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GEOMAGNETIC EULER POTENTIALS

BY

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ABSTRACT

It is well known that magnetic fields may be represented by the cross product of the gradients of two scalars α and β ("Euler Potentials") and that such a representation has considerable use in the description of field lines and guiding center motion. Due to practical difficulties, however, such potentials are known only for relatively simple configurations, including the field of a magnetic dipole. A perturbation scheme is developed here by which, given the spherical harmonic expansion of the geomagnetic scalar potential and treating it as representing a perturbed dipole field, Euler potentials can be derived to any desired order of accuracy. Its first-order solutions involve trigonometric integrals $V_n^m(\theta)$ and $t_n^m(\theta)$ previously defined by Pennington, which may be expressed in closed form; tables of coefficients enabling one to obtain these integrals up to n = 6 are included. The results are tested for the internally generated geomagnetic field by mapping lines of constant α and β on the earth's surface; conjugate points (which have the same α and β) are obtained with typical accuracy of 1-2 degrees. The method is also applied to a magnetospheric model in which external sources are added to represent the effect of the solar wind; although this model clearly breaks down at large distances, it gives fairly good results within about 10 earth radii, including a "magnetopause" on which two neutral points are located.

GEOMAGNETIC EULER POTENTIALS

THE SCALAR GEOMAGNETIC POTENTIAL

Magnetic fields may be represented by several methods. The most prevalent one, and the one most directly connected with the field's sources, is by means of a vector potential **A**. In source-free regions, however, the field may be more simply represented as the gradient of a scalar potential, and this is the way in which the geomagnetic field is customarily described. We shall denote here the geomagnetic scalar potential by γ, assuming it to be given by a spherical harmonic expansion in spherical coordinates r, θ and ϕ with origin at the earth's center (a is the earth's radius):

$$\mathbf{B} = -\nabla\gamma \tag{1}$$

$$\gamma = a \sum_{n=1} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_n^m(\theta) \{g_n^m \cos m\phi + h_n^m \sin m\phi\} \tag{2}$$

Of the various terms in the expansion of γ, the dipole part, corresponding to $n = 1$, is the dominant one. It will be assumed that the polar axis of the coordinate system has been suitably rotated (giving "tilted dipole" coordinates) so that this part is expressed by g_1^0 alone, while g_1^1 and h_1^1 are both zero (Stern, 1965, Table 1). In this work we investigate quantities which are easily derived for the dipole case, but not for arbitrary magnetic fields: it is then useful to regard the field as that of a perturbed dipole, and carry out a perturbation calculation to various orders.

In any perturbation calculation it is necessary to introduce a "small parameter" ϵ, providing an estimate of accuracy and helping one to decide, at any stage, which terms to keep and which to neglect. Since g_1^0 exceeds the next largest coefficient in the expansion (namely, g_2^1) by a factor of about 10, the ratio between these two terms may be chosen as ϵ. In the perturbation calculation one then splits up γ into sections of different order

$$\gamma = \gamma_0 + \gamma_1 + \gamma_2 + \cdots \tag{3}$$

where γ_0 is the dipolar part

$$\gamma_0 = a g_1^0 \left(\frac{a}{r}\right)^2 \cos\theta \tag{4}$$

and where γ_1 contains coefficients ranging between ϵg_1^0 and $\epsilon^2 g_1^0$, γ_2 contains those between $\epsilon^2 g_1^0$ and $\epsilon^3 g_1^0$, and so forth. The perturbation is by no means "very small" (as is often the case in perturbation problems), for ϵ is of the order of 0.1 and there exist 9 coefficients in the range between ϵg_1^0 and $\epsilon g_1^0/3$.

EULER POTENTIALS

The preceding representation suffers from the drawback that it does not afford an easy description of magnetic field lines. A representation which allows such a description, originally devised by Euler to describe the velocity field of incompressible fluids (Euler, 1769, § 26, §49; Truesdell, 1954, § 13), utilizes two scalar functions which are conserved along field lines and has the form

$$\mathbf{B} = \nabla\alpha \times \nabla\beta \tag{5}$$

In what follows, such functions will be termed Euler Potentials. They are not uniquely defined but may be replaced by any pair $u(\alpha, \beta)$ and $v(\alpha, \beta)$, provided

$$\frac{\partial (u, v)}{\partial (\alpha, \beta)} = 1 \tag{6}$$

For further properties of these potentials, including considerations of their single-valuedness and superposition, the reader is referred to more detailed work (Stern, 1966).

Both α and β are independent solutions of the linear partial differential equation (here written for α)

$$\nabla\alpha \cdot \mathbf{B} = 0 \tag{7}$$

On the other hand, an arbitrary independent pair of functions u and v solving this equation does not necessarily satisfy (5). Since the general solution of (7) is an arbitrary function of two independent solutions, functional relations then exist of the form

$$\alpha = \alpha(u, v)$$

$$\beta = \beta(u, v)$$

and substitution into (5) gives

$$\mathbf{B} = \frac{\partial(\alpha, \beta)}{\partial(u, v)} (\nabla u \times \nabla v) = \chi(u, v) (\nabla u \times \nabla v) \tag{8}$$

which, unless (6) happens to hold, is not of the same form. The solutions u and v are conserved on field lines and may still be used to label them, but their cross product no longer gives the field's magnitude correctly. Any solution of (7) will be called here an Euler potential (the justification of this is given in the next paragraph) while a pair of such solutions which also satisfies (5) will be termed a matched pair of potentials.

Given two nonmatching potentials [u, v], it is always possible (at least in principle) to derive from them a matching pair [u, δ(u , v)] (e.g. Phillips, 1933, §49). Demanding

$$\mathbf{B} = \nabla u \times \nabla \delta$$

one obtains

$$\frac{\partial \delta}{\partial v} = \frac{\mathbf{B}}{\nabla u \times \nabla v}$$

where denominator and enumerator are parallel vectors and therefore have a well defined scalar ratio. Consider now the family of surfaces u = constant. On each such surface points may be labelled by v and by some other coordinate, e.g. γ or, more generally, the arc length s measured along a field line. Integrating on such surfaces along lines of constant γ (or s), from some reference point $v = v_R$, one obtains (compare Ray, 1963)

$$\delta = \int_{v_R}^{v} \frac{\mathbf{B}}{\nabla u \times \nabla v'} dv' + \delta'(u) \tag{9}$$

Euler potentials have been extensively used in the theory of the motion of charged particles trapped by the geomagnetic field. However, there exists no analytic representation for them equivalent to equation (2) giving γ. The reason for this omission is probably the lack of any easy method by which such functions may be derived, and it is the purpose of this work to explore means by which this situation may be improved, for the case of the geomagnetic field.

EXPANSION OF THE POTENTIALS

Because the dipole component of the field possesses axial symmetry, it is possible to derive its Euler potentials, namely

$$\alpha_0 = a g_1^0 \left(\frac{a}{r}\right) \sin^2 \theta \tag{10}$$

$$\beta_0 = a \phi \tag{11}$$

This suggests that the Euler potentials corresponding to the perturbed dipole field can be expanded

$$\left.\begin{aligned} \alpha &= \alpha_0 + \alpha_1 + \alpha_2 + \cdots \\ \beta &= \beta_0 + \beta_1 + \beta_2 + \cdots \end{aligned}\right\} \tag{12}$$

Substituting these together with the expansion (3) into equation (5) and equating individual orders gives

$$(\nabla \alpha_1 \times \nabla \beta_0) + (\nabla \alpha_0 \times \nabla \beta_1) = -\nabla \gamma_1 \tag{13}$$

$$(\nabla \alpha_2 \times \nabla \beta_0) + (\nabla \alpha_1 \times \nabla \beta_1) + (\nabla \alpha_0 \times \nabla \beta_2) = -\nabla \gamma_2 \tag{14}$$

and so forth. Because γ_2 represents the limit of accuracy to which the geomagnetic field is determined, there actually exists no practical need for approximations of order higher than the second. Each of the above vector equations is equivalent to a set of three scalar equations, obtained by forming its scalar product with 3 independent vector fields. Let two of these be $\nabla \alpha_0$ and $\nabla \beta_0$; the equations then obtained are

$$(\nabla \alpha_1 \cdot \nabla \gamma_0) + (\nabla \alpha_0 \cdot \nabla \gamma_1) = 0 \tag{15}$$

$$(\nabla \beta_1 \cdot \nabla \gamma_0) + (\nabla \beta_0 \cdot \nabla \gamma_1) = 0 \tag{16}$$

and

$$\nabla\alpha_2 \cdot \nabla\gamma_0 + \nabla\alpha_1 \cdot \nabla\gamma_1 + \nabla\alpha_0 \cdot \nabla\gamma_2 = 0 \tag{17}$$

$$\nabla\beta_2 \cdot \nabla\gamma_0 + \nabla\beta_1 \cdot \nabla\gamma_1 + \nabla\beta_0 \cdot \nabla\gamma_0 = 0 \tag{18}$$

These (and corresponding equations of higher orders) are the same as would have been obtained from an expansion of (7) to its various orders, and therefore, the expanded solutions (12) obtained by considering them alone yield Euler potentials which are not necessarily matched. Nevertheless, because the equations in this form are linear and involve α and β separately, we shall start by solving them, leaving the problem of matching to be considered later.

It is evident that the functions α_1, β_1, α_2 and β_2 derived by equations (15) to (18) are not unique, for these equations still hold if an arbitrary function of α_0 and β_0 is added to any of them. This suggests that we choose a new set of independent variables which includes α_0 and β_0. For convenience, we will use ϕ rather than β_0 and retain θ as the third independent coordinate, even though one may more generally consider γ_0 or the arc length s_0 along a dipole field line for this role. One thus writes

$$\alpha = \alpha_0 + \alpha_1 (\alpha_0, \theta, \phi) + \alpha_2 (\alpha_0, \theta, \phi) + \cdots \tag{19}$$

$$\beta = a\phi + \beta_1 (\alpha_0, \theta, \phi) + \beta_2 (\alpha_0, \theta, \phi) + \cdots \tag{20}$$

In these variables,

$$\nabla\alpha_1 = \frac{\partial\alpha_1}{\partial\alpha_0}\nabla\alpha_0 + \frac{1}{a}\frac{\partial\alpha_1}{\partial\phi}\nabla\beta_0 + \frac{\partial\alpha_1}{\partial\theta}\frac{\hat{\boldsymbol{\theta}}}{r} \tag{21}$$

so that

$$\nabla\alpha_1 \cdot \nabla\gamma_0 = \frac{1}{r^2}\frac{\partial\gamma_0}{\partial\theta}\frac{\partial\alpha_1}{\partial\theta}\bigg|_{\alpha_0,\phi} = -\frac{\partial\alpha_1}{\partial\theta} g_1^0 \frac{a^3}{r^4}\sin\theta \tag{22}$$

where the partial derivative of γ_0, it should be noted, is evaluated with r held constant.

Let $\Sigma_{(1)}$ indicate summation over those terms included in γ_1. Substitution in the equation (15) yields

$$\frac{\partial\alpha_1}{\partial\theta} = a\sum_{(1)} \left(\frac{a}{r}\right)^n \left\{(n+1)\sin\theta P_n^m(\theta) + 2\cos\theta \frac{dP_n^m}{d\theta}\right\}\{g_n^m \cos m\phi + h_n^m \sin m\phi\} \tag{23}$$

Before this can be integrated, the right hand side has to be expressed in terms of α_0, θ and ϕ. Using

$$\left(\frac{a}{r}\right) = \left(\frac{\alpha_0}{a g_1^0}\right) \sin^{-2} \theta \tag{24}$$

one gets

$$\frac{\partial \alpha_1}{\partial \theta} = a \sum_{(1)} \left(\frac{\alpha_0}{a g_1^0}\right)^n \sin^{-2n} \theta \left\{(n+1) \sin \theta P_n^m(\theta) + 2 \cos \theta \frac{dP_n^m}{d\theta}\right\} \{g_n^m \cos m\phi + h_n^m \sin m\phi\} \tag{25}$$

and hence

$$\alpha_1 = a \sum_{(1)} \left(\frac{\alpha_0}{a g_1^0}\right)^n V_n^m(\theta) \{g_n^m \cos m\phi + h_n^m \sin m\phi\} + \xi(\alpha_0, \phi) \tag{26}$$

where the function

$$V_n^m(\theta) = \int_{\pi/2}^{\theta} \sin^{-2n} \Theta \left\{(n+1) \sin \Theta P_n^m(\Theta) + 2 \cos \Theta \frac{dP_n^m}{d\Theta}\right\} d\Theta \tag{27}$$

is a trigonometric integral which may be evaluated in closed form. This result is equivalent to one previously derived by Pennington (1966) and tables giving $V_n^m(\theta)$ are available (Pennington, 1961; Stern, 1965). It also has been derived, in a different way, by Birmingham and Northrop (private communication). The lower limit of integration has been arbitrarily chosen as $\pi/2$ (dipole equator); a different choice merely leads to a different additive function ξ.

In a similar manner, the equation for β_1 gives

$$\beta_1 = \frac{a}{g_1^0} \sum_{(1)} \left(\frac{\alpha_0}{a g_1^0}\right)^{n-1} t_n^m(\theta) \{h_n^m \cos m\theta - g_n^m \sin m\phi\} + \eta(\alpha_0, \phi) \tag{28}$$

with

$$t_n^m(\theta) = m \int_{\pi/2}^{\theta} \sin^{-(2n+1)} \Theta \, P_n^m(\Theta) \, d\Theta \tag{29}$$

This, too, is equivalent to one of Pennington's results. For the explicit form of $t_n^m(\theta)$, see Tables 1 and 2.

Table 1
Coefficients of the Trigonometric Integrals $t_n^m(\theta)$ for odd m

n	m							
n+m	odd	$\sin^{-1}\theta$	$\sin^{-3}\theta$	$\sin^{-5}\theta$	$\sin^{-7}\theta$	$\sin^{-9}\theta$	$\sin^{-11}\theta$	1
n+m	even	Multiply Above Factors by $\cos\theta$						-
1	1	-1.						
2	1		-.57735					.57735
3	1	.73485	.36742	-.48990				
3	3	-1.58114	-.79057					
4	1			1.10680	-.45175			-.65504
4	3			-1.25499				1.25499
5	1	-.79919	-.39959	-.29970	1.44468	-.43033		
5	3	1.79284	.89642	.67231	-1.79284			
5	5	-1.87083	-.93541	-.70156				
6	1				-2.70044	2.29128	-.41660	.82576
6	3				4.26977	-2.41522		-1.85455
6	5				-1.66201			1.66201

To derive the second order term α_2, it is best to split it into two parts

$$\alpha_2(\alpha_0, \phi, \theta) = \alpha_2' + \alpha_2''$$

where

$$(\partial \alpha_2' / \partial \theta)\, g_1^0\, a^3\, r^{-4} \sin\theta = \nabla \alpha_0 \cdot \nabla \gamma_2 \tag{30}$$

Table 2
Coefficients of the Trigonometric Integrals $t_n^m(\theta)$ for even m

n	m						
n+m	odd	$\sin^{-2}\theta$	$\sin^{-4}\theta$	$\sin^{-6}\theta$	$\sin^{-8}\theta$	$\sin^{-10}\theta$	1
n+m	even	Multiply Above Factors by cos					$\log\tan(\theta/2)$
2	2	-.86603					.86603
3	2		-.96825				.96825
4	2	.83853	.55902	-1.11803			-.83853
4	4	-1.10926	-.73951				1.10926
5	2			2.56174	-1.28087		-1.28087
5	4			-1.47902			1.47902
6	2	-1.01892	-.67928	-.54343	3.80398	-1.44913	1.01892
6	4	1.39522	.93015	.74412	-2.48039		-1.39522
6	6	-1.25942	-.83961	-.67169			1.25942

$$(\partial \alpha_2''/\partial \theta)\, g_1^0\, a^3\, r^{-4} \sin\theta = \nabla\alpha_1 \cdot \nabla\gamma_1 \tag{31}$$

When added together, the above equations yield the equation (17); similar expressions may be derived for β_2.

The equation for α_2' resembles that for α_1 and presents no difficulty; indeed, one can easily include the calculation of this part with the calculation of α_1. The equation for α_2'', on the other hand, requires considerable effort: both α_1 and γ_1 are linear in the set of coefficients (g_n^m, h_n^m) summed by $\Sigma_{(1)}$ and therefore the right-hand side of (31) – and consequently α_2'' – contains one term for each binary combination of these coefficients. The result can still be integrated in closed form, but it is rather lengthy and will not be further discussed here.

THE MATCHING OF SOLUTIONS

The corrections α_i and β_i added to the zero order Euler potentials are all derived by integration within an arbitrary function of α_0 and β_0. The question arises whether the freedom afforded by such a function is sufficient to enable one to make the solutions match, within the order to which they are derived. It will be shown in what follows – with no attempt at mathematical rigor, however — that this indeed is the case.

Let a subscript such as n characterize quantities of the n-th order, while a superscript (n) will refer to a variable or product of variables evaluated to the n-th order, i.e. to its perturbation expansion summed up to and including the n-th order term. Suppose that matching Euler potentials $\alpha^{(n)}$ and $\beta^{(n)}$ of the n-th order have been derived by perturbing α_0 and β_0, i.e.

$$\{\nabla \alpha^{(n)} \times \nabla \beta^{(n)}\}^{(n)} = - \nabla \gamma^{(n)} + 0\,(\epsilon^{n+1}) \tag{32}$$

Starting with $\alpha^{(n)}$ and $\beta^{(n)}$ one may then proceed and derive higher orders, and we assume that the procedure converges, tending to certain limits α and β

$$\alpha = \alpha^{(n)} + \alpha_{n+1} + \alpha_{n+2} + \cdots$$

$$\beta = \beta^{(n)} + \beta_{n+1} + \beta_{n+2} + \cdots$$

In general α and β, or their approximations to an order higher than the n-th, do not match. However, it will be shown that a suitable function $k_{n+1}(\alpha_0, \beta_0)$ can be found which, when added to β_{n+1}, causes the (n + 1) order approximation to match as well (alternatively, such a function may be derived for α_{n+1}). The process may then be repeated, allowing matching to any desired order.

It has been shown in equation (9) that if α and β are nonmatching Euler potentials, a function $\delta\,(\alpha, \beta)$ matching α does exist, i.e.

$$\nabla \alpha \times \nabla \delta = \frac{\partial \delta}{\partial \beta}(\nabla \alpha \times \nabla \beta) = - \nabla \gamma \tag{33}$$

Here we are not as much concerned with δ as with its derivative

$$\partial \delta / \partial \beta = \lambda\,(\alpha, \beta)$$

We assume that like α and β, the function $\lambda\,(\alpha, \beta)$ – and therefore also δ – can be expanded to various orders

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \cdots$$

and so can, therefore, equation (33). Now equation (32) expresses the given fact that up to order n, the expansions of α and β match. When this equation is compared to (33), we see that at least to order n, $\lambda\ (\alpha, \beta)$ equals unity

$$\lambda^{(n)} = 1 + 0\,(\epsilon^{n+1})$$

and hence

$$\lambda^{(n+1)} = 1 + h_{n+1}\,(\alpha, \beta)$$

where $h_{n+1}(\alpha, \beta)$ is some function of order (n + 1). Integration gives, apart from an irrelevant function of α alone

$$\delta^{(n+1)} = \beta + k_{n+1}\,(\alpha, \beta)$$

$$= \beta^{(n+1)} + k_{n+1}\,(\alpha, \beta) + 0\,(\epsilon^{n+2})$$

Now comes the important step: if one substitutes (α_0, β_0) for (α, β) in a term of order (n + 1), the error introduced is only of order (n + 2). In particular

$$\delta^{(n+1)} = \beta^{(n+1)} + k_{n+1)}\,(\alpha_0, \beta_0) + O\,(\epsilon^{n+2})$$

This is exactly the desired result, for its states that a solution $\delta^{(n+1)}$ matching α to the (n + 1) order may be obtained by adding to the nonmatching solution $\beta^{(n+1)}$ a function $k_{n+1}\ (\alpha_0, \beta_0)$ of α_0 and β_0.

MATCHING FIRST ORDER SOLUTIONS

As has been shown in the preceding section, the first order Euler potentials may be matched by a proper choice of the functions $\xi\,(\alpha_0, \beta_0)$ and $\eta\,(\alpha_0, \beta_0)$ appearing in equations (26) and (28). It is sufficient for this purpose to consider only solutions of the type

$$\alpha_1 = a\sum_{(1)} \left(\frac{\alpha_0}{a g_1^0}\right)^n [V_n^m(\theta) + \xi_n^m]\ \{g_n^m \cos m\phi + h_n^m \sin m\phi\}$$

$$\beta_1 = \frac{a}{g_1^0}\sum_{(1)} \left(\frac{\alpha_0}{a g_1^0}\right)^{n-1} [t_n^m(\theta) + \eta_n^m]\ \{h_n^m \cos m\phi - g_n^m \sin m\phi\}$$

where ξ_n^m and η_n^m are constants remaining to be determined.

The third of the scalar equations equivalent to (13) may be derived by scalar multiplication with the unit vector $\hat{\boldsymbol{\theta}}$. This gives

$$- r^{-1} (\partial \gamma_1 / \partial \theta) = (\nabla \beta_0 \times \hat{\boldsymbol{\theta}}) \cdot \nabla \alpha_1 + (\hat{\boldsymbol{\theta}} \times \nabla \alpha_0) \cdot \nabla \beta_1$$

$$= - (a / r \sin \theta) (\hat{r} \cdot \nabla \alpha_1) - (\partial \alpha_0 / \partial r) (\hat{\boldsymbol{\varphi}} \cdot \nabla \beta_1)$$

$$= - \frac{1}{r \sin \theta} \frac{\partial \alpha_0}{\partial r} \left\{ a \frac{\partial \alpha_1}{\partial \alpha_0} + \frac{\partial \beta_1}{\partial \phi} \right\}$$

or

$$\frac{\partial \gamma_1}{\partial \theta} + \frac{\alpha_0^2}{a^2 g_1^0 \sin^3 \theta} \left\{ a \frac{\partial \alpha_1}{\partial \alpha_0} + \frac{\partial \beta_1}{\partial \phi} \right\} = 0 \tag{34}$$

Expanding, and reconverting the first term to (α_0, θ, ϕ) variables, yields

$$a \sum_{(1)} \left(\frac{\alpha_0}{a g_1^0} \right)^{n+1} (g_n^m \cos m\phi + h_n^m \sin m\phi) \sin^{-3} \theta \, F_n^m (\theta) = 0$$

where

$$F_n^m (\theta) = \sin^{-(2n-1)} \theta \frac{dP_n^m}{d \theta} + n \left[V_n^m (\theta) + \xi_n^m \right] - m \left[t_n^m (\theta) + \eta_n^m \right]$$

By making use of Legendre's associated equation it may be shown that

$$\frac{dF_n^m}{d \theta} = 0$$

and therefore $F_n^m(\theta)$ is a constant. It only remains to choose the adjustable part of F_n^m

$$n \, \xi_n^m - m \, \eta_n^m$$

so that this constant vanishes. For instance, we may choose

$$\eta_n^m = 0$$

and evaluating ξ_n^m at $\theta = \pi/2$ we then get

$$\xi_n^m = -\frac{1}{n}\frac{dP_n^m(\pi/2)}{d\theta} \tag{35}$$

Table 3 lists the constant parts of $[V_n^m(\theta) + \xi_n^m]$ for $n \leq 6$, with $V_n^m(\theta)$ defined as in (27) and ξ_n^m satisfying (35). These values should replace the constant parts listed by Stern (1965; Tables 2 and 3) if one wishes to calculate matching potentials in the manner described here.

Table 3
Integration Constants Added to

n = \ m =	0	1	2	3	4	5	6
1	1.0	0					
2		.28867	0				
3	0	0	.64550	0			
4	0	-.16377	0	.94124	0		
5	0	0	-.51235	0	1.18322	0	
6	0	.13761	0	-.92727	0	1.38500	0

The above are the θ-independent portions of $V_n^m(\theta)$, selected so that $V_n^m(\pi/2) = (-1/n)\,[dP_n^m(\pi/2)/d\theta]$

CONJUGATE POINTS

In geomagnetic nomenclature, two points (usually chosen on the earth's surface) are termed conjugate to each other if they are connected by a magnetic field line. The identification of such pairs of points is of considerable interest, because observations made at them of effects propagating along magnetic field lines – e.g. hydromagnetic waves – tend to be correlated (Wescott, 1966). A large number of such pairs has been tabulated (Roederer et al., 1966) by numerically tracing field lines of the geomagnetic field.

The accuracy of first-order Euler potentials may be tested by deriving a map of α and β on the earth's surface (Figure 1). On such a map, pairs of conjugate points should be identified by the same α and β, and the extent to which they fail to do so gauges the accuracy of the approximation.

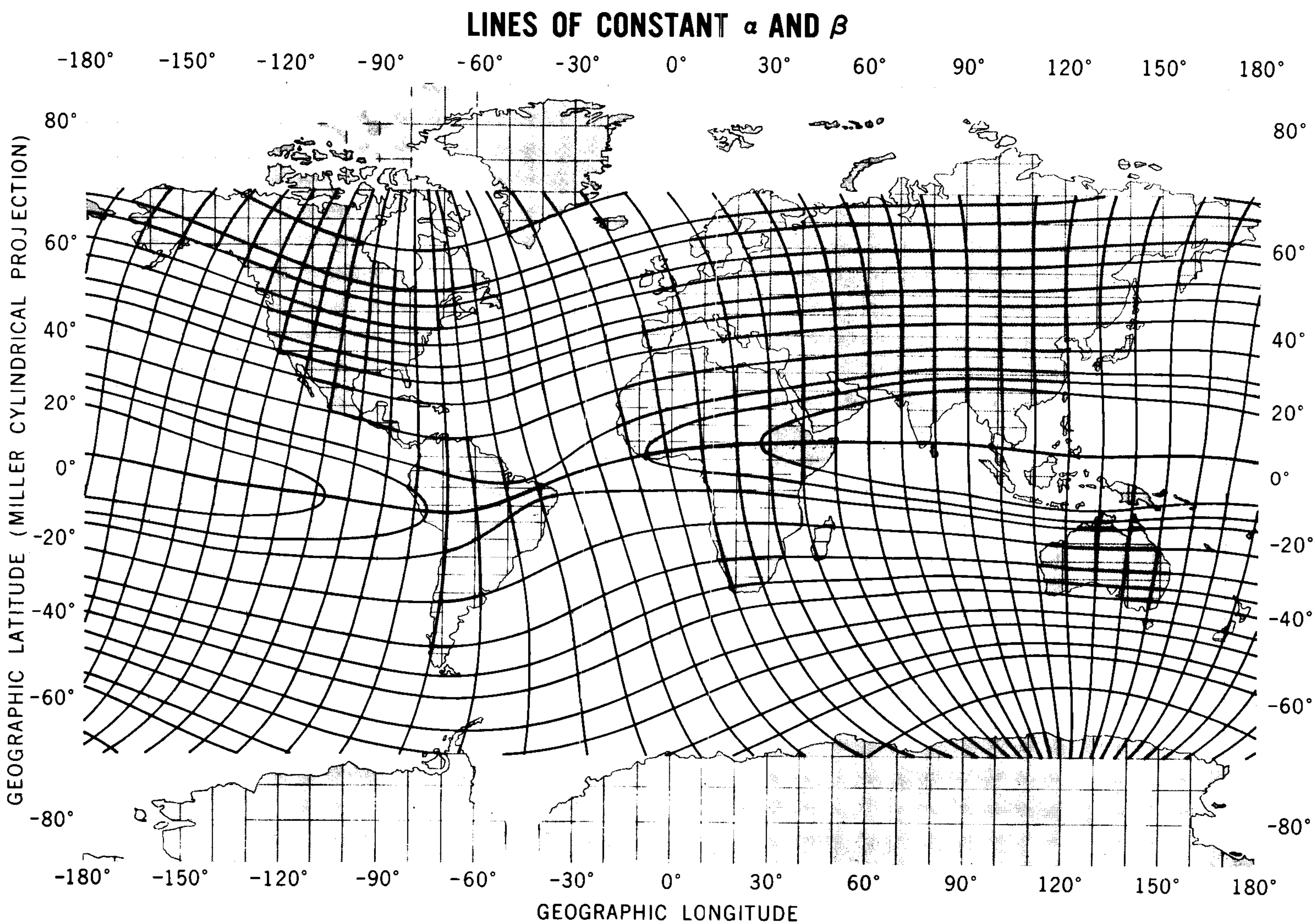

Figure 1–Lines of constant first-order Euler potentials on the earth's surface. The quantities plotted are β/a in 10-degree intervals and $\lambda = ag_1^0/\alpha$ for the values (increasing with distance from the geomagnetic equator, which is also shown) 1, 1.05, 1.1, 1.25, 1.5, 1.75, 2, 2.5, 3, 4, 5 and 10.

In Table 4 the conjugates to a number of points on the earth's surface are derived using first order Euler potentials (and assuming a spherical earth), and they are compared with those derived numerically by Roederer et.al. (1966). A deviation averaging about one degree in latitude and a little more than that in longitude is evident: it can probably be attributed to the omission of higher-order terms.

Table 4
Comparison of Conjugate Points

			This Work		Roederer et.al.	
	Latitude	Longitude	Conjugate Latitude	Conjugate Longitude	Conjugate Latitude	Conjugate Longitude
Adelaide	-34.95	138.53	52.98	145.73	53.2	143.3
Anchorage	61.17	-149.98	-53.46	172.63	-53.6	172.2
Antofagasta	-23.65	-70.42	-4.55	-68.80	-3.1	-69.7
Archangelsk	64.58	40.5	-52.00	72.07	-51.7	69.1
Athens	37.97	23.72	-21.25	31.00	-19.7	29.3
Baghdad	33.35	44.38	-16.53	48.21	-16.4	46.4
Bangkok	13.75	100.55	5.21	101.46	5.4	100.4
Belem	-1.00	-49.00	-18.24	-43.67	-17.2	-44.6
Bismark	46.82	-100.77	-63.18	133.44	-62.6	-135.3
Bogota	4.63	-74.08	-32.00	-76.11	-30.5	-77.2
Bombay	19.00	72.83	-0.94	74.73	-1.2	74.3
Capetown	-34.15	18.32	44.52	2.83	45.0	-0.6
Deep River	46.10	-77.5	-72.61	-97.61	-72.5	-98.0
Haifa	32.83	35.1	-15.15	38.85	-14.3	37.0
Johannesburg	-26.20	28.03	41.38	19.71	42.5	18.3
Kyoto	35.02	135.78	-18.18	135.30	-18.4	134.7
Kiruna	67.83	20.43	-58.85	66.17	-58.7	63.6
Kodaikanal	10.23	77.48	8.41	78.55	8.5	77.6
London	51.53	0.10	-46.27	26.14	-44.3	25.8
Midway	6.97	158.22	5.84	159.10	6.6	158.2
Port Moresby	-9.20	147.15	24.61	150.36	24.9	149.1
Seattle	47.75	-122.42	-54.18	-153.73	-53.7	-155.2
Torino	45.20	7.65	-33.67	23.87	-32.0	23.3
Trinidad	10.60	-61.2	-38.54	-55.57	-37.8	-56.3
Washington, D.C.	38.73	-77.13	-66.03	-90.70	-65.7	-91.8

EXTERNAL SOURCES

The approximate derivation of Euler potentials by the method outlined here, performed in geomagnetic tilted-dipole coordinates (Stern, 1965, Table 1), improves with increasing distance from the earth, because the field's nondipole components fall off faster than the dipole part. However, at distances larger than about 3a an additional factor begins perturbing the field, namely the compression and sweeping of the geomagnetic field by action of the solar wind. It is a matter of controversy whether an appreciable current density ("ring current") exists in this region; here such currents will be neglected and the external influence will be taken into account by adding radially increasing terms to the scalar potential expansion

$$\gamma = a \sum_{n=1} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} P_n^m(\theta) \{g_n^m \cos m\phi + h_n^m \sin m\phi\}$$

$$+ a \sum_{n=1} \sum_{m=0}^{n} \left(\frac{r}{a}\right)^{n} P_n^m(\theta) \{\bar{g}_n^m \cos m\phi + \bar{h}_n^m \sin m\phi\} \qquad (36)$$

Following Mead (1964), a fair approximation to the external perturbation is obtained by retaining the terms involving $\bar{g}_1^0$ and $\bar{g}_2^1$ along; these coefficients turn out to be of the order $10^{-3} g_1^0$ and $10^{-4} g_1^0$ respectively. Obviously the field can be regarded as a perturbed dipole only up to some (not too well defined) limiting distance, of the order of 10a. While the present perturbation method will diverge outside this limit, one may use it for the region inside, leading to a first order contribution to α

$$\bar{\alpha}_1 = a \sum \left(\frac{a g_1^0}{\alpha_0}\right)^{n+1} [\bar{V}_n^m(\theta) + \bar{\xi}_n^m] \{\bar{g}_n^m \cos m\phi + \bar{h}_n^m \sin m\phi\} \qquad (37)$$

where

$$\bar{V}_n^m(\theta) = \int_{\pi/2}^{\theta} \sin^{2(n+1)}\Theta \left\{2\cos\Theta \frac{dP_n^m}{d\Theta} - n \sin\Theta\, P_n^m(\Theta)\right\} d\Theta \qquad (38)$$

In both cases, the results resemble previously derived ones, with $-(n+1)$ replacing n. As before, the constants $\bar{\xi}_n^m$ and $\bar{\eta}_n^m$ may be chosen so that the potentials match. Starting with equation (34), one finds

$$a \sin^{-3}\theta \sum (a g_1^0/\alpha_0)^n \{\bar{g}_n^m \cos m\phi + \bar{h}_n^m \sin m\phi\} \bar{F}_n^m(\theta) = 0$$

where

$$\bar{F}_n^m(\theta) = \sin^{2n+3}\theta \frac{dP_n^m}{d\theta} - (n+1)[\bar{V}_n^m + \bar{\xi}_n^m] - m[\bar{t}_n^m + \bar{\xi}_n^m] \tag{39}$$

It may again be shown, by means of Legendre's associated equation, that $\bar{F}_n^m(\theta)$ is a constant. If $\bar{\xi}_n^m$ and $\bar{\eta}_n^m$ are chosen so as to satisfy

$$(n+1)\bar{\xi}_n^m + m\bar{\eta}_n^m = \frac{dP_n^m}{d\theta}(\pi/2)$$

the value of this constant is zero and the matching condition is satisfied. Unlike in the case of internal sources, however, additional factors may have to be taken into account here when choosing the adjustable constants.

The reason is that the expansions of α and β derived here are valid only if $\bar{\alpha}_1$ and $\bar{\beta}_1$ turn out to be relatively small compared to α_0 and β_0. Because of the presence of α_0 in the denominators of (37) and (38), this condition will not be met, in general, near the poles ($\theta = 0, \pi$), at which α_0 vanishes. To reduce and possibly remove this source of trouble, it is advantageous to choose ξ_n^m in such a way (if possible) that $[\bar{V}_n^m + \bar{\xi}_n^m]$ vanishes at the poles. From Equation (39) it follows that $[\bar{t}_n^m + \bar{\eta}_n^m$ then also tends to zero, so that such a choice also reduces the corresponding divergence in $\bar{\beta}_1$.

For $m = 0$, equation (39) shows that this condition is always satisfied. In other cases, $\bar{\xi}_n^m$ may be chosen so that the condition holds at one of the poles, e.g. the northern one. Whether it then holds for the southern pole as well depends on the parity of $\bar{V}_n^m(\theta)$ with respect to the equator. If (n + m) is odd, $\bar{V}_n^m(\theta)$ and $\bar{t}_n^m(\theta)$ have both even parity (with appropriate added constant)

$$\bar{V}_n^m(\theta) = \bar{V}_n^m(\pi - \theta)$$

and hence the condition will hold for both poles. In other cases, the parity (with suitable additive constants) is odd and $\bar{V}_n^m(\theta)$ assumes different values at opposite poles (unless it vanishes there). The condition then cannot be met simultaneously for both poles: fortunately, this does not occur in the simple model which will now be considered, for which only $\bar{g}_1^0$ and $\bar{g}_2^1$ are taken into account and all parities are even. In that case

$$P_1^0 = \cos\theta \qquad P_2^1 = \sqrt{3}\sin\theta\cos\theta$$

$$\bar{V}_1^0 + \bar{\xi}_1^0 = -\frac{1}{2}\sin^6\theta \qquad \bar{V}_2^1 + \bar{\xi}_2^1 = 2\sqrt{3}\,\{(\sin^7\theta/7) - (\sin^9\theta/3)\} \tag{40}$$

$$\bar{t}_1^0 = \bar{\eta}_1^0 = 0 \qquad \bar{t}_2^1 + \bar{\eta}_2^1 = (\sqrt{3}/7)\sin^7\theta$$

Let $\bar{G}_n^m = \bar{g}_n^m/g_1^0$. The approximate solution is then

$$\alpha/ag_1^0 = \left\{(a/r) - \frac{1}{2}\bar{G}_1^0(r/a)^2\right\}\sin^2\theta + 2\sqrt{3}\,\bar{G}_2^1(r/a)^3\{(\sin\theta/7) - (\sin^3\theta/3)\}\cos\phi \tag{41}$$

$$\beta/a = \phi - (\sqrt{3}/7)\,\bar{G}_2^1(r/a)^4\sin^{-1}\theta\sin\phi \tag{42}$$

Let A = 10a. Following Mead (1964) we choose

$$\bar{g}_1^0 = -0.2511\cdot 10^{-3} \qquad \bar{G}_1^0 = 0.8\cdot 10^{-3}$$

$$\bar{g}_2^1 = 0.1242\cdot 10^{-4} \qquad \bar{G}_2^1 = 0.3954\cdot 10^{-4}$$

giving

$$\alpha/ag_1^0 = 0.1\{(A/r) - 0.4(r/A)^2\}\sin^2\theta + 0.1(r/A)^3(0.1957\sin\theta - 0.4566\sin^3\theta)\cos\phi \tag{43}$$

$$\beta/a = \phi - 0.09784\,(r/A)^4\sin^{-1}\theta\sin\phi \tag{44}$$

A plot of $\alpha/a\,g_1^0$ in the $\phi = 0,\pi$ plane (corresponding to the noon-midnight meridional plane) is given in Figure 2, while Figure 3 gives β/a in the equatorial plane.

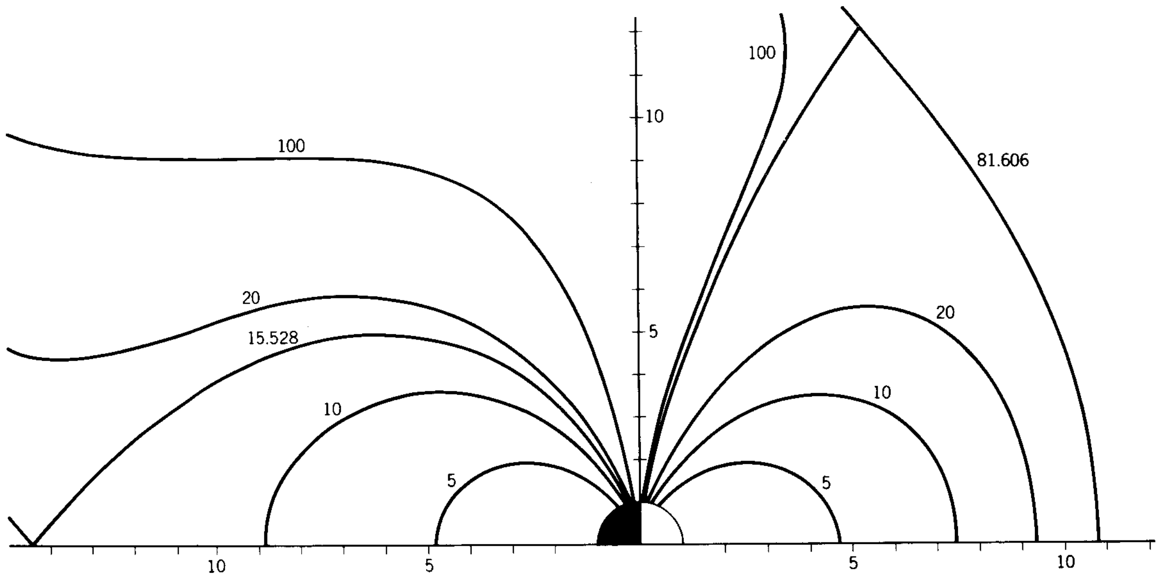

Figure 2—Lines of constant α (which also are field lines) in the noon-midnight meridian of the first-order magnetospheric model. Values given are of $\lambda = ag_1^0/\alpha$, which in a dipole field equals the equatorial crossing distance, in earth radii, of the corresponding field line.

It is evident from these figures that up to a distance of about 10a this model quite closely approximates our ideas of the magnetosphere's configuration. It is therefore not particularly useful to regard it any more, at this stage, as an approximation of the 3-parameter scalar potential model from which it was derived (and from which it deviates considerably near r = 10a) because that model, in its turn, is no more than a simplified representation of the actual field. Instead, we may regard the field represented by (41) and (42) as an independent model of the magnetosphere, expressed by means of Euler potentials and containing two adjustable parameters. The properties of this model – comparison to observations, motion of trapped charged particles, etc. – will be explored in a separate article.

We now investigate a little further the meaning of equation (41). If $\bar{g}_2^1$ is zero, this solution for α – which will be denoted by α_c, the subscript standing for "compressed dipole" – is not merely correct to the first order but holds exactly, being the well-known solution for a dipole aligned with an external homogeneous field (Figure 4). In that case the points at which $\alpha = 0$ trace the z-axis and also cover the sphere

$$r = a\,(2/\bar{G}_1^0)^{1/3}$$

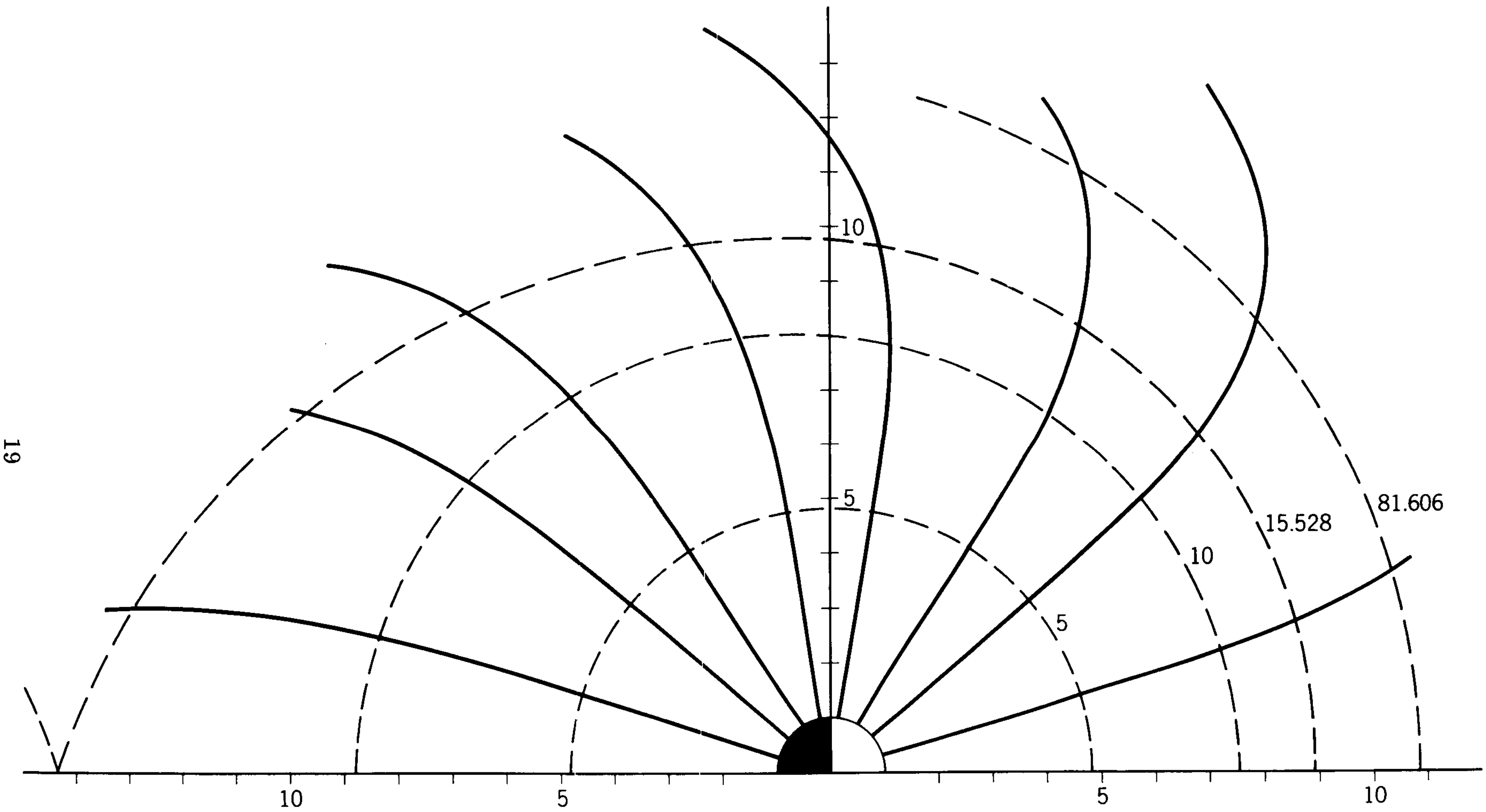

Figure 3–Lines of constant β (solid) and constant α (broken) in the meridional plane of the first-order model magnetosphere. Numbers given are values of $\lambda = ag_1^0/\alpha$.

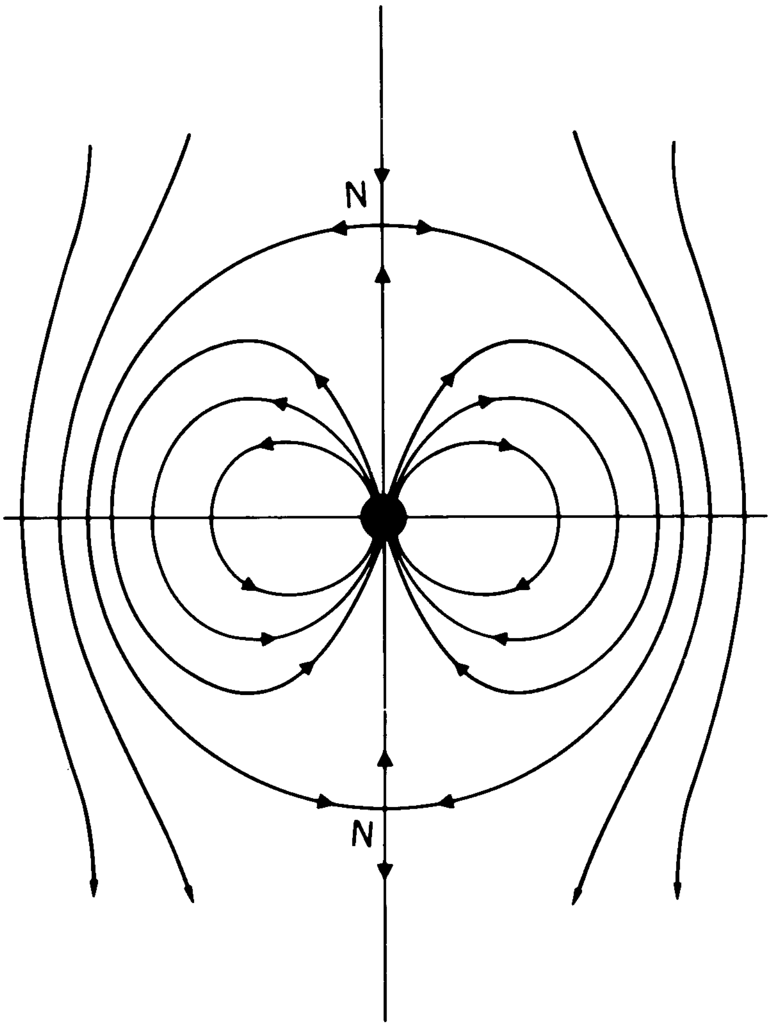

Figure 4—Lines of constant α (which are also field lines) in the meridional plane of a compressed dipole. Letter N indicates neutral points.

which separates field lines attached to the dipole from those reaching to infinity. There exist two neutral points where the z-axis meets the sphere, at which the field's intensity vanishes and its direction is therefore not uniquely defined.

When the first-order contribution of $\bar{g}_2^1$ is added to this configuration, the surface of demarcation opens up on one side (the night side), permitting some of the lines originating in the dipole to extend to infinity. The neutral points shift towards the opposite side and the value of α corresponding to them and to the demarcation surface now differs from zero, so that the field lines connecting them to the dipole are shifted to the front side of the polar field line. The properties of the surface of demarcation thus resemble those believed to exist for the magnetosphere, the surface observed to separate field lines originating at the earth from those originating in the solar wind or in its interaction with the geomagnetic field.

In principle one could perform the preceding perturbation expansion with the joint contribution α_c of g_1^0 and $\bar{g}_1^0$ as the zero-order potential. In practice, this is not feasible, because the inversion

$$r = r(\alpha_c, \theta)$$

which has to be substituted in the expression for $\partial \alpha_1 / \partial \theta$ prior to integration cannot be readily derived. In addition, the denominator $\partial \gamma_c / \partial \theta$ of such an expression would be too complicated to permit integration in closed form.

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