterion of Popov</u>

The V. M. Popov theorem gives sufficient conditions for the system of Equations 2-1 to 2-3 to be absolutely stable. This theorem is given below:

For the system of Equations 2-1 to 2-3 to be absolutely stable in the sector [0, K] for the principal case, and in the sector (0, K] for the particular cases it is sufficient that there exist a finite real number q such that for all $\omega \geq 0$

$$\mathrm{Re}[(1 + jq\omega)G(j\omega)] + \frac{1}{K} > 0 \qquad (2\text{-}7)$$

and, additionally for the particular cases, that the conditions for stability in-the-limit be satisfied.

A purely geometric formulation of the Popov theorem can be obtained from the above analytic formulation (Aizerman and Gantmacher, 1964). A modified frequency response, $W(j\omega)$, is used where $\mathrm{Re}[W(j\omega)] = \mathrm{Re}[G(j\omega)] = X$ and $\mathrm{Im}[W(j\omega)] = \omega\,\mathrm{Im}[G(j\omega)] = Y$. Then

$$\mathrm{Re}[(1 + jq\omega)G(j\omega)] = X - qY.$$

Condition 2-7 can now be written as

$$X - qY + \frac{1}{K} > 0. \qquad (2\text{-}8)$$

On the W-plane the limiting condition of Equation 2-8 is the equation of a straight line with slope $\frac{1}{q}$ which passes through the point $-\frac{1}{K}$ on the real axis. This line is called the Popov line. Condition (2-8) requires that the plot of the modified frequency response lie entirely in the half plane to the right of the Popov line. Thus the geometric formulation of the V. M. Popov theorem is as follows:

In order that the system defined by Equations 2-1 to 2-3 be absolutely stable in the sector [0, K] for the principal case, or in the sector (0, K] for the particular cases, it is sufficient that there exist in the W-plane a straight line, passing through the point on the real axis with abscissa $-\frac{1}{K}$, such that the modified frequency response $W(j\omega)$ lies strictly to the right of it, and additionally, that for the particular cases the conditions for stability-in-the-limit be satisfied.

Figure 2-2 illustrates the geometric interpretation of the Popov theorem. In the example of Figure 2-2c, the Popov line can be drawn through the origin. Thus the stability sector [0, K] includes the complete first and third quadrants. Figure 2-2d illustrates a case where the stability sector for nonlinear systems as found from the Popov theorem is less than the stability sector for linear systems.

In a practical system, it is reasonable to expect the slope of the nonlinearity to be limited. The question therefore arises as to whether the stability sector of the nonlinear system can be increased by placing restrictions on the slope of the nonlinearity in systems such as that illustrated by Figure 2-2d. This problem has been investigated and criteria developed for extending the stability sector in such cases by Yakubovich (1965a, 1965b), Brockett and Willems (1965a, 1965b), Dewey and Jury (1966), and Dewey (1966). The results of Dewey and Jury and Yakubovich are essentially the same although they were obtained by different approaches. Brockett and Willems' results are in a different form and, though they appear to be more general, are not as easily interpreted.

Dewey's criterion is given here in order to indicate the nature of the results obtained when restrictions are placed on the slope of the nonlinearity. The results reported in the other references have the same general form. Conditions 2-4 (or 2-5) and 2-6 still apply along with the additional restrictions that

1. $|f(\sigma)| \leq M$ (2-9)

2. $-K_1 < \dfrac{df(\sigma)}{d\sigma} < K_2$ (2-10)

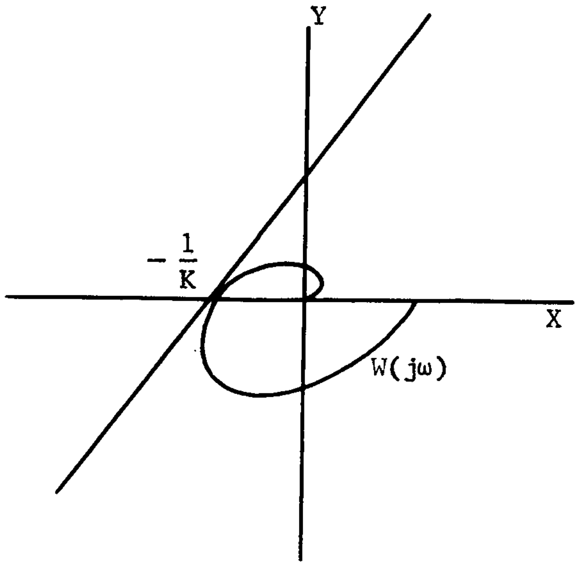


Figure 2-2a. $q > 0$.

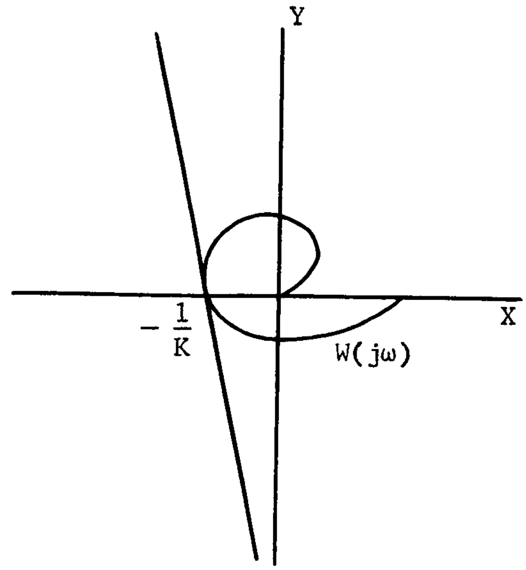


Figure 2-2b. $q < 0$.

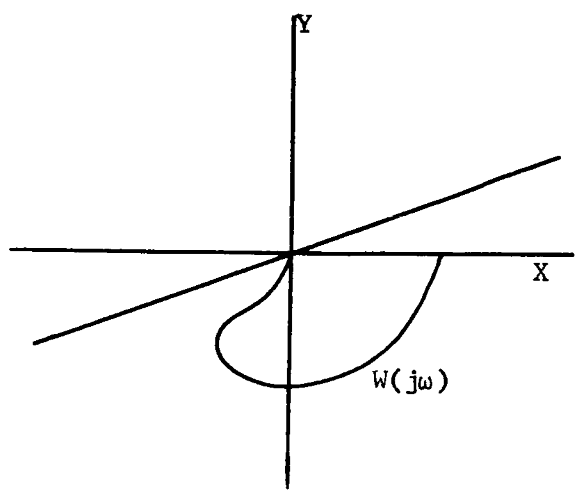


Figure 2-2c. Popov Line Goes Through the Origin. $K = \infty$.

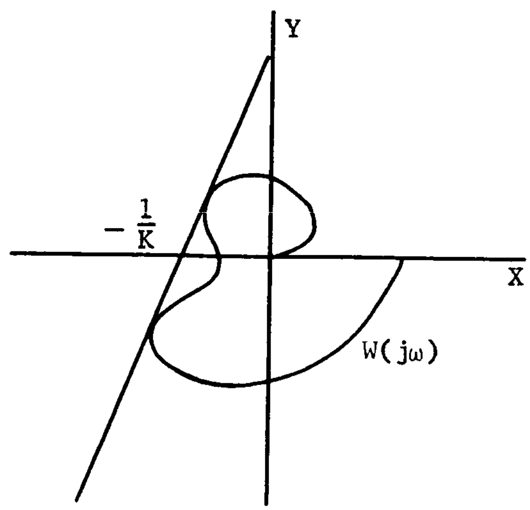


Figure 2-2d. The Popov Stability Sector is Less Than That for a Linear Gain.

Figure 2-2. Illustrations of the Geometric Interpretation of the Popov Criterion.

The restriction that $F(\sigma)$ be bounded is not required in the other papers. This restriction makes possible a simpler criterion. The theorem is as follows:

Theorem: For the system shown in Figure 1, if there exists a finite number q such that for all $\omega \geq 0$,

$$\text{a)}\quad H(\omega) = \mathrm{Re}[j\omega q G(j\omega)] + \omega^2\{1 + (K_2-K_1)\mathrm{Re}G(j\omega)-K_1K_2|G(j\omega)|^2\} \geq 0 \qquad (2\text{-}11)$$

$$\text{b)}\quad G(j\omega) \neq -\frac{1}{K},\ G(0) > -\frac{1}{K}, \qquad (2\text{-}12)$$

then in the principal case, for all nonlinearities with slope restriction $(-K_1, K_2)$ in the sector $[0, K]$ and for all initial states, the response is bounded on $[0,\infty)$ and tends to zero as $t \to \infty$. In the simplest particular case, the theorem remains true for all nonlinearities $f(\sigma)$ in the sector $[\varepsilon, K]$ such that $f(\sigma) - \varepsilon\sigma$ is bounded on $(-\infty, \infty)$ where $\varepsilon > 0$ is arbitrarily small.

Corollary 1: With the slope restriction $f' > -1$, condition a) becomes

$$\mathrm{Re}[j\omega q G(j\omega)] + \omega^2\{\mathrm{Re}G(j\omega)-K_1|G(j\omega)|^2\} \geq 0. \qquad (2\text{-}13)$$

Corollary 2: With the slope restriction $f' > K_2$, condition a) becomes

$$\mathrm{Re}[j\omega q G(j\omega)]-\omega^2\{\mathrm{Re}G(j\omega)+ K_2|G(j\omega)|^2\} \geq 0. \qquad (2\text{-}14)$$

Corollary 3: With the slope restriction $(0, K_2)$, condition a) becomes

$$\mathrm{Re}[j\omega q G(j\omega)] + \omega^2\{\mathrm{Re}G(j\omega) + \frac{1}{K_2}\} \geq 0. \qquad (2\text{-}15)$$

Corollary 4: With the slope restriction $f' > 0$, condition a) becomes

$$\mathrm{Re}[(j\omega q + \omega^2)G(j\omega)] \geq 0. \qquad (2\text{-}16)$$

Remark: In the particular cases, inequalities 2-11, 2-13, and 2-14 can only be satisfied for the simplest particular case. Inequalities 2-15 and 2-16 can be considered for all the particular cases.

The Popov theory has been extended to time-varying systems by Rozenvasser (1963) for the principal case. In the time-varying systems,

$f(\sigma)$ becomes $f(\sigma,t)$. The result is extended to the simplest particular case by Higgins (1966). The stability criterion is

$$\mathrm{Re}G(j\omega) + \frac{1}{K} > 0. \qquad (2\text{-}9)$$

Of the stability theory presented here, this dissertation makes use primarily of the V. M. Popov theorem and its extension to time-varying systems. The extensions of the Popov theory in which the slope of the nonlinearity is considered are used to indicate how the design procedure might be modified in cases where more restrictions on the nonlinearity can be tolerated. It is noted that in the Popov criterion and all its extensions, the object in the analysis is to determine the value of one or more constants which indicate the maximum stability sector for a given G(s). This determines the constraints which the nonlinearity must satisfy if the system is to be absolutely stable. In this dissertation, the approach is to modify G(s) so that no constraints on the nonlinearity are required in order to satisfy the Popov criterion for absolute stability.

CHAPTER III

CLOSED LOOP DESIGN OF LINEAR SYSTEMS VIA STATE VARIABLE FEEDBACK

Introduction

The purpose of this chapter is to present a method for the design of scalar input, scalar output, linear control systems via the state variable feedback (SVF) method. First, the procedure for designing for a desired closed loop transfer function is presented. This procedure is then used as the basis for developing a method for designing linear gain insensitive systems which is extended to certain nonlinear and/or time-varying systems in Chapter 4.

The procedure for designing for a desired closed loop response is formulated from the matrix approach. This procedure is discussed in detail by Schultz (1966) from the standpoint of the block diagram. Although the block diagram approach is more familiar to many control engineers, the matrix approach is more general and does not require the manipulation of the block diagram into any special form. When the system is represented by a block diagram of the form assumed by Schultz, the two methods are equivalent.

After most of this chapter was written, it was discovered by the author that Morgan (1963, 1966) has also proposed the use of state variable feedback for designing linear systems to have a desired closed loop transfer function. He presents the matrix formulation for the design procedure, but the approach is somewhat different from that presented here. The procedure presented here provides for a combination of state variable feedback as called for by modern control theory and series compensation as practiced in classical control theory, thereby combining the advantages of the two and increasing the versatility of the design procedure. While state variable feedback alone can change neither the order of the system nor the location of the zeros, the method developed here can do both.

The design procedure makes use of the fact that the closed loop poles of a linear system may be forced to occur anywhere in the s-plane by feeding back all the state variables in the proper linear combination (Brockett, 1965). The requirement that all the state variables be fed back indicates that the state variables should be chosen to agree with actual physical variables. When it is not possible, or practical, to feed back all the state variables, the calculated values of the feedback coefficients can be used to determine suitable minor loop compensation. When only the output can be fed back, the resulting configuration is similar to that obtained by the Guillemin procedure (Truxal, 1955).

The SVF method is a systematic, completely analytic design procedure in which the analytical work is relatively simple, requiring the solution of n linear algebraic equations, where n is the order of the system. It differs in basic philosophy from both the classical and modern design procedures which are in common use. The common practice in the classical approach to the design of linear control systems is to modify the open loop system in such a way that when the loop is closed the performance is satisfactory. That is, the system is compensated with a view towards realizing certain open loop characteristics which, in general, lead to desirable closed loop performance. Among the extensive literature on this subject are the books by Bower and Schultheiss (1958) and D'Azzo and Houpis (1960). In the SVF method, the system is compensated so as to realize a desired closed loop transfer function which is determined from the performance specifications. Since, ultimately, desirable closed loop response is the goal of the designer, a method of designing for desired closed loop characteristics provides an advantage over one of designing for desired open loop characteristics.

Although the motivation for the SVF method of design stems from a result of modern control theory, the approach is quite different. In the matrix formulation of modern control theory, a system is represented by a set of equations as follows:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{b}u \tag{3-1}$$

$$y = \underset{\sim}{c}^T\underset{\sim}{x} \tag{3-2}$$

Here $\underset{\sim}{x}$ is an n-dimensional state vector, $\underset{\sim}{A}$ is an (n x n) plant matrix, y is the scalar output, $\underset{\sim}{c}$ is an n-dimensional output vector, $\underset{\sim}{b}$ is an n-dimensional control vector, and u is a scalar control. In the optimum control problem the design is based on minimizing a quadratic performance index of the form

$$V(x) = \int_0^\infty (x^TQx + pu^2)dt = \int_0^\infty (x^Trr^Tx + pu^2)dt. \tag{3-3}$$

The well known solution to this problem (Kalman, 1964) is that the optimal control is

$$u = -\underset{\sim}{k}^T\underset{\sim}{x} . \tag{3-4}$$

That is, the optimal control consists of a linear combination of all the state variables. This linear combination may be found by solving the reduced Matrix Ricatti equation,

$$\underset{\sim}{A}^T \underset{\sim o}{R} + \underset{\sim o}{R}\underset{\sim}{A} - \underset{\sim o}{R}\underset{\sim}{b} p^{-1} \underset{\sim}{b}^T \underset{\sim o}{R} + \underset{\sim}{r}\underset{\sim}{r}^T = 0;$$

$\underset{\sim}{k}$ is then found from $\underset{\sim}{k} = \underset{\sim o}{R}\underset{\sim}{b}$. This result suggests a system configuration quite different from that of the classical method of series compensation with unity feedback from the output. In fact, using the frequency domain criterion for optimality as developed by Kalman (1964), it can be shown that very few systems designed by the classical method are optimal for any quadratic performance index. Systems designed by the SVF method may or may not satisfy this criterion. This is discussed further in the section on gain insensitive systems.

A major difficulty in the optimum control approach to design arises from the lack of suitable criteria for specifying the performance index. The SVF method makes use of the system configuration suggested by Equation 3-4, but the design criteria is a desired closed loop transfer function rather than a performance index.

The system configuration resulting from SVF design is illustrated in Figure 3-1a. $G_p(s)$ represents the plant to be controlled, $G_c(s)$ the series compensation, and K an unspecified linear gain. G(s) is defined as $G(s) = G_c(s)G_p(s)$ and KG(s) is the forward transfer function. Since it is usually desirable that $k_1 = 1$ in order that the output will follow the input with as small a steady state error as possible, this value is used throughout this dissertation. The loss of this variable parameter is compensated for by providing the unspecified gain, K, preceding G(s).

Since the SVF method requires that the state variables correspond to actual physical variables which can be measured and fed back, it sometimes happens that a term involving $\dot{u}$ must be included in Equation 3-1. For example, in Figure 3-1b, $\dot{x}_n = -bx_n + aKu + K\dot{u}$, and Equation 3-1 becomes

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{b}u + \underset{\sim}{d}\dot{u}, \tag{3-5a}$$

or

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}\underset{\sim}{u}, \tag{3-5b}$$

where

$$\underset{\sim}{B} = \left[\begin{array}{c|c} \underset{\sim}{b} & \underset{\sim}{d} \end{array}\right]$$

with $\underset{\sim}{b}$ and $\underset{\sim}{d}$ n-dimensional vectors and

$$\underset{\sim}{u} = \left.\begin{matrix} u \\ \dot{u} \end{matrix}\right] .$$

Systems represented by Equation 3-1 will be referred to in this dissertation as Class I and those represented by Equation 3-5a or 3-5b as Class II.

The remainder of this chapter is arranged in the following order:

1. The matrix formulation of SVF design is developed.
2. A design procedure for gain insensitive systems is developed.
3. The procedure to be followed when all the state variables cannot be fed back is discussed.
4. The results are summarized.

The SVF Method - Matrix Formulation

The general procedure is developed on the basis of Class II systems since the results can easily be extended to Class I systems by letting $\underset{\sim}{d} = \underset{\sim}{0}$.

From Figure 3-1b,

$$u = -\underset{\sim}{k}^T \underset{\sim}{x} + r. \tag{3-6}$$

This is the same as Equation 3-4 except for the term r, which represents the scalar input to the closed loop system. Substituting Equation 3-6 into Equation 3-5a, transforming, and solving for $\underset{\sim}{X}(s)$ gives

$$\underset{\sim}{X}(s) = [s\underset{\sim}{I} + \underset{\sim}{d}\underset{\sim}{k}^T s - (\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{k}^T)]^{-1}(\underset{\sim}{b} + \underset{\sim}{d}s)R(s). \tag{3-7}$$

This is combined with Equation 3-2 to give the closed loop transfer function in terms of factors of the form Kk_i (From Figure 3-1, it is evident that K is a factor in each element of $\underset{\sim}{b}$ and $\underset{\sim}{d}$).

$$\frac{Y}{R}(s) = \underset{\sim}{c}^T[s(\underset{\sim}{I} + \underset{\sim}{d}\underset{\sim}{k}^T) - (\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{k}^T)]^{-1} (\underset{\sim}{b} + \underset{\sim}{d}s). \tag{3-8}$$

This transfer function can now be compared with the desired transfer function, and the n algebraic equations that result from equating corresponding coefficients of s can be solved for the Kk_i. With the previous assumption that $k_1 = 1$, K and the other k_i can then be found. Actually, only the denominators of the two transfer functions need to be compared in order to determine the k_i and K, as state variable feedback does not affect the zeros of the transfer function: i.e., the zeros of the closed loop transfer function are the same as the zeros of the open loop transfer function. This can be shown from the equivalent system of Figure 3-1c. The expression for the equivalent feedback transfer function, $H_{eq}(s)$, is derived in the section on gain insensitive systems, and it is shown there that the poles of $H_{eq}(s)$ are also zeros of $G(s)$. Thus the zeros of the closed loop transfer function,

$$\frac{Y}{R}(s) = \frac{KG(s)}{1+KG(s)H_{eq}(s)},$$

must be the same as the zeros of $G(s)$. If the numerator of $G(s)$ is not equal to the numerator of the desired $\frac{Y}{R}(s)$, series compensation is necessary to make the two compatible. The case where series compensation is not necessary is called the simplest case, and the case where series compensation is required is called the general case.

If $G(s)$ is not known, the numerator must be found in order to compare it with the numerator of the desired closed loop transfer function. From Figure 3-1, $KG(s) = \frac{Y(s)}{U(s)}$. Transforming Equations 3-2 and 3-5, combining, and solving for $\frac{Y(s)}{U(s)}$ gives

$$KG(s) = \frac{Y(s)}{U(s)} = \underset{\sim}{c}^T[s\underset{\sim}{I}-\underset{\sim}{A}]^{-1}(\underset{\sim}{b} + \underset{\sim}{d}s). \tag{3-9}$$

This can be written in the form

$$KG(s) = \frac{\underset{\sim}{c}^T\underset{\sim}{F}^a(s)(\underset{\sim}{b} + \underset{\sim}{d}s)}{\det \underset{\sim}{F}(s)} \tag{3-10}$$

where $\underset{\sim}{F}(s)$ is the matrix $[s\underset{\sim}{I}-\underset{\sim}{A}]$ and $\underset{\sim}{F}^a(s)$ is the adjoint of $\underset{\sim}{F}(s)$. Since the adjoint has no poles (only positive powers of s occur in the matrix), it follows that the poles of the transfer function must be zeros of det $\underset{\sim}{F}(s)$. The converse does not necessarily hold, since one or more zeros of the determinant may be cancelled by zeros in the numerator. The necessary and sufficient conditions that the converse hold are that the system be both controllable and observable (Brockett, 1965). It is not usually necessary to find all the elements of $\underset{\sim}{F}^a(s)$ in order to determine the numerator of G(s). For example, if the non-zero elements of $\underset{\sim}{c}$ are designated c_i and the non-zero elements of $\underset{\sim}{d}$ are designated d_j, the only elements of the adjoint matrix that affect the numerator of G(s) are the elements in the i^{th} rows and the j^{th} columns. In systems where $y = x_1$, the only non-zero element of $\underset{\sim}{c}$ is c_1. Therefore no elements in any row other than the first affect the numerator of G(s). If it is desirable to determine the complete inverse matrix, the Leverrier algorithm (Gantmacher, 1959) provides an orderly procedure for the simultaneous computation of the coefficients of the characteristic polynomial and the adjoint matrix, and is adaptable to machine computation.

Equation 3-8 is now written in the form

$$\frac{Y}{R}(s) = \frac{\underset{\sim}{c}^T \underset{\sim}{F}_k^a(s)(\underset{\sim}{b} + \underset{\sim}{d}s)}{\det \underset{\sim}{F}_k(s)} \tag{3-11}$$

where $\underset{\sim}{F}_k(s)$ is the matrix $[s(\underset{\sim}{I} + \underset{\sim}{d}\underset{\sim}{k}^T) - (\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{k}^T)]$ and $\underset{\sim}{F}_K{}^a(s)$ is the adjoint of $\underset{\sim}{F}_k(s)$. Assuming that the numerator of G(s) is known, only the denominator of Equation 3-11 must be determined in order to find $\frac{Y}{R}(s)$ in terms of the k_i and K, and this is equal to det $\underset{\sim}{F}_k(s)$.

The above discussion indicates the general procedure of SVF design. Before outlining the specific procedure it is shown that the algebraic equations which must be solved for the Kk_i are always linear. The proof makes use of the following theorem (Nering, 1963):

> If A' is the matrix obtained from A by adding a multiple of one row (or column) to another, then det A' = det A.

Since K is a factor of the elements of both $\underset{\sim}{b}$ and $\underset{\sim}{d}$, these vectors are written as

$$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} Kb'_1 \\ \vdots \\ Kb'_n \end{bmatrix}, \qquad \underset{\sim}{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} Kd'_1 \\ \vdots \\ Kd'_n \end{bmatrix}.$$

The matrix of Equation 3-8 has the form

$$\begin{bmatrix} [s(1+Kk_1d'_1)-a_{11}+Kk_1b'_1] & [sKk_2d'_1-a_{12}+Kk_2b'_1] & \cdots & [sKk_nd'_1-a_{1n}+Kk_nb'_1] \\ [Kk_1d'_2s-a_{21}+Kk_1b'_2] & [s(1+Kk_2d'_2)-a_{22}+Kk_2b'_2] & \cdots & [sKk_nd'_2-a_{2n}+Kk_nb'_2] \\ \vdots & & & \\ [Kk_1d'_ns-a_{n1}+Kk_1b'_n] & [Kk_2d'_ns-a_{n2}+Kk_2b'_n] & \cdots & [s(1+Kk_nd'_n)-a_{nn}+Kk_nb'_n] \end{bmatrix}$$

where the a_{ij} are the elements of the $\underset{\sim}{A}$ matrix. Now the n^{th} row multiplied by $\dfrac{sd'_1 + b'_1}{sd'_n + b'_n}$ is subtracted from the first row. If $\delta_{ij} = 0$ for $i \neq j$ and 1 for $i = j$, then the j^{th} element of the first row is given by the following expression:

$$s\left[\delta_{1j} - \delta_{nj}\frac{sd'_1 + b'_1}{sd'_n + b'_n}\right] - a_{1j} + a_{nj}\frac{sd'_1 + b'_1}{sd'_n + b'_n}$$

This process is repeated for each of the first (n-1) rows, with the n^{th}

row multiplied by $\frac{sd_i' + b_i'}{sd_n' + b_n'}$ being subtracted from the i^{th} row. Then the j^{th} element of the i^{th} row is given by the following expression:

$$s\left[\delta_{ij} - \delta_{nj}\frac{sd_i' + b_i'}{sd_n' + b_n'}\right] - a_{ij} + a_{nj}\frac{sd_i' + b_i'}{sd_n' + b_n'} \tag{3-12}$$

According to the above theorem, the determinant is not changed by these operations. Since Kk_i is not a factor in the terms of Equation 3-12, it follows that the factors Kk_i appear only in the n^{th} row of the new matrix. Thus the determinant contains only first order terms in the Kk_i and has the form

$$\begin{aligned}\det[s(\underset{\sim}{I} + \underset{\sim}{d}\underset{\sim}{k}^T) - (\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{k}^T)] &= f_n(Kk_1,\cdots,Kk_n)s^n \\ &+ f_{n-1}(Kk_1,\cdots Kk_n)s^{n-1} + \cdots + f_1(Kk_1,\cdots,Kk_n)s \\ &+ f_0(Kk_1,\cdots,Kk_n),\end{aligned} \tag{3-13}$$

where the functions $f_0,\cdots,f_n$ are linear in the Kk_i. The denominator of the desired closed loop transfer function can be written in the form,

$$P(s) = s^n + P_{n-1}s^{n-1} + \cdots + P_1 s + P_0. \tag{3-14}$$

Equating corresponding coefficients of Equations 3-13 and 3-14 gives a set of n linear algebraic equations which are linear in Kk_i and can be solved for K and $\underset{\sim}{k}$ (k_1 is usually set equal to unity as noted previously, so only n unknowns occur in the n equations).

In the general case, it is not required that the numerators of $G_p(s)$ and $\frac{Y}{R}(s)$ be compatible, although $\frac{Y}{R}(s)$ may never have a pole-zero excess less than that of $G_p(s)$, as this would require a compensator in which the numerator is of higher order than the denominator. Series compensation is used in addition to state variable feedback in order to realize the desired response. This usually means adding pole-zero pairs, with the zeros being required to shape the closed loop frequency response curve or to provide the required velocity error coefficient (Truxal, 1955).

The zeros of the compensator are assumed to be known since, unless they cancel a pole in $G_p(s)$, they will also be zeros of $\frac{Y}{R}(s)$. Each pole-zero pair increases the order of the system, except in those cases where poles or zeros are cancelled, and the number of state variables by one. This means that each pole-zero pair adds two new parameters, the pole location and the feedback coefficient. One of these must be chosen arbitrarily and the other determined along with the other k_i. If the new feedback coefficients are chosen to be zero, a series compensator results, and the locations of the poles must be determined along with the value of the non-zero k_i. If the pole locations are chosen, the additional feedback coefficients must be determined along with the feedback coefficients from the original system.

On the basis of the above discussion, the following design procedures are suggested:

<u>The Simplest Case</u>

1. Describe the system in terms of meaningful, physical state variables and assume these are all available and are fed back through constant gain elements.
2. Choose the desired closed loop response, $\frac{Y}{R}(s)$.
3. From Equation 3-8, find $\frac{Y}{R}$ (s) in terms of K and the k_i.

4. Equate the expressions for the denominator of $\frac{Y}{R}(s)$ from steps 2 and 3 and solve for K and the k_i by equating like powers of s.

5. Use the known values of the system parameters to realize a final system configuration. If all the state variables are not available to feed back, use the calculated values of k to determine suitable minor loop compensation.

The General Case

1. Same as step 1 of the simplest case.

2. Same as step 2 of the simplest case.

3. Add a sufficient number of pole-zero pairs to make $G(s) = G_c(s)G_p(s)$ compatible with the desired $\frac{Y}{R}(s)$. Assume the number added is p.

4. Choose p arbitrary pole and/or feedback coefficients associated with the p new state variables introduced.

5. Same as step 3 for the simplest case.

6. Equate the expressions for the denominator of $\frac{Y}{R}(s)$ from steps 2 and 5 and solve for K and the k_i by equating like powers of s.

7. Same as step 5 for the simplest case.

The following examples illustrate the design procedures. Example 3-1 represents the simplest case of a Class I system. Example 3-2 represents the general case of a Class II system.

Example 3-1: In Figure 3-1a, let $G_c(s) = 1$, and assume that the system is represented by the set of differential equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + 2x_3$$
$$\dot{x}_3 = -4x_3 + 5Ku$$
$$u = -\underset{\sim}{k}^T \underset{\sim}{x} + r$$
$$y = \underset{\sim}{c}^T \underset{\sim}{x} = x_1$$
$$k_1 = 1 .$$

Comparing this with Equation 3-1 yields

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix}, \qquad \underset{\sim}{b} = \begin{bmatrix} 0 \\ 0 \\ 5k \end{bmatrix}, \qquad \underset{\sim}{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} .$$

The desired $\frac{Y}{R}(s)$ is chosen as

$$\frac{Y}{R}(s) = \frac{260}{(s^2 + 4s + 13)(s + 20)} = \frac{260}{s^3 + 24s^2 + 93s + 260} . \qquad (3\text{-}15)$$

From the $\underset{\sim}{A}$ matrix and the $\underset{\sim}{b}$ vector, the matrix of Equation 3-8 is found to be

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s + 1 & -2 \\ 5K & 5Kk_2 & s + 4 + 5\ Kk_3 \end{bmatrix} .$$

Substituting this matrix and the $\underset{\sim}{b}$ and $\underset{\sim}{c}$ vectors into Equation 3-11 with $\underset{\sim}{d} = \underset{\sim}{0}$ gives

$$\frac{Y}{R}(s) = \underset{\sim}{c}^T \, [s\underset{\sim}{I} - (\underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{k}^T)]^{-1}\underset{\sim}{b}$$

$$= \frac{10K}{s^3+(5+5Kk_3)s^2+(4+5Kk_3+10Kk_2)s+10K} . \qquad (3\text{-}16)$$

Equating Equation 3-16 to the desired transfer function of Equation 3-15 gives the following set of linear algebraic equations:

$$10K = 260$$

$$5+5Kk_3 = 24$$

$$4+5Kk_3+10Kk_2 = 93.$$

These equations can be solved for Kk_i, giving

$$K = 26$$

$$Kk_3 = \frac{19}{5}$$

$$Kk_2 = 7.$$

These equations are easily solved for the proper set of feedback coefficients to give the desired closed loop transfer function. The result is given below:

$$K = 26$$

$$k_2 = 0.269$$

$$k_3 = 0.146.$$

<u>Example 3-2</u>: Assume that the system to be controlled is represented by the equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2+u$$

$$u = -\underset{\sim}{k}^T\underset{\sim}{x}+r$$

$$y = \underset{\sim}{c}^T\underset{\sim}{x} = x_1$$

$$k_1 = 1.$$

For this system,

$$G_p(s) = \underline{|1 \quad 0|} \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s(s+2)}. \qquad (3\text{-}17)$$

It is assumed that the desired closed loop transfer function is

$$\frac{Y}{R}(s) = \frac{5(s+4)}{(s^2+2s+2)(s+10)}.$$

The realization of this desired transfer function requires, in addition to state variable feedback, a compensator of the form $\frac{(s+4)}{s+\alpha}$ as shown in Figure 3-2a with the value of either α or the feedback coefficient k_3 to be chosen arbitrarily. If α is chosen to be 10, the system equations become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2+x_3$$

$$\dot{x}_3 = -10x_3+4ku+k\dot{u}$$

$$u = -\underset{\sim}{k}^T\underset{\sim}{x}+r$$

$$y = \underset{\sim}{c}^T\underset{\sim}{x} = x_1$$

$$k_1 = 1.$$

Comparing these equations with Equation 3-5 leads to

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -10 \end{bmatrix}, \qquad \underset{\sim}{b} = \begin{bmatrix} 0 \\ 0 \\ 4K \end{bmatrix},$$

$$\underset{\sim}{d} = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}, \qquad \underset{\sim}{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From the $\underset{\sim}{A}$ matrix and the $\underset{\sim}{b}$ and $\underset{\sim}{d}$ vectors, the matrix of Equation 3-8 is found to be

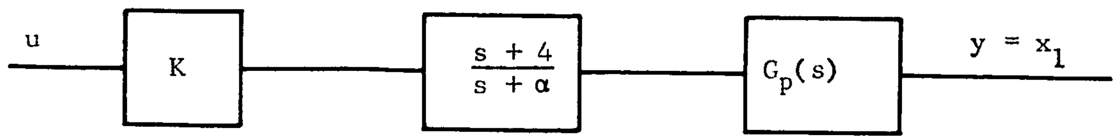


Figure 3-2a. The Plant of Example 3-2 with Series Compensation.

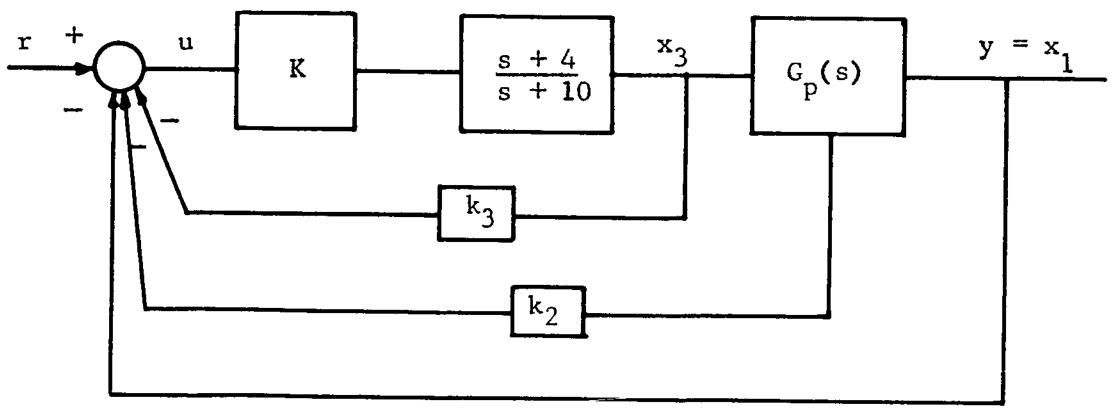


Figure 3-2b. Method of Controlling the Plant of Example 3-2.

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s+2 & -1 \\ Ks+4K & Kk_2s+4Kk_2 & s+Kk_3s+10+4Kk_3 \end{bmatrix} .$$

Substituting this matrix and vectors $\underset{\sim}{b}$, $\underset{\sim}{c}$, and $\underset{\sim}{d}$ into Equation 3-11 gives

$$\frac{Y}{R}(s) = \frac{K(s+4)}{(1+Kk_3)s^3+(12+6Kk_3+Kk_2)s^2+(20+8Kk_3+4Kk_2+K)s+4K} . \quad (3\text{-}19)$$

Equating Equation 3-19 to the desired transfer function of Equation 3-18 gives

$$\frac{4K}{1+Kk_3} = 20$$

$$\frac{20+8Kk_3+4Kk_2+K}{1+Kk_3} = 22$$

$$\frac{12+6Kk_3+Kk_2}{1+Kk_3} = 12 .$$

The solution to this set of equations is

$$k_3 = -\frac{1}{20}$$

$$k_2 = -\frac{3}{10}$$

$$K = 4 .$$

This completes the design procedure. The final system configuration is shown in Figure 3-2b.

Applications in Nuclear Reactor Control System Design

In this section several examples of the application of state variable feedback design to reactor control are presented. Since this new design method makes use of all the system variables, it is particularly amenable to the control of multiregion reactors as well as single region models. With the recent emphasis on spacial kinetics, this method of control is apropos.

Two Temperature Region Reactor

The block diagram for the linear two-temperature region reactor with state variable feedback control and neglecting delayed neutrons, is shown in Fig. 3.3 where

a_i = heat removal coefficients of i^{th} region

k_i = feedback coefficients

K = gain constant of controller

K_i = proportionality constant between power and temperature of the i^{th} region

α_i = temperature coefficient of reactivity of the i^{th} region

x_1 = neutron density or power

x_2 = temperature in region 1

x_3 = temperature in region 2

x_4 = reactivity input from controller

B_{12} = temperature coupling coefficient from region 1 to 2

B_{21} = temperature coupling coefficient from region 2 to 1

γ = reciprocal of the controller time constant

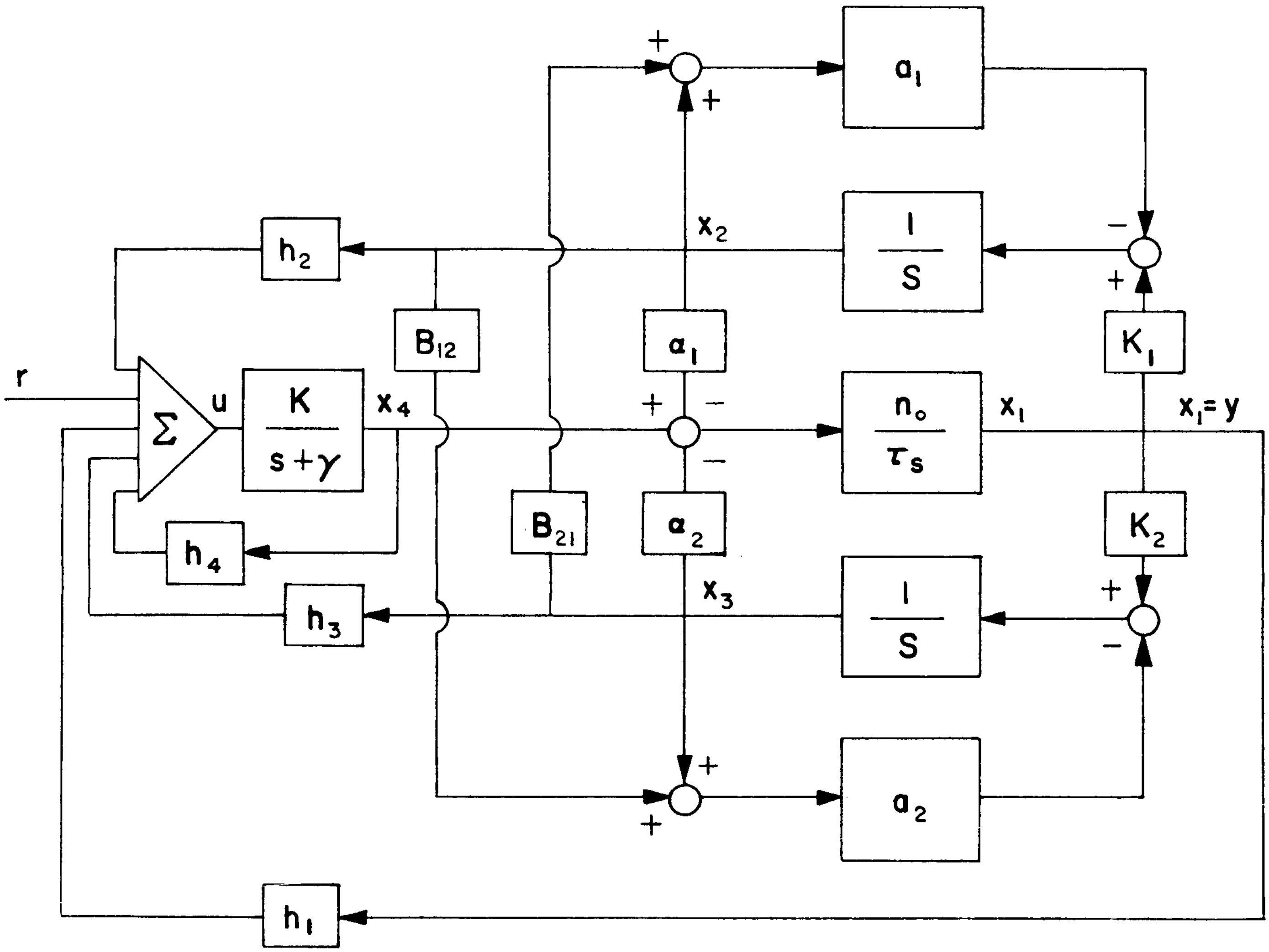


Figure 3.3 Block Diagram of Two Temperature Region Reactor with State Variable Feedback Control

n_o = steady state neutron density or power

τ = effective neutron generation time

The differential equations defining the system are

$$\dot{x}_1 = \frac{-n_o}{\tau}\alpha_1 x_2 - \frac{n_o}{\tau}\alpha_2 x_3 + \frac{n_o}{\tau} x_4$$

$$\dot{x}_2 = K_1 x_1 - a_1(x_2 + B_{21}x_3)$$

$$\dot{x}_3 = K_2 x_1 - a_2(B_{12}x_2 + x_3)$$

$$\dot{x}_4 = -\gamma x_4 + Ku$$

Referring to the equations above, the terms in Eq. (3-8) are given by

$$A = \begin{bmatrix} 0 & \frac{-n_o}{\tau}\alpha_1 & \frac{-n_o}{\tau}\alpha_2 & \frac{n_o}{\tau} \\ K_1 & -a_1 & -a_1 B_{21} & 0 \\ K_2 & -a_2 B_{12} & -a_2 & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} \qquad c^T = \lfloor 1 \quad 0 \quad 0 \quad 0 \rfloor$$

$$k^T = \lfloor k_1 \quad k_2 \quad k_3 \quad k_4 \rfloor$$

where d = 0.

Then

$$[s\tilde{I} - \tilde{A} - \tilde{b}\tilde{h}^T] = \begin{bmatrix} s & \frac{n_o}{\tau}\alpha_1 & \frac{n_o}{\tau}\alpha_2 & \frac{-n_o}{\tau} \\ -K_1 & s+a_1 & a_1 B_{21} & 0 \\ -K_2 & a_2 B_{12} & s+a_2 & 0 \\ Kk_1 & Kk_2 & Kk_3 & s+\gamma+Kk_4 \end{bmatrix} = \tilde{F}$$

Letting the co-factors of $\underset{\sim}{F}$ be F_{ij} and the determinate of $\underset{\sim}{F}$ be det $\underset{\sim}{F}$, the inverse of $\underset{\sim}{F}$ is

$$\underset{\sim}{F}^{-1} = \frac{[F_{ij}]^T}{\det \underset{\sim}{F}}$$

and Eq. (3-8) becomes

$$\frac{Y(s)}{R(s)} = \underset{\sim}{c}^T \underset{\sim}{F}^{-1} \underset{\sim}{b}$$

After some matrix algebra, the transfer function for the case in point is

$$\frac{Y(s)}{R(s)} = \frac{KF_{41}}{\det F}$$

and in terms of the system parameters

$$F_{41} = \frac{n_o}{\tau} [s^2+(a_1+a_2)s+a_1a_2(1-B_{12}B_{21})]$$

and

$$\det \underset{\sim}{F} = s^4(a_1+a_2+\gamma+k_4K)s^3+[(a_1+a_2)(\gamma+Kk_4)+a_1a_2(1-B_{12}B_{21}) + \frac{K_2\alpha_2n_o}{\tau}$$

$$+ \frac{K_1\alpha_1n_o}{\tau} + \frac{n_o}{\tau} k_1K]s^2+\{a_1a_2(\gamma+k_4K)(1-B_{12}B_{21}) + \frac{n_o}{\tau} [-\alpha_1a_1B_{21}K_2$$

$$- \alpha_2a_2B_{12}K_1+K_2\alpha_2(a_1+\gamma+k_4K)+K_1\alpha_1(a_2+\gamma+k_4K)+Kk_1(a_1+a_2)+K_1k_2K+K_2k_3K]\}s$$

$$+ \frac{n_o}{\tau}\{[-\alpha_1a_1B_{21}K_2-\alpha_2B_{12}K_1+K_2\alpha_2a_1+K_1\alpha_1a_2](\gamma+k_4K)+K_1k_3Ka_2B_{12}+k_1Ka_1a_2$$

$$+a_1B_{21}K_2k_2K+k_1K(a_1a_2)B_{12}B_{21}+K_1k_2Ka_2+K_2k_3Ka_1\}$$

As in the previous example it is seen that the zeroes of the system transfer function are independent of k_i, while the pole locations are a function of k_i. Therefore the form of the system time response can be chosen at will by selecting proper values for the feedback coefficients k_i.

Example 3-3: As an example, consider a system that has the following constants.

$$n_o = 10^5 \text{ watts} \qquad K_1 = 2\text{x}10^{-5} \text{degrees/watt sec}$$

$$\gamma = 10 \text{ sec}^{-1} \qquad K_2 = 10^{-5} \text{ degrees/watt sec}$$

$$a_1 = 0.01 \text{ sec}^{-1} \qquad K = 1.0$$

$$a_2 = 0.05 \text{ sec}^{-1} \qquad \tau = 0.1 \text{ sec}$$

$$\alpha_1 = 10^{-3} \text{ per degree} \qquad B_{12} = -0.2$$

$$\alpha_2 = 10^{-4} \text{ per degree} \qquad B_{21} = -1.0$$

For these values of the system parameter

$$F_{41} = 10^6(s^2+0.06s+4\text{x}10^{-4})$$

Therefore the system has two zeroes close to the origin. Suppose that the desired dynamics of the system is given by the second order transfer function

$$\left[\frac{Y(s)}{R(s)}\right]_d = \frac{10^6}{s^2+20s+200}$$

which has well-behaved transient characteristics with a dampening ratio of 0.707 and desirable frequency response. To realize these desired system characteristics, Eq. 3-8 must equal

$$\frac{Y(s)}{R(s)} = \frac{10^6(s^2+0.06s+4\text{x}10^{-4})}{(s^2+20s+200)(s^2+0.06s+4\text{x}10^{-4})}$$

or

$$\frac{Y(s)}{R(s)} = \frac{10^6(s^2+0.06s+4\text{x}10^{-4})}{s^4+20s^3+201s^2+12s+0.08}$$

Equating the coefficients of like powers of s in the denominator of the equation above to det $\underset{\sim}{F}$ and solving the resultant linear algebraic simultaneous Eqs. for k_i yields

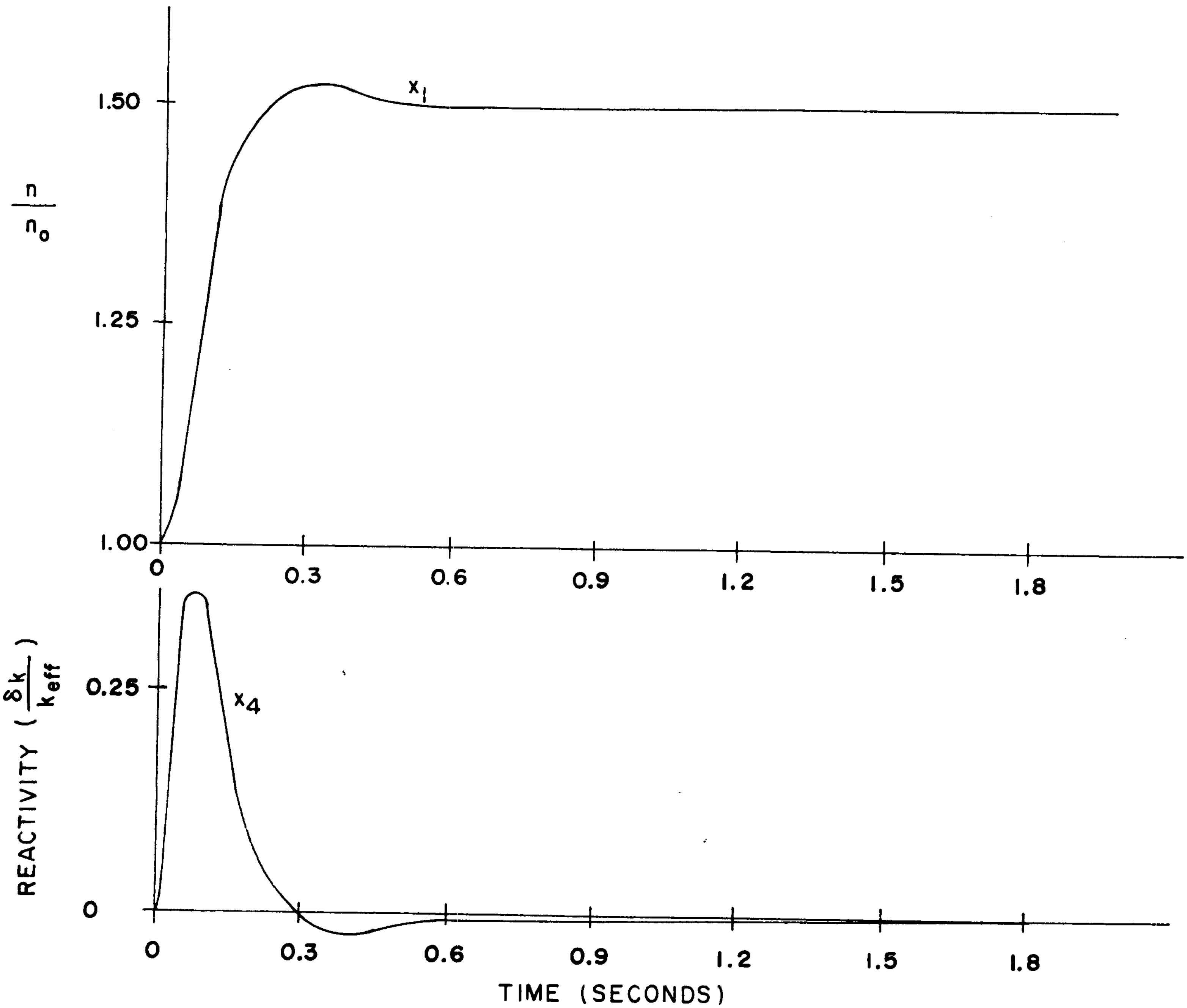


Figure 3.4 State Variable Response to Step Demand in Power for a Two Temperature Region Reactor

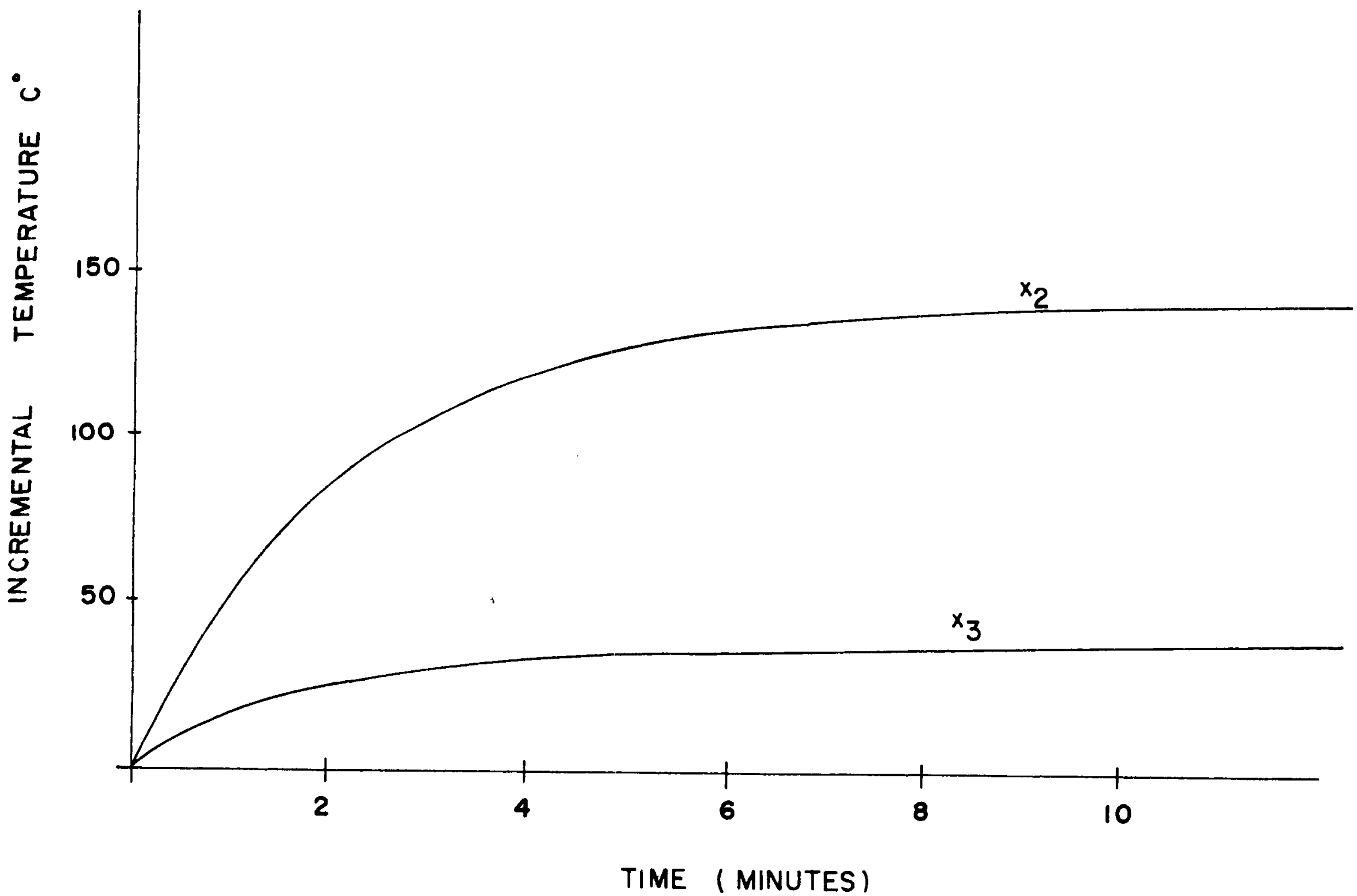


Figure 3.5 State Variable Response to Step Demand in Power for a Two Temperature Region Reactor

$k_1 = +1.98\text{x}10^{-4}$ $\qquad$ $k_3 = -0.002$

$k_2 = -0.02$ $\qquad$ $k_4 = +10.0$

Figures 3.4 and 3.5 give the response of the system variables to a step demand in power. From the response curve of x_1, it is seen that the transient response behavior corresponds exactly to that expected from the desired system transfer function. Further from simulation studies, variations in the feedback coefficients k_2 and k_3 corresponding to the temperature state variables, had virtually no effect on the system dynamics. Neglecting these two feedbacks essentially did not alter the step response. A $\pm$ 20% change in the control rod position feedback coefficient had no noticeable effect on the transient response. Setting this feedback constant equal to zero gave a transient response with a damping ratio of about 0.25. Changes in the steady state power level resulting from a step demand in power were sensitive to variations in k_1, the output state variable feedback coefficient. As k_1 decreased, the steady state power level increased and the system became more damped. For increasing values of k_1 the steady state power level decreased and the system became less damped. From the above it can be concluded that for the case in point only the two feedback coefficients k_1 and k_4 are significant in determining the system dynamics.

Coupled Core Reactors

Another good illustration of the application of this new design technique is the control of a coupled core reactor, a block diagram of which, along with the feedback coefficients, is shown in Fig. 3.6. For convenience, delayed neutrons have been neglected, and the cores are assumed to be identical with the same neutron coupling coefficient. In this model the symbols are as denoted in the previous example except

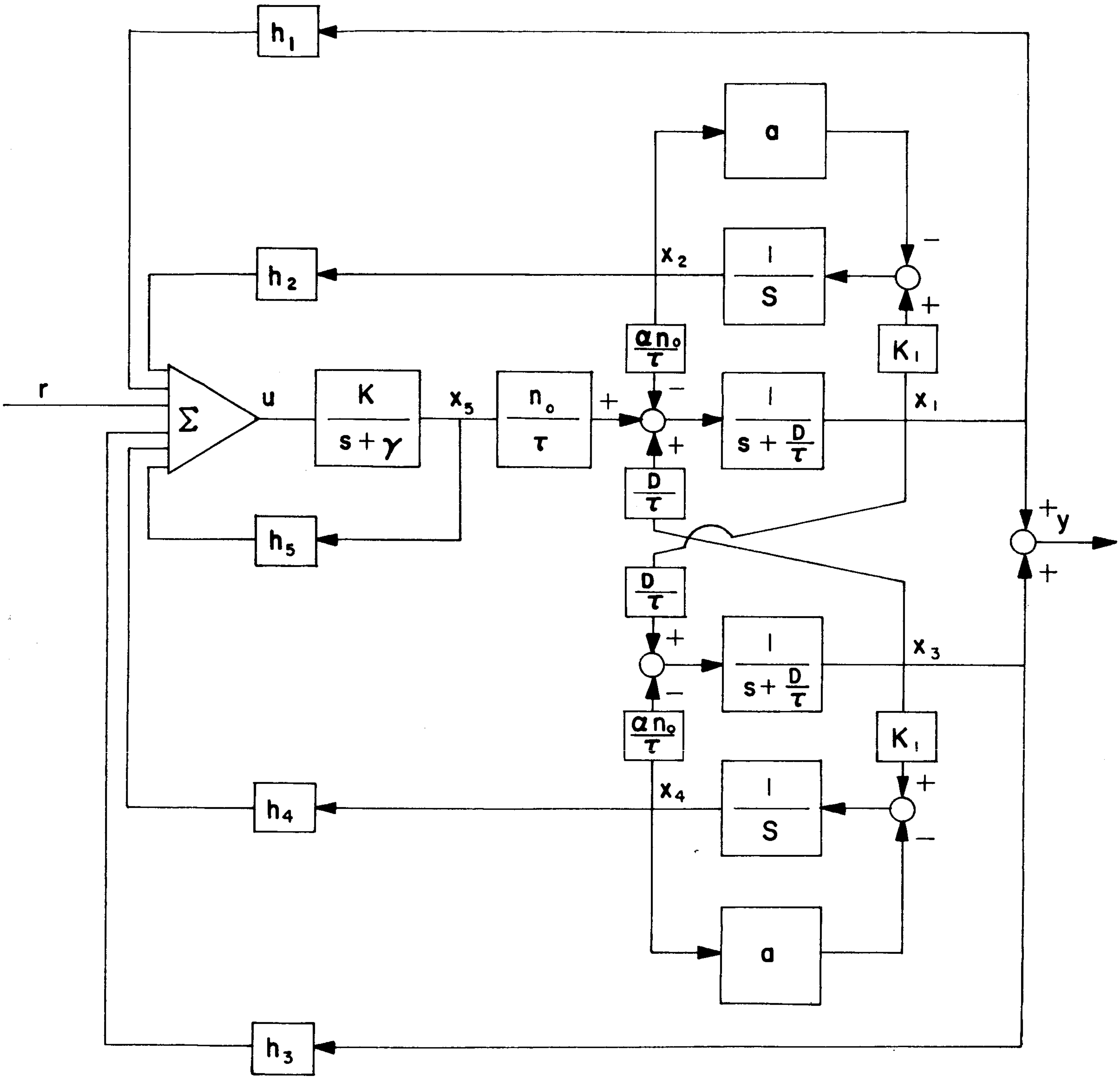


Figure 3.6 Block Diagram of Coupled Core Reactor with State Variable Feedback Control

x_1 = neutron density or power in core 1

x_2 = temperature in core 1

x_3 = neutron density or power in core 2

x_4 = temperature in core 2

D = neutron coupling coefficient

y = total neutron density or power of combined cores

x_5 = reactivity input from controller

The differential equations for this system are

$$\dot{x}_1 = -\frac{D}{\tau}x_1 + \frac{D}{\tau}x_3 - \frac{\alpha n_o}{\tau}x_2 + \frac{n_o}{\tau}x_5$$

$$\dot{x}_2 = K_1x_1 - ax_2$$

$$\dot{x}_3 = -\frac{D}{\tau}x_3 + \frac{D}{\tau}x_1 - \frac{\alpha n_o}{\tau}x_4$$

$$\dot{x}_4 = K_1x_3 - ax_4$$

$$\dot{x}_5 = -\gamma x_5 + Ku$$

$$y = x_1 + x_3$$

In this example

$$[s\underset{\sim}{I} - \underset{\sim}{A} - \underset{\sim}{b}\underset{\sim}{k}^T] = \begin{bmatrix} s+\frac{D}{\tau} & \frac{\alpha n_o}{\tau} & \frac{-D}{\tau} & 0 & -\frac{n_o}{\tau} \\ -K_1 & s+a & 0 & 0 & 0 \\ \frac{-Dn_o}{\tau} & 0 & s+\frac{D}{\tau} & \frac{\alpha n_o}{\tau} & 0 \\ 0 & 0 & -K_1 & s+a & 0 \\ Kk_1 & Kk_2 & Kk_3 & Kk_4 & s+\gamma+Kk_5 \end{bmatrix} = \underset{\sim}{F}$$

$$\underset{\sim}{c}^T = \lfloor 1 \quad 0 \quad 1 \quad 0 \quad 0 \rfloor$$

and the closed loop transfer function is

$$\frac{Y(s)}{R(s)} = c^T F^{-1} b = \frac{K(F_{51}+F_{53})}{\det \underset{\sim}{F}}$$

where again F_{ij} are the co-factors of $\underset{\sim}{F}$ and det $\underset{\sim}{F}$ denotes the determinate of $\underset{\sim}{F}$.

Example 3-4: To demonstrate the above, assume the following values for the system parameters

$$n_o = 10^5 \text{ watts} \qquad K_1 = 10^{-5} \text{ degrees/watt sec}$$

$$a = 10^{-2} \text{ sec}^{-1} \qquad \tau = 0.1 \text{ sec}$$

$$\alpha = 10^{-3} \text{ per degree} \qquad D = 10^{-3}$$

$$\gamma = 10 \text{ sec}^{-1} \qquad K = 1.0$$

and specify the same system transfer function as in the previous example. To realize this desired system transfer function Eq. (3-8) must equal

$$\frac{Y(s)}{R(s)} = \frac{10^6(s+0.01)(s^2+1000s+10)}{(s+0.01)(s^2+1000s+10)(s^2+20s+200)}$$

Following the same procedure as before equating the coefficients of like powers of s in the denominator of the equation above to the corresponding coefficients of powers of s in the denominator of det $\underset{\sim}{F}$ and solving the linear algebraic simultaneous equations thus formed, the values for k_i are determined.

$$k_1 = +2 \times 10^{-4} \qquad k_3 = +2 \times 10^{-4}$$

$$k_2 = -2 \times 10^{-2} \qquad k_4 = -1.92 \times 10^{-2}$$

$$k_5 = +10$$

Figure 3.7 shows the response of the system to a step demand in power for a coupling coefficient D = 0.001, 0.01, and 0.1. As indicated in the figure,

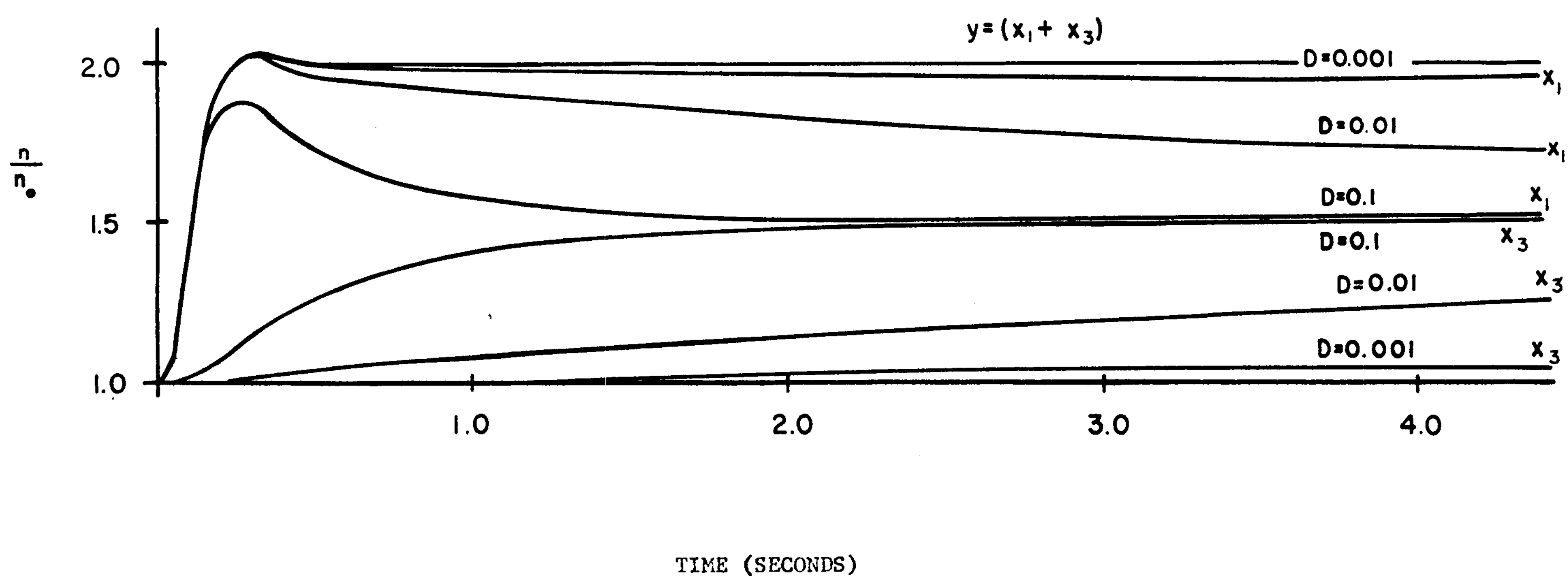


Figure 3.7 Power Response to Step Demand for Various Coupling Coefficients

the desired system response y(t) is insensitive to changes in D. Obviously, the response of the individual cores will depend on D.

As in the previous example, the temperature feedbacks had little influence on the system dynamics and neglecting them made only a very slight change in the system response. Again the final value of the system power response was affected by variations in the power feedback coefficients, increasing for decreasing values in k_1 or k_3. Neglecting the control rod position state variable feedback yields a step response with about a 0.3 damping ratio, and again variations in $k_5 \pm 20\%$ had little effect on the power response. It can be concluded therefore that only the feedback coefficients k_1, k_3 and k_5 have appreciable effects on the system behavior.

Inaccessible State Variables

In the examples presented, it was assumed that all state variables were available. If delayed neutrons are included in the system model, obviously it is impossible to measure neutron precursor concentration for control purposes, and thus this state variable is not available. However, it can easily be generated provided it can be determined from a mathematical relationship. To demonstrate, consider the block diagram of a reactor system with delayed neutrons and state variable feedback, as shown in Fig. 3.8a.

Obviously, the state variable x_1 cannot be measured; however, it can be generated by moving the line at x_1 to x_2 as shown in Fig. 3.8b. Fig. 3.8b reduces to the form in Fig. 3.8c.

Clearly, from the discussion above, state variables that are not available can be generated by placing a frequency dependent element in the feedback path as demonstrated. If, for some reason, one of the state variables cannot be fed back or cannot be generated, then only n-1 poles of the desired closed-loop transfer function can be specified, where n is the order of the system, the other pole falls where it may.

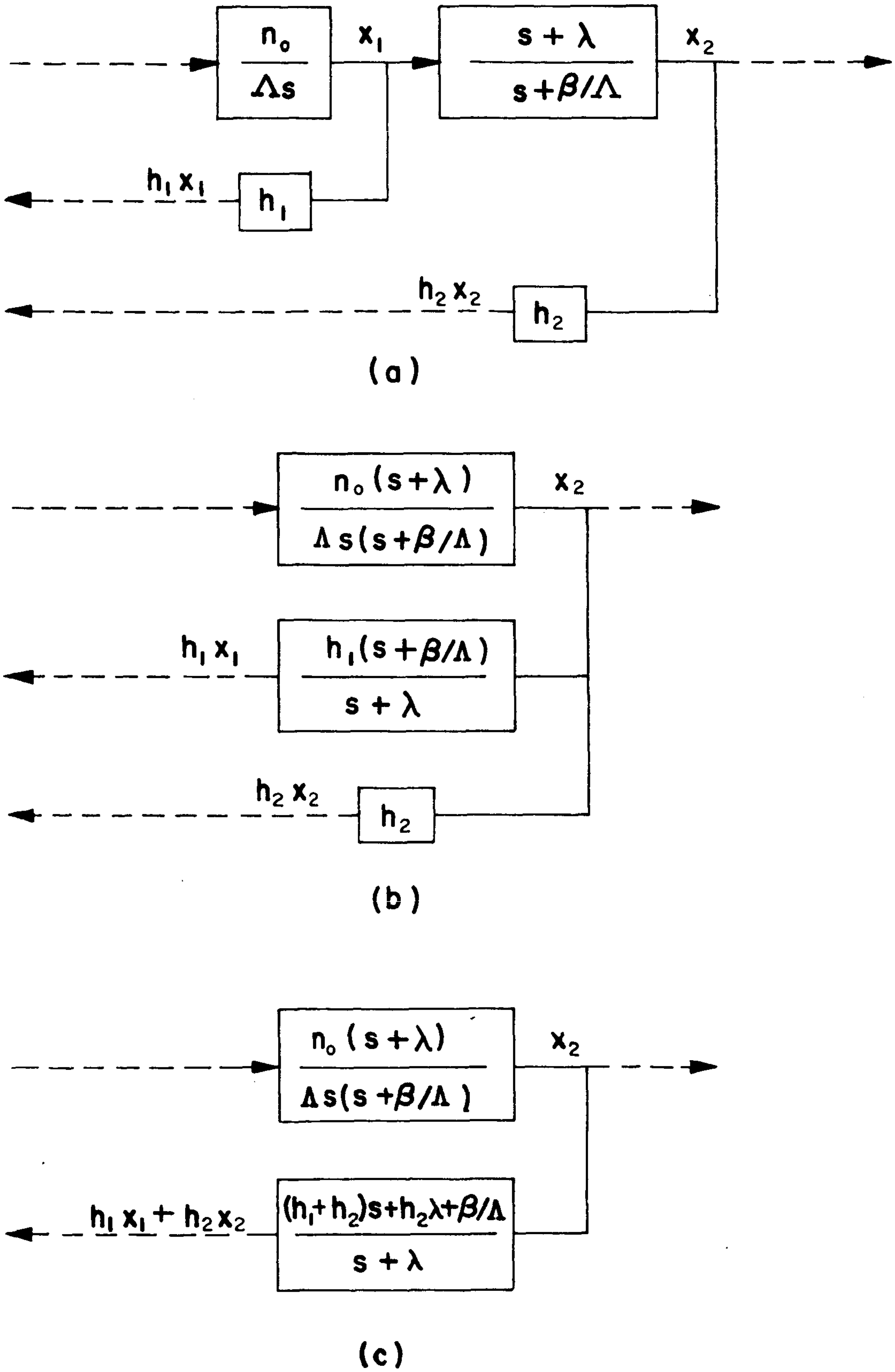


Figure 3.8 Block Diagram of State Variable Generation

Gain Insensitive Systems

A system designed by the method proposed in this section is shown to possess two interesting characteristics: 1) The closed loop transfer function is essentially independent of a gain K located as shown in Figure 3-1b. This is the characteristic that leads to the design of certain nonlinear and time-varying systems in later chapters. 2) The frequency domain criterion for an optimal control subject to a quadratic performance index of the form of Equation 3-3 is always satisfied for some performance index. As mentioned previously, control systems in general do not have this property.

The second item is considered first. A frequency domain criterion for an optimal control subject to a performance index of the form of Equation 3-3 has been shown by Kalman (1964) to be (for a controllable system)

$$|1+\underset{\sim}{k}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}|^2 = 1+|\underset{\sim}{\gamma}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}|^2 \ . \tag{3-20}$$

Here $\underset{\sim}{\Phi}(s) = [s\underset{\sim}{I}-\underset{\sim}{A}]^{-1}$ is the resolvent matrix of the plant to be controlled plus any series compensation. Equations 3-1 and 3-2 are solved for the open loop transfer function $\frac{Y}{U}(s) = KG(s)$. The result is

$$\frac{Y(s)}{U(s)} = KG(s) = \underset{\sim}{c}^T \frac{\underset{\sim}{X}(s)}{U(s)} = \underset{\sim}{c}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}. \tag{3-21}$$

From Figure 3-1c and Equation 3-21,

$$H_{eq}(s) = \frac{\underset{\sim}{k}^T\underset{\sim}{X}(s)}{Y(s)} = \frac{\underset{\sim}{k}^T\underset{\sim}{X}(s)}{\underset{\sim}{c}^T\underset{\sim}{X}(s)} = \frac{\underset{\sim}{k}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}}{\underset{\sim}{c}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}} \ . \tag{3-22}$$

Then,

$$KG(s)H_{eq}(s) = \frac{\underset{\sim}{c}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}\underset{\sim}{k}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}}{\underset{\sim}{c}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}} = \underset{\sim}{k}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b} \tag{3-23}$$

Equation 3-20 can now be written in the form

$$|1+KG(s)H_{eq}(s)|^2 = 1+|\underset{\sim}{\gamma}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}|^2,$$

or

$$|1+KG(s)H_{eq}(s)|^2 > 1. \tag{3-24}$$

Thus for a system to be optimal for some quadratic performance index, it is necessary that inequality 3-24 be satisfied. This is rarely the case in a system designed in the classical manner, as can be seen from Figure 3-9, where curve a represents a typical open loop function, $KG(s)H_{eq}(s)$. In order to be an optimum system, the plot of $KG(s)H_{eq}(s)$ must remain outside the unit circle with center at -1. Such a system is represented by curve b. It is shown below that systems designed by the method of this section always satisfy this condition.

Since the design procedure of this section is extended to the case where the gain K is nonlinear and/or time-varying in Chapter 4, it is desirable that K not appear in the transfer functions used to derive the expression for $H_{eq}(s)$. In order to accomplish this, u' is defined as shown in Figure 3-1c (u' = Ku) and the relationship b = Kb' is used. Equation 3-21 is replaced with

$$\frac{Y(s)}{U'(s)} = G(s) = \underset{\sim}{c}^T\,\frac{X(s)}{U'(s)} = \underset{\sim}{c}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}', \tag{3-21}$$

and Equation 3-22 with

$$H_{eq}(s) = \frac{\underset{\sim}{k}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}'}{\underset{\sim}{c}^T\underset{\sim}{\Phi}(s)\underset{\sim}{b}'} \tag{3-22'}$$

Equation 3-22' is now written in the form

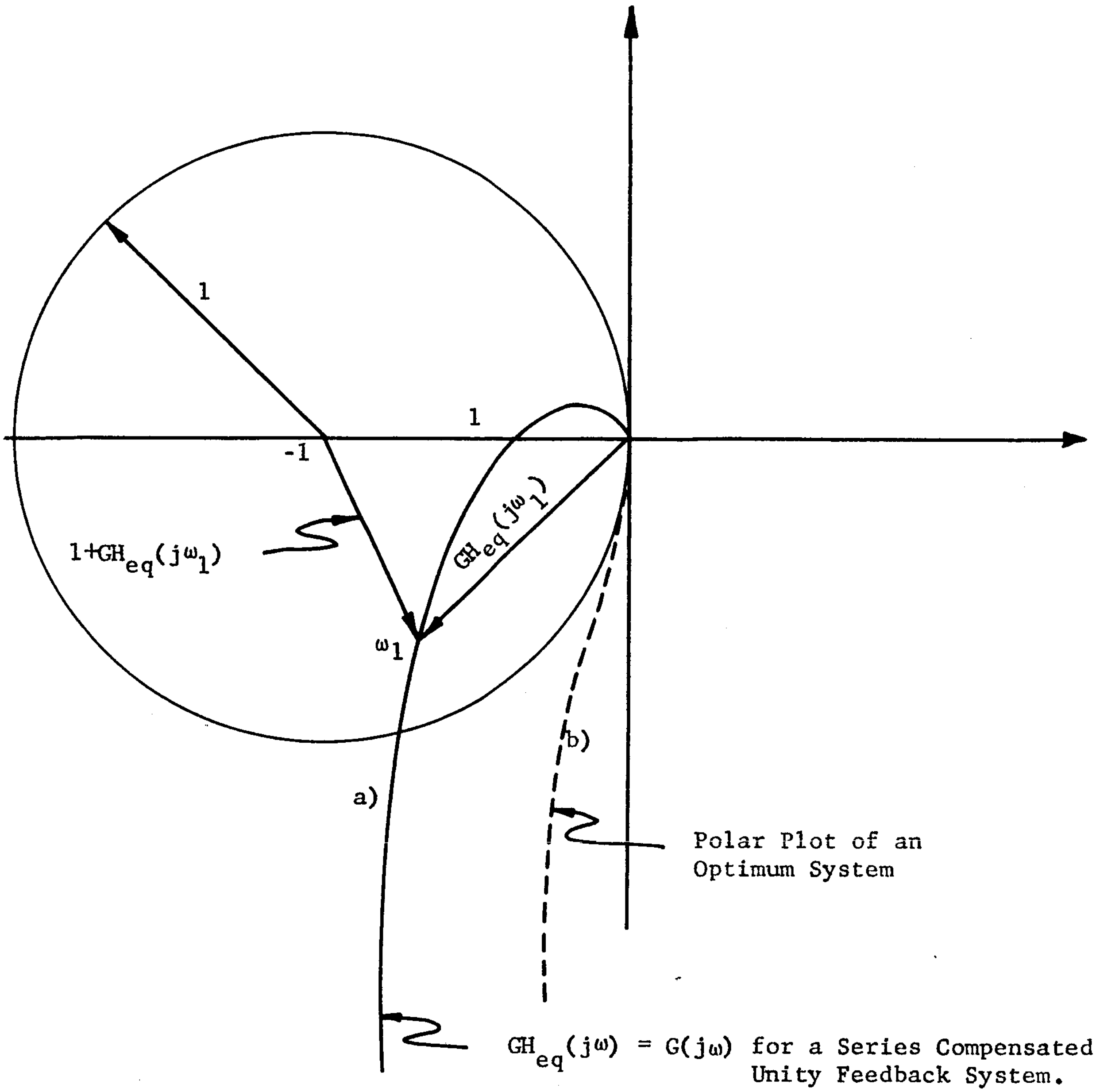


Figure 3-9. Nyquist Diagram of $GH_{eq}(j\omega)$ for a System that Would be Considered Satisfactory in Terms of Conventional Criteria.

$$H_{eq}(s) = \frac{\dfrac{\underset{\sim}{k}^T \underset{\sim}{F}^a(s)\underset{\sim}{b}'}{\det \underset{\sim}{F}(s)}}{\dfrac{\underset{\sim}{c}^T \underset{\sim}{F}^a(s)\underset{\sim}{b}'}{\det \underset{\sim}{F}(s)}} = \frac{\underset{\sim}{k}^T \underset{\sim}{F}^a(s)\underset{\sim}{b}'}{\underset{\sim}{c}^T \underset{\sim}{F}^a(s)\underset{\sim}{b}'} \qquad (3\text{-}25)$$

where $\underset{\sim}{F}^a(s)$ is the adjoint of the resolvent matrix $\underset{\sim}{F}(s)$. The elements of $\underset{\sim}{F}^a(s)$ contain terms of order (n-1) and lower in the numerator and have no poles. If all the k_i are non-zero, it follows that the numerator of $H_{eq}(s)$ will have (n-1) zeros and, since the denominator of Equation 3-25 is the numerator of G(s), that the poles of $H_{eq}(s)$ are equal to zeros of G(s). For Class II systems, $\underset{\sim}{X}(s) = [s\underset{\sim}{I}-\underset{\sim}{A}]^{-1}(\underset{\sim}{b}'+\underset{\sim}{d}'s)u'(s)$ and Equation 3-25 becomes

$$H_{eq}(s) = \frac{\underset{\sim}{k}^T \underset{\sim}{F}^a(s)(\underset{\sim}{b}'+\underset{\sim}{d}'s)}{\underset{\sim}{c}^T \underset{\sim}{F}^a(s)(\underset{\sim}{b}'+\underset{\sim}{d}'s)} \,. \qquad (3\text{-}26)$$

This is given here in order to show that for a system configuration such as that of Figure 3-10, where $\underset{\sim}{b}'$ and $\underset{\sim}{d}'$ are related by a constant, there will be a cancellation in Eq. 3-26 and $H_{eq}(s)$ will not have a pole corresponding to the zero of G(s) at $s = -\frac{1}{\tau_1}$.

The desired $H_{eq}(s)$ is chosen so thât the (n-1) zeros are equal to (n-1) of the n poles of G(s). From the above discussion, it then follows that

$$G(s)H_{eq}(s) = \frac{K'}{s+a} \,, \qquad (3\text{-}27)$$

except for a system configuration such as that shown in Figure 3-10. In this case, since $H_{eq}(s)$ does not have a pole corresponding to the zero of G(s) at $s = -\frac{1}{\tau_1}$,

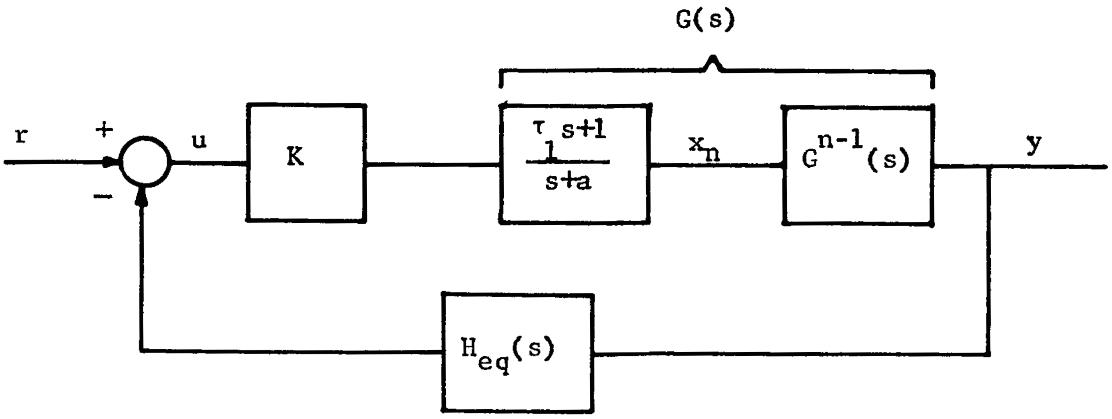


Figure 3-10. System in which $\underset{\sim}{b}$ and $\underset{\sim}{d}$ are Related by a Constant, τ_1.

$$G(s)H_{eq}(s) = \frac{K'(\tau_1 s+1)}{s+a}. \tag{3-28}$$

In the above equations, $-a$ is the pole of $G(s)$ for which there is no corresponding zero in $H_{eq}(s)$, $-\frac{1}{\tau_1}$ is the zero of $G(s)$ for which there is no corresponding pole in $H_{eq}(s)$, and K' and $K'\tau_1$ are the open loop gains for the systems represented by Equations 3-27 and 3-28 respectively.

From Equations 3-27 and 3-28, it follows immediately that the frequency criterion for optimal control is always satisfied, since the polar plot of $KG(s)H_{eq}(s)$ never crosses into the left half plane, and therefore remains outside the unit circle of Figure 3-9.

From Figure 3-1c and Equation 3-27,

$$\frac{KG(s)}{1+KG(s)H_{eq}(s)} = \frac{KN(s)}{\frac{D(s)}{s+a}\,[KK'+s+a]}, \tag{3-29}$$

where $\frac{N(s)}{D(s)} = G(s)$, and $D(s)$ has a zero at $s = -a$. For the case represented by Equation 3-28, this becomes

$$\frac{Y}{R}(s) = \frac{KN(s)}{\frac{D(s)}{s+a}[s+a+KK'\tau_1 s+KK']}. \tag{3-30}$$

In both these equations, it is seen that the $(s+a)$ factor in $D(s)$ is cancelled. It follows that for all cases $(n-1)$ of the closed loop poles are the same as $(n-1)$ of the open loop poles. The n^{th} closed loop pole will have little effect on the nature of the response if K is made large enough, since the residue and time constant associated with it become negligible as it moves far out from the origin.

If the state variables are defined in such a way that the block diagram of Figure 3-11 results, the expression for $H_{eq}(s)$ in Equations 3-25 and 3-26 can be written in the same form as that resulting from the

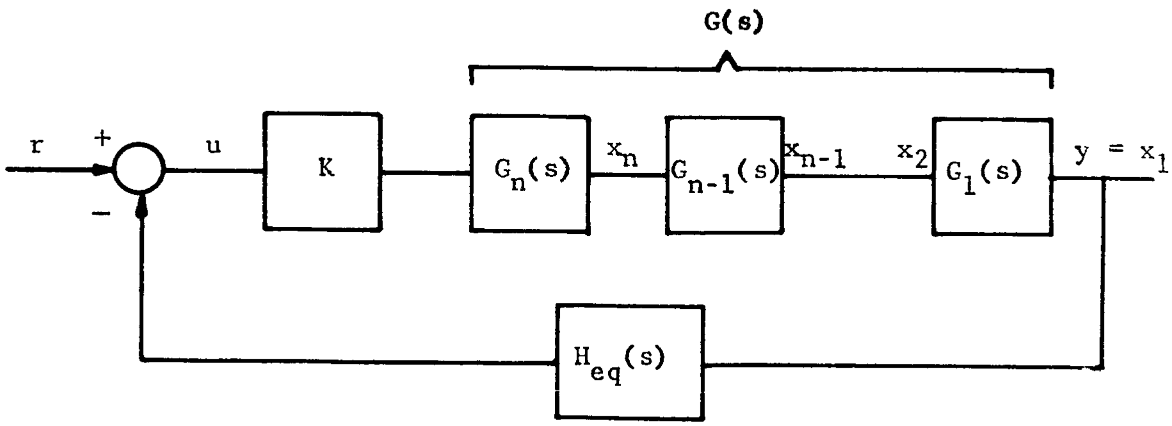


Figure 3-11. System with n First Order Transfer Functions in Series in the Forward Path.

block diagram formulation. Since b' (and d' if Equation 3-26 applies) is non-zero only in the n^{th} element, $H_{eq}(s)$ becomes

$$H_{eq}(s) = \frac{k^T F_n{}^a(s)}{c^T F_n{}^a(s)}, \tag{3-31}$$

where $F_n{}^a(s)$ is the n^{th} column of $F^a(s)$. With the previous assumptions that $c^T = \lfloor 1 \quad 0 \cdot \cdot \cdot \cdot \cdot 0 \rfloor$ and that $k_1 = 1$, Equation 3-31 reduces further to

$$H_{eq}(s) = \frac{f_{n1}{}^a(s)+k_2 f_{n2}{}^a(s)+\cdots+k_n f^a_{nn}(s)}{f_{n1}{}^a(s)}$$

$$= 1+ \frac{k_2 f_{n2}{}^a(s)}{f_{n1}{}^a(s)} +\cdots+ \frac{k_n f_{nn}{}^a(s)}{f_{n1}{}^a(s)} \tag{3-32}$$

where $f_{ni}{}^a(s)$ is the i^{th} element of $F_n{}^a(s)$. Since for this configuration

$$\frac{f_{ni}{}^a(s)}{f_{n1}{}^a(s)} = \left[\frac{X_1(s)}{X_i(s)}\right]^{-1},$$

Equation 3-32 can be written as

$$H_{eq}(s) = 1+k_2 \left[\frac{X_1(s)}{X_2(s)}\right]^{-1} +\cdots+k_n \left[\frac{X_1(s)}{X_n(s)}\right]^{-1}$$

$$= 1+ \frac{1}{G_1(s)} k_2 +\cdots+ \frac{1}{G_1(s)\cdots G_{n-1}(s)} k_n. \tag{3-33}$$

This has the same form as the expression for $H_{eq}(s)$ obtained from the block diagram formulation by Schultz (1966). Thus the block diagram formulation is seen to represent a special case of the more general matrix formulation.

Some restrictions on the $H_{eq}(s)$ that can be realized by feeding back all the state variables are obvious. For example, in Figure 3-11, the zeros of $H_{eq}(s)$ cannot be made equal to the poles of $G_{n-1}(s) \cdots G_1(s)$ since, as seen from Equation 3-33, this would require that $k_n = \infty$. Also, since the output is fed back directly to the input, it follows that $H_{eq}(s)$ must always have a constant term and therefore cannot have a zero root. In a system with one integration, the zeros of $H_{eq}(s)$ are forced to equal the non-zero poles of G(s). In a system with two or more integrations, $H_{eq}(s)$ cannot have (n-1) zeros equal to (n-1) poles of G(s).

On the basis of the above discussion, the following procedure is proposed for the design of gain insensitive systems:

1. Describe the system in terms of meaningful, physical state variables and assume that these are all available and are fed back through constant gain elements.
2. Choose the desired closed loop response $\frac{Y}{R}(s)$.
3. Use a combination of series compensation and feedback to insure that all but one of the open loop poles correspond to the desired closed loop poles. Normally, a pole at the origin is left undisturbed so as not to change the type of the system.
4. Use state variable feedback to force the zeros of $H_{eq}(s)$ to correspond to the altered open loop poles, which are the desired closed loop poles. The required values of the k_i can be found by calculating $H_{eq}(s)$ in terms of the k_i

by any of the above methods and equating this to the desired $H_{eq}(s)$.

5. If all state variables are not available, use the calculated values of $\underset{\sim}{k}$ to determine suitable minor loop compensation.

The following example illustrates the design procedure.

Example 3-5: The system of Example 3-1 is used. It is assumed that the desired locations of two of the closed-loop poles are at $s = -2 \pm j2$. The other closed-loop pole is not specified, but as shown above, moves along the negative-real axis as K is varied. In Example 3-1, the numerator of G(s) was found to be 10. The denominator is

$$\det [s\underset{\sim}{I}-\underset{\sim}{A}] = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & -2 \\ 0 & 0 & s+4 \end{bmatrix} = s(s+1)(s+4).$$

This gives

$$G(s) = \frac{10K}{s(s+1)(s+4)}.$$

Some configuration for G(s) must be assumed before proceeding to step 3 of the design procedure. This is shown in Figure 3-12a. It is necessary to feed back x_2 and x_3 as shown in Figure 3-12b to have open loop poles at the location of the desired closed loop poles of $s = -2 \pm j2$. This requires that

$$s^2+4s+8 = \det \begin{bmatrix} s+1 & -2 \\ K_1 & s+4+K_1k_2' \end{bmatrix}$$

$$= s^2+(5+K_1k_2')s+4+K_1k_2'+2K_1.$$

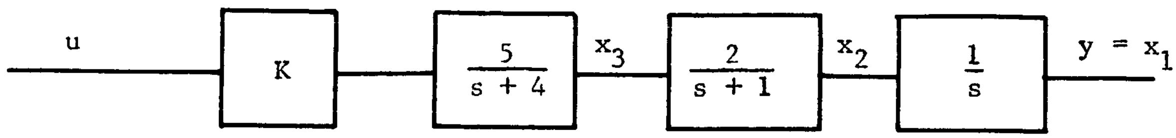


Figure 3-12a. The Plant to be Controlled in Example 3-5.

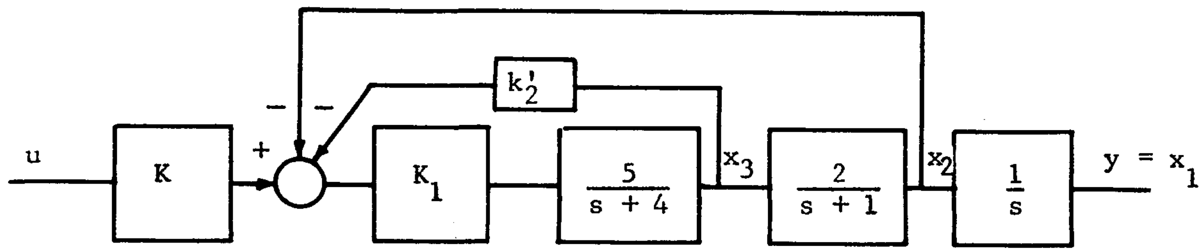


Figure 3-12b. Feedback Configuration Used to Force the Open Loop Poles to be at the Location of the Desired Closed Loop Poles.

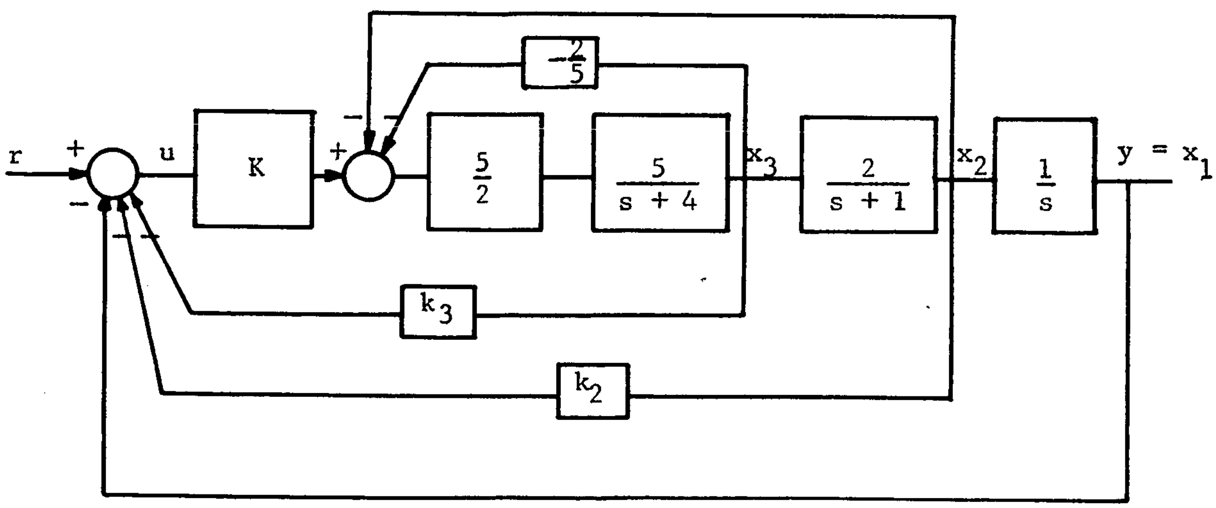


Figure 3-12c. Method of Controlling the Plant of Example 3-5.

Equating coefficients and solving the resulting equations gives

$$K_1 = \frac{5}{2}$$

$$k_2' = -\frac{2}{5}.$$

Proceeding to step 4 of the design procedure, the final feedback configuration required by Step 4 of the design procedure is shown in Figure 3-12c.

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s+1 & -2 \\ 0 & \frac{5}{2} & s+3 \end{bmatrix}.$$

Substituting into Equation 3-25 gives

$$H_{eq}(s) = \frac{k_3}{2}\,[s(s+1)+ \frac{2k_2}{k_3}\,s+ \frac{2}{k_3}\,]. \tag{3-34}$$

Equating this to the desired value of $H_{eq}(s)$,

$$H_{eq}(s) = \frac{k_3}{2}\,[s^2+4s+8],$$

gives

$$\frac{2}{k_3} = 8$$

$$\frac{2k_2}{k_3} = 4\ .$$

Solving this set of equations yields

$$k_3 = \frac{1}{4}$$

$$k_2 = \frac{1}{2}.$$

Applications of Gain Insensitive Design to Reactor Control

In this section, the gain insensitive design procedure outlined in the previous section is applied to the control of the nuclear reactor model with reactivity feedback due to temperature, shown in Fig. 3.13 where

ρ_i = input reactivity

ρ_f = feedback reactivity

n = reactor power

n_o = steady state power level

λ = weighted neutron precursor decay constant

Λ = neutron generation time

K_T = constant relating power to reactivity feedback

a = heat removal coefficient

Observing Fig. 3.13 it is noted that the gain term, $\frac{n_o}{\Lambda}$, is a function of the steady state power level and therefore will vary as the power level changes. This variation will change the pole locations of the closed loop transfer function, $\frac{n(s)}{\rho_i(s)}$ and thus the system dynamics. The design criterion is to determine a state variable feedback control such that the system dynamics is independent of power level. For purpose of illustration the following values are assumed for the system parameters

$$\lambda = 0.1 \quad \frac{\beta}{\Lambda} = 6.4 \quad a = 2 \quad K_T = 5 \times 10^{-9}$$

It is further assumed that the controller dynamics can be neglected in comparison to the time constants of the reactor. The assumption of a perfect controller with a transfer function equal to 1 is reasonable in space reactor systems where the controllers have a very fast time response.

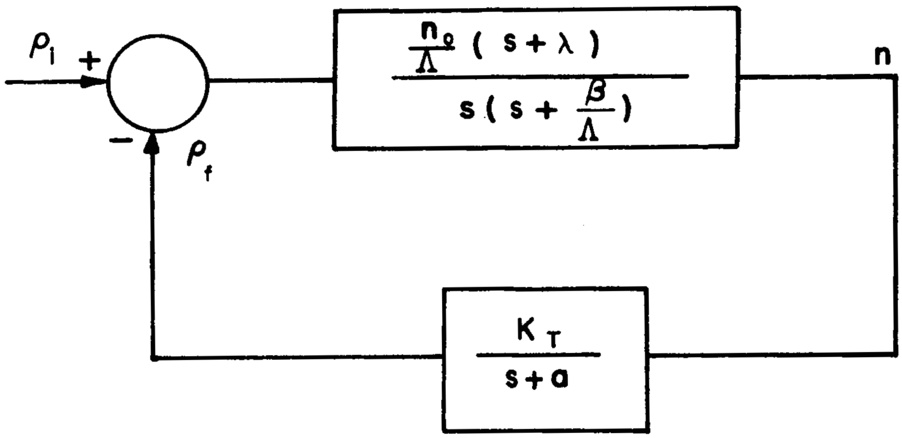


Figure 3.13 Point Reactor with Reactivity Feedback

Applying state variable feedback control, under the assumptions above, to the reactor model in Fig. 3.13 gives a system of the form shown in Fig. 3.14a, which reduces to the system in Fig. 3.14b, where

$$H_{eq}(s) = \frac{k_2 s}{s+0.1} + k_1 + \frac{5\times10^{-9}}{s+2} \tag{3-35}$$

If k_1 and k_2 are large compared to 10^{-9} then the term involving 10^{-9} can be neglected and Eq. (3-35) reduces to

$$H_{eq}(s) = \frac{(k_1+k_2)s+0.1k_1}{s+0.1} \tag{3-36}$$

Letting $k_1 = 1$ and assuming the desired $H_{eq}(s)$ is

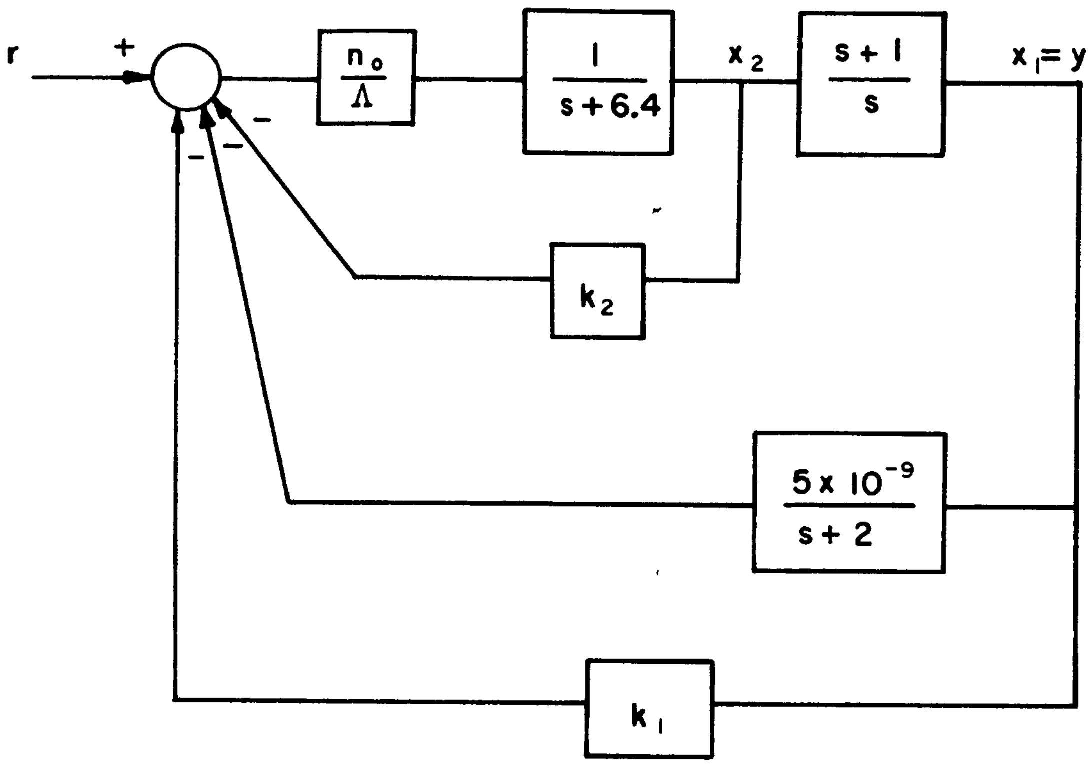
$$H_{eq}(s) = \frac{(k_1+k_2)(s+6.4)}{s+0.1} \tag{3-37}$$

then comparing Eqs. (3-36) and (3-37) gives

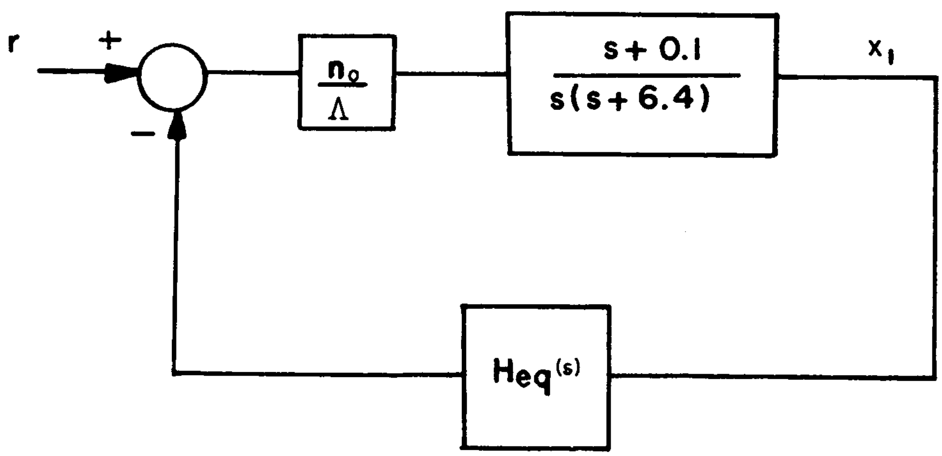
$$k_2 = -\frac{63}{64}$$

Clearly with these values for k_1 and k_2, the assumption of neglecting the terms involving 10^{-9} is valid. The system in Fig. 3.14a now becomes of the form shown in Fig. 3.15.

The transfer function of the system in Fig. 3.15, $\frac{Y(s)}{R(s)}$ has poles at $s = -6.4$ and $s = -\frac{n_o}{64\Lambda}$ and a zero at $s = -0.1$. For values of $\frac{n_o}{\Lambda}$ greater than 10^4, the pole at $-\frac{n_o}{64\Lambda}$ has very little effect on system dynamics. For a neutron generation time of $\Lambda = 10^{-3}$ sec the corresponding power level is 10 watts. Therefore, for power reactors, the system dynamics is virtually independent of the gain $\frac{n_o}{\Lambda}$ and thus the reactor power level. Since it is impossible to measure the state variable x_2, the control is realized by feeding back the reactor output through a lead lag network



(a)



(b)

Figure 3.14 Point Reactor with State Variable Feedback

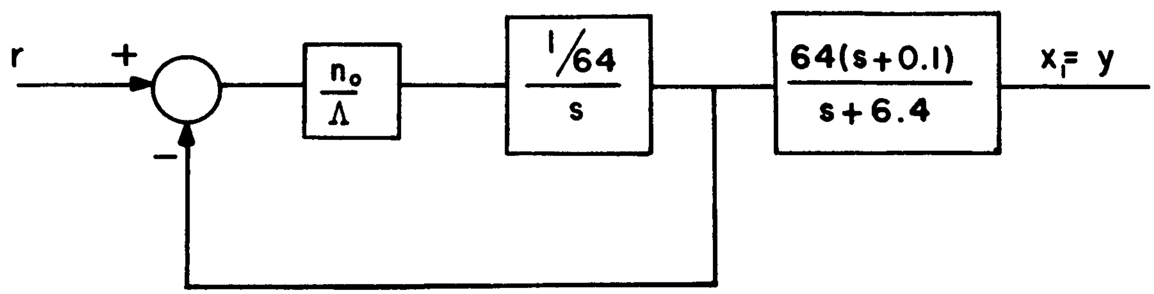


Figure 3.15 Equivalent Block Diagram for System in Figure 3.14

given by Eq. (3-37). The resultant design is shown in Fig. 3.16. Unfortunately, the design of Fig. 3.16 may not be the transfer function desired, even though it is independent of the reactor power level.

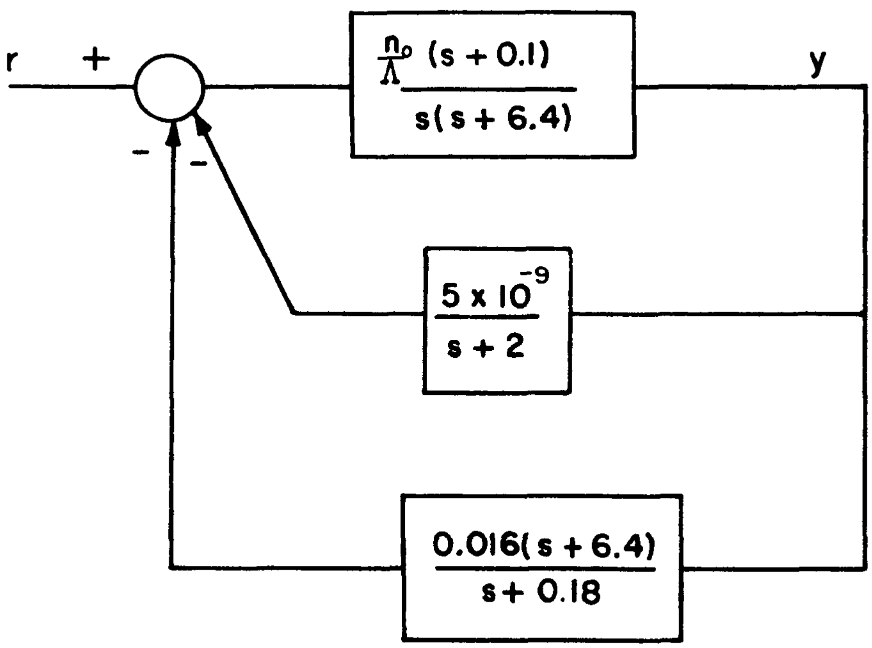


Figure 3.16 Final Reactor Control System Design

This is the desired result and, with the assumption that all the state variables are available to be fed back, completes the design procedure. The closed loop transfer function is

$$\frac{Y}{R}(s) = \frac{25K}{(s^2+4s+8)(s+\frac{25}{2}Kk_3)}.$$

This has two poles at the location of the modified open loop poles and a third pole at $s = -\frac{25}{2}Kk_3$. Thus if K is large, the third pole will have little effect on the system response, because of the fast time constant and small residue associated with it. The response is thus primarily dependent on the pair of complex poles and independent of K.

Procedure When All State Variables Cannot Be Fed Back

This topic was discussed in Examples 3.4 and 3.5 and is further emphasized in this section by working an example. In example 3-1, it is assumed that x_2 cannot be fed back. The coefficients are determined as before and x_3 is fed back through a feedback function $k_3 + k_2 \frac{X_2(s)}{X_2(s)}$. This ratio of $\frac{X_2(s)}{X_3(s)}$ can be found by noting that $\frac{X_1(s)}{U(s)}$ can be found in the manner discussed previously by letting $y = x_1$ in Equation 3-2. This simply means that c_1 is the only non-zero element in $\underset{\sim}{c}$. Performing this operation in the problem under consideration gives $\frac{X_2(s)}{X_3(s)} = \frac{2}{s+1}$. Therefore, x_3 is fed back through $k_3 + k_2\ (\frac{2}{s+1}) = \frac{k_3(s+1)+2k_2}{s+1}$. This feedback function can be realized with a passive network. It is next assumed that only the output can be fed back in Example 3-1. Again, the feedback coefficients are determined as before, and x_4 is defined as shown in Figure 3-7a and fed back through $H_1(s)$, where

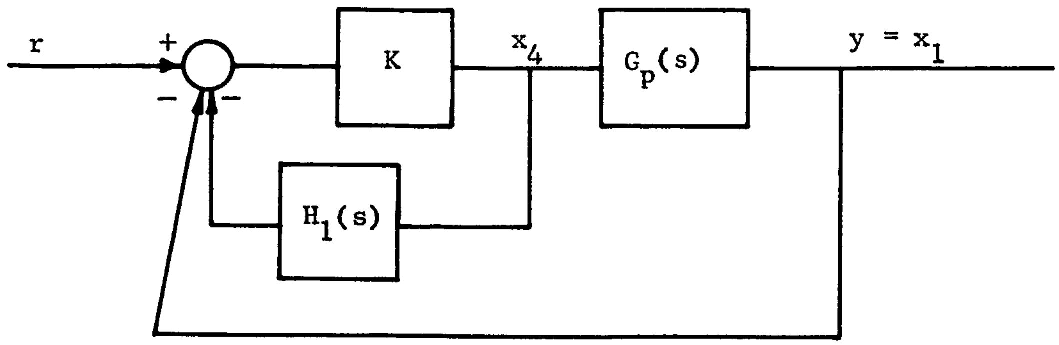


Figure 3-7a. Feedback Configuration When Only the Output Variable Can be Fed Back.

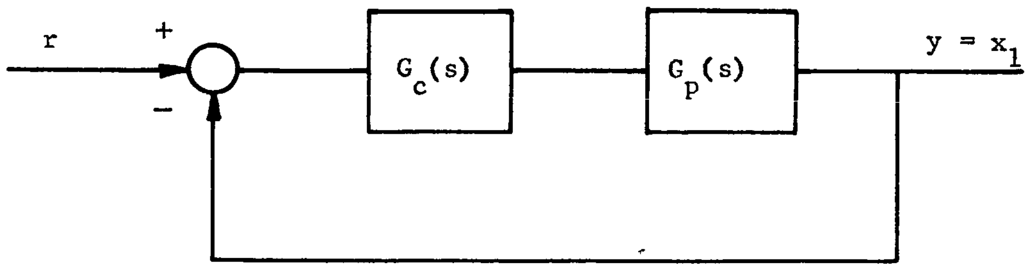


Figure 3-7b. An Equivalent System Using Series Compensation.

$$G_c(s) = \frac{K}{1+KH_1(s)} \ .$$

$$H_1(s) = k_2 \frac{X_2(s)}{X_4(s)} + k_3 \frac{X_3(s)}{X_4(s)} \, .$$

From the values of the example, this is found to be

$$H_1(s) = k_2 \frac{10}{(s+1)(s+4)} + k_3 \frac{5}{s+4} = \frac{10k_2+5k_3(s+1)}{(s+1)(s+4)} \, .$$

This feedback function can also be realized with a passive network. In fact, it can be combined with K to obtain a series compensation network as shown in Figure 3-7b. This configuration is similar to that obtained by the Guillemin method.

On the basis of the above discussion, the procedure to be followed when one or more state variables cannot be fed back is to calculate the feedback coefficients as though all state variables could be fed back and to use the results in determining physically realizable feedback functions. This is not the same as feeding back all the state variables for two reasons: 1) It assumes that the transfer function between the two states is known exactly and can be reproduced exactly. This is never true. 2) Changes in the system parameters would affect the two feedback configurations differently.

<u>Summary</u>

A method for designing linear systems for a desired closed loop response by feeding back all the state variables has been developed from the matrix representation of such systems. The procedure is straightforward and requires only elementary matrix operations and the solution of n linear algebraic equations. It parallels the procedure of Schultz (1966) which is based on the block diagram system representation. It is used to develop a procedure for designing gain insensitive systems which leads to the design method for certain nonlinear and/or time-varying systems developed in the following chapter.

Although the design procedure is referred to as the SVF method, it incorporates the classical technique of series compensation with that of feeding back all the state variables. The use of series compensation makes it possible to add poles and zeros, thereby increasing the order of the system and increasing the flexibility of the design method. It does not make use of the information provided by the state variables. The use of state variable feedback does make use of the information provided by all the state variables, and this information is fed back through constant gain elements as suggested by the results of modern control theory. An interesting characteristic of the gain insensitive systems designed by the method proposed here is that they always satisfy the Kalman frequency condition for optimality. Most systems designed by classical techniques do not satisfy this criterion.

CHAPTER IV

DESIGN OF NONLINEAR AND/OR TIME-VARYING CONTROL SYSTEMS VIA STATE VARIABLE FEEDBACK

Introduction

In this chapter, a proposed method of synthesis for single-input, single-output systems containing a single nonlinear and/or time-varying gain which satisfies conditions 2-4 (or 2-5) and 2-6 is developed. This is accomplished by compensating the constant linear portion of the system in such a way that the Popov stability criterion discussed in Chapter 2 is satisfied. The design procedure is a logical extension of the state variable feedback design of linear gain insensitive systems as developed in Chapter 3. Systems designed by the proposed method are shown to have an infinite stability sector and to have a bounded output for bounded inputs.

The degree of success with which the method can be used is dependent upon the form of the particular system. This lack of generality is characteristic of analysis and design procedures for nonlinear systems. In its basic form, the method is limited to systems with one nonlinear and/or time-varying gain located as shown in Figure 4-1. The limitation to systems having no more than one integration noted in the discussion of linear gain insensitive systems applies here also. It is shown in the next chapter that the method is not applicable to particular cases other than the simplest particular case due to structural stability problems.

Thus the method in its basic form is applicable only for the principal case and the simplest particular case. Modifications in the design procedure which remove some of these restrictions in certain cases are considered in Chapter 5.

The proposed method is applicable, in a practical sense, to systems of any order. The determination of the feedback coefficients requires the solution of (n-1) linear algebraic equations, where n is the order of the linear system G(s). Both the matrix and block diagram formulations of the procedure are discussed.

The organization of the remainder of this chapter is as follows: 1) For systems containing nonlinear gains, the basic design procedure is developed, and the absolute stability properties and the closed loop response of the resulting system are discussed. 2) These same three topics are discussed with respect to time-varying (or nonlinear and time-varying) systems. 3) The significant features of systems designed by the proposed method are summarized.

The SVF Method for Nonlinear Systems

Figure 4-1 illustrates the basic feedback configuration of the compensated system. The system consists of a single nonlinear gain, N, in series with a stable linear system, G(s). The nonlinear gain may appear in the plant to be controlled as an undesirable characteristic, or it may be intentionally introduced in order to achieve a desired result. For example, a saturation element might be used to prevent signals in some part of the system from becoming excessive. The system equations are given in Chapter 2 and are repeated here.

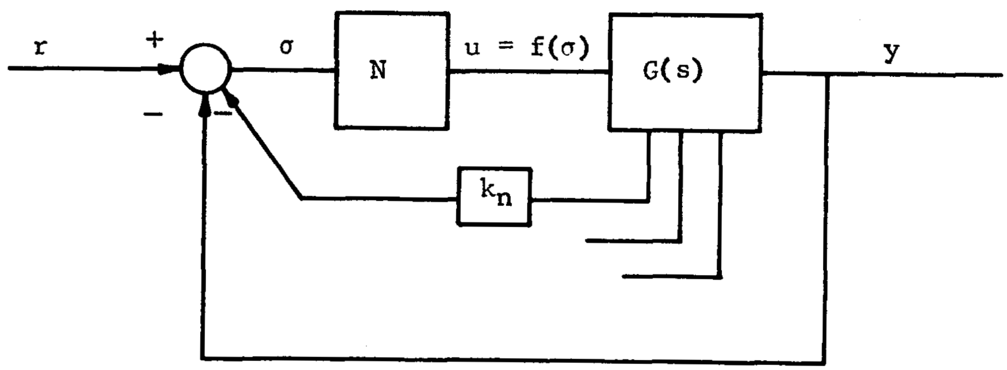


Figure 4-1. Basic Configuration for Controlling a Plant Containing a Single Nonlinear and/or Time-Varying Gain.

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x}+\underset{\sim}{b}u \tag{4-1}$$

$$u = f(\sigma) \tag{4-2}$$

$$\sigma = -\underset{\sim}{k}^T\underset{\sim}{x}+r \tag{4-3}$$

$$y = \underset{\sim}{c}^T\underset{\sim}{x} \tag{4-4}$$

As in the case of the linear systems of Chapter 3, since the state variables are required to correspond to physical variables, Equation 4-1 might have the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x}+\underset{\sim}{b}u+\underset{\sim}{d}\dot{u}. \tag{4-5}$$

The matrix formulation of the basic design procedure is exactly the same as that developed for linear gain insensitive systems in the previous chapter and is therefore not repeated here. It is noted that the gain K appears nowhere in the transfer functions used in the calculation of the k_i for gain insensitive systems and therefore does not affect $H_{eq}(s)$. Thus the fact that K is now assumed to be nonlinear has no effect on the procedure for finding $H_{eq}(s)$. The only equations in the discussion of Chapter 3 which cannot be applied in the nonlinear case are those for the closed loop transfer function. The reference to closed loop poles in the procedure is justified for the nonlinear case in the section on closed loop response.

The block diagram formulation of the design procedure requires that the block diagram be manipulated into the series form shown in Figure 4-2a, where the $G_i(s)$ are first order transfer functions. The feedback configuration is shown in Figure 4-2b, and $H_{eq}(s)$ can easily be determined by comparing this with the equivalent system of Figure 4-2c. The result is

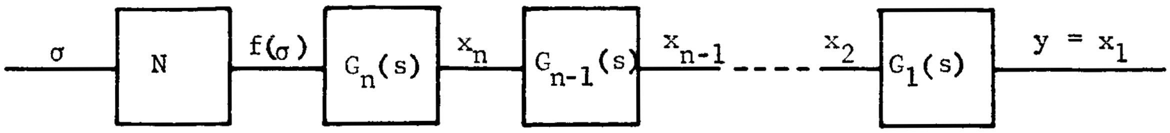


Figure 4-2a. Block Diagram in the Form of a Series of First Order Transfer Functions.

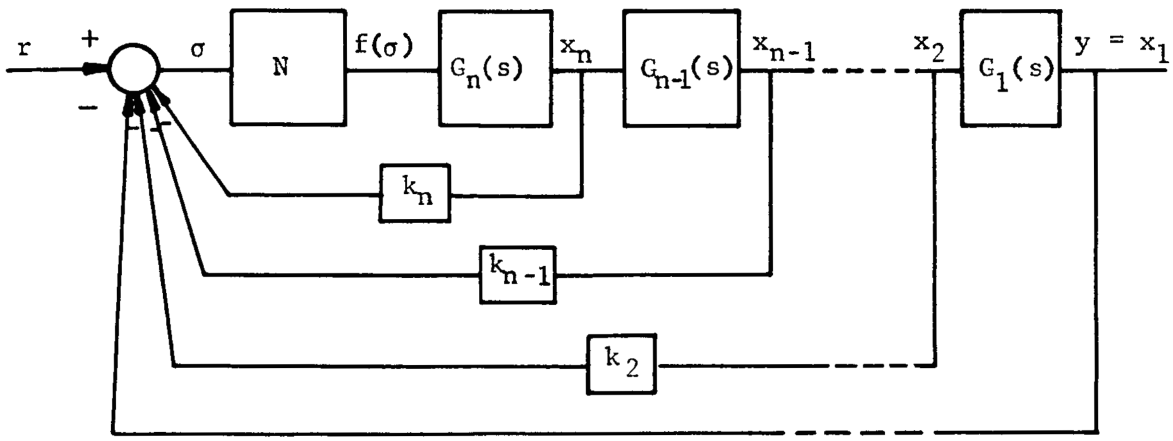


Figure 4-2b. Method of Controlling the System of Figure 4-2a.

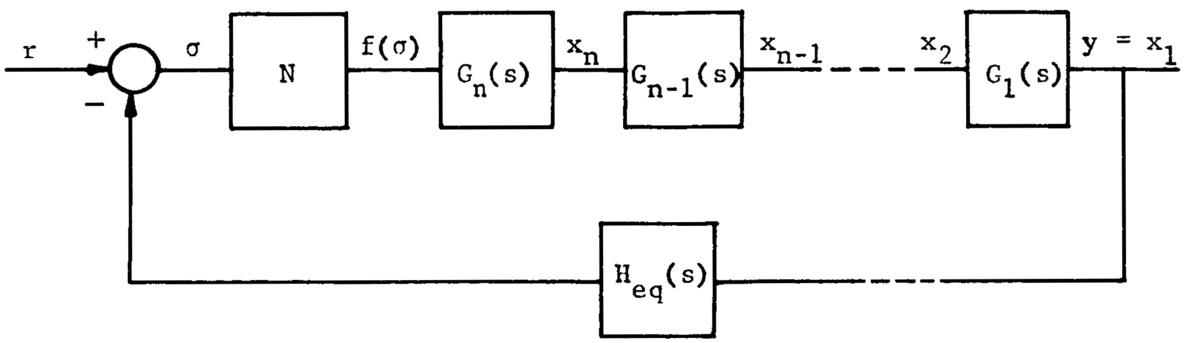


Figure 4-2c. An Equivalent System for Figure 4-2b.

$$H_{eq}(s) = 1+k_2\left[\frac{X_1(s)}{X_2(s)}\right]^{-1} +\ldots+k_n\left[\frac{X_1(s)}{X_n(s)}\right]^{-1}, \qquad (4\text{-}6)$$

which is the same as Equation 3-33 developed from the matrix representation for this particular configuration.

Absolute Stability of the Resulting System

The two possible open loop transfer functions for a system designed by the proposed method are given in Equations 3-27 and 3-28 and are repeated here:

$$G(s)H_{eq}(s) = \frac{K'}{s+a} \qquad (3\text{-}27)$$

$$G(s)H_{eq}(s) = \frac{K'(\tau_1 s+1)}{s+a}. \qquad (3\text{-}28)$$

The modified plot used in the interpretation of the Popov criterion never crosses into the left half plane for either of these functions. Therefore, the Popov line can be drawn through the origin, indicating that the sector in which the system is absolutely stable for the type nonlinearity being considered includes the entire first and third quadrants.

Closed Loop Response of the Resulting System

A mathematically equivalent system is derived for studying input-output relations of the closed loop system. The linear system $G(s)$ is rearranged as shown in Figure 4-3a, where $G_a(s)$ is the first order transfer function of Equation 3-27 or 3-28, $G'(s)$ is an $(n-1)$st order transfer function given by

$$G'(s) = \frac{G(s)}{K''G_a(s)}, \qquad (4\text{-}7)$$

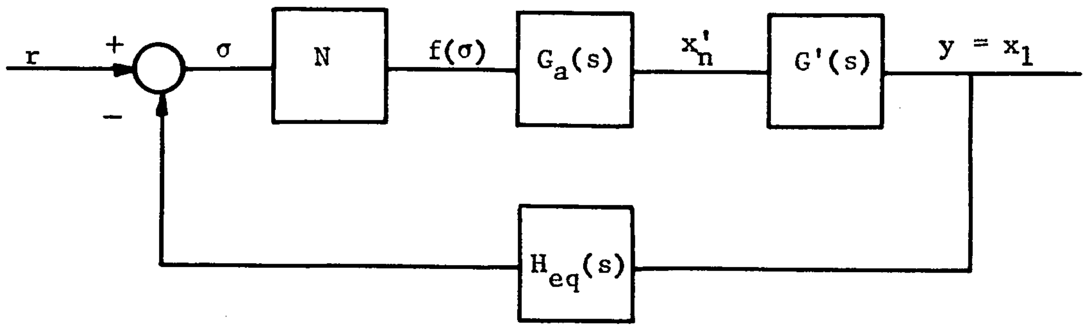


Figure 4-3a. The Equivalent System of Figure 4-1 Showing $H_{eq}(s)$ and with the Linear Part Rearranged.

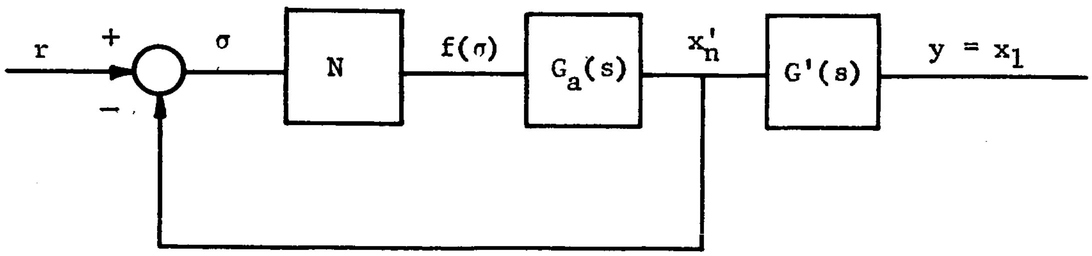


Figure 4-3b. The Equivalent System of Figures 4-3a and 4-1 for Determining Input-Output Relations.

and K" is the coefficient of s^{n-1} in $H_{eq}(s)$. Figure 4-3a is next represented by the equivalent system of Figure 4-3b, which is used in the analysis that follows. The following observations are made concerning the equivalent system:

1. The state variables $x_2', \ldots, x_n'$ are different from those of Figure 4-1. Only σ, $f(\sigma)$, and the input and output variables are the same.
2. Changes in system parameters will affect the systems of Figures 4-1 and 4-3 differently.
3. The equivalent system consists of a first order nonlinear system with unity feedback in series with an (n-1) st order stable linear system. Therefore, if it can be shown that the output of the nonlinear portion is bounded for a bounded input, it follows that the output, x_1, of the actual system represented by Figure 4-1 is also bounded.
4. The equivalent system indicates that the (n-1) poles of G'(s) are also poles of the closed loop system.

The Second Method of Liapunov is used to show that the output is bounded for a bounded input. For generality, it is assumed that $G_a(s)$ has the form of Equation 3-28. The system equations for the nonlinear portion of the equivalent system are then

$$\dot{x}_n' = -ax_n' + K'f(\sigma) + K'\tau_1\dot{f}(\sigma) \qquad (4\text{-}8)$$

$$\sigma = r - x_n' . \qquad (4\text{-}9)$$

A Liapunov function of the form

$$V = x_n'^2 \tag{4-10}$$

is chosen. It follows that

$$\dot{V} = 2x_n' \, [-ax_n'+K'f(\sigma)+K'\tau_1\dot{f}(\sigma)]. \tag{4-11}$$

From Equation 4-9 and the nature of the nonlinearity as given by conditions 2-4 through 2-6, it follows that $\dot{V}$ will always become negative when r is bounded if the magnitude of x_n' becomes large enough. This is also true for the cases where $G_a(s)$ has no zero ($\tau_1 = 0$) and where $G_a(s)$ has an integration ($a = 0$), or both. Thus the output of the nonlinear portion of the equivalent system is bounded for all cases when the input is bounded. Since the system from x_n' to the output y of the equivalent system is a stable linear system, it follows that the output of the closed loop system of Figure 4-1 is bounded for bounded inputs. Moreover, since the output of the nonlinearity is the same in the equivalent system of Figure 4-3b and in the actual system of Figure 4-1, all the physical state variables are bounded, or the system is Lagrange stable.

Under certain quite restrictive conditions, an analytical solution for the output of the nonlinear portion of the equivalent system can be obtained. A sufficient condition for obtaining such a solution is that the variables in Equation 4-8 can be separated and the resulting integration carried out. This can be accomplished in the simplest particular case ($a = 0$) with $\tau_1 = 0$ if the input is a step function and $f(\sigma)$ is such that $\frac{dx}{f(M-x)}$ can be integrated, where M is the magnitude of the step input. The following example illustrates the procedure.

Example 4-1: It is assumed that $r = Mu(t)$, $f(\sigma) = \sigma^3$, $a = 0$, and $\tau_1 = 0$. Then Equation 4-8 may be written in the form

$$\int_{x_n'(0)}^{x_n'(t)} \frac{dx}{(M-x_n')^3} = \int_0^t dt. \tag{4-12}$$

Carrying out this integration gives

$$\frac{1}{2(M-x_n')^2} - \frac{1}{2(M-x_n'(0))^2} = t.$$

Rearranging and solving for x_n' gives the result

$$x_n' = M \pm \frac{1}{2}\left[4M^2-4(M^2-\frac{1}{2t+\frac{1}{(M-x_n'(0))^2}}\right]^{\frac{1}{2}}. \tag{4-13}$$

As t becomes large, $x_n' \rightarrow M$, the equilibrium state of the system for this particular input.

The conditions necessary for obtaining an analytical solution for x_n' are so restrictive that some other means of determining this output of the nonlinear part of the equivalent system is desirable. Since the nonlinear part is a first order system, it is always possible to find the output by use of the basic graphical procedure known as the isocline method when the input and the nature of the nonlinearity are known. This is illustrated by the following example.

Example 4-2: It is assumed that $G_a(s) = \frac{1}{s}$ and that $f(\sigma)$ is a saturation type gain described by Figure 4-4a. The nonlinear portion of the equivalent system is shown in Figure 4-4b. This is all the information needed since this example is concerned only with determining the response

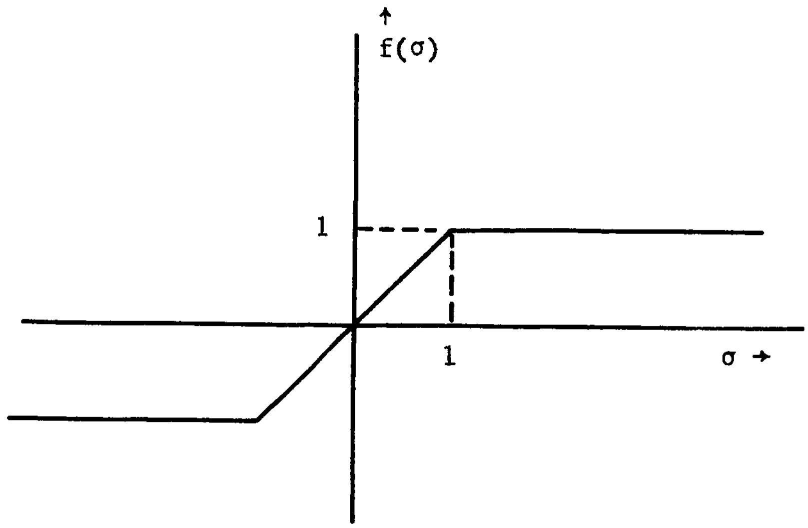


Figure 4-4a. Nonlinear Gain Characteristic for Example 4-2.

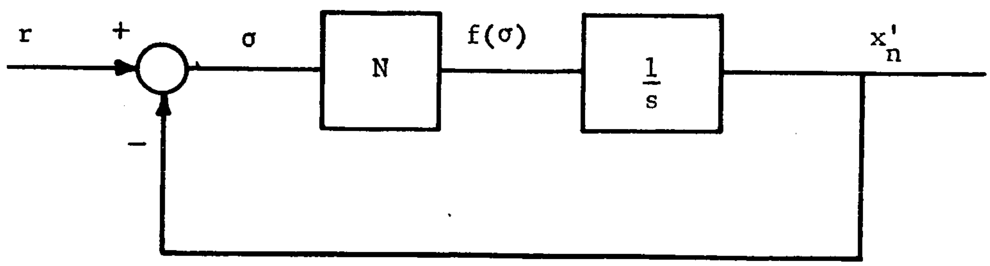


Figure 4-4b. Nonlinear Portion of the Equivalent System for Example 4-2.

of the nonlinear portion of the equivalent system. The isocline method is used to determine this response for a sinusoidal input, $r = \pi \sin t$. From Figure 4-4b,

$$\sigma = \pi \sin t - x_n' , \tag{4-14a}$$

$$\dot{x}_n' = f(\sigma) . \tag{4-14b}$$

From these relationships and the nonlinear characteristics shown in Figure 4-4a, the following values of $\dot{x}_n'$ are determined:

1. When $x_n' > 1 + \pi \sin t$, $\dot{x}_n' = -1$.
2. When $x_n' < -1 + \pi \sin t$, $\dot{x}_n' = 1$.
3. When $x_n' = h + \pi \sin t$ for $-1 \le h \le 1$, $\dot{x}_n' = -h$.

These values of $\dot{x}_n'$ are used to determine x_n' graphically as shown in Figure 4-5. This response is the input to the linear portion of the equivalent system, and the output can now be found by linear methods.

The ability to show that the closed loop system has a bounded output for bounded inputs and to determine the output for a specific input is a significant advantage, as this is not generally possible with nonlinear systems. The next example illustrates the application of the basic design procedure and the calculation of the closed loop response of the resulting system.

<u>Example 4-3</u>: The block diagram of the system to be controlled is shown in Figure 4-6a. This system might represent, for example, a DC motor driven by an amplifier which saturates for large inputs. In this case x_1 represents position, x_2 velocity, and x_3 the field current. The saturation level might be inherent in the amplifier or it might be built

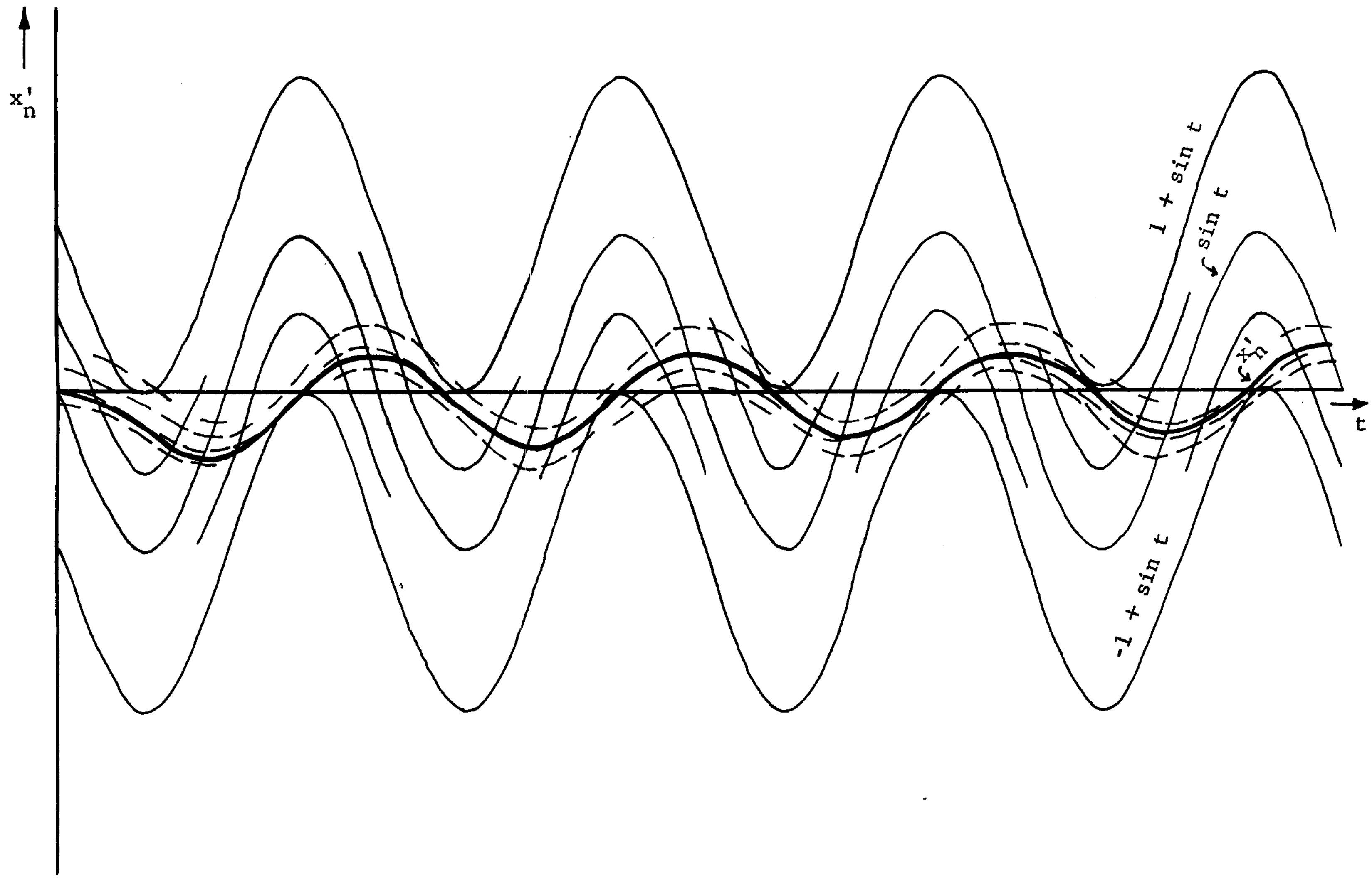


Figure 4-5. Output of the Nonlinear Portion of the Equivalent System for Example 4-2.

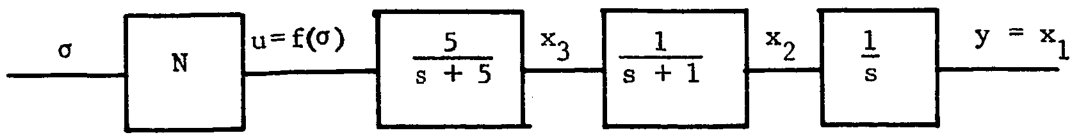


Figure 4-6a. System to be Controlled in Example 4-3.

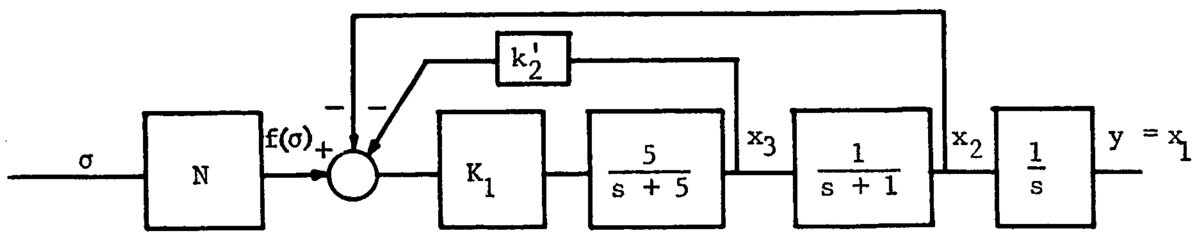


Figure 4-6b. Compensation of the System to Force the Open Loop Poles to be Equal to the Desired Closed Loop Poles.

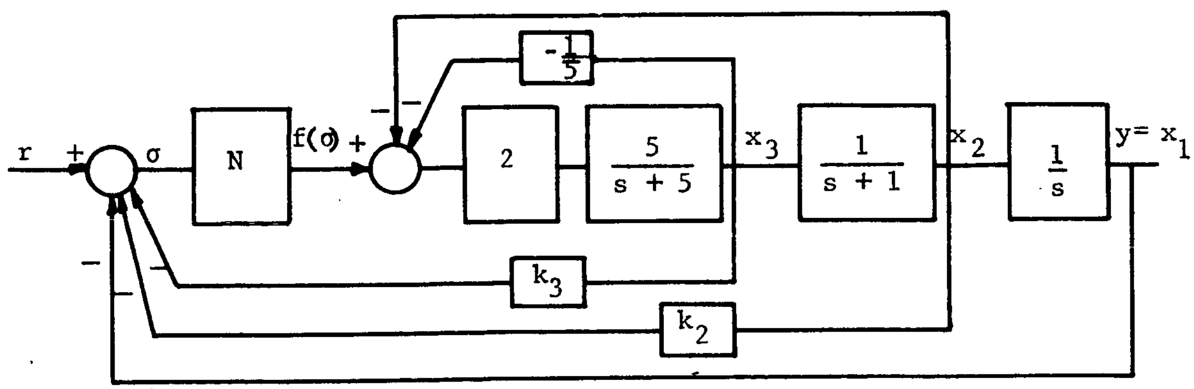


Figure 4-6c. Method of Controlling the Plant of Example 4-3.

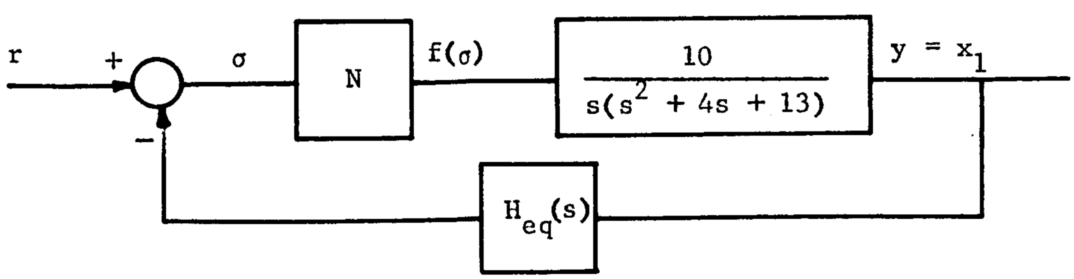


Figure 4-6d. Equivalent System for Figure 4-5c.

into the amplifier to prevent the velocity from exceeding some maximum value. An application for such a system might be to position an antenna in a satellite tracking system. The activating signal, σ, in such a system can be very large during the process of locating the target, thus driving the amplifier well into the saturation region. Once the target is located and being tracked, $|\sigma|$ is assumed to be small enough that the system operates in the linear region. Thus the system must be stable for any activating signal and should respond quickly for small activating signals. In order to achieve the desired response, it is assumed that the gain of the amplifier is high in the linear region and that the closed loop system has poles at $s = -2 \pm j3$. The first step in the design procedure is to compensate the system as shown in Figure 4-6b so that the open loop poles are at the desired location of the closed loop poles. By the method of Chapter 3, it is determined that $k_2' = -\frac{1}{5}$ and $K_1 = 2$ result in the open loop poles having the desired locations. The state variables are then fed back as indicated in Figure 4-6c. In the equivalent system of Figure 4-6d,

$$H_{eq}(s) = k_3 s(s+1) + k_2 s + 1 = k_3 \left[s^2 + \left(1+\frac{k_2}{k_3}\right)s + \frac{1}{k_3}\right]. \qquad (4\text{-}15)$$

In order to realize the desired closed loop poles, the required $H_{eq}(s)$ must have zeros at $s = -2 \pm j3$. This gives

$$H_{eq}(s) = k_3 \left[s^2 + 4s + 13\right]. \qquad (4\text{-}16)$$

Equating the coefficients of Equations 4-15 and 4-16 gives

$$1 + \frac{k_2}{k_3} = 4$$

$$\frac{1}{k_3} = 13.$$

Thus the required feedback coefficients are

$$k_3 = \frac{1}{13}$$

$$k_2 = \frac{3}{13}.$$

It has already been shown that the system when designed in the above manner will be absolutely stable regardless of the gain in the linear region of operation. In addition, the location of the two complex poles is independent of the gain. Therefore, the gain can be made as high as desired in the linear region, and the resulting system has the desirable characteristics of stability for any error signal and the desired response in the linear region of operation.

While the system is in the linear tracking mode, the closed loop transfer function is

$$\frac{Y}{R}(s) = \frac{10K}{(s^2+4s+13)+\frac{10}{13}K(s^2+4s+13)}$$

$$= \frac{10K}{(s^2+4s+13)(s+\frac{10}{13}K)}, \qquad (4\text{-}17)$$

where K is the amplifier gain in the linear region. This indicates that the closed loop system has poles at $s = -2 \pm j3$ and $s = -\frac{10}{13}K$. Both the time constant and the residue associated with the real pole are very small, and the nature of the system response is therefore determined by the pair of complex conjugate poles if K is large. The velocity error coefficient is also determined to a good approximation by the complex conjugate poles if the gain in the linear region is high. This is given by (Truxal, 1955)

$$\frac{1}{K_v} = \sum_{j=1}^{3} \frac{1}{P_j} = \frac{4}{13} + \frac{13}{10K}. \qquad (4\text{-}18)$$

This indicates that as K becomes large, K_v is determined by the location of the complex conjugate poles. The significance of these results is that the system can be designed for the desired characteristics in the linear region without being bothered by stability problems when operating in the nonlinear region.

The equivalent system for studying the closed loop response is shown in Figure 4-7a. For the purpose of illustrating the procedure for calculating the response from this equivalent system, it is assumed that the amplifier has the characteristics shown in Figure 4-7b. It is also assumed that at t = 0, the antenna is pointing in a direction 10° ahead of the satellite in the line of travel and that the satellite is moving with respect to the antenna at the rate of one degree per minute. The location of the satellite is taken as the reference position. Thus the system begins operating in the saturation region, with an activating signal of -10°. The output, x_3', of the nonlinear portion of the equivalent system is found by the isocline method and is shown in Figure 4-8. The reference signal, r = t, is also shown in this figure. When $-1 + t \leq x_3' \leq 1 + t$, $|\sigma| \leq 1$ and the system is operating in the linear region. The graph shows that the system operates in the linear region after approximately 0.4 seconds. With the x_3' of Figure 4-8 as the input to the linear portion of the equivalent system, the output of the closed loop system can be found. One way of calculating this output is to approximate the input to the linear portion of the equivalent system by a piece-wise linear function. Due to the low-pass characteristics of most control systems, this approach will usually yield a good approximation to the output. Where this approach is not practical, graphical convolution can be used.

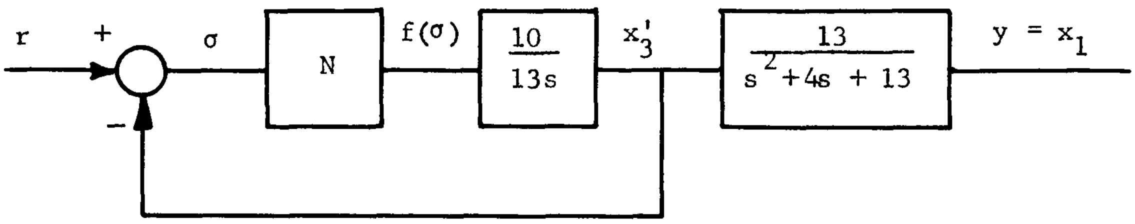


Figure 4-7a. Equivalent System for Determining the Closed Loop Response in Example 4-3.

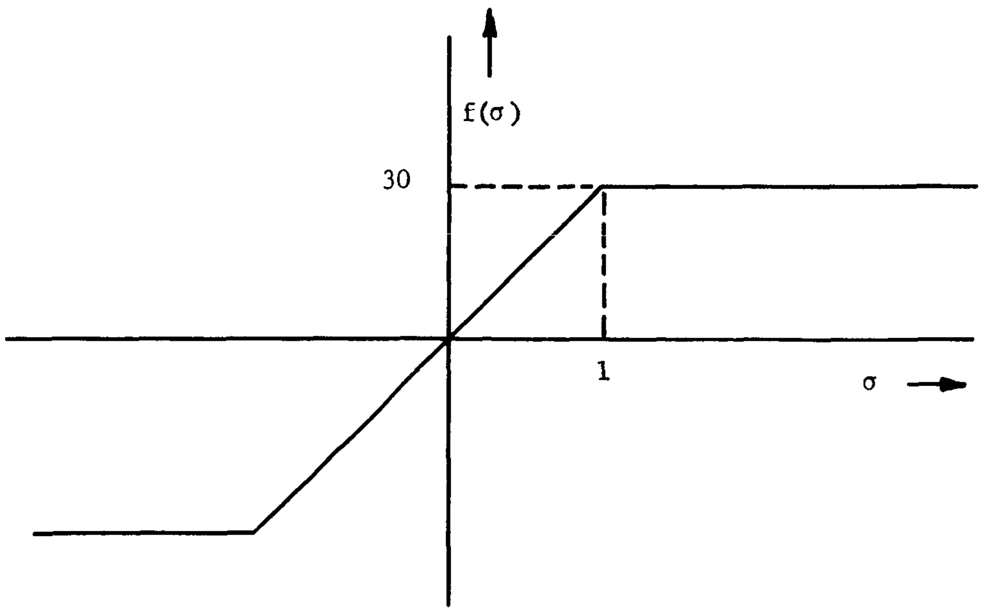


Figure 4-7b. Nonlinear Amplifier Characteristics for Example 4-3.

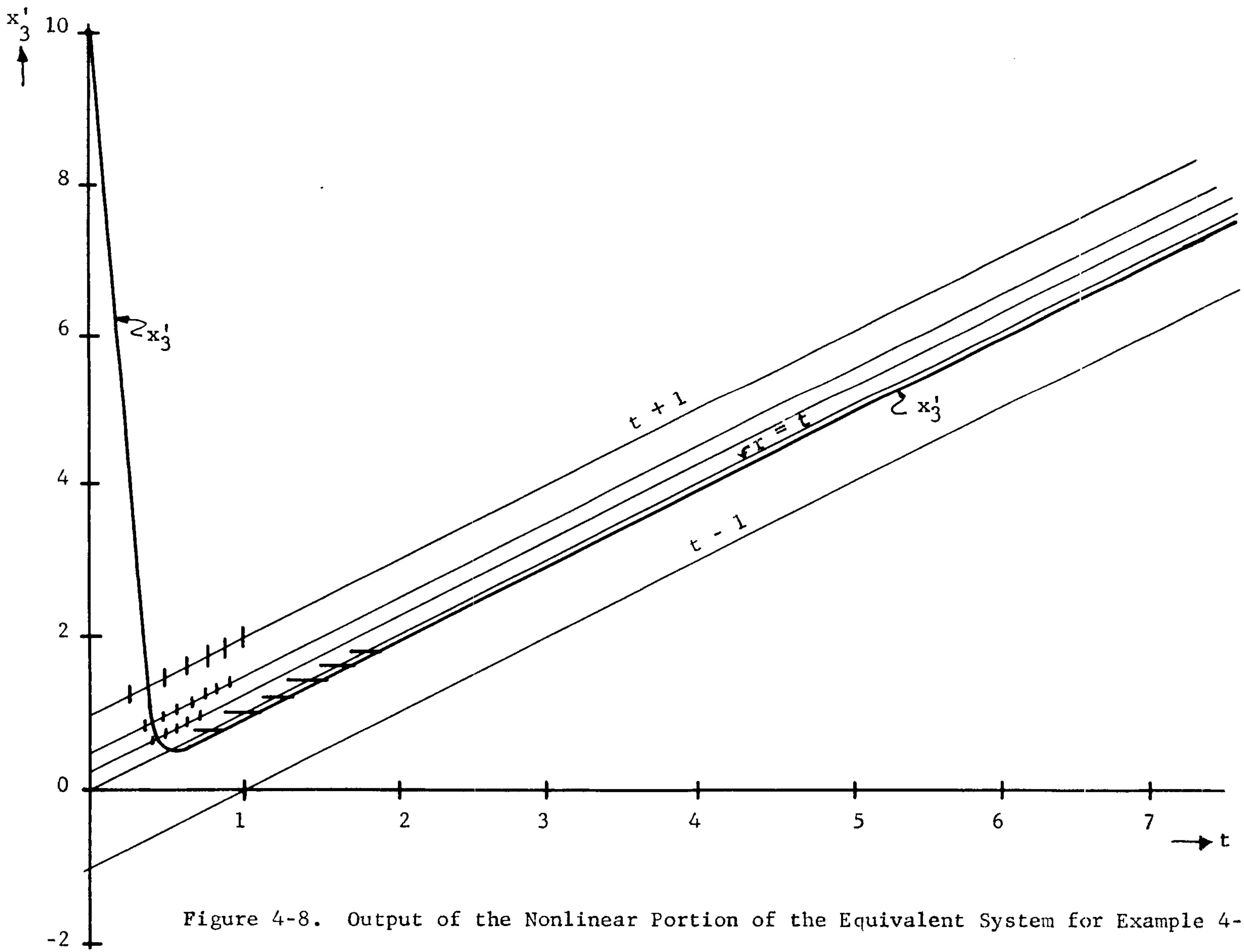


Figure 4-8. Output of the Nonlinear Portion of the Equivalent System for Example 4-3.

In this example, x_3', can be approximated by the function

$$x_3'(t) = 10u(t)-23tu(t) + 24(t-.42)u(t-.42). \tag{4-19}$$

(It is noted that u(t) represents a unit step function and is not the same u as is used for the system control function throughout this dissertation.) From Equation 4-19,

$$X_3'(s) = \frac{10}{s} - \frac{23}{s^2} + \frac{24e^{-.42s}}{s^2}. \tag{4-20}$$

Combining this expression with the transfer function of the linear portion of the system and noting that both $x_3'(t)$ and $y(t)$ are equal to 10^o at $t = 0$ gives, after rearranging,

$$Y(s) = -\frac{23}{s^2} + \frac{17.1}{s} - \frac{5.4+7.1s}{s^2+4s+13} + \frac{24e^{-.42s}}{s^2} - \frac{7.4e^{-.42s}}{s} + \frac{(5.6+7.4s)e^{-.42s}}{s^2+4s+13}. \tag{4-21}$$

From this equation, after combining terms and simplifying,

$$\begin{aligned} y(t) = {} & 17.1u(t)-7.4u(t-.42)-23tu(t)+24(t-.42)u(t-.42) \\ & - 7.68e^{-2t}\cos(3t+22.5^o)u(t) \\ & + 8.26e^{-2(t-.42)}\cos[3(t-.42)+22.5^o]u(t-.42). \end{aligned} \tag{4-22}$$

Equation 4-22 is the approximate output of the system operating under the assumed conditions. The graph of this output is shown in Figure 4-9.

The system of Example 4-3 was simulated on an analog computer, and the resulting response shown in Figure 4-10 agrees reasonably well with the calculated response shown in Figure 4-9. The gain in the linear region was varied between 12 and 700 with the only noticeable change in the response being a small increase in the overshoot as the gain was increased. Thus the results of the simulation indicate that the system is indeed insensitive to large variations in the gain.

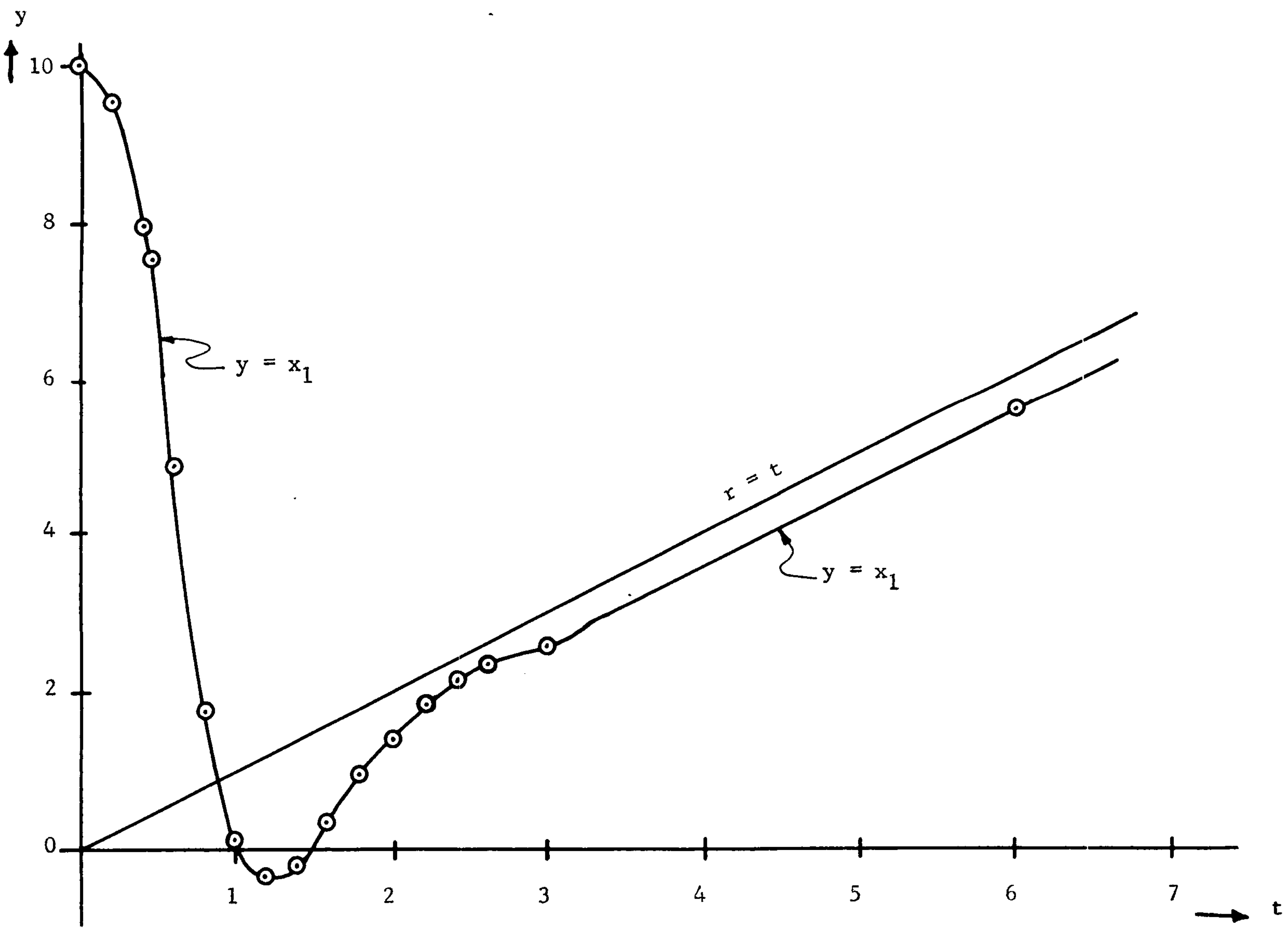


Figure 4-9. Calculated Closed Loop Response for Example 4-3.

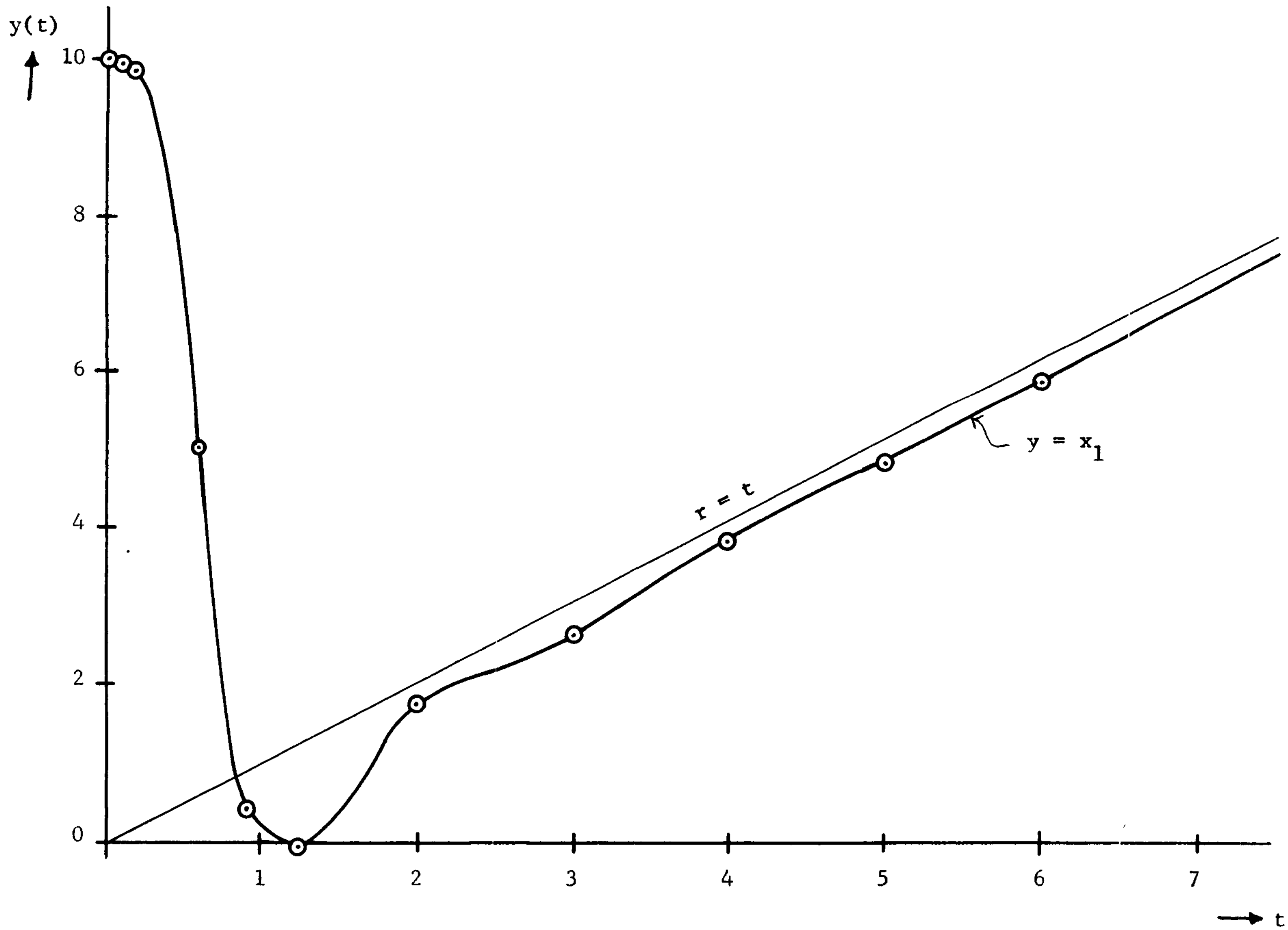


Figure 4-10. Output of the System of Example 4-3 as Obtained from the Analog Simulation.

The SVF Method for Time-Varying Systems

In this section, the application of the proposed design procedure to time-varying (or time varying and nonlinear) gains is considered. The system equations are the same as in the nonlinear case except that Equation 4-2 becomes

$$u = f(\sigma, t). \tag{4-2'}$$

The design procedure for time-varying systems is exactly the same as that for nonlinear systems. Equations 3-27 and 3-28 indicate that the modified plot of $G(s)H_{eq}(s)$ for the resulting system never crosses into the second or third quadrants. Therefore the Popov criterion for time-varying systems as given in Equation 2-17 is always satisfied, and the system is absolutely stable for all time-varying (or time-varying and nonlinear) gains of the type being considered. An equivalent system for input-output relations can be derived in the same manner as for nonlinear systems, with the only difference being that the gain in the system of Figure 4-3b will be time-varying. Because of the restrictions on the time varying gain as given by conditions 2-4 through 2-6, Equation 4-10 can again be used as a V function to show that the output is bounded for bounded inputs.

The system configuration of Example 4-3 might also be used as an illustration of the application of the design procedure to time-varying systems. In fact, the feedback coefficients calculated in this example would be the same for any nonlinear and/or time varying gain (with the exception noted below) as long as the desired location of the closed loop poles remain constant. The only exception is that the gain cannot be equal to zero for $\sigma \neq 0$. This restriction follows from the Popov stability criterion for the simplest particular case.

Summary

The method of designing linear gain insensitive systems has been extended to develop a design procedure for a certain class of non-linear and/or time-varying systems. The procedure is based on realizing a desired equivalent feedback transfer function, $H_{eq}(s)$, by feeding back all the state variables in the proper linear combination. The resulting system has the following significant features:

1. The absolute stability sector, as determined from the Popov theorem, includes the entire first and third quadrants.
2. The output of the closed loop system is bounded for bounded inputs.
3. The closed loop system can be represented by an equivalent system for input-output relations consisting of a first order nonlinear and/or time varying portion in series with an (n-1) st order stable linear portion. This equivalent system can be used to find the approximate output for a given input.
4. In certain cases, the dominant time constant, band width, and overshoot of the closed loop system for any input are determined primarily by the constant linear portion of the equivalent system and are essentially independent of the non-linear and/or time varying gain.
5. In systems where the operation is linear for normal control signals but nonlinear for large control signals, it is possible to design for the desired operation in the linear region without having to worry about system stability for large control signals.

6. The closed loop system is linearized to the extent that (n-1) of the closed loop poles are equal to (n-1) of the open loop poles of the linear plant, independent of the nonlinear gain.

The design procedure, unlike most nonlinear and/or time-varying design methods, presents no real computational difficulties. It requires the solution of n linear algebraic equations where n is the order of the linear part of the system G(s). It is therefore applicable, in a practical sense, to high order systems.

In general, series compensation cannot be used in the basic design procedure unless such compensation can be located in the system so that the nonlinear and/or time varying gain remains in the relative position in the system shown in Figure 4-1. In the next chapter, consideration is given to some practical aspects of the design procedure and to its possible extension to include systems to which the basic procedure cannot be applied.

CHAPTER V

PRACTICAL LIMITATIONS AND EXTENSIONS OF THE DESIGN PROCEDURE

Introduction

The design procedure of the previous chapter is based on the assumption that all the state variables can be measured and fed back in the proper linear combination to force the zeros of the resulting $H_{eq}(s)$ to be exactly equal to n-1 of the poles of G(s). Since exact pole-zero cancellation is never possible in a physical system, the effect of small differences between the zero locations of $H_{eq}(s)$ and the corresponding pole locations of G(s) is of interest. This question is considered in the section on structural stability at the beginning of this chapter.

The last part of this chapter is concerned with possible methods of extending the design procedure to certain systems where the basic procedure cannot be used. First, possible methods by which the basic procedure can be extended to systems in which the nonlinear and/or time-varying gain is not located in the position shown in Figure 4-1 are discussed. Then the possibility of using the design procedure to design systems having a finite stability sector in certain cases where an infinite sector of stability cannot be achieved is considered. Such a reduction in the sector of stability results in less stringent constraints on the open loop gain, $G(s)H_{eq}(s)$.

Structural Stability of the System

The discussion of stability in the previous chapter is based on the premise that the zeros of $H_{eq}(s)$ can be made to occur at the exact locations of the poles of $G(s)$. Since perfect pole-zero cancellation cannot be achieved in a physical system, there will be differences in the locations of the zeros of $H_{eq}(s)$ and the corresponding poles of $G(s)$. As a consequence, the expressions for $G(s)H_{eq}(s)$ given in Equations 3-27 and 3-28 will have n-1 additional poles and zeros. These additional critical frequencies have the property that the zeros are almost equal to the poles. Thus the question of interest becomes: "How useful is the approximation of Equations 3-27 and 3-28 in a physical system?" The answer to this question lies in the structural stability properties of the system. If it can be shown that small changes in the system parameters do not radically affect the nature of the system response, then results obtained from Equations 3-27 and 3-28 should be valid approximations to the actual system response.

The effect of small differences in the corresponding poles and zeros of $G(s)$ and $H_{eq}(s)$ on the absolute stability of the system is considered first. Using the Popov stability criterion, analytical results are obtained which indicate that the absolute stability is not greatly affected by small changes in system parameters except in those cases where closed loop poles occur on or near the imaginary axis. These results indicate how much variation in the pole and zero locations can be tolerated without the absolute stability properties of the system being seriously affected.

The question of absolute stability is investigated from the standpoint of the additional phase shift introduced into $G(s)H_{eq}(s)$ by small differences in corresponding pole and zero locations. Zeros of $H_{eq}(s)$ and the corresponding poles of $G(s)$ to which they are nearly equal are referred to as pole-zero pairs in the discussion which follows.

Only the effect of a single pole-zero pair is considered for poles and zeros on the real axis. For complex conjugate poles, two pole-zero pairs must be considered. The results can then be used to determine the effect of differences in the zero and pole locations of any pole-zero pair in the system.

The diagram of Figure 5-1 is used to determine general results for arbitrary pole-zero locations which are then used in discussing more specific cases. From Figure 5-1 the total phase shift of the two pairs of complex conjugate poles and zeros is

$$\theta = \alpha_1-\beta_1-\alpha_2+\beta_2 = \tan^{-1}\frac{\ell}{a} - \tan^{-1}\frac{\ell-c}{b} - \tan^{-1}\frac{d-\ell}{a} + \tan^{-1}\frac{e-\ell}{b}. \quad (5\text{-}1)$$

The first two terms represent the phase shift contributed by the pole-zero pair in the upper half plane, and the last two terms represent the phase shift contributed by the lower half plane pole-zero pair. Since the expressions for the phase shift contributed by the two pole-zero pairs are similar, the last two terms are neglected for the present, giving

$$\theta_1 = \alpha_1-\beta_1 = \tan^{-1}\frac{\ell}{a} - \tan^{-1}\frac{\ell-c}{b}. \quad (5\text{-}2)$$

From this,

$$\frac{d\theta_1}{d\ell} = \frac{a}{a^2+\ell^2} - \frac{b}{b^2+(\ell-c)^2} = \ell^2(a-b)-2ac\ell+a(b^2+c^2-ab). \quad (5\text{-}3)$$

Setting Equation 5-3 equal to zero gives the following expression for the values of ℓ at which θ_1 reaches its extreme positive and negative values:

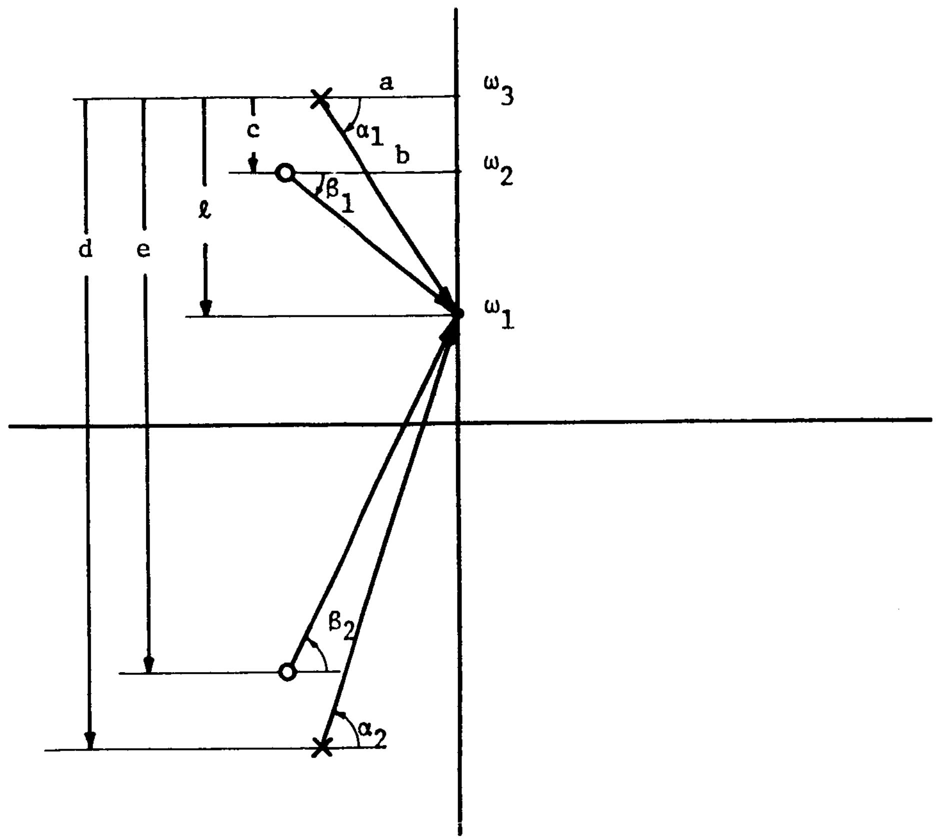


Figure 5-1. Diagram Used in Determining the Phase Shift Contributed by the Pole-Zero Pairs.

$$\ell = \frac{ac \pm \sqrt{ab(a-b)^2+abc^2}}{a-b} \quad . \tag{5-4}$$

The maximum phase shift due to the complex pole-zero pair can be found by substituting the value of ℓ found from Equation 5-4 into Equation 5-2. The maximum total phase shift contributed by the two complex conjugate pole-zero pairs will always be less than twice this value. With c = 0, Equations 5-4 and 5-2 can also be used to determine the maximum phase shift due to a pole-zero pair on the real axis.

Equations 5-2 through 5-4 are now used to investigate specific cases of pole and zero locations.

Case I: The pole and zero are real with a = 0.9b, giving a 10% difference between the pole and zero locations. From Equation 5-4 the maximum phase shift occurs at

$$\ell = \pm \sqrt{0.9b^2} = \pm 0.95b.$$

Since only positive values of frequency are of interest, only the negative value of ℓ is considered. Substituting into Equation 5-2 gives

$$\theta_{max} = -\tan^{-1}(1.06) + \tan^{-1}(0.95) = -3.2^{\circ}. \tag{5-5}$$

For b = 0.9a, the maximum phase shift has the same magnitude but opposite sign. Thus the maximum phase shift for a 10% difference in the locations of a real axis pole and zero is $\pm$ 3.2°.

Case II: There are two complex conjugate pole-zero pairs with c = 0 and the ratio between a and b = 0.9. Here, the magnitude of the total phase shift from the two pole-zero pairs is always less than twice that of Equation 5-5, or 6.4°. This is a conservative upper limit for the phase shift, as it will in general be considerably less, since the maximum from the two pole-zero pairs will not occur at the same frequency.

Case III: There are two complex conjugate pole-zero pairs with $a = 0.9b$ and $c = 0.1b$. From Equation 5-4, the maximum phase shift caused by this difference in the upper half plane pole-zero pair is found to occur at $\ell = 0.44b$. Substituting this into Equation 5-2 indicates a maximum phase shift of 7.2°. Since the maximum phase shift from the pole-zero pair in the lower half plane is no more than this, the maximum phase shift contributed by the complex conjugate pole-zero pairs is seen to be less than 14.4°. Again this is a conservative upper limit for the maximum phase shift, which will in general be considerably less.

Case IV: There are two imaginary pole-zero pairs with $a = b = 0$. The total phase shift for positive frequencies comes from the pole-zero pair in the upper half plane. From Equation 5.1, this is

$$\theta = \lim_{a,b \to 0} \left[\tan^{-1} \frac{\ell}{a} - \tan^{-1} \frac{\ell - c}{a}\right]. \tag{5-6}$$

This is always zero except for frequencies between the pole and zero, where $\theta = \pm 180^{\circ}$.

The absolute stability of a system designed by the proposed method can now be considered without the unrealistic assumption that the zeros of $H_{eq}(s)$ can be made exactly equal to poles of $G(s)$. From the nature of the ideal open loop transfer function, $G(s)H_{eq}(s)$, as given by Equations 3-27 and 3-28, it is apparent that not only is the Popov stability criterion of Equations 2-7 or 2-8 satisfied for any nonlinear gain of the type being considered, but it will continue to be satisfied for any change in $G(s)H_{eq}(s)$ that results in a change in the phase shift of $G(s)H_{eq}(s)$ of less than 90°. In fact, the Popov criterion for nonlinear systems can be satisfied in some cases for maximum changes greater than 90° if this maximum change does not occur at the same frequency as the maximum phase shift in the ideal case, or if the sign of the change is such as to reduce the total phase shift.

Figures 5-2a through 5-2e illustrate further the effect of differences in the locations of the pole and zero of a pole-zero pair. A third order system with one integration is used in these examples. The pole-zero plot is shown on the left and the corresponding modified frequency plot on the right. Figures 5-2d and 5-2e indicate that the relative displacement of the pole and zero is important for pole-zero pairs near the imaginary axis. Figure 5-2d also indicates the possibility of designing for absolute stability in a finite sector where it is not possible to include the entire first and third quadrants in the stability sector.

Systems with time-varying gains must satisfy stronger conditions for stability than those for nonlinear gains. Equation 2-17 indicates that the modified plot must never cross into the second or third quadrants if the stability sector of the system is to be infinite. Figures 5-2b and 5-2c suggest two possible approaches in the time-varying case·
1) Intentionally displace the zeros of $H_{eq}(s)$ as shown in Figure 5-2b.
2) Design for a finite stability sector. In most cases, infinite stability sectors are not required.

The above discussion leads to the following conclusions concerning the absolute stability of a physical system designed by the proposed method:

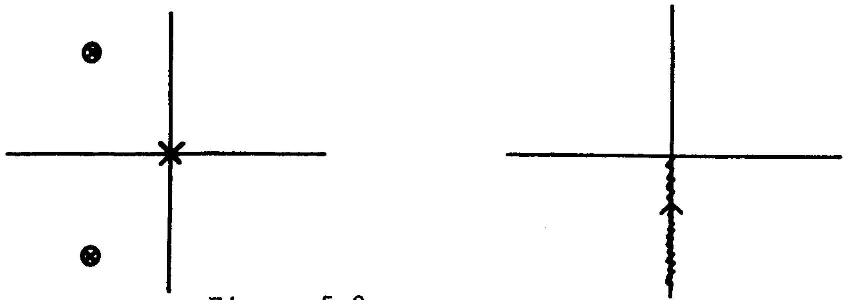


Figure 5-2a

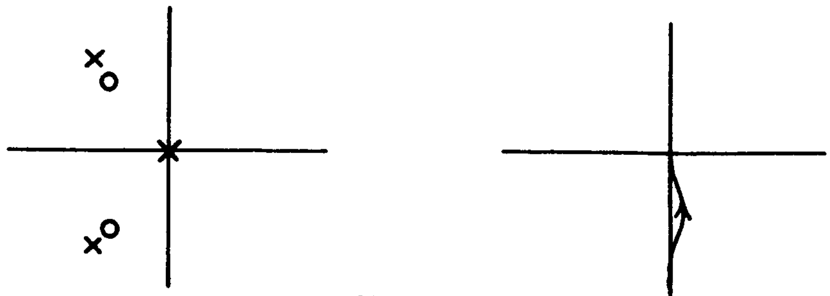


Figure 5-2b

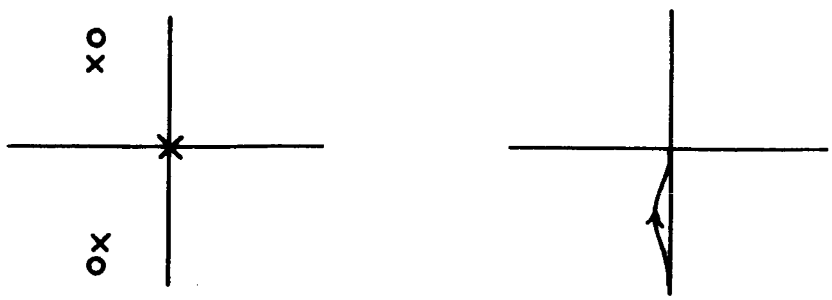


Figure 5-2c

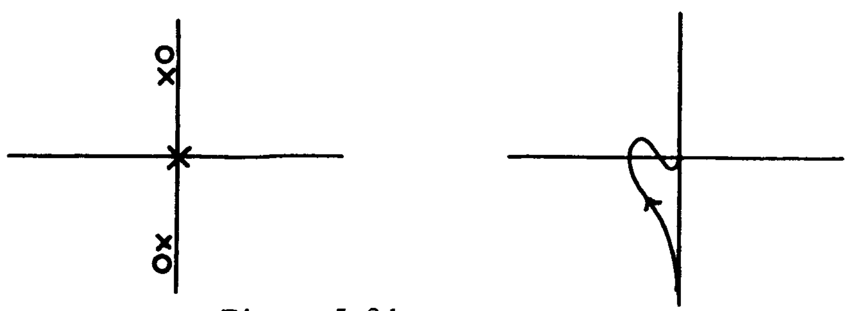


Figure 5-2d

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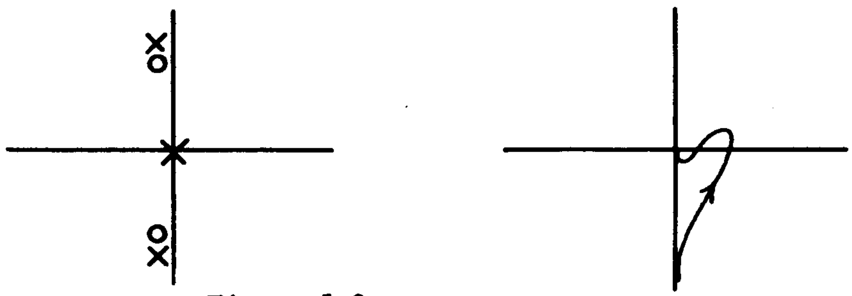

Figure 5-2e

Figure 5-2. Illustrations of the Effect of Differences in the Locations of the Pole and Zero of a Pole-Zero Pair. The Pole-Zero Diagram of a Third Order System is shown on the Left, and the Corresponding Modified Frequency Plot is Shown on the Right.

1. Except for systems with pole-zero pairs on or near the imaginary axis, the absolute stability sector for systems with nonlinear gains is infinite.
2. Where absolute stability cannot be assured for an infinite sector because of pole-zero pairs near the imaginary axis it might be possible to design for a finite absolute stability sector.
3. In the time-varying case, it is necessary to displace the zeros of $H_{eq}(s)$ in the proper direction to assure an infinite absolute stability sector.

The closed loop response was determined in Chapter 4 by making use of an equivalent system consisting of an (n-1)st order linear part and a first order nonlinear and/or time varying part in series. When the zeros of $H_{eq}(s)$ are not exactly equal to the poles of G(s), the nonlinear and/or time-varying part of this equivalent system has the form shown in Figure 5-3. In this equivalent system, the zeros are almost equal to the corresponding poles, and the question of structural stability again becomes important in determining whether the output of the equivalent system is a valid approximation to the closed loop output of the actual system. Because the nonlinear and/or time-varying gain cannot be separated into a first order part of the equivalent system, it does not appear possible to determine general analytical results which assure that the equivalent system of Chapter 4 is valid for a physical system. However, all the known properties of the system indicate that the equivalent system is a good approximation to the physical system. Certain analytical results

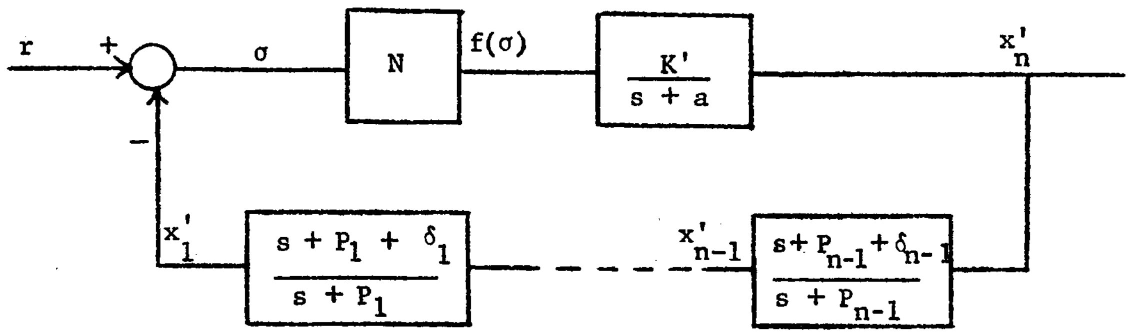

Figure 5-3. The Form of the Nonlinear Portion of the System When the Zeros of $H_{eq}(s)$ are not Exactly Equal to the Poles of G(s). The δ_i are Much Smaller Than the P_i.

can be obtained for specific systems. For example, a result of Zames (1966) is used to show that the output of the system is bounded for bounded inputs if the stability sector is not required to be infinite and if only the principal case is considered.

The results reported by Zames give sufficient conditions for a bounded closed loop output when the input is bounded. The conditions apply to the type system considered in this dissertation if all the poles of G(s) are in the left-half plane. The result of interest is called the Circle Theorem and is given here in the following form:

> If the nonlinearity is inside a sector $\{K_1, K_2\}$, and if the frequency response of G(s) avoids a "critical region" in the complex plane, then the closed loop output is bounded for bounded inputs: if $K_1 > 0$ then the "critical region" is a disk whose center is halfway between the points $-\frac{1}{K_1}$ and $-\frac{1}{K_2}$, and whose diameter is greater than the distance between these points. If $K_1 = 0$, the condition corresponding to this "critical region" is $\mathrm{Re}[G(j\omega)] \geq -\frac{1}{K_2} + \delta$, where $\delta > 0$.

A graphical illustration of the above theorem is given in Figure 5-4. If $f(\sigma)$ vs. σ and $G(j\omega)$ lie in the shaded regions, then the closed loop response is bounded for bounded inputs. In the discussion of the Popov criterion in Chapter 2, K_1 was taken to be zero and K corresponds to K_2.

From the above theorem and preceding discussion it follows that a system designed by the proposed procedure, in which G(s) has no poles on the imaginary axis, will have a bounded closed loop response for bounded inputs if the nonlinearity is confined to an appropriate finite sector. Since the gains in practical systems are not infinite, the

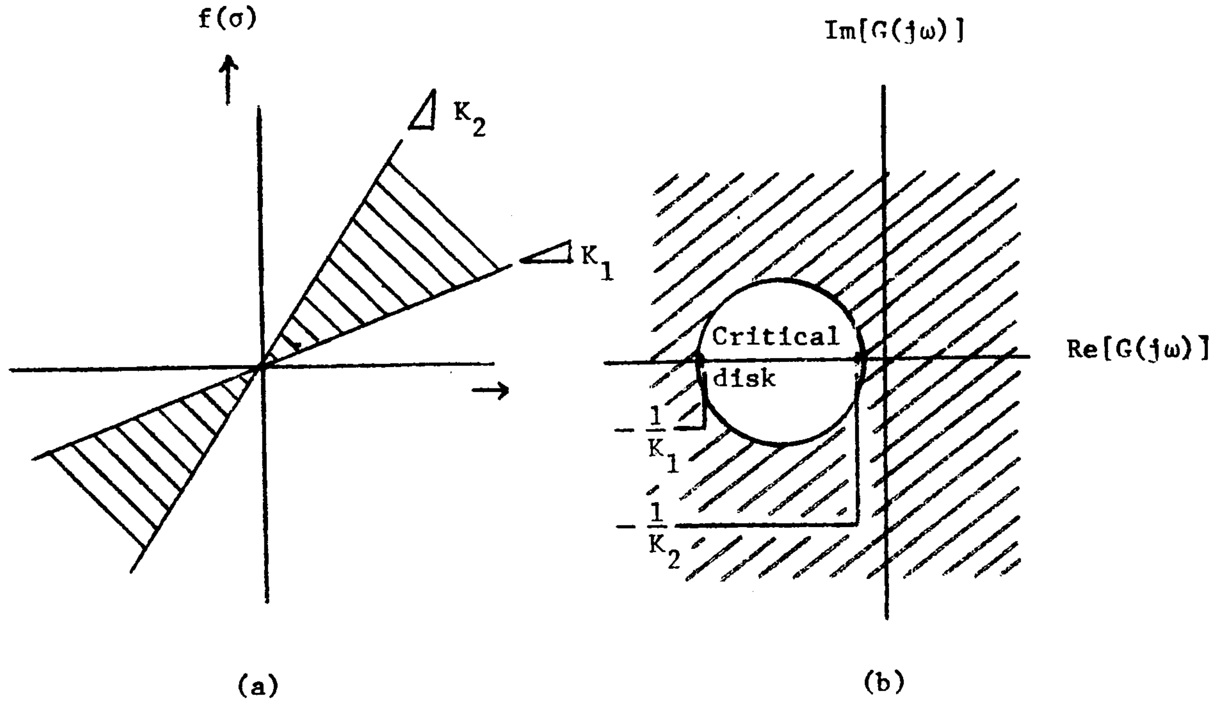

Figure 5-4. Illustration of the Circle Theorem. If $f(\sigma)$ vs. σ and $G(j\omega)$ lie in the Shaded Regions and if the Nyquist Diagram of $G(j\omega)$ does not Encircle the Critical Disc, the Closed Loop is Bounded.

sector determined from this theorem with appropriate tolerances placed on the pole and zero locations will be all that is required in many cases.

The equivalent system of Figure 5-3 will now be considered. It is assumed that none of the complex conjugate pole-zero pairs are near enough to the imaginary axis to cause excessive overshoot to occur or to cause the linear system to go unstable for small changes in the pole and zero locations. From this assumption and the fact that the transfer function between x_n' and x_1' is linear, stable, and approximately equal to unity, one is led to expect the response at x_1' to be very nearly equal to the output, x_n', of the nonlinear portion of the equivalent system. But this is exactly the case in the equivalent system of Chapter 4, so it appears possible to use the equivalent system with a high degree of confidence except in those cases having closed loop poles on or near the imaginary axis. The confidence in this conclusion is increased still further in those cases where both the input and the nonlinear and/or time-varying characteristic are relatively smooth. The requirement that the closed loop poles be constrained to be away from the imaginary axis is in agreement with the results of the discussion on absolute stability.

The final conclusion from the above discussion is that systems designed by the proposed method can be expected to be structurally stable except in those cases having closed loop poles on or near the imaginary axis. Analytical results are given which indicate the extent of the effect of small changes in the pole and zero locations on the absolute stability of the system. General analytical results were not obtained to indicate the extent of the effect of such changes in the case of the closed loop response, but reasons are given for expecting the equivalent system developed in Chapter 4 to give a good approximation to the actual closed loop response. The results of analog computer simulations, one of which is reported in Example 4-3, support this conclusion.

Effect of the Location of the Nonlinear and/or Time-Varying Gain

The basic design procedure of feeding back all the state variables in order to realize a desired $H_{eq}(s)$ is not applicable when the nonlinear and/or time-varying gain is located arbitrarily in the system. The reason for this is illustrated by Figure 5-5a. In this system an equivalent feedback function $H_{eq}(s)$ from the output to the input cannot be obtained independently of the gain N. No general procedure for handling such problems appears to be best for all systems. Consequently, it is treated here by suggesting the following possible procedures based on the systems shown in Figures 5-5a through 5-5c.

<u>Method 1</u>: If the gain N is located as shown in Figure 5-5b and its output can be measured and fed back, the equivalent feedback function is

$$H_{eq}(s) = k_4(s+b)(s+c)(s+d) + k_3\ (s+c)(s+d) + k_2\ (s+d) + 1,$$

and the result is the same as when the gain is located as shown in Figure 4-1. Therefore the basic procedure applies.

<u>Method 2</u>: If the gain N is a saturation type, it is in some cases possible to introduce another saturation type gain at the input which will prevent the original nonlinearity from saturating for any input. The system is then equivalent to a system with one nonlinearity at the proper location for applying the proposed design procedure. This method could also be

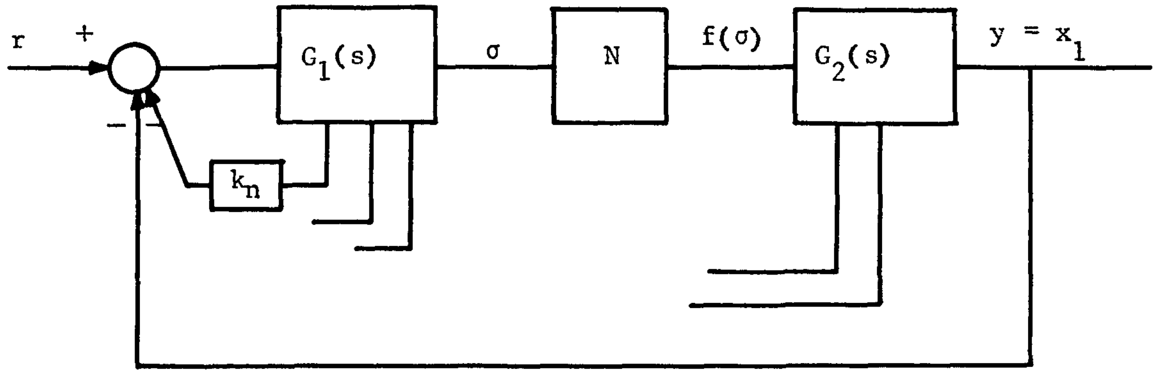

Figure 5-5a. System in Which $H_{eq}(s)$ is not Independent of the Nonlinear Gain, N. $(G(s) = G_1(s)G_2(s))$.

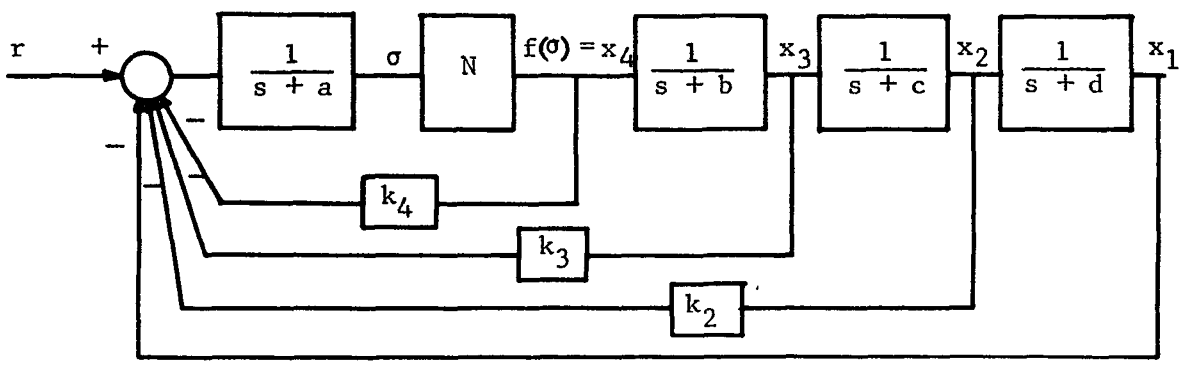

Figure 5-5b. System Configuration for Method 1.

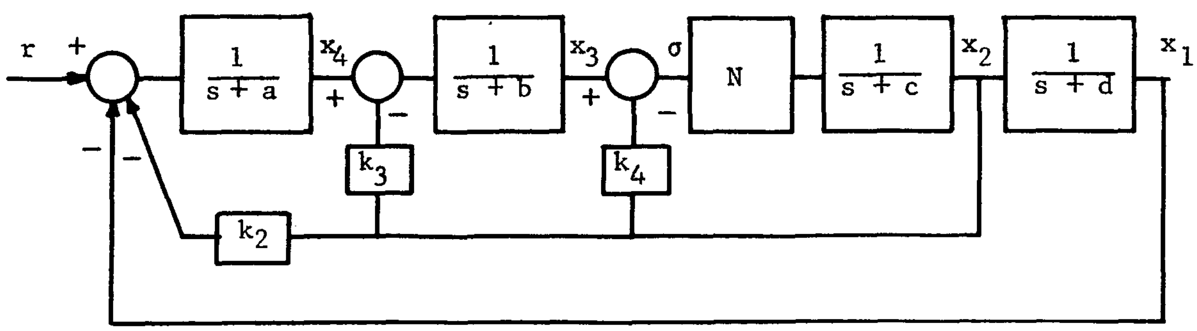

Figure 5-5c. System Configuration for Method 2.

used in some cases to extend the design procedure to systems with multiple saturation type nonlinearities. This approach could not be used when pure integrations are present between the input and the nonlinearity.

Method 3: For a feedback configuration such as that shown in Figure 5-5c, the equivalent feedback function is

$$H_{eq}(s) = k_4(s+a)(s+b)(s+d) + k_3(s+a)(s+d) + k_2(s+d) + 1.$$

This is the same type of equation as that obtained from the basic feedback configuration of Figure 4-1, so that the k_i can be found in the same manner as before. From Figure 5-5c and the equation above, it is apparent that $H_{eq}(s)$ cannot have zeros equal to the (n-1) poles of G(s) that are outside the inner loop with k_4 as the feedback element, as this would require that $k_4 = \infty$. The objection to this method is that it is not usually possible to feed back into the required points in the system. Also, since $u \neq -\underset{\sim}{k}^T\underset{\sim}{x}$, the matrix formulation of the basic procedure cannot be used here. An advantage is that it allows the extension of the basic procedure to systems where series compensation is desirable and must be placed in front of the nonlinear and/or time-varying gain. In this case it will often be possible to feed the output back to any point in the compensation network.

An equivalent system for input-output relations similar to that developed above can be derived for all these methods. The only difference from the previous equivalent system is that the single order nonlinear and/or time-varying part will appear in the middle of the linear part for Methods 1 and 3.

The determination of the feedback coefficients in Methods 1 and 2 is no different from the basic procedure. Once the expression for $H_{eq}(s)$ has been determined from the block diagram in Method 3, the calculation of the k_i is accomplished as in the basic design procedure. It is interesting to note that in the basic procedure information concerning all the state variables is fed back to the input in order to establish the actuating signal, while in Method 3 the output is fed back so as to establish desired control signals at certain points in the system.

In many actual design problems where the basic procedure cannot be used, a combination of the above methods might be useful. The following example illustrates the procedure of Method 3.

Example 5-1: The fixed plant of Figure 5-6a is used to illustrate the procedure of Method 3. It is assumed that closed loop poles are desired at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. The first step in the design is to compensate the fixed plant as indicated in Figure 5-6b so that the open loop poles will be at the desired location of the closed loop poles. The required feedback configuration is shown in Figure 5-6c. Comparing this with the equivalent system of Figure 5-6d gives

$$H_{eq}(s) = k_3(s^2 + s) + k_2 s+1 . \tag{5-7}$$

The desired transfer function must have the form

$$H_{eq}(s) = k_3[s^2 + s+1] . \tag{5-8}$$

Equating the corresponding coefficients of Equations 5-7 and 5-8 and solving the resulting equations gives

$$k_3 = 1$$

$$k_2 = -1 .$$

This set of feedback coefficients produces the desired closed loop poles

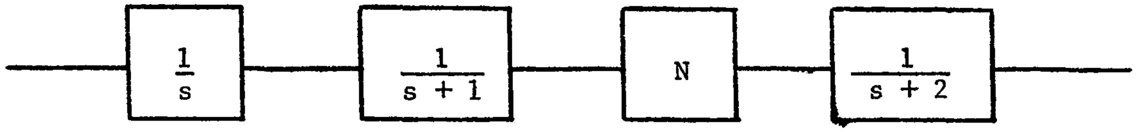

Figure 5-6a. The Plant to be Controlled in Example 5-1.

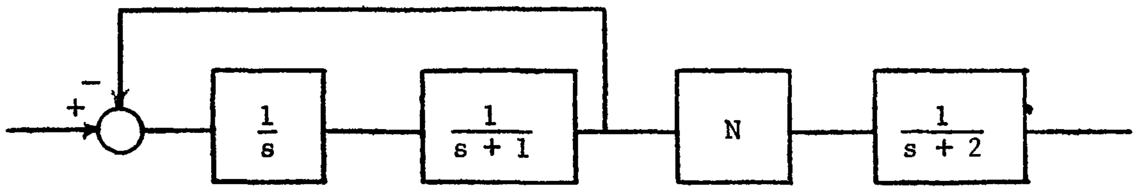

Figure 5-6b. Compensation of the Plant of Example 5-1 to Force the Open Loop Poles to be at the Desired Location of the Closed Loop Poles.

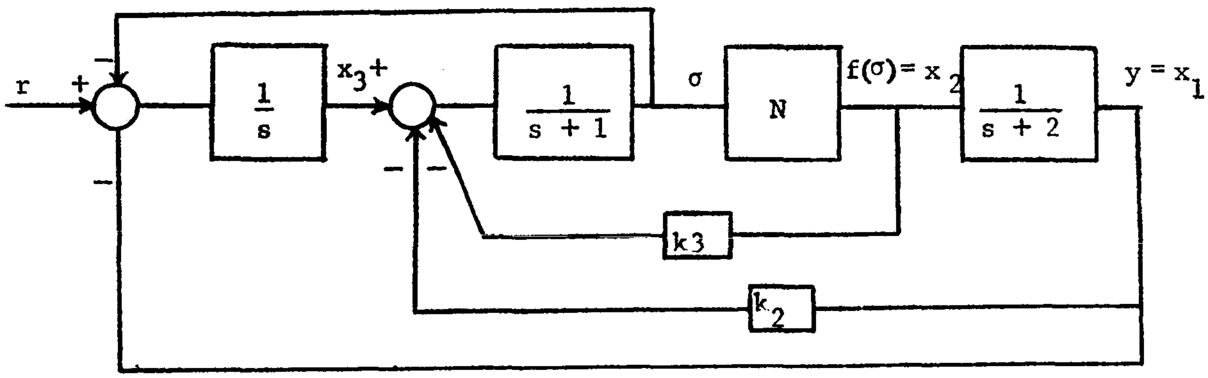

Figure 5-6c. Method of Controlling the Plant of Example 5-1.

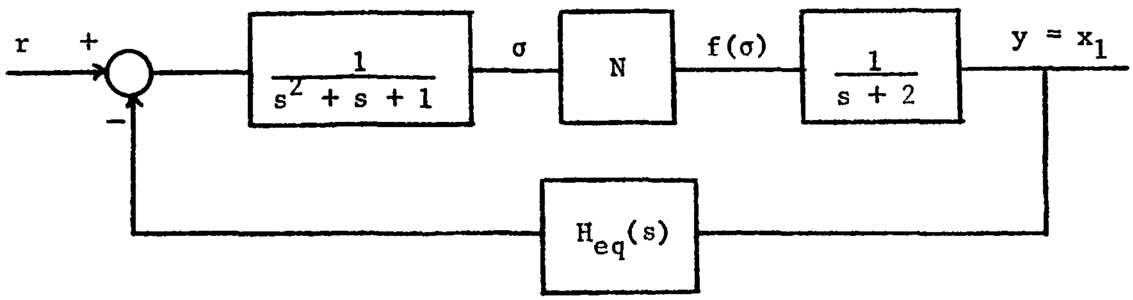

Figure 5-6d. An Equivalent System for Figure 5-6c.

at $s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, independent of the nonlinear and/or time-varying gain.

One point of interest is that the feedback shown in Figure 5-6b changes the type of the system. If zero steady state error is desired, another integrator must be added to the system. This would require an additional feedback path as it increases the order of the system and the number of state variables by one.

Design for Finite Sectors of Stability

All the previous discussion is based on the premise that all the state variables can be fed back to the desired points in the system to be compensated. The systems which result from the proposed design procedure then have infinite stability sectors. It is often not possible to feed back all the state variables as desired. Also, it is not usually necessary that the stability sector be infinite. In this section, the possibility of designing for a finite stability sector when all the state variables cannot be fed back is discussed. (The procedure suggested in Chapter 3 for those cases where all the state variables cannot be fed back can be used in the nonlinear case as well. The suggestions here provide additional possible porcedures.) Because of the many different situations that can arise in nonlinear systems, no attempt is made to provide a general solution. Instead, specific cases of nonlinear systems are considered. The same ideas can be applied to time-varying systems, but the conditions for absolute stability are more strict.

Case I: It is assumed that in an n^{th} order system, one of the state variables cannot be fed back. If the other (n-1) state variables are fed back, it follows from the discussion of Chapters 3 and 4 that (n-2) of the zeros of $H_{eq}(s)$ can be located as desired by feeding back through the proper constant gains. If these (n-2) zeros are made equal to poles of G(s), the open loop transfer function has the form

$$G(s)H_{eq}(s) = \frac{K'}{(s+a)(s+b)} \tag{5-9}$$

or

$$G(s)H_{eq}(s) = \frac{K'(\tau_1 s+1)}{(s+a)(s+b)} \,. \tag{5-10}$$

Theoretically, the stability sector is infinite because the phase shift can never be greater than 180^o, so the Popov line can always be drawn through the origin. Practically, the results of the first part of this chapter indicate that the stability sector might become finite because of the effect of the zeros of $H_{eq}(s)$ not being exactly equal to poles of G(s). The equivalent system for input-output relations is similar to that developed in Chapter 4 except that the nonlinear portion here is second order and the linear portion is of order (n-2). The configuration of this equivalent system is shown in Figure 5-7. It is possible to determine the output of the nonlinear portion of the equivalent system by using phase-plane analysis. With this as the input to the linear portion of the equivalent system, the output of the closed loop system can be found as in Example 4-3.

For the case under consideration, it is seen that absolute stability can be assured for a finite sector and that the output can be found for a given input. It is obvious that the effect of the nonlinearity cannot be controlled to the extent that it can in the case where all the

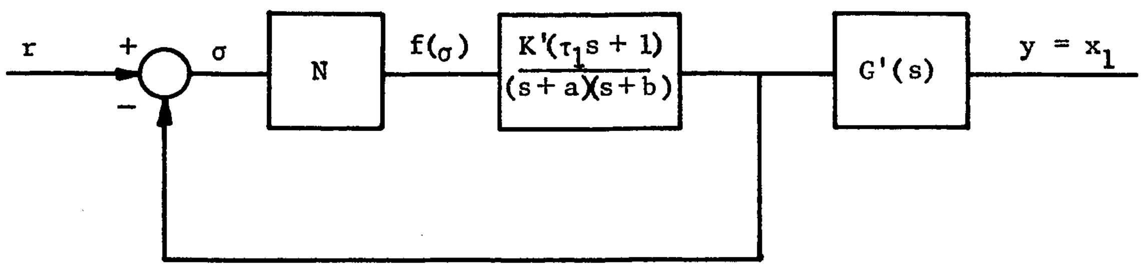

Figure 5-7. Equivalent System When n-1 State Variables are Fed Back as in Case I. G'(s) is of Order n-2.

state variables are fed back, because now only (n-2) of the closed loop poles can be made independent of the nonlinear gain.

Case II: It is assumed that in an n^{th} order system, m of the state variables cannot be fed back. In this case the location of (n-m-1) of the closed loop poles can be made independent of the nonlinear gain. The open loop transfer function will have (m + 1) poles and the nonlinear part of the equivalent system will be of order (m + 1). Thus, although a finite sector of stability can be realized and though the order of the nonlinear portion of the equivalent system is lower than the order of the actual system by (n-m-1), many of the advantages of the design procedure are lost. It is no longer possible to determine the output of the system by the relatively simple procedures used when the nonlinear portion of the equivalent system was of order 1 or 2. Neither is it possible to show in general that the output is bounded for a bounded input. Because of this, it appears that this procedure might be more useful in the case of a regulator system.

Case III: Another possible approach to be used when one or more of the state variables cannot be fed back is to realize $H_{eq}(s)$ by actually inserting a transfer function, $H_{eq}'(s)$, in the feedback path to produce the desired zeros. In order to make this transfer function realizable, poles must be added so that $H_{eq}'(s)$ will have as many poles as zeros. If these poles are added far enough out on the real axis, they will not appreciably affect the system response except for high gains.

The extreme example of this case occurs when only the output can be fed back. Then $H_{eq}'(s)$ must have (n-1) zeros and poles. Depending on the extent to which the effect of the poles of $H_{eq}'(s)$ can be ignored in the region of operation, this method is an approximation to feeding back the output and its (n-1) derivatives (or the phase variables) in a linear combination. An advantage of the method is that the location of the fixed closed loop poles will be independent of gains located anywhere in the system. The following example is an illustration of this procedure.

Example 5-2: Here the same fixed plant is considered as in Example 5-1, with N representing a nonlinear gain. However, it is assumed that only the output can be fed back, and that it can be fed back only to the input. If the stability sector is not required to be infinite, the realization of $H_{eq}'(s)$ as the feedback transfer function can be used to get similar results, with the primary difference in the two systems being the same as those pointed out in the above discussion of this method. For example, $H_{eq}'(s)$ might be chosen as

$$H_{eq}'(s) = \frac{625(s^2+s+1)}{(s+25)^2} . \tag{5-9}$$

The open loop transfer function of the linear part of the system then becomes

$$G(s)H_{eq}'(s) = \frac{625}{(s+2)(s+25)^2} . \tag{5-10}$$

It has been shown by Brockett and Willems (1965a) that the stability sector of a third order nonlinear system with no zeros is the same as the linear stability sector. From this, it follows that the stability sector of the above system is (0,58.3]. By making the gain of $H_{eq}'(s)$ less than that indicated by Equation 5-9, this stability sector can be increased by the same ratio.

Summary

In this chapter the properties of the systems which result from the basic design procedure are investigated from a practical viewpoint. In particular, the effect of the zeros of $H_{eq}(s)$ not being exactly equal to the poles of G(s) is considered. It is concluded that the systems are structurally stable insofar as this effect is concerned except in those cases having closed loop poles on or near the imaginary axis.

Possible procedures that might be useful when the basic design method is not applicable, either because of the location of the nonlinearity in the system or because all the state variables cannot be fed back, are discussed. Although no general solution to this problem is obtained, there are several approaches suggested. The best approach will depend upon the particular system to be controlled and the required performance of the system.

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

Conclusions

Design procedures for single input, single output systems based on the modern control theory concept of feeding back all the state variables have been developed, and the characteristics of the resulting systems studied. These procedures are based on the concept of the SVF method as developed by Schultz (1966). They utilize both the concept of series compensation as practiced in classical control theory and state variable feedback as suggested by modern control theory. First, a procedure for designing linear systems for a desired closed loop transfer function is developed from the matrix representation of the system. From this, a procedure for linear gain insensitive systems is developed. This procedure for gain insensitive systems is then used to develop the principal results of this dissertation, a method for designing certain nonlinear and/or time-varying systems.

The basic design procedure applies to systems with a single memoryless nonlinear and/or time-varying gain whose input-output graph is confined to the first and third quadrants. This is the class of nonlinear systems to which the Popov stability criterion applies. In its basic form, it is limited to systems with the nonlinear and/or time-varying gain located as shown in Figure 4-1. It is applicable only to systems classified as the principal case and the simplest particular case. Modifications of the basic procedure which can be used to overcome

The requirement concerning the location of the nonlinearity are discussed.

Significant features of the basic design procedure and the resulting systems are as follows:

1. The procedure is applicable, in a practical sense, to systems of any order. Basic matrix or block diagram manipulations and the solution of a linear algebraic equations are required in carrying it out.
2. The sector of absolute stability is infinite.
3. The output of the system is bounded for bounded inputs.
4. The closed loop system can be represented by an equivalent system for studying input-output relations which consists of a first order nonlinear and/or time-varying portion in series with an (n-1)st order stable linear portion. This equivalent system can be used to determine the closed loop response for known inputs.
5. The closed loop system is linearized to the extent that (n-1) of the closed loop poles are equal to (n-1) of the open loop poles, independent of the nonlinear gain.
6. In systems where the operation is linear for normal control signals but nonlinear for large control signals, it is possible to design for desired performance in the linear region without having to worry about system stability for large control signals.

Suggestions for Further Work

Since the proposed design procedure appears to have immediate practical applications, one obvious area of further work is in the application of the method to actual control problems. It is difficult to predict what type of problems might be encountered until this is done.

There are several theoretical questions concerning the method which need to be investigated. The sensitivity of the system response to parameter changes in the linear plant need to be investigated. Morgan (1963, 1966) has studied this problem, and Bob White, a graduate student at the University of Arizona, is currently investigating the subject of sensitivity in systems with state variable feedback.

The equivalent system allows the calculation of only the output variable and the control signal, as the other state variables in this equivalent system are not physical variables. Indications are that the magnitude of these state variables will in general probably not be excessive compared to corresponding magnitudes in a system with unity feedback from the output only. However, this is largely supposition, and the subject needs to be investigated. Since the output of the non-linear element can be found from the equivalent system, this could be used in the actual system block diagram to determine the value of any desired state variable.

There needs to be more of a comparison between the performance of systems designed by the proposed method and those designed by classical methods (or by other modern control methods). A criterion for this comparison must be chosen. This problem will be resolved at least in part by the success with which the procedure can be applied to actual systems.

The modifications to the basic procedure suggested in Chapter 5 need to be investigated more thoroughly. For example, if all the state variables cannot be fed back, is the method of determining feedback transfer functions suggested in Chapter 3 or that of realizing an $H_{eq}'(s)$ as suggested in Chapter 5 better for a particular application?

More information on how to choose a desired closed loop transfer function in the linear case would be helpful. Likewise in the nonlinear case, more information on how to choose the location of the closed loop poles that are made independent of the nonlinear gain would be helpful.

It is shown that the linear gain insensitive systems always satisfy Kalman's frequency condition for some performance index. It would be interesting, and perhaps useful, to know something about the performance index for which this condition is satisfied and whether it can be related to the closed loop transfer function.

More information concerning the extent of the effect of small changes in the pole and zero locations on the validity of the closed loop response as calculated from the equivalent system is needed. Additional analog computer simulations would be helpful at this point and would also provide other useful information concerning systems designed by the proposed method.

In conclusion, although several aspects of systems designed by the proposed method still need to be investigated, it is felt that the analysis of such systems presented in this dissertation is enough to indicate the possible usefulness of this approach to the design of control systems with nonlinear and/or time-varying gains.

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