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BIOGRAPHICAL NOTE

Dr. Kozai received his degree from the Tokyo University in 1958.

Kozai has been associated with the Tokyo Astronomical Observatory since 1952, and has held concurrent positions as staff astronomer with that observatory and consultant to the Smithsonian Astrophysical Observatory since 1958.

He has specialized in celestial mechanics, his research at SAO being primarily in the determination of zonal coefficients in the earth's gravitational potential by using precisely reduced Baker-Nunn observations. He is also interested in the seasonal variations of the earth's potential.

ABSTRACT

In this paper lunisolar short-periodic perturbations and lunar perturbations that depend on the lunar mean longitude are given insofar as their amplitudes in the expressions of satellite orbital elements are larger than 10^{-7}.

LUNISOLAR PERTURBATIONS WITH SHORT PERIODS

Yoshihide Kozai

1. INTRODUCTION

The principal secular and long-periodic parts of the disturbing function for the satellite motion due to the lunisolar gravitational attractions are given in an earlier paper (Kozai, 1959).

However, since the disturbing factors are $1.65 \times 10^{-5}/n^2$ and $0.75 \times 10^{-5}/n^2$ for the moon and the sun, respectively, when the mean motion n of the satellite is expressed in revolutions per day, it is found that in order to compute the position of the satellite with seven significant figures, lunisolar short-periodic perturbations must be included even if the mean motion is as large as 10 revolutions per day.

Of the long-periodic terms, periods of terms depending on the lunar mean longitude are not very long and some of them are as short as 10 days. Hence these terms should be taken into account to derive mean orbital elements (Gaposchkin, 1964).

In this paper lunisolar short-periodic perturbations and lunar perturbations that depend on the lunar mean longitude are derived insofar as their amplitudes are larger than 10^{-7}. It is found that the parallactic terms in the disturbing function can be neglected for this purpose when the eccentricity of the satellite is less than 0.20 and the mean motion is larger than 10 revolutions per day.

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2. DISTURBING FUNCTION

For the sake of convenience, the following primed symbols denote a lunar reference: m', Ω', i', M', v', λ', ζ', n', a', r'. The disturbing function due to the moon is written as,

$$
\begin{aligned}
R = n'^{2}\, m'\, a^{2}\left(\frac{r}{a}\right)^{2}\left(\frac{a'}{r}\right)^{3}\Bigg[& \frac{1}{4}\left(1 - \frac{3}{2}\sin^{2} i\right)\left(1 - \frac{3}{2}\sin^{2} i'\right) \\
& + \frac{3}{16}\sin 2i \sin 2i' \cos(\Omega - \Omega') \\
& + \frac{3}{16}\sin^{2} i \sin^{2} i' \cos 2(\Omega - \Omega') \\
& + \frac{3}{8}\sin^{2} i'\left(1 - \frac{3}{2}\sin^{2} i\right)\cos 2(\zeta' - \Omega') \\
& + \frac{3}{8}\sin^{2} i \cos^{4}\frac{i'}{2}\cos 2(\zeta' - \Omega) \\
& - \frac{3}{8}\sin 2i \sin i' \cos^{2}\frac{i'}{2}\cos(2\zeta' - \Omega - \Omega') \\
& + \frac{3}{8}\sin^{2} i \sin^{4}\frac{i'}{2}\cos 2(\zeta' - 2\Omega' + \Omega) \\
& + \frac{3}{8}\sin 2i \sin i' \sin^{2}\frac{i'}{2}\cos(2\zeta' + \Omega - 3\Omega') \\
& + \frac{3}{4}\cos^{4}\frac{i}{2}\cos^{4}\frac{i'}{2}\cos 2(\zeta' - L - \Omega) \\
& + \frac{3}{8}\sin^{2} i\left(1 - \frac{3}{2}\sin^{2} i'\right)\cos 2L \\
& + \frac{3}{8}\cos^{4}\frac{i}{2}\sin^{2} i' \cos 2(L + \Omega - \Omega') \\
& + \frac{3}{4}\sin^{4}\frac{i}{2}\cos^{4}\frac{i'}{2}\cos 2(L - \Omega + \zeta') \\
& + \frac{3}{4}\cos^{4}\frac{i}{2}\sin^{4}\frac{i'}{2}\cos 2(L + \Omega + \zeta' - 2\Omega')
\end{aligned}
$$

$$+\frac{3}{4}\sin^4\frac{i}{2}\ \sin^4\frac{i'}{2}\cos 2\,(\zeta' - 2\Omega' - L + \Omega)$$

$$+\frac{9}{32}\sin^2 i\,\sin^2 i'\cos 2\,(\zeta' - \Omega' - L)$$

$$+\frac{9}{32}\sin^2 i\,\sin^2 i'\cos 2\,(\zeta' + L - \Omega')$$

$$+\frac{3}{4}\sin i\cos^2\frac{i}{2}\sin i'\cos^2\frac{i'}{2}\cos\,(2\zeta' - \Omega' - 2L - \Omega)$$

$$-\frac{3}{8}\sin i\cos^2\frac{i}{2}\ \sin 2\,i'\cos\,(2L + \Omega - \Omega')$$

$$+\frac{3}{8}\sin^4\frac{i}{2}\ \sin^2 i'\cos 2\,(L - \Omega + \Omega')$$

$$+\frac{3}{8}\sin i\sin^2\frac{i}{2}\ \sin 2\,i'\cos\,(2L - \Omega + \Omega')$$

$$-\frac{3}{4}\sin i\sin^2\frac{i}{2}\ \sin i'\cos^2\frac{i'}{2}\cos\,(2\zeta' + 2L - \Omega - \Omega')$$

$$-\frac{3}{4}\sin i\cos^2\frac{i}{2}\ \sin i'\sin^2\frac{i'}{2}\cos\,(2\zeta' + 2L + \Omega - 3\Omega')$$

$$\left.+\frac{3}{4}\sin i\sin^2\frac{i}{2}\ \sin^2 i'\sin^2\frac{i'}{2}\cos\,(2\zeta' - 3\Omega' - 2L + \Omega)\right], \qquad (1)$$

where the orbital elements are referred to the equator of the earth, ζ' is the lunar true longitude, and L is the argument of latitude, that is, $L = v + \omega$.

The disturbing function due to the sun can be derived by putting

$$\left.\begin{array}{llll} n' = n_{\odot}, & r' = r_{\odot}, & a' = a_{\odot}, & m' = 1 \\ i' = \epsilon, & \Omega' = 0, & \zeta' = \zeta_{\odot}, & \end{array}\right\} \qquad (2)$$

in (1), where ϵ is the obliquity.

In the expression (1) the lunar inclination i' to the equator is not constant and Ω' cannot be expressed as a linear function of time. However, the lunar inclination J to the ecliptic is almost constant and is $5\overset{\circ}{.}15$, and the longitude of the ascending node N, referred to the ecliptic, can be approximated by the linear function of time. The relations between them are written as

$$\left.\begin{aligned} \sin i' \sin \Omega' &= \sin J \sin N = (0.086 + 0.004) \sin N, \\ \sin i' \cos \Omega' &= \sin \epsilon \cos J + \cos \epsilon \sin J \cos N \\ &= 0.396 + (0.086 - 0.004) \cos N \quad . \end{aligned}\right\} \tag{3}$$

For the expansion of the disturbing function (1), the following expressions derived from (3) are necessary:

$$\sin^2 i' = 0.164 + 0.065 \cos N - 0.001 \cos 2N,$$

$$\cos^4 \frac{i'}{2} = 0.916 - 0.034 \cos N,$$

$$\left.\begin{aligned} \sin 2i' \sin \Omega' &= \phantom{0.721 + {}}(0.143 + 0.021) \sin N - 0.003 \sin 2N, \\ \sin 2i' \cos \Omega' &= 0.721 + (0.143 - 0.021) \cos N - 0.003 \cos 2N, \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin i' \cos^2 \frac{i'}{2} \sin \Omega' &= \phantom{0.378 + {}}(0.079 + 0.007) \sin N - 0.001 \sin 2N, \\ \sin i' \cos^2 \frac{i'}{2} \cos \Omega' &= 0.378 + (0.079 - 0.007) \cos N - 0.001 \cos 2N, \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin^2 i' \sin 2\Omega' &= \phantom{0.156 + {}}(0.068 + 0.003) \sin N + 0.007 \sin 2N, \\ \sin^2 i' \cos 2\Omega' &= 0.156 + (0.068 - 0.003) \cos N + 0.007 \cos 2N, \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin i' \sin^2 \frac{i'}{2} \sin 3\Omega' &= \phantom{0.016 + {}} 0.011 \sin N + 0.002 \sin 2N, \\ \sin i' \sin^2 \frac{i'}{2} \cos 3\Omega' &= 0.016 + 0.011 \cos N + 0.002 \cos 2N, \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin^4 \frac{i'}{2} \sin 4\Omega' &= \phantom{0.002 + {}} 0.002 \sin N, \\ \sin^4 \frac{i'}{2} \cos 4\Omega' &= 0.002 + 0.002 \cos N. \end{aligned}\right\} \tag{4}$$

In the expression (1), $(a'/r')^3$ and ζ' should be expanded into power series of the lunar eccentricity, e', which is assumed to be 0.055, by means of the following formulas, with the lunar mean anomaly M' as argument:

$$\left(\frac{a'}{r'}\right)^3 = \left(1 - e'^2\right)^{-3/2} + 3\, e' \cos M' + \frac{9}{2} e'^2 \cos 2M'$$

$$= 1.005 + 0.165 \cos M' + 0.014 \cos 2M' ,$$

$$\left(\frac{a'}{r'}\right)^3 \cos 2\,(v' - M')$$

$$= 1 - \frac{5}{2} e'^2 + 3\, e' \cos M' + \frac{17}{2} e'^2 \cos 2M'$$

$$= 0.992 + (0.192 - 0.027) \cos M' + 0.026 \cos 2M' ,$$

$$\left(\frac{a'}{r'}\right)^3 \sin 2\,(v' - M')$$

$$= 4\, e' \sin M' + \frac{17}{2} e'^2 \sin 2M'$$

$$= (0.192 + 0.027) \sin M' + 0.026 \sin 2M' . \tag{5}$$

Now the disturbing function can be written as follows:

$$
\begin{aligned}
R = n'^2 m' r^2 \Big\{ \left(1 - \tfrac{3}{2}\sin^2 i\right) \Big[\, & 0.189 - 0.024 \cos N \\
& + 0.031 \cos M' - 0.002 \cos (M'+N) - 0.002 \cos (M' - N) \\
& + 0.003 \cos 2M' + 0.059 \cos 2\lambda' + 0.026 \cos (2\lambda' - N) \\
& + 0.003 \cos 2(\lambda' - N) + 0.011 \cos (2\lambda' + M') \\
& + 0.005 \cos (2\lambda' + M' - N) \Big] \\
+ \sin 2 i \Big[\, & 0.136 \cos \Omega + 0.027 \cos (\Omega - N) - 0.004 \cos (\Omega + N) \\
& + 0.011 \cos (\Omega + M') + 0.011 \cos (\Omega - M') \\
& + 0.002 \cos (\Omega + M' - N) + 0.002 \cos (\Omega - M' - N) \\
& - 0.141 \cos (\Omega - 2\lambda') - 0.030 \cos (\Omega + N - 2\lambda') \\
& - 0.003 \cos (\Omega - N - 2\lambda') - 0.027 \cos (\Omega - 2\lambda' - M') \\
& - 0.006 \cos (\Omega + N - 2\lambda' - M') + 0.004 \cos (\Omega - 2\lambda' + M') \\
& - 0.004 \cos (\Omega - 2\lambda' - 2M') + 0.006 \cos (\Omega + 2\lambda') \\
& + 0.004 \cos (\Omega + 2\lambda' - N) \Big] \\
+ \sin^2 i \Big[\, & 0.029 \cos 2\Omega + 0.013 \cos (2\Omega - N) \\
& + 0.002 \cos (2\Omega + M') + 0.002 \cos (2\Omega - M') \\
& + 0.341 \cos 2 (\Omega - \lambda') - 0.006 \cos (2\Omega - 2\lambda' - N)
\end{aligned}
$$

$$- 0.006 \cos (2\Omega - 2\lambda' + N) + 0.066 \cos (2\Omega - 2\lambda' - M')$$

$$- 0.009 \cos (2\Omega - 2\lambda' + M') + 0.009 \cos (2\Omega - 2\lambda' - 2M')\Big]$$

$$+ \cos^4 \frac{i}{2}\Big[0.682 \cos 2 (L + \Omega - \lambda') - 0.013 \cos (2L + 2\Omega - 2\lambda' - N)$$

$$- 0.013 \cos (2L + 2\Omega - 2\lambda' + N) + 0.132 \cos (2L + 2\Omega - 2\lambda' - M')$$

$$-0.019 \cos (2L + 2\Omega - 2\lambda' + M') + 0.018 \cos 2(L + \Omega - \lambda' - M')$$

$$+ 0.058 \cos 2(L + \Omega) + 0.025 \cos (2L + 2\Omega - N)$$

$$+ 0.003 \cos 2(L + \Omega - N) + 0.004 \cos (2L + 2\Omega + M')$$

$$+ 0.004 \cos (2L + 2\Omega - M') + 0.002 \cos (2L + 2\Omega - N + M')$$

$$+ 0.002 \cos (2L + 2\Omega - N - M') - 0.002 \cos (2L + 2\Omega - N - 2\lambda' - M')$$

$$- 0.002 \cos (2L + 2\Omega + N - 2\lambda' - M')\Big]$$

$$+ \sin^4 \frac{i}{2}\Big[0.682 \cos 2(L - \Omega + \lambda') - 0.013 \cos (2L - 2\Omega + 2\lambda' - N)$$

$$- 0.013 \cos (2L - 2\Omega + 2\lambda' + N) + 0.132 \cos (2L - 2\Omega + 2\lambda' + M')$$

$$- 0.019 \cos (2L - 2\Omega + 2\lambda' - M') + 0.018 \cos (2L - 2\Omega + 2\lambda' + 2M')$$

$$- 0.002 \cos (2L - 2\Omega + 2\lambda' + M' - N) - 0.002 \cos (2L - 2\Omega + 2\lambda' + M' + N)$$

$$+ 0.058 \cos 2(L - \Omega) + 0.025 \cos (2L - 2\Omega + N)$$

$$+ 0.003 \cos 2(L - \Omega + N) + 0.004 \cos (2L - 2\Omega + M')$$

$$+ 0.004 \cos(2L - 2\Omega - M') + 0.002 \cos(2L - 2\Omega + N + M')$$

$$\left. + 0.002 \cos(2L - 2\Omega + N - M') \right]$$

$$+ \sin^2 i \left[0.280 \cos 2L - 0.018 \cos(2L + N) - 0.018 \cos(2L - N) \right.$$

$$+ 0.023 \cos(2L + M') + 0.023 \cos(2L - M')$$

$$+ 0.002 \cos 2(L + M') + 0.002 \cos 2(L - M')$$

$$+ 0.044 \cos 2(L - \lambda') + 0.020 \cos(2L - 2\lambda' + N)$$

$$+ 0.002 \cos 2(L - \lambda' + N) + 0.008 \cos(2L - 2\lambda' - M')$$

$$+ 0.004 \cos(2L - 2\lambda' - M' + N) + 0.044 \cos 2(L + \lambda')$$

$$+ 0.020 \cos(2L + 2\lambda' - N) + 0.002 \cos 2(L + \lambda' - N)$$

$$\left. + 0.008 \cos(2L + 2\lambda' + M') + 0.004 \cos(2L + 2\lambda' + M' - N) \right]$$

$$+ \sin i \cos^2 \frac{i}{2} \left[0.281 \cos(2L + \Omega - 2\lambda') + 0.059 \cos(2L + \Omega + N - 2\lambda') \right.$$

$$- 0.005 \cos(2L + \Omega - N - 2\lambda') + 0.055 \cos(2L + \Omega - 2\lambda' - M')$$

$$+ 0.011 \cos(2L + \Omega + N - 2\lambda' - M') - 0.008 \cos(2L + \Omega - 2\lambda' + M')$$

$$+ 0.008 \cos(2L + \Omega - 2\lambda' - 2M') - 0.268 \cos(2L + \Omega)$$

$$- 0.053 \cos(2L + \Omega - N) + 0.007 \cos(2L + \Omega + N)$$

$$- 0.022 \cos(2L + \Omega - M') - 0.022 \cos(2L + \Omega + M')$$

$$- 0.004 \cos (2L + \Omega - N - M') - 0.004 \cos (2L + \Omega - N + M')$$

$$- 0.002 \cos (2L + \Omega - 2M') - 0.002 \cos (2L + \Omega + 2M')$$

$$- 0.012 \cos (2L + \Omega + 2\lambda') - 0.008 \cos (2L + \Omega + 2\lambda' - N)$$

$$- 0.002 \cos (2L + \Omega + 2\lambda' + M')\Big]$$

$$+ \sin i \sin^2 \frac{i}{2} \Big[- 0.281 \cos (2L - \Omega + 2\lambda') - 0.059 \cos (2L - \Omega + 2\lambda' - N)$$

$$+ 0.005 \cos (2L - \Omega + 2\lambda' + N) - 0.055 \cos (2L - \Omega + 2\lambda' + M')$$

$$- 0.011 \cos (2L - \Omega - N + 2\lambda' + M') + 0.008 \cos (2L - \Omega + 2\lambda' - M')$$

$$- 0.008 \cos (2L - \Omega + 2\lambda' + 2M') + 0.268 \cos (2L - \Omega)$$

$$+ 0.053 \cos (2L - \Omega + N) - 0.007 \cos (2L - \Omega - N)$$

$$+ 0.022 \cos (2L - \Omega - M') + 0.022 \cos (2L - \Omega + M')$$

$$+ 0.004 \cos (2L - \Omega + N - M') + 0.004 \cos (2L - \Omega + N + M')$$

$$+ 0.002 \cos (2L - \Omega - 2M') + 0.002 \cos (2L - \Omega + 2M')$$

$$+ 0.012 \cos (2L - \Omega - 2\lambda') + 0.008 \cos (2L - \Omega - 2\lambda' + N)$$

$$+ 0.002 \cos (2L - \Omega - 2\lambda' - M')\Big]\Big\}, \qquad (6)$$

where λ' is the lunar mean longitude.

3. LONG-PERIODIC PERTURBATIONS DEPENDING ON λ' AND/OR M'

Since the following relations exist,

$$\left.\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 dM = 1 + \frac{3}{2}e^2 \quad , \\ &\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2f \, dM = \frac{5}{2}e^2 \quad , \\ &\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \sin 2f \, dM = 0 \quad , \end{aligned}\right\} \tag{7}$$

secular and long-periodic parts of the disturbing function are derived if we replace r^2 by $a^2(1 + 3e^2/2)$ in the first three parts independent of L, that is, parts having factors $[1 - (3 \sin^2 i)/2]$, $\sin 2i$, and $\sin^2 i$ in (6), and r^2 by $5e^2 a^2/2$ and L by ω in the last five parts. Of the long-periodic terms thus derived terms depending on λ' and/or M' are picked up. The perturbations due to these terms can be computed by the following equations of variation:

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M} ,$$

$$\frac{de}{dt} = \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial M} - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial \omega} ,$$

$$\frac{di}{dt} = \frac{\cos i}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \Omega} ,$$

$$\frac{d\omega}{dt} = - \frac{\cos i}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial e} ,$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial i} ,$$

$$\frac{dM}{dt} = n - \frac{1 - e^2}{na^2 e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} . \tag{8}$$

Of course, the semimajor axis is not disturbed. The expressions for the eccentricity and for the inclination can be easily derived in the following forms:

$$\delta e = - \frac{10^{-6}}{n} e\sqrt{1 - e^2} \left[B_1' \cos^4 \frac{i}{2} + B_2' \sin^4 \frac{i}{2} \right.$$

$$\left. + B_3' \sin^2 i + B_4' \sin i \cos^2 \frac{i}{2} + B_5' \sin i \sin^2 \frac{i}{2} \right] ,$$

$$\delta i = - \frac{10^{-6}}{n\sqrt{1 - e^2}} \left\{ \left(1 + \frac{3}{2} e^2\right) (A'_2 \cos i + A'_3 \sin i) \right.$$

$$+ \frac{1}{2} e^2 \left[B'_1 \sin i \cos^2 \frac{i}{2} - B'_2 \sin i \sin^2 \frac{i}{2} - B'_3 \sin 2\,i \right.$$

$$\left.\left. + B'_4 (1 - 2 \cos i) \cos^2 \frac{i}{2} - B'_5 (1 + 2 \cos i) \sin^2 \frac{i}{2} \right] \right\} , \qquad (9)$$

where n is expressed in revolutions per day and A'_j and B'_j are given below.

Direct parts of perturbations in the longitude of the ascending node, the argument of perigee, and the mean anomaly are written as follows:

$$\sin i \, d\Omega_d = \frac{10^{-6}}{\sqrt{1 - e^2}\, n} \left[\left(1 + \frac{3}{2} e^2\right) \left(- \frac{1}{2} A_1 \sin 2i + A_2 \cos 2i + A_3 \sin^2 i\right) \right.$$

$$+ \frac{e^2}{2} \left(- B_1 \cos^2 \frac{i}{2} \sin i + B_2 \sin^2 \frac{i}{2} \sin i + B_3 \sin 2i \right.$$

$$\left.\left. + B_4 \frac{\cos i + \cos 2i}{2} + B_5 \frac{\cos i - \cos 2i}{2} \right) \right] ,$$

$$\delta\omega_d = \frac{10^{-6}}{n} \sqrt{1 - e^2} \left[A_1 \left(1 - \frac{3}{2} \sin^2 i\right) + \frac{3}{2} A_2 \sin 2i + \frac{3}{2} A_3 \sin^2 i \right.$$

$$+ B_1 \cos^4 \frac{i}{2} + B_2 \sin^4 \frac{i}{2} + B_3 \sin^2 i + B_4 \sin i \cos^2 \frac{i}{2}$$

$$\left. + B_5 \sin i \sin^2 \frac{i}{2} \right] - \cos i \, \delta\Omega_d ,$$

$$\delta M_d = -\frac{10^{-6}}{n}\left\{\left(7 + 3e^2\right)\left[\frac{A_1}{3}\left(1 - \frac{3}{2}\sin^2 i\right) + \frac{A_2}{2}\sin 2i + \frac{A_3}{2}\sin^2 i\right] + (1 + e^2)\left[B_1 \cos^4 \frac{i}{2} + B_2 \sin^4 \frac{i}{2} + B_3 \sin^2 i + B_4 \sin i \cos^2 \frac{i}{2} + B_5 \sin i \sin^2 \frac{i}{2}\right]\right\} . \tag{10}$$

In addition to the direct parts, there are indirect parts in $\delta\Omega$, $\delta\omega$, and δM that come from interaction between J_2 secular terms and δe and δi, and they are written as

$$\delta\Omega = \delta\Omega_d + \frac{4e}{1 - e^2}\beta \cdot \overline{\delta e} - \beta \tan i \cdot \overline{\delta i} ,$$

$$\delta\omega = \delta\omega_d + \frac{4e}{1 - e^2}\alpha \cdot \overline{\delta e} + 5\beta \sin i \cdot \overline{\delta i} ,$$

$$\delta M = \delta M_d + 3e\sqrt{1 - e^2}\,\frac{2 - 3\sin^2 i}{4 - 5\sin^2 i}\,\alpha \cdot \overline{\delta e} + 3\beta\sqrt{1 - e^2}\sin i \cdot \overline{\delta i} , \tag{11}$$

where α and β are secular motions of ω and Ω expressed in terms of the lunar mean motion, that is,

$$\alpha = \dot{\omega}/n' ,$$

$$\beta = \dot{\Omega}/n' , \tag{12}$$

and $\overline{\delta e}$ and $\overline{\delta i}$ are derived by integrating $n'\delta e$ and $n'\delta i$, respectively, with respect to time t. The expressions for $\overline{\delta e}$ and $\overline{\delta i}$ are derived by replacing A_j' and B_j' by $\overline{A}_j$ and $\overline{B}_j$ in δe and δi.

The expressions for A_j and B_j are

$$A_1 = 42 \sin M' - 3 \sin (M' + N) - 3 \sin (M' - N)$$
$$+ 2 \sin 2M' + 40 \sin 2\lambda' + 18 \sin (2\lambda' - N)$$
$$+ 2 \sin 2(\lambda' - N) + 5 \sin (2\lambda' + M') + 2 \sin (2\lambda' + M' - N) \quad ,$$

$$A_2 = \frac{1}{\beta + 1}\left[10 \sin (\Omega + M') + 2 \sin (\Omega + M' - N)\right]$$
$$+ \frac{1}{\beta - 1}\left[10 \sin (\Omega - M') + 2 \sin (\Omega - M' - N) + 4 \sin (\Omega - 2\lambda' + M')\right]$$
$$- \frac{1}{\beta/2 - 1}\left[64 \sin (\Omega - 2\lambda') + 14 \sin (\Omega + N - 2\lambda') + \sin (\Omega - N - 2\lambda')\right]$$
$$+ \frac{1}{\beta/2 + 1}\left[3 \sin (\Omega + 2\lambda') + 2 \sin (\Omega + 2\lambda' - N)\right]$$
$$- \frac{1}{\beta/3 - 1}\left[8 \sin (\Omega - 2\lambda' - M') + 2 \sin (\Omega + N - 2\lambda' - M')\right]$$
$$- \frac{1}{\beta/4 - 1} \sin (\Omega - 2\lambda' - 2M') \quad ,$$

$$A_3 = \frac{2}{2\beta+1} \sin(2\Omega + M') + \frac{1}{2\beta - 1}\left[2 \sin(2\Omega - M') - 8 \sin(2\Omega - 2\lambda' + M')\right]$$

$$+ \frac{1}{\beta - 1}\left[154 \sin 2(\Omega - \lambda') - 3 \sin(2\Omega - 2\lambda' - N) - 3 \sin(2\Omega - 2\lambda' + N)\right]$$

$$+ \frac{20}{2\beta/3 - 1} \sin(2\Omega - 2\lambda' - M')$$

$$+ \frac{2}{\beta/2 - 1} \sin(2\Omega - 2\lambda' - 2M') \quad ,$$

$$B_1 = \frac{1}{\alpha+\beta-1}\left[769 \sin 2(\omega + \Omega - \lambda') - 14 \sin(2\omega + 2\Omega - 2\lambda' - N)\right.$$

$$\left. - 14 \sin(2\omega + 2\Omega - 2\lambda' + N)\right]$$

$$+ \frac{1}{2(\alpha+\beta) - 1}\left[- 42 \sin(2\omega + 2\Omega - 2\lambda' + M') + 10 \sin(2\omega + 2\Omega - M')\right.$$

$$\left. + 5 \sin(2\omega + 2\Omega - N - M')\right]$$

$$+ \frac{1}{2(\alpha+\beta)+1}\left[10 \sin(2\omega + 2\Omega + M') + 5 \sin(2\omega + 2\Omega - N + M')\right]$$

$$+ \frac{99}{2(\alpha+\beta)/3 - 1} \sin(2\omega + 2\Omega - 2\lambda' - M')$$

$$+ \frac{10}{(\alpha+\beta)/2 - 1} \sin 2(\omega + \Omega - \lambda' - M') \quad ,$$

$$B_2 = \frac{1}{\alpha - \beta + 1}\Big[769 \sin 2(\omega - \Omega + \lambda') - 14 \sin(2\omega - 2\Omega + 2\lambda' - N)$$

$$- 14 \sin(2\omega - 2\Omega + 2\lambda' + N)\Big]$$

$$+ \frac{1}{2(\alpha - \beta) + 1}\Big[- 42 \sin(2\omega - 2\Omega + 2\lambda' - M') + 10 \sin(2\omega - 2\Omega + M')$$

$$+ 5 \sin(2\omega - 2\Omega + N + M')\Big]$$

$$+ \frac{1}{2(\alpha + \beta) - 1}\Big[10 \sin(2\omega - 2\Omega - M') + 5 \sin(2\omega - 2\Omega + N - M')\Big]$$

$$+ \frac{99}{2(\alpha - \beta)/3 + 1} \sin(2\omega - 2\Omega + 2\lambda' + M')$$

$$+ \frac{10}{(\alpha - \beta)/2 + 1} \sin(2\omega - 2\Omega + 2\lambda' + 2M') \quad ,$$

$$B_3 = \frac{52}{2\alpha + 1} \sin(2\omega + M') + \frac{52}{2\alpha - 1} \sin(2\omega - M')$$

$$+ \frac{1}{\alpha + 1}\Big[3 \sin 2(\omega + M') + 50 \sin 2(\omega + \lambda')$$

$$+ 22 \sin(2\omega + 2\lambda' - N) + 3 \sin 2(\omega + \lambda' - N)\Big]$$

$$+ \frac{1}{\alpha - 1}\Big[3 \sin 2(\omega - M') + 50 \sin 2(\omega - \lambda')$$

$$+ 22 \sin(2\omega - 2\lambda' + N) + 3 \sin 2(\omega - \lambda' + N)\Big]$$

$$+ \frac{1}{2\alpha/3 - 1}\Big[6 \sin(2\omega - 2\lambda' - M') + 3 \sin(2\omega - 2\lambda' - M' + N)\Big]$$

$$+ \frac{1}{2\alpha/3 + 1}\Big[6 \sin(2\omega + 2\lambda' + M') + 3 \sin(2\omega + 2\lambda' + M' - N)\Big] \quad ,$$

$$B_4 = \frac{1}{\alpha + \beta/2 - 1}\left[317 \sin(2\omega + \Omega - 2\lambda') + 67 \sin(2\omega + \Omega + N - 2\lambda')\right.$$

$$\left. - 6 \sin(2\omega + \Omega - N - 2\lambda') - 3 \sin(2\omega + \Omega - 2M')\right]$$

$$- \frac{1}{\alpha + \beta/2 + 1}\left[3 \sin(2\omega + \Omega + 2M') + 14 \sin(2\omega + \Omega + 2\lambda')\right.$$

$$\left. + 9 \sin(2\omega + \Omega + 2\lambda' - N)\right]$$

$$+ \frac{1}{(2\alpha + \beta)/3 - 1}\left[41 \sin(2\omega + \Omega - 2\lambda' - M') + 8 \sin(2\omega + \Omega + N - 2\lambda' - M')\right]$$

$$- \frac{1}{2\alpha + \beta - 1}\left[17 \sin(2\omega + \Omega - 2\lambda' + M') + 51 \sin(2\omega + \Omega - M')\right.$$

$$\left. + 10 \sin(2\omega + \Omega - N - M')\right]$$

$$- \frac{1}{2\alpha + \beta + 1}\left[51 \sin(2\omega + \Omega + M') + 10 \sin(2\omega + \Omega - N - M')\right]$$

$$+ \frac{4}{(2\alpha + \beta)/4 - 1} \sin(2\omega + \Omega - 2\lambda' - 2M') \quad ,$$

$$B_5 = \frac{1}{\alpha - \beta/2 + 1}\left[- 317 \sin(2\omega - \Omega + 2\lambda') - 67 \sin(2\omega - \Omega - N + 2\lambda')\right.$$

$$\left. + 6 \sin(2\omega - \Omega + N + 2\lambda') + 3 \sin(2\omega - \Omega + 2M')\right]$$

$$+ \frac{1}{\alpha - \beta/2 - 1}\left[3 \sin(2\omega - \Omega - 2M') + 14 \sin(2\omega - \Omega - 2\lambda')\right.$$

$$\left. + 9 \sin(2\omega - \Omega + N - 2\lambda')\right]$$

$$- \frac{1}{(2\alpha+\beta)/3+1}\left[41 \sin(2\omega - \Omega + 2\lambda' + M') + 8 \sin(2\omega - \Omega - N + 2\lambda' + M')\right]$$

$$+ \frac{1}{2\alpha-\beta+1}\left[17 \sin(2\omega - \Omega + 2\lambda' - M') + 51 \sin(2\omega - \Omega + M')\right.$$

$$\left. + 10 \sin(2\omega - \Omega + N + M')\right]$$

$$+ \frac{1}{2\alpha-\beta-1}\left[51 \sin(2\omega - \Omega - M') + 10 \sin(2\omega - \Omega + N - M')\right]$$

$$- \frac{4}{(2\alpha-\beta)/4+1} \sin(2\omega - \Omega + 2\lambda' + 2M') \quad . \qquad (13)$$

The expressions for A_j' and B_j' are derived by replacing sine by cosine in A_j and B_j of (13).

The expressions for $\overline{A}_j$ and $\overline{B}_j$ are written as

$$\overline{A}_2 = \frac{1}{(\beta+1)^2}\left[10 \sin(\Omega + M') + 2 \sin(\Omega - N + M')\right]$$

$$+ \frac{1}{(\beta-1)^2}\left[10 \sin(\Omega - M') + 2 \sin(\Omega - N - M') + 4 \sin(\Omega - 2\lambda' + M')\right]$$

$$- \frac{1}{(\beta/2-1)^2}\left[32 \sin(\Omega - 2\lambda') + 7 \sin(\Omega + N - 2\lambda')\right]$$

$$- \frac{3}{(\beta/3-1)^2} \sin(\Omega - 2\lambda' - M') + \frac{2}{(\beta/2+1)^2} \sin(\Omega + 2\lambda') \quad ,$$

$$\overline{A}_3 = \frac{2}{(2\beta + 1)^2} \sin (2\Omega + M') + \frac{1}{(2\beta - 1)^2} \Big[2 \sin (2\Omega - M') - 8 \sin (2\Omega - 2\lambda' + M') \Big]$$

$$+ \frac{1}{(\beta - 1)^2} \Big[77 \sin 2(\Omega - \lambda') - 2 \sin (2\Omega - N - 2\lambda') - 2 \sin (2\Omega + N - 2\lambda') \Big]$$

$$+ \frac{7}{(2\beta/3 - 1)^2} \sin (2\Omega - 2\lambda' - M') \quad ,$$

$$\overline{B}_1 = \frac{1}{(\alpha + \beta - 1)^2} \Big[384 \sin 2(\omega + \Omega - \lambda') - 7 \sin (2\omega + 2\Omega - N - 2\lambda') - 7 \sin (2\omega + 2\Omega + N - 2\lambda') \Big]$$

$$+ \frac{1}{[2(\alpha + \beta) - 1]^2} \Big[- 42 \sin (2\omega + 2\Omega - 2\lambda' + M') + 10 \sin (2\omega + 2\Omega - M') + 5 \sin (2\omega + 2\Omega - N - M') \Big]$$

$$+ \frac{1}{[2(\alpha + \beta) + 1]^2} \Big[10 \sin (2\omega + 2\Omega + M') + 5 \sin (2\omega + 2\Omega - N + M') \Big]$$

$$+ \frac{33}{[2(\alpha + \beta)/3 - 1]^2} \sin (2\omega + 2\Omega - 2\lambda' - M')$$

$$+ \frac{3}{[(\alpha + \beta)/2 - 1]^2} \sin 2(\omega + \Omega - \lambda' - M') \quad ,$$

$$\overline{B}_2 = \frac{1}{(\alpha - \beta + 1)^2}\Big[384 \sin 2(\omega - \Omega + \lambda') - 7 \sin (2\omega - 2\Omega - N + 2\lambda')$$

$$- 7 \sin (2\omega - 2\Omega + N + 2\lambda')\Big]$$

$$+ \frac{1}{[2(\alpha - \beta) + 1]^2}\Big[- 42 \sin (2\omega - 2\Omega + 2\lambda' - M') + 10 \sin (2\omega - 2\Omega + M')$$

$$+ 5 \sin (2\omega - 2\Omega + N + M')\Big]$$

$$+ \frac{1}{[2(\alpha - \beta) - 1]^2}\Big[10 \sin (2\omega - 2\Omega - M') + 5 \sin (2\omega - 2\Omega + N - M')\Big]$$

$$+ \frac{33}{[2(\alpha - \beta)/3 + 1]^2} \sin (2\omega - 2\Omega + 2\lambda' + M')$$

$$+ \frac{3}{[(\alpha - \beta)/2 + 1]^2} \sin 2(\omega - \Omega + \lambda' + M') \quad ,$$

$$\overline{B}_3 = \frac{52}{(2\alpha + 1)^2} \sin (2\omega + M') + \frac{52}{(2\alpha - 1)^2} \sin (2\omega - M')$$

$$+ \frac{1}{(\alpha - 1)^2}\Big[25 \sin 2(\omega - \lambda') + 11 \sin (2\omega + N - 2\lambda')\Big]$$

$$+ \frac{1}{(\alpha + 1)^2}\Big[25 \sin 2(\omega + \lambda') + 11 \sin (2\omega - N + 2\lambda')\Big] \quad ,$$

$$\overline{B}_4 = \frac{1}{(\alpha + \beta/2 - 1)^2}\Big[159 \sin (2\omega + \Omega - 2\lambda') + 34 \sin (2\omega + \Omega + N - 2\lambda')$$

$$- 3 \sin (2\omega + \Omega - N - 2\lambda')\Big]$$

$$+ \frac{1}{[(2\alpha + \beta)/3 - 1]^2} \left[14 \sin (2\omega + \Omega - 2\lambda' - M') \right.$$

$$\left. + 3 \sin (2\omega + \Omega + N - 2\lambda' - M') \right]$$

$$- \frac{1}{(2\alpha + \beta - 1)^2} \left[17 \sin (2\omega + \Omega - 2\lambda' + M') + 51 \sin (2\omega + \Omega - M') \right.$$

$$\left. + 10 \sin (2\omega + \Omega - N - M') \right]$$

$$- \frac{1}{(2\alpha + \beta + 1)^2} \left[51 \sin (2\omega + \Omega + M') + 10 \sin (2\omega + \Omega - N + M') \right]$$

$$- \frac{1}{(\alpha + \beta/2 + 1)^2} \left[7 \sin (2\omega + \Omega + 2\lambda') + 5 \sin (2\omega + \Omega - N + 2\lambda') \right] ,$$

$$\overline{B}_5 = - \frac{1}{(\alpha - \beta/2 + 1)^2} \left[159 \sin (2\omega - \Omega + 2\lambda') + 34 \sin (2\omega - \Omega - N + 2\lambda') \right.$$

$$\left. - 3 \sin (2\omega - \Omega + N + 2\lambda') \right]$$

$$- \frac{1}{[(2\alpha - \beta)/3 + 1)]^2} \left[14 \sin (2\omega - \Omega + 2\lambda' + M') + 3 \sin (2\omega - \Omega - N + 2\lambda' + M') \right]$$

$$+ \frac{1}{(2\alpha - \beta + 1)^2} \left[17 \sin (2\omega - \Omega + 2\lambda' - M') + 51 \sin (2\omega - \Omega + M') \right.$$

$$\left. + 10 \sin (2\omega - \Omega + N + M') \right]$$

$$+ \frac{1}{(2\alpha - \beta - 1)^2} \left[51 \sin (2\omega - \Omega - M') + 10 \sin (2\omega - \Omega + N - M') \right]$$

$$+ \frac{1}{(\alpha - \beta/2 - 1)^2} \left[7 \sin (2\omega - \Omega - 2\lambda') + 5 \sin (2\omega - \Omega + N - 2\lambda') \right] .$$

When the mean motion is far less than 10 revolutions per day, then, in order to compute the perturbations with the same accuracy, perturbations in the lunar motion should be taken into account.

4. SHORT-PERIODIC PERTURBATIONS

The short-periodic part of the disturbing function is written as

$$R_p = n'^2 m' a^2 \left\{ \left[\left(\frac{r}{a}\right)^2 - 1 - \frac{3}{2} e^2 \right] A + \left[\left(\frac{r}{a}\right)^2 \cos 2v - \frac{5}{2} e^2 \right] B - \left(\frac{r}{a}\right)^2 \sin 2v \; B' \right\} , \tag{15}$$

where A represents the first three parts independent of L in (6), and B consists of the remaining five parts in which L is replaced by ω. We derive B' by replacing cosine by sine in B.

Short-periodic perturbations in the orbital elements are easily derived from the following relations, with the assumption that A, B, and B' are constant:

$$\int \frac{\partial R_p}{\partial M} \, dM = R_p ,$$

$$\int \left[\left(\frac{r}{a}\right)^2 - 1 - \frac{3}{2} e^2 \right] dM = \int \left[\left(\frac{r}{a}\right)^2 - 1 - \frac{3}{2} e^2 \right] \frac{r}{a} \, dE$$

$$= \left(-2e + \frac{3}{4} e^3 \right) \sin E + \frac{3}{4} e^2 \sin 2E - \frac{e^3}{12} \sin 3E ,$$

$$\int \left[\left(\frac{r}{a}\right)^2 \cos 2v - \frac{5}{2} e^2 \right] dM = -\frac{5}{2} e \left(1 - \frac{e^2}{2}\right) \sin E + \frac{1}{2} \left(1 + \frac{e^2}{2}\right) \sin 2E - \frac{e}{6} \left(1 - \frac{e^2}{2}\right) \sin 3E ,$$

$$\int \left(\frac{r}{a}\right)^2 \sin 2v \, dM = \sqrt{1 - e^2} \left[\frac{5}{2} e \cos E - \frac{1}{2} (1 + e^2) \cos 2E + \frac{e}{6} \cos 3E \right] , \tag{16}$$

where E is the eccentric anomaly.

We find that the perturbations in the inclination and the ascending node are entirely negligible. The perturbations in the other four elements are transformed to those in the radius and the argument of latitude. The perturbations due to the sun are also computed.

The transformation formulas are

$$dr = \frac{r}{a}\, da + a\left(\frac{e}{\sqrt{1 - e^2}} \sin v \, dM - \cos v \, de\right) ,$$

$$dL = d\omega + \frac{1}{\sqrt{1 - e^2}} \left(\frac{a}{r}\right)^2 dM + \frac{2 + e \cos v}{1 - e^2}\, de \quad . \tag{17}$$

At the final stage we find that the eccentricity of the satellite is negligible if it is less than 0. 2, and the results are written as

$$\begin{aligned}
\frac{\delta r}{a} = \frac{10^{-5}}{n^2} \Big[& 0.9\left(1 - \frac{3}{2}\sin^2 i\right) - 0.5 \sin 2i \cos 2(\lambda' - \Omega) \\
& + 1.1 \sin^2 i \cos 2(\lambda' - \Omega) + 0.5 \sin^2 i \cos 2(\lambda_\odot - \Omega) \\
& + 1.5 \cos^4 \frac{i}{2} \cos 2(L + \Omega - \lambda') + 0.7 \cos^4 \frac{i}{2} \cos 2(L + \Omega - \lambda_\odot) \\
& + 1.5 \sin^4 \frac{i}{2} \cos 2(L - \Omega + \lambda') + 0.7 \sin^4 \frac{i}{2} \cos 2(L - \Omega + \lambda_\odot) \\
& + 0.6 \sin i \cos^2 \frac{i}{2} \cos (2L + \Omega - 2\lambda) \\
& - 0.6 \sin i \sin^2 \frac{i}{2} \cos (2L - \Omega + 2\lambda') \\
& + 0.9 \sin^2 i \cos 2L - 0.9 \sin i \cos^2 \frac{i}{2} \cos (2L + \Omega) \\
& + 0.9 \sin i \sin^2 \frac{i}{2} \cos (2L - \Omega) \Big] ,
\end{aligned}$$

$$\delta L = \frac{10^{-5}}{n^2}\Big[2\cos^4\frac{i}{2}\sin 2(L+\Omega-\lambda') + \cos^4\frac{i}{2}\sin 2(L+\Omega-\lambda')$$

$$+ 2\sin^4\frac{i}{2}\sin 2(L-\Omega+\lambda') + \sin^4\frac{i}{2}\sin 2(L-\Omega+\lambda_\odot)$$

$$+ \sin i\cos^2\frac{i}{2}\sin(2L+\Omega-2\lambda') - \sin i\sin^2\frac{i}{2}\sin(2L-\Omega+2\lambda')$$

$$+ \sin^2 i\sin 2L - \sin i\cos^2\frac{i}{2}\sin(2L+\Omega)$$

$$+ \sin i\sin^2\frac{i}{2}\sin(2L-\Omega)\Big] , \qquad (18)$$

where n is also expressed in revolutions per day.

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