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CONSTRUCTION OF THE LYAPUNOV FUNCTION FOR A CASE OF STABILITY WITH CONSTANTLY ACTING DISTURBANCES IN NONLINEAR SPACES

a translation of

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Notice

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1. Let us consider the differential equation:

$$\frac{dx}{dt} = f(t,x) \tag{1}$$

where $x = x(t) \in L$ is an unknown function (vector) of the independent variable t, $f(t,x) \in L$, where L is a nonlinear space introduced by Persidskii[1].

Let us assume that the function $f(t,x)$ is given in the region h:

$$t \geq 0; \quad ||x||_L \leq R \tag{h}$$

and is represented by means of the following sum

$$f(t,x) = \ell(t,x) + \psi(t,x) + \phi(t,x) \tag{2}$$

In (2) $\ell(t,x)$ is defined for all $t \geq 0$; for any $x \in L(||x|| < \infty)$ and satisfies the following conditions:

I. $\ell(t,\theta) = \theta$, where θ is the zero element of the space L;

II. $\ell(t,x)$ is continuous in t (in the sense of the metric defined on L);

III. $\ell(t,x)$ satisfies for the variable x the Cauchy condition

$$||\ell(t,x_1) - \ell(t,x_2)|| \leq K\, ||x_1 - x_2||.$$

with the constant $K > 0$.

The function $\psi(t,x)$ satisfies in the region h the following conditions:

1. $\psi(t,\theta) = \theta$
2. $\psi(t,x)$ is continuous in t (in the sense of the metric defined on L).

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1. K. P. Persidskii, "Differential Equations in Nonlinear Spaces," < Izvestnia AH. Kazak SSR > Seria Fiziko - Matematicheskikh nauk, No. 1, 1965.

3. $\psi(t,x)$ satisfies the inequality:

$$||\psi(t,x)||_L \leq ||x|| \; \gamma(||x||),$$

where $\gamma(||x||) \to 0$ for $||x|| \to 0$;

4. $\psi(t,x)$ satisfies the Cauchy condition

$$||\psi(t,x_1) - \psi(t,x_2)||_L \leq \beta(t)||x_1 - x_2||_L. \tag{2}$$

where $\beta(t)$ is a continuous function in t.

In the same region h the function $\phi(t,x)$ satisfies the following conditions.

a. $\phi(t,\theta)$ generally speaking is not the zero element of the space L.

b. $\phi(t,x)$ is continuous in t (in the sense of the metric defined on L)

c. $\phi(t,x)$ satisfies a Cauchy condition similar to that satisfied by $\psi(t,x)$

d. In the region h, $\phi(t,x)$ satisfies the inequality

$$||\phi(t,x)|| \leq \rho.$$

where the value $\rho > 0$ can be freely chosen.

In the following the function $\phi(t,x)$ will be taken as a constantly acting disturbance.

2. Let us assume, that for an arbitrary initial point (t_o,x_o) the solution $x = y(t, t_o, x_o)$ of the differential equation

$$\frac{dx}{dt} = \ell(t,x) \tag{3}$$

satisfies the condition

$$||y(t,t_o,x_o)|| \leq B||x_o||\ell^{-\alpha(t-t_o)} \tag{4}$$

for all $t \geq t_o \geq 0$, where $B \geq 1$ and $\alpha > 0$ are some constants.

Let us first of all notice, that the null solution $x = \theta$ of the differential equation (without disturbance)

$$\frac{dx}{dt} = \ell(t,x) + \psi(t,x) \tag{5}$$

is stable for any small constantly acting disturbance $\phi(t,x)$, if the solution $X = X(t,t_o,x_o)$, going through the point $(t_o,x_o) \in L$ of the differential equation

$$\frac{dx}{dt} = \ell(t,x) + \psi(t,x) + \phi(t,x) \tag{6}$$

satisfies the following condition:

For any given $\varepsilon > 0$ and for any given initial time $t_o \geq 0$ there exist two numbers $r = r(\varepsilon,t_o) > 0$ and $\rho = \rho(\varepsilon,t_o) > 0$

such that

$$||x_o||_L \leq r \text{ and } ||\phi(t,x)||_L \leq \rho$$

implies that for all $t \geq t_o$

$$||X(t,t_o,x_o)||_L \leq \varepsilon$$

For specific problems, it will be considered that

$$\varepsilon < R.$$

3. It is well known, that if the space L is a Banach space uncer condition (4) the null solution $x = \theta$ of the differential equation (5) is stable for a constantly acting disturbance $\phi(t,x)$. This is also true in the nonlinear space L, namely:

> If condition (4) is fulfilled, then the null solution $x = \theta$ of the differential equation (5) will be stable for constantly acting disturbances $\phi(t,x)$ in the nonlinear space L.

But the method, which ordinarily is used for the proof of showing the stability of the null solution of the differential equation (5) in a Banach space, does not apply for the space L, because of its nonlinearity. Therefore our proof will be based on the Second Method of Lyapunov.

For this purpose let us consider the real function $v(t,x)$ defined by the following equality

$$v(t,x) = \int_0^\infty ||y(\tau,t,x)||_L dt \tag{7}$$

and constructed from the solution $x = y(t,t_o,x_o)$ of the differential equation (3).

In order to fulfill condition (4) the function $v(t,x)$ has to be the solution of the functional equation

$$\lim_{\Delta t \to 0} \frac{v[t + \Delta t, x + \Delta t\, \ell(t,x)] - v(t,x)}{\Delta t} = - ||x||_L \tag{8}$$

and has to be positive definite, with an infinitely small upper bound. (In the region $t \geq 0$, $||x||_L < \infty$).

In addition, the function $v(t,x)$ satisfies the Cauchy condition

$$|v(t,x_1) - v(t,x_2)| \leq H \; ||x_1 - x_2||_L \quad , \tag{9}$$

where H is some constant.

Let us also notice, that on the basis of (8) the total derivative of the function $v(t,x)$, with respect to the differential equation (3) is a negative definite function for all $t \geq 0$ and for all $x \in L$ and is equal to $-||x||_L$.

Next let us make an estimate for the total derivative of the function $v(t,x)$ with respect to the differential equation (6) one will have:

$$\begin{aligned} &|v[t + \Delta t, \; x + \Delta t(\ell(t,x) + \psi(t,x) + \phi(t,x))] - v[t + \Delta t, \; x + \Delta t \; \ell(t,x)]| \leq \\ &\leq H||\Delta t[\ell(t,x) + \psi(t,x) + \phi(t,x)] - \Delta t \; \ell(t,x)|| \end{aligned} \tag{10}$$

On the basis of one of the properties of the space L, given in [1] one can write:

$$\begin{aligned} I = &||\Delta t[\ell(t,x) + \psi(t,x) + \phi(t,x)] - \Delta t \; \ell(t,x)|| \leq \\ &\leq A|\Delta t| \; ||\psi(t,x) + \phi(t,x)|| \end{aligned} \tag{11}$$

where A is a certain constant in the region h.

Taking into account the conditions imposed on the functions $\psi(t,x)$ and $\phi(t,x)$, the inequality (11) gives the following inequality:

$$I \leq A|\Delta t|(||x||\gamma \; (||x||) + \rho) \tag{12}$$

From (10) and (12) one will have

$$\left|\{v[t + \Delta t,\, x + \Delta t(\ell(t,x) + \psi(t,x) + \phi(t,x))] - v(t,x)\} - \right.$$

$$\left. - \{v[t + \Delta t,\, x + \Delta t\ \ell(t,x)] - v(t,x)\}\right| \leq \tag{13}$$

$$H\ A\ |\Delta t|(||x||\ \gamma(||x||) + \rho)$$

Therefore the total derivative (more precisely, the upperlimit of the total derivative) of the function $v(t,x)$ with respect to the differential equation (6) will satisfy the inequality

$$\begin{aligned} v'(t,x) &\leq -||x|| + H\ A||x||\gamma(||x||) + H\ A\ \rho = \\ &= -\ ||x||(1 - H\ A\ \gamma(||x||)) + H\ A\ \rho \end{aligned} \tag{14}$$

We will consider that a given number $\varepsilon > 0$ can be chosen so small that

$$H\ A\ \gamma(\varepsilon) \leq 1/4$$

and let us take a value $\rho > 0$ so small that

$$H\ A\ \rho \leq 1/4r$$

Then on the basis of (14), for all values of $t \geq 0$ and for

$$r \leq ||x|| \leq \varepsilon \tag{15}$$

we will have

$$v'(t,x) \leq -1/2\ ||x|| \tag{16}$$

Let us now turn to the choice of the number $r > 0$. For this purpose denote by m, the lowest value of the function $v(t,x)$ for $||x|| = \varepsilon$ and $t > 0$. Let us choose the number $r > 0 (r < \varepsilon)$ such, that the largest value of $v(t,x)$ will be less than m, for $||x|| \leq r$ and $t \geq 0$.

The existence of such a number $r > 0$ follows from the fact that the function $v(t,x)$ is positive definite, and possessed of an infinitely small upper bound. Hence on the basis of (16) it is easily shown that the solution $x = x(t,t_o,x_o)$ of the differential equation (6) will satisfy for all times $t \geq t_o$ the inequality

$$||x(t,t_o,x_o)|| < \varepsilon$$

whenever

$$||x_o|| \leq r$$

and

$$||\phi(t,x)|| \leq \rho$$

where $r > 0$ and $\rho > 0$ are chosen as indicated above.

Indeed in the ring-shaped region

$$r \leq ||x||_L \leq \varepsilon$$

the derivative $v'(t,x)$ will satisfy the inequality (16), and therefore along the trajectories of the differential equation (6) in the indicated ring the function $v(t,x)$ itself can only decrease since the norm

$$||x(t,t_o,x_o)||$$

cannot become equal to the number ε, since m is the highest value of the function $v(x,t)$ for $||x|| = r$ and $t \geq 0$. Thus in satisfying condition (4) the function $v(t,x)$ determined by the equation (7), is a Lyapunov function and guarantees the stability of differential equation (6) for members of the class $\psi(t,x)$ with small higher order terms and for constantly acting disturbances $\phi(t,x)$.

Since the choice of the number $r > 0$ and that of the number $\rho > 0$ does not depend on the selection of the initial time $t_o \geq 0$, it is shown that the stability is uniform.