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NUMERICAL SOLUTION OF THE FLOW FIELD IN THE THROAT REGION OF A NOZZLE

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A B S T R A C T

A time dependent finite difference technique was used to compute the flow field in a nozzle throat region. Steady state results are obtained as an asymptotic solution in time. This report describes the finite difference solution and presents comparisons with both experimental data and the theoretical work of Sims (Ref. 3).

Approved

R. W. Donnelly

R. W. Donnelly, Manager
Engine Systems Branch

Approved

B H Neuffer

B. H. Neuffer, Manager
Propulsion Department

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NUMERICAL SOLUTION OF THE FLOW FIELD IN THE THROAT REGION OF A NOZZLE

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Prepared By

Lee M. Saunders

Engine Systems Analysis Section
Propulsion Department
Space Vehicle Division
Brown Engineering Company, Inc.

for

Engine Systems Branch
Propulsion Division
Propulsion and vehicle Engineering Laboratory
George C. Marshall Space Flight Center
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LIST OF SYMBOLS

x,r	Cylindrical coordinates parallel and perpendicular to the nozzle symmetry axis, respectively
X,Y	Coordinates defined by equation (9)
s	r coordinate of nozzle wall
ρ	Non-dimensional density
P	Non-dimensional pressure
V	Non-dimensional velocity
u,v	Components of V in the x, r directions, respectively
W, F, G, H	Variables defined by equations (5)
γ	Ratio of specific heats
θ	Flow direction angle measured with respect to the symmetry axis
Q	Mass flow
A	Cross sectional area of nozzle
c	Non-dimensional sound speed
a	Dimensional sound speed
t	Non-dimensional time
M	Mach number
$( )^*$	Refers to critical values

S U M M A R Y

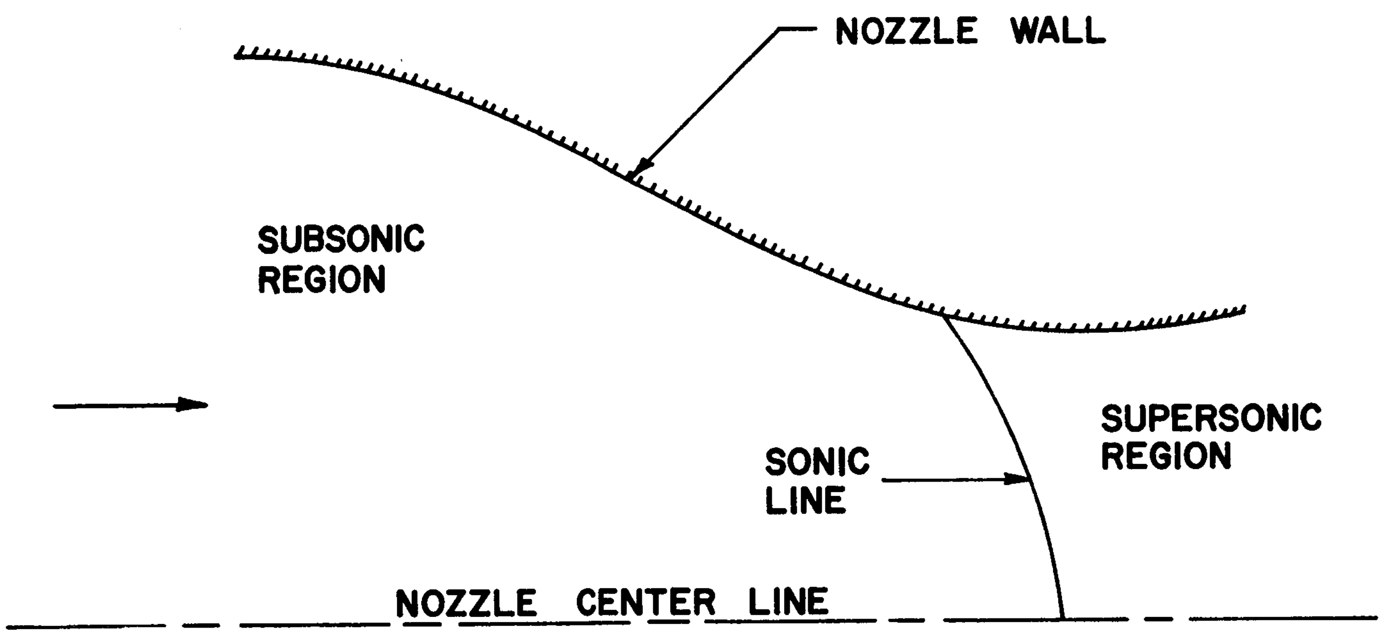
A method was developed to compute the entire flow field in the throat region of a converging-diverging nozzle. The time-dependent equations describing the flow of an inviscid, ideal, non-heat-conducting gas were solved by a numerical finite difference technique described by Thommen (Ref. 5). Steady state results are obtained as an asymptotic solution of the time dependent equations of motion.

A FORTRAN program was written for the CDC 3200 digital computer and two check cases were run. The first case was compared with the solutions of Sims (Ref. 3) and the second was compared with experimental data (Ref. 6). Comparison with experimental data was excellent.

INTRODUCTION

Regions of mixed subsonic-supersonic flow have generally escaped accurate theoretical analysis due mainly to the mixed nature of the governing differential equations; the equations being elliptic, parabolic, and hyperbolic in the subsonic, sonic, and supersonic portions of the flow, respectively. In supersonic flow, the well known method of characteristics provides a fast, accurate method of attack provided that an initial data line is known. This then, is the goal of most transonic flow solutions. That is, to compute the subsonic portion of the field, through the sonic transition region, and far enough downstream of the sonic line to provide an accurate, supersonic initial data line for continuation of the flow calculation by the method of characteristics.

A classic example of the mixed flow problem is in the throat region of a nozzle (Fig. 1). In earlier works, the problem was attacked by series expansion of the equation for the potential function. Coefficients of the series were then obtained by some type of numerical analysis, such as the iterative procedure devised by Oswatitsh and Rothstein (Ref. 1). Unfortunately, the iteration requires numerical differentiation, and the procedure is unstable due to the elliptic nature of the flow equations. If the nozzle throat is of small radius of curvature, several iterations may be required with possibly disastrous results.



GENERAL DESCRIPTION OF FLOW FIELD

FIG. I

Another solution is that of Sauer (Ref. 2) which is a series expansion about the critical line. Sauer's method contains only two terms of the series and should be used only for nozzles with large radii of curvature at the throat wall.

In reference 3, Sims extended the method of Sauer by adding more terms to the series expansion. This should provide more accurate results, especially for a throat with small radius of curvature. Addition of more terms to the series presents a most laborious task, however, and it is quite clear from Sims' results that the series is not yet closed for nozzles having very small radii of curvature.

Since instability is a major drawback to most numerical solutions of the elliptic flow equations, a solution is posed here which circumvents the basic instability problems.

The basic idea is to numerically integrate the time dependent flow equations, imposing steady state boundary conditions at the wall, and obtaining a steady state solution asymptotically with increasing time. The differential equations describing the flow are replaced by finite difference equations and integrated with respect to time. This scheme was shown to be at least conditionally stable (see Refs. 4, 5) and is of second order accuracy.

In this study the gas was assumed to be ideal, inviscid, and non heat-conducting. Also, this program applies only to smooth nozzles; i.e. there can be no discontinuities in the wall slope. The flow is axisymmetric or two dimensional.

THEORY & EQUATIONS

The equations governing the axisymmetric flow are given with respect to a cylindrical coordinate system x, r, ϕ as follows:

$$W_t = F_x + G_r + H \tag{1}$$

Where W, F, G and H are the column vectors

$$W = \begin{pmatrix} \rho r \\ \rho ur \\ \rho vr \\ Er \end{pmatrix} \quad F = \begin{pmatrix} -\rho ur \\ -\rho u^2 r - Pr \\ -\rho uvr \\ -u\left[E + P\right] r \end{pmatrix} \quad G = \begin{pmatrix} -\rho vr \\ -\rho uvr \\ -\rho v^2 r - Pr \\ -v\left[E + P\right] r \end{pmatrix} \quad H = \begin{pmatrix} 0 \\ 0 \\ P \\ 0 \end{pmatrix}$$

The subscripts t, x, r denote partial differentiation, and

$$E = \frac{1}{\gamma - 1} P + \tfrac{1}{2}\rho V^2$$

Where ρ is the density, P the pressure, V is the velocity of the gas, u and v are the components of V in the x, r directions respectively, and γ is the ratio of specific heats.

All lengths are made dimensionless by the r coordinate of the wall at the throat. Flow parameters are non-dimensionalized by

$$\rho = \frac{\rho'}{\rho_*} \qquad P = \frac{P'}{\gamma P^*} \qquad V = \frac{V'}{a^*}$$

where the primes denote dimensional quantities and the asterisk (*) refers to the sonic values.

The above equations are written for axisymmetric flow. Two dimensional flow equations are obtained by replacing r by r^ϵ, where

$$\epsilon = \begin{cases} 0 & \text{for two-dimensional flow} \\ 1 & \text{for axisymmetric flow} \end{cases}$$

The vector H for two-dimensional flow is identically zero.

COMPUTATION SCHEME

Assuming that W is known over the entire flow field considered at $t = t_o$, then a truncated Taylor series in t yields

$$W(t_o + \Delta t) = W(t_o) + W_t \Delta t + W_{tt} \frac{\Delta t^2}{2} + 0(\Delta t^3) \tag{2}$$

Let W at any point in the field be denoted by $W^{l}_{m,n}$

where

$t = l\Delta t \qquad x = n\Delta x \qquad r = m\Delta r$

Then W_t can be evaluated at (l, m, n) by replacing equation (1) with the difference analog

$$W_t = \frac{1}{2\Delta x}\left[F^{l}_{m,n+1} - F^{l}_{m,n-1}\right] + \frac{1}{2\Delta r}\left[G^{l}_{m+1,n} - G^{l}_{m-1,n}\right] + H^{l}_{m,n} \tag{3}$$

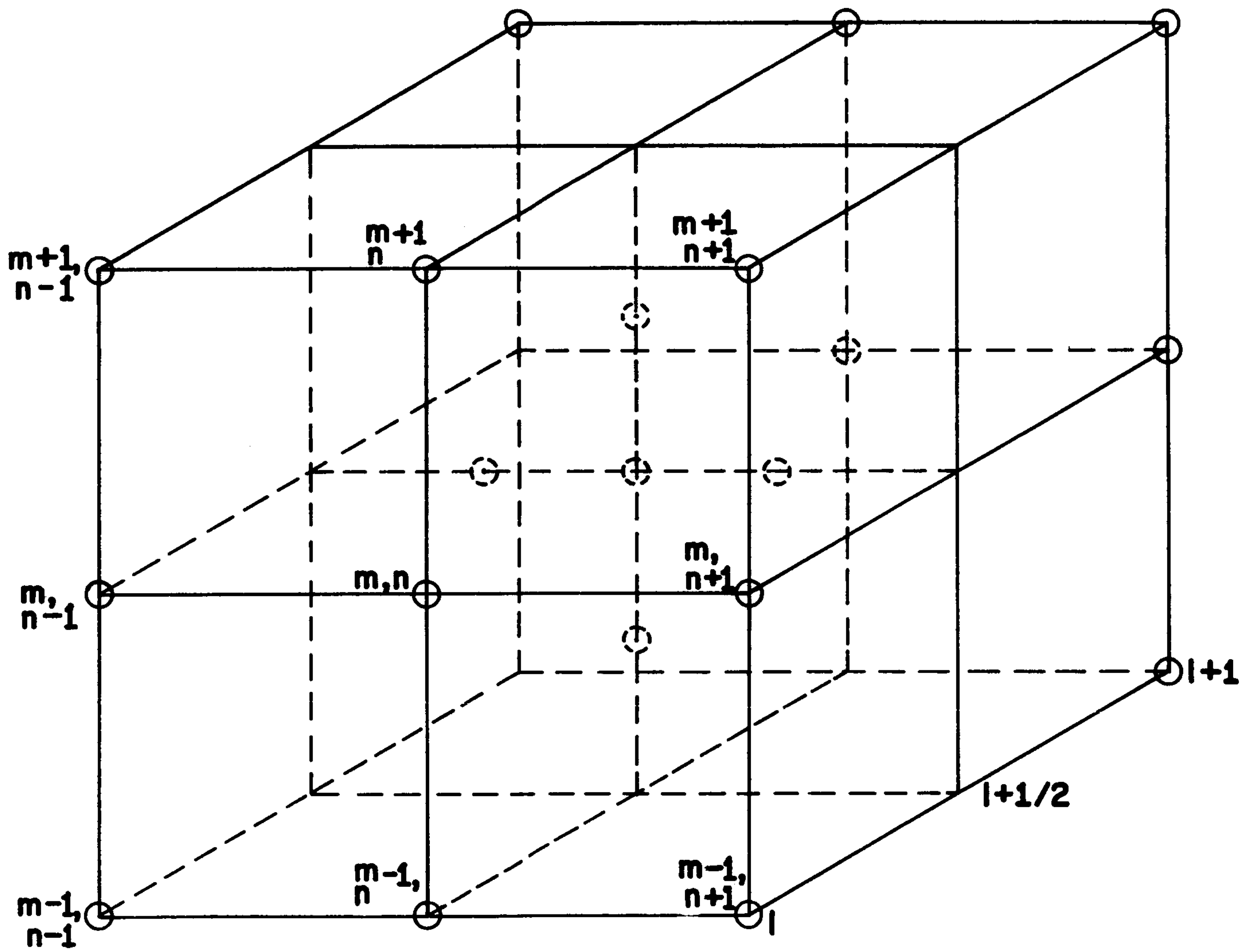
The value of W_{tt} may be computed by differentiating equation (1).

$$W_{tt} = F_{xt} + G_{rt} + H_t \tag{4}$$

Now

$$\begin{array}{lll}
W_1 = \rho r & F_1 = -W_2 & G_1 = -W_3 \\
W_2 = \rho ur & F_2 = -\left[\dfrac{W_2^{\,2}}{W_1} + Pr\right] & G_2 = \dfrac{-W_2W_3}{W_1} \\
W_3 = \rho vr & F_3 = \dfrac{-W_2W_3}{W_1} & G_3 = -\left[\dfrac{W_3}{W_1} + Pr\right] \\
W_4 = Er & F_4 = \dfrac{-W_2}{W_1}\left[W_4 + Pr\right] & G_4 = \dfrac{-W_3}{W_1}\left[W_4 + Pr\right]
\end{array} \tag{5}$$

$$Pr = (\gamma - 1)\left[W_4 - \tfrac{1}{2}\,\frac{W_2{}^2 + W_3{}^2}{W_1}\right]$$



FINITE DIFFERENCE GRID

FIG. 2

$$\frac{\partial F_i}{\partial t} = \alpha i,j \ \frac{\partial W_j}{\partial t}$$

$$\frac{\partial G_i}{\partial t} = \beta i,j \ \frac{\partial W_j}{\partial t}$$

$$\frac{\partial H_i}{\partial t} = \delta i,j \ \frac{\partial W_j}{\partial t}$$

where repeated subscripts, (j), require summation over that subscript, and

$$\alpha i,j = \frac{\partial F_i}{\partial W_j} \qquad \beta i,j = \frac{\partial G_i}{\partial W_j} \qquad \delta i,j = \frac{\partial H_i}{\partial W_j}$$

The derivative $\frac{\partial W_i}{\partial t}$ may be evaluated as in equation (3). Finally, the second order term, W_{tt}, is evaluated with a finite difference analog of

$$W_{tt} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right) + \frac{\partial}{\partial r}\left(\frac{\partial G}{\partial t}\right) + \frac{\partial H}{\partial t}$$

This scheme requires the evaluation of the Jacobians $\alpha_{i,j}$, $\beta_{i,j}$, $\delta_{i,j}$ and subsequent matrix multiplication at each point.

A simpler scheme of equal accuracy as outlined by Thommen (Ref. 5) was used for the present program. The finite difference equations and computing sequence are outlined as follows:

(Step 1) Compute W at $(1 + \frac{1}{2}, m \pm \frac{1}{2}, n \pm \frac{1}{2})$ and $(1 + \frac{1}{2}, m,n)$ from the difference analog of equation (1)

$$\begin{aligned} W(1 + \tfrac{1}{2}, m \pm \tfrac{1}{2}, n) = {} & \tfrac{1}{2}\,[W(1,m,n) + W(1,m \pm 1,n)] \\ & + \frac{\Delta t}{2}\Big\{ \frac{1}{4\Delta x}\,[F(1,m \pm 1,n + 1) - F(1,m \pm 1, n-1) \\ & + F(1,m,n + 1) - F(1,m,n - 1)] \pm \frac{1}{\Delta r}\,[G(1,m \pm 1,n) \\ & - G(1,m,n)] + \tfrac{1}{2}\,[H(1,m \pm 1,n) + H(1,m,n]\Big\} \quad (6) \end{aligned}$$

$$W(1+\tfrac{1}{2},m,n\pm\tfrac{1}{2}) = \tfrac{1}{2}[W(1,m,n)+W(1,m,n\pm 1)] + \frac{\Delta t}{2}\Big\{\pm\frac{1}{\Delta x}[F(1,m,n\pm 1) - F(1,m,n)] + \frac{1}{4\Delta r}[G(1,m+1,n\pm 1) - G(1,m-1,n\pm 1) + G(1,m+1,n) - G(1,m-1,n)] + \tfrac{1}{2}[H(1,m,n\pm 1) + H(1,m,n)]\Big\} \qquad (7)$$

$$W(1+\tfrac{1}{2},m,n) = \tfrac{1}{4}[W(1,m,n-1)+W(1,m,n+1)+W(1,m+1,n) + W(1,m-1,n)] + \frac{\Delta t}{2}\Big\{\frac{1}{2\Delta x}[F(1,m,n+1) - F(1,m,n-1)] + \frac{1}{2\Delta r}[G(1,m+1,n) - G(1,m-1,n)] + H(1,m,n)]\Big\} \qquad (8)$$

(Step 2) Compute $F(1+\frac{1}{2},m,n\pm\frac{1}{2})$, $G(1+\frac{1}{2},m\pm\frac{1}{2},n)$, $H(1+\frac{1}{2},m,n)$ from the W computed in equations (6) - (8), referring to the algebraic forms of W, F, G and H in equation (5).

(Step 3) Finally, compute $W(1+1,m,n)$ with W_t centered at $(t+\Delta t, m, n)$ from

$$W(1+1,m,n) = W(1,m,n) + \Delta t\Big\{\frac{1}{\Delta x}[F(1+\tfrac{1}{2},m,n+\tfrac{1}{2}) - F(1+\tfrac{1}{2},m,n-\tfrac{1}{2})] + \frac{1}{\Delta r}[G(1+\tfrac{1}{2},m+\tfrac{1}{2},n) - G(1+\tfrac{1}{2},m-\tfrac{1}{2},n)] + H(1+\tfrac{1}{2},m,n)\Big\}$$

COORDINATE CHANGE

The above equations were derived for a cylindrical coordinate system. In order to obtain a grid with equal mesh size in the r direction at the wall, a slight change of coordinates was made.

Let s = s(x) be a function describing the wall boundary, then the coordinate change was made as follows.

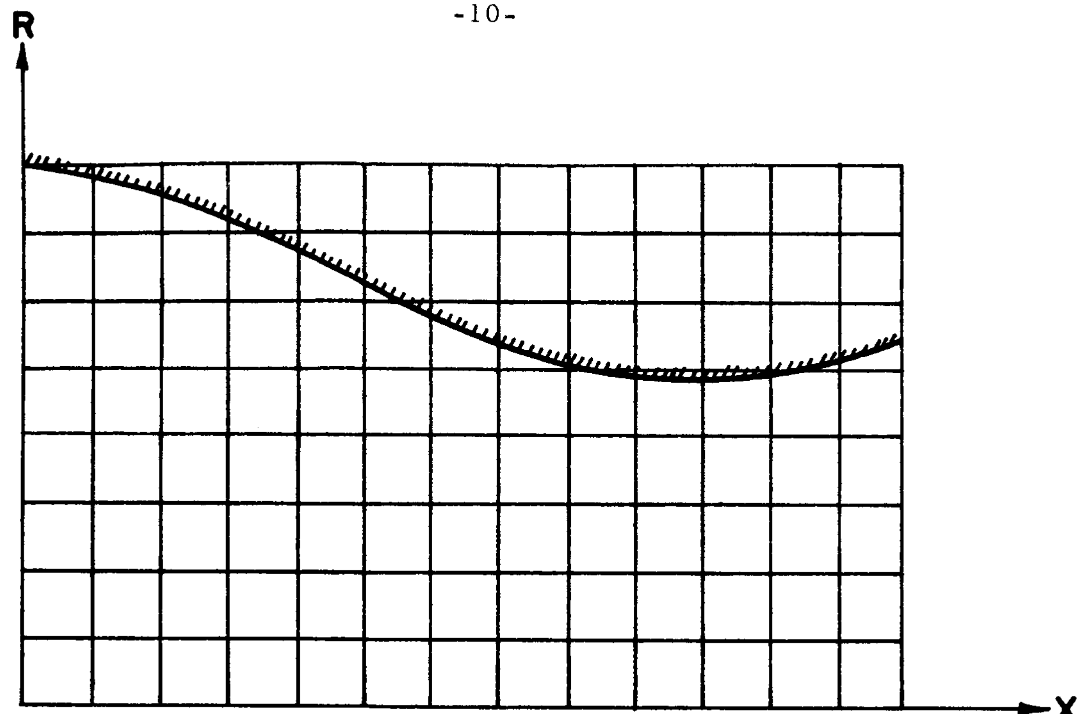
$$X = x, \quad Y = \frac{r}{s(x)} \tag{9}$$

Equation (1) now has the form:

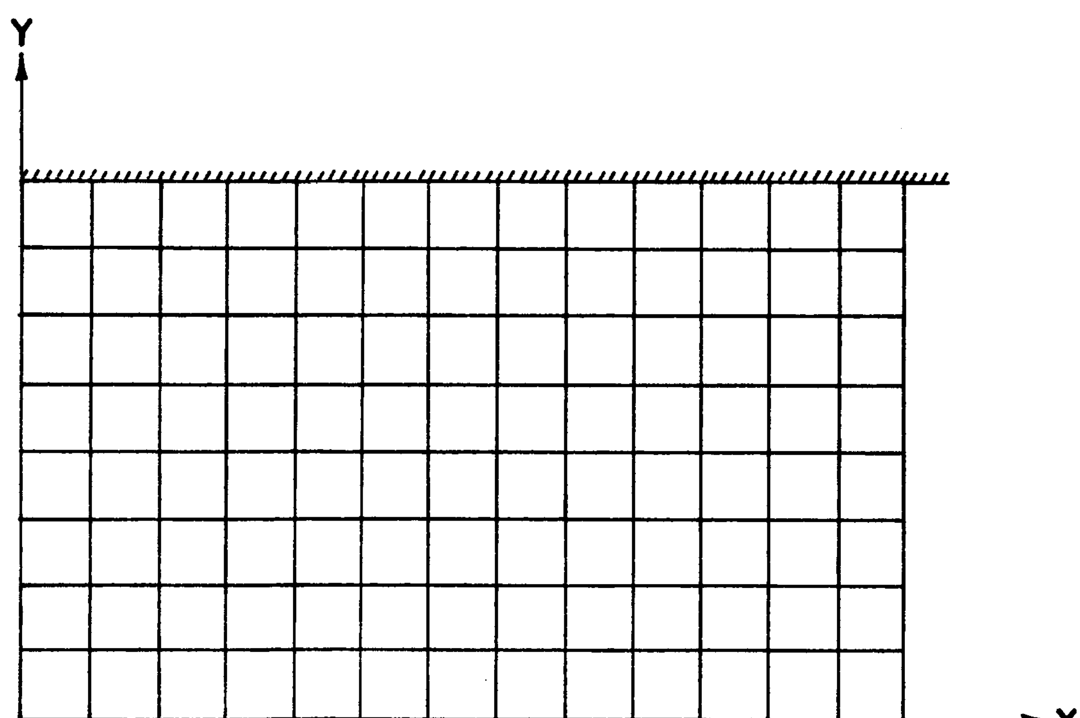
$$W_t = F_X - Y \frac{s'}{s} F_Y + \frac{1}{s} G_Y + H$$

where the prime denotes differentiation with respect to x. Although the basic equations have become slightly more complicated due to the coordinate change, the advantage gained is considerable.

It is easily seen that the logic required in a computer program is considerably reduced using the X, Y coordinate system. A more important consideration, however, lies in the possibility of local instabilities in the finite difference scheme if the mesh size at the wall should be much smaller than the standard mesh. In the X, Y coordinate system, a constant value of ΔY can be maintained throughout the entire field.



GRID MESH IN CYLINDRICAL COORDINATES



GRID MESH IN X,Y COORDINATE SYSTEM

COORDINATE SYSTEM

FIG. 3

BOUNDARY CONDITIONS

Wall Boundary

Let m = M denote the nozzle wall boundary. Assuming that for any n, the points M - 1, M - 2, M - 3 have been computed, the wall point is computed as follows.

Using the three preceding points M - 1, M - 2, and M - 3, extrapolate W_1, W_4, and P to m = M.

Evaluate V from

$$V = \left[\frac{2}{W_1}\left(W_4 - \frac{pr}{\gamma-1}\right)\right]^{\frac{1}{2}}$$

Then

$$W_2 = W_1 V \cos\theta$$

$$W_3 = W_1 V \sin\theta \qquad \text{Where } \theta = \tan^{-1}(s')$$

A second order extrapolation equation for a general function Z is:

$$Z_M = 3(Z_{M-1} - Z_{M-2}) + Z_{M-3}$$

Axis Boundary

For axis symmetric flow, the axis boundary conditions are quite simple.

Since $W_1 = \rho r \qquad W_3 = \rho v r$

$W_2 = \rho u r \qquad W_4 = E r$

then $W_i = 0$ for i = 1, 2, 3, 4. The pressure, P, at the axis is determined as follows.

Evaluate Pr at all points in the field m = 2, -- M at any M from

$$Pr = (\gamma - 1)\left[W_4 - \frac{1}{2}\,\frac{W_2^2 + W_3^2}{W_1}\right]$$

then

$$P = \frac{Pr}{r}$$

Of course Pr = 0 at the axis, so that it is necessary to interpolate P from points off the axis. The pressure at points below the axis are reflected from corresponding points above the axis in order to utilize a centered interpolation formula.

Let the x - axis be denoted by m = 1. Then an interpolation formula for any function Z yields:

$$Z\ (1) = 1/3 \left[4Z\ (2) - Z\ (3)\right]$$

For two-dimensional flow, a different procedure is used at the axis. Here, use is made of the flow symmetry about the x - axis. The axis is designated by m = 2, and the line m = 1 is a mirror image of the line m = 3. For the W_i at m = 1

$$W_1\ (1,1,n) = W_1\ (1,3,n)$$
$$W_2\ (1,1,n) = W_2\ (1,3,n)$$
$$W_3\ (1,1,n) = -W_3\ (1,3,n)$$
$$W_4\ (1,1,n) = W_4\ (1,3,n)$$

The axis points are computed with the same general procedure as regular flow field points.

Downstream Boundary

A coordinate line x = constant at n = N is chosen as the downstream boundary several grid stations downstream of the sonic line. No errors generated along this line can be propagated upstream from this line due to the principal of forbidden signals. A linear extrapolation is therefore sufficient to obtain flow properties at the downstream boundary. The extrapolation equation used in the present program is

$$Z_N = 2\ Z_{N-1} - Z_{N-2}$$

for any function Z.

Upstream Boundary

To insure good upstream boundary conditions, the inlet to the nozzle is assumed to be a straight pipe of sufficient length that the flow may be considered uniform in the straight section of the nozzle at some station $n = 1$.

Supersonic flow is established downstream of the throat by insuring that the mass flow Q at $n = 1$ is equal to the critical mass flow.

Let A = cross sectional area of the nozzle, then

$$Q = (\rho'u'A)_1 = \rho^* a^* A^*$$

At each time step, the mass flow is numerically integrated across the throat section.

$$Q = 2\pi\int_0^{S_{NT}} W_2 dr = A^* \qquad W_2 = (\rho ur)_{n = NT}$$

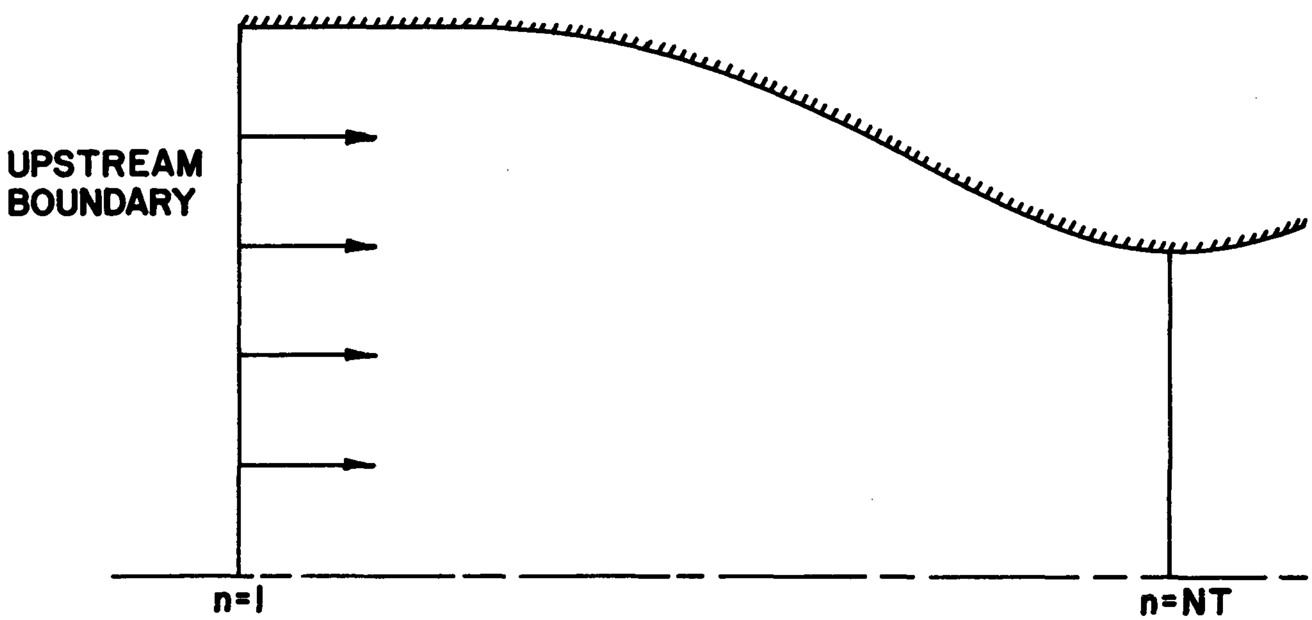
So that the area ratio, AR, is computed from

$$AR = \frac{2}{S_1^{\,2}}\int_0^{S_{NT}} W_2 dr = \frac{A^*}{A_1}$$

The flow properties at $n = 1$ are computed from one-dimensional area ratio equations and

$$W_1 = \rho_1 r \qquad W_3 = 0$$

$$W_2 = \rho_1 V_1 r \qquad W_4 = Er$$



UPSTREAM BOUNDARY

FIG. 4

LOCAL FLOW PARAMETERS

The variables W_i are used in the program to simplify the computation scheme. Local flow properties may be computed with the following equations.

$$\rho = \frac{W_1}{r^\epsilon} \qquad u = \frac{W_2}{W_1} \qquad v = \frac{W_3}{W_1}$$

$$V^2 = \frac{W_2^{\,2} + W_3^{\,2}}{W_1^{\,2}} \qquad P = \frac{(\gamma - 1)}{r^\epsilon} \left[W_4 - \tfrac{1}{2} \frac{W_2^{\,2} + W_3^{\,2}}{W_1} \right]$$

$$\theta = \tan^{-1} \left(\frac{W_3}{W_2} \right)$$

For axis-symmetric flow, $W_1 = W_2 = W_3 = W_4 = 0$ at the axis. Flow parameters on the axis must be computed by interpolating from points off the axis, as was done for P in the axis boundary calculations.

STABILITY

It was mentioned earlier that the finite difference scheme presented here is conditionally stable. A linear analysis was performed by a number of investigators to establish a stability criterion for this type of technique. In particular, Burstein (Ref. 4) shows a "safe" step size relative to local flow conditions to be given by

$$\frac{\Delta t}{\Delta} < \frac{(M + 1)^{-1}}{c \sqrt{8}}$$

where

M = local Mach number

c = local speed of sound

Δ = spacial mesh size

This stability criterion was followed in the present work, the step size Δt being computed initially based on sonic flow and kept constant throughout the entire run.

RESULTS

This problem was programmed for the CDC 3200 digital computer, and two test cases have been run. Figure 5 is a sketch of the nozzle indicating the final sonic line position for the first case.

The initial data at $t = 0$ for this case was computed from one-dimensional area ratio equations, and a record was kept at each time step of the value of W_1 at the throat wall position. After about 450 time steps, W_1 at this point was damped to within .05% of the final value (Fig. 6). The mesh size for this run was $M \times N = 11 \times 23$ points.

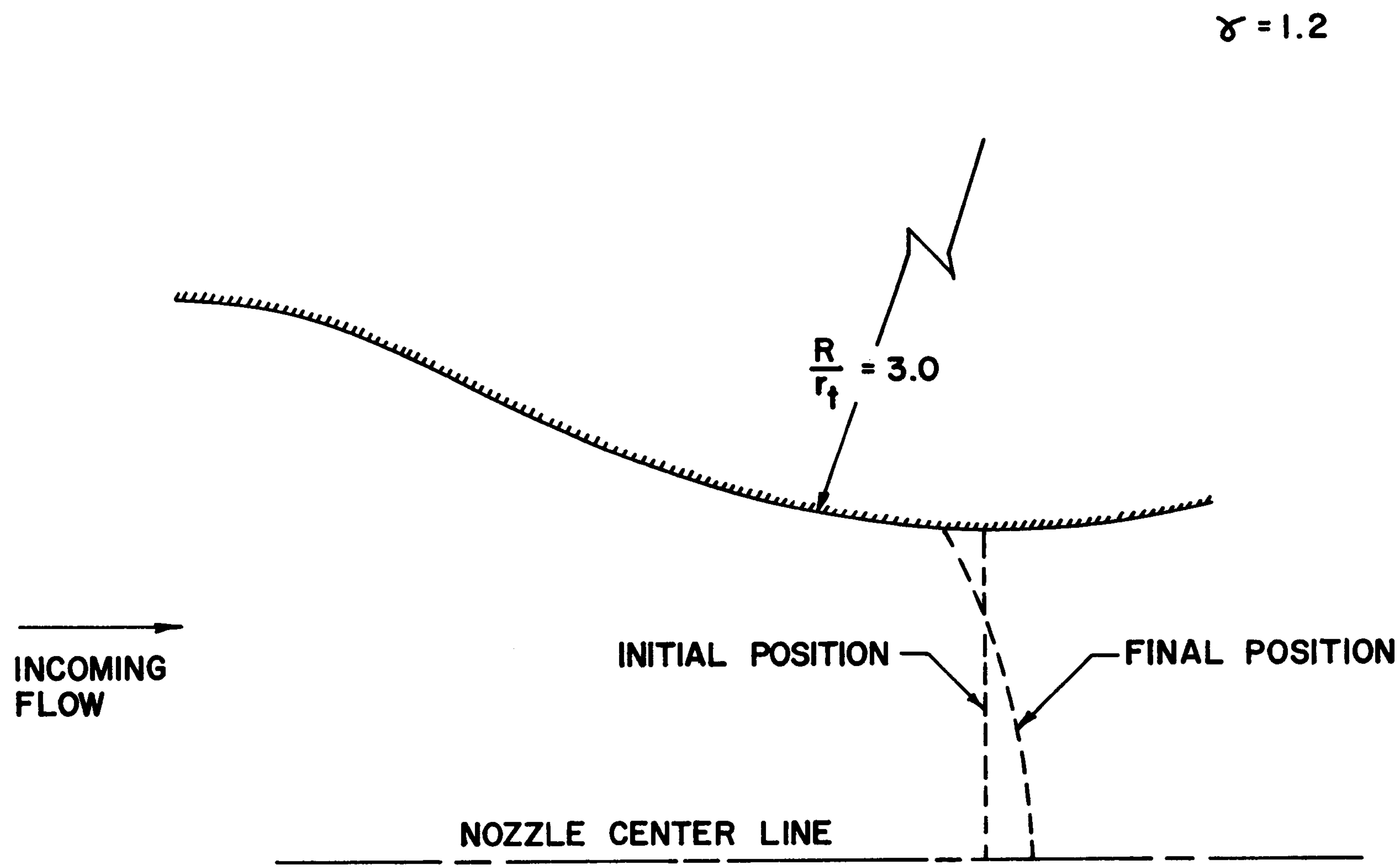
Comparison of the sonic line position measured from the x-coordinate of the throat with that computed by Sims (Ref. 3) yields agreement of about 2% at the nozzle axis and 60% at the nozzle wall. Little explanation can be given by the author for the relatively poor agreement at the wall. It may be pointed out, however, that Sims' work, which is an extension of Sauer's series expansion method, is based on the velocity gradient on the nozzle center line at the sonic point. It seems likely that the two methods should agree better at the intersection of the sonic line and symmetry axis.

Figure 8 shows the comparison of the second case with the experimental data of reference 6. Agreement is quite good, especially in the vicinity of the sonic line.

CONCLUSION

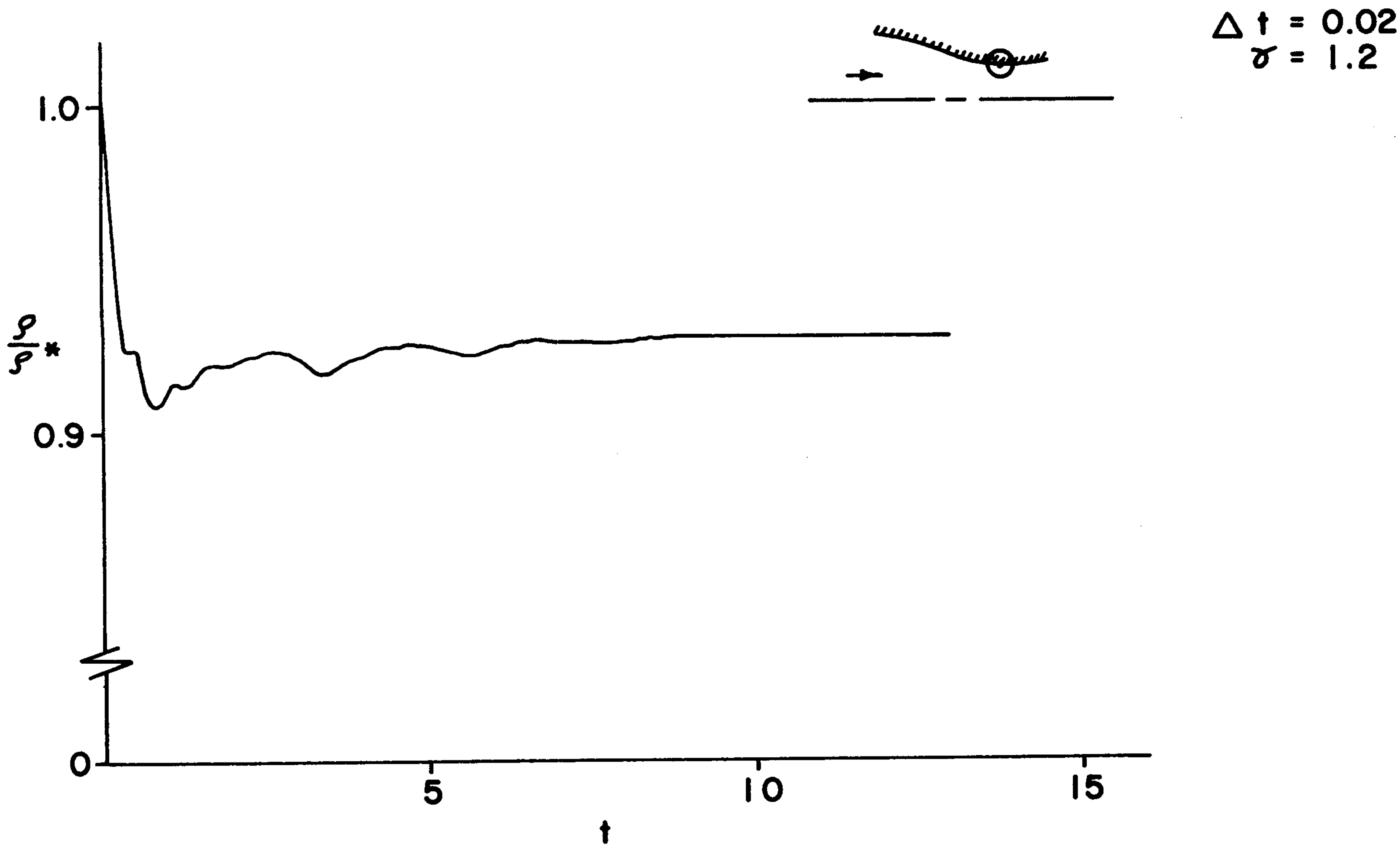
The time dependent finite difference technique is a relatively simple, straightforward method of solution for the nozzle throat problem. Restrictions on the flow equations are not severe and could be relaxed in various ways. It is possible, in principle, to add the viscosity terms and some real gas effects. As a matter of fact, Thommen (Ref. 5) has computed the viscous flow field for a straight wall. Of course the mesh size must be considerably reduced near the wall, which in turn increases computer run time. With the present mesh, the program computes about ten time steps per minute.

As the program is now written, it can handle only axisymmetric flow. It is a simple matter to incorporate an option to compute two dimensional flow, and this will be done in the near future.



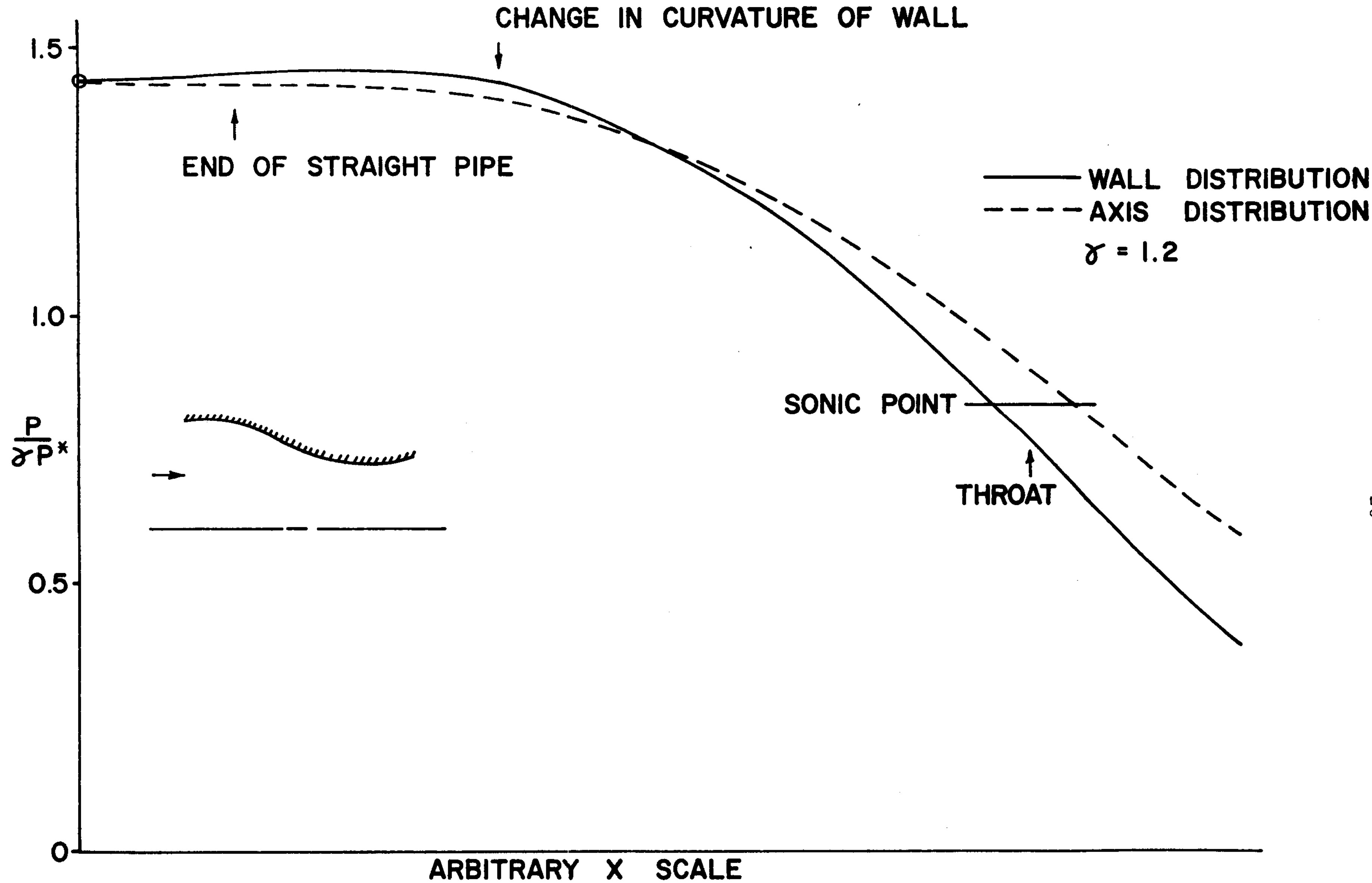
SONIC LINE POSITION FOR FIRST CASE

FIG. 5



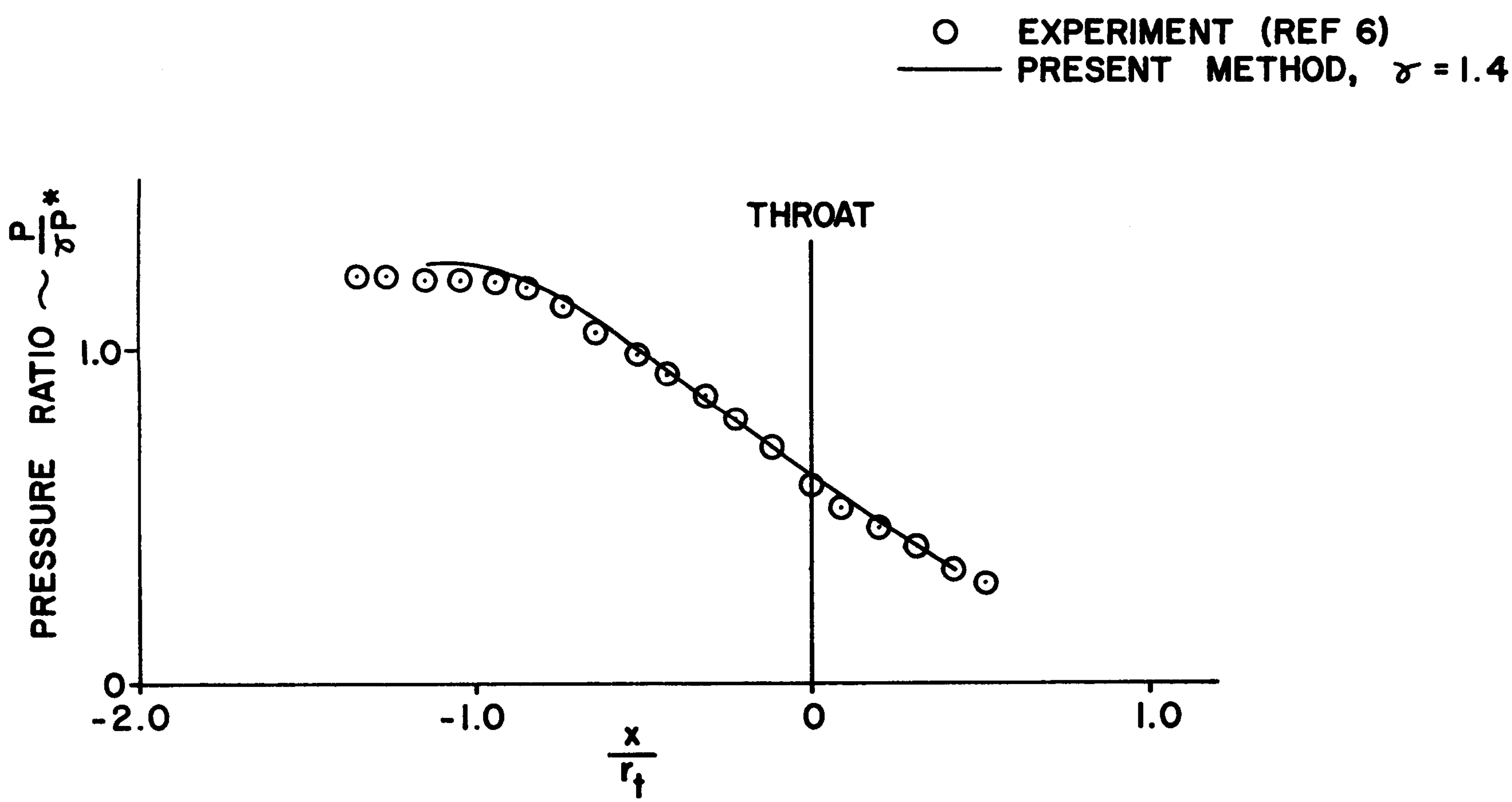
TYPICAL PARAMETER VARIATION WITH TIME

FIG. 6



PRESSURE DISTRIBUTION ON WALL & AXIS FOR FIRST CASE

FIG. 7



COMPARISON OF SECOND CASE WITH EXPERIMENT

FIG. 8

R E F E R E N C E S

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