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A Two-Variable Asymptotic Solution for Three-Dimensional, Solar-Powered, Low-Thrust Trajectories in the Vicinity of the Ecliptic Plane

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Abstract

An approximate analytic solution is derived for the variables which describe a three-dimensional, solar-powered, heliocentric, low-thrust trajectory in the vicinity of the ecliptic plane. This approximate solution consists of the first three terms of an asymptotic series, valid in the limit of vanishing thrust. To obtain an extended domain of validity, the two-variable method is used.

It is assumed that the solar-powered spacecraft is sun-oriented. Direction and magnitude of the thrust acceleration may change discontinuously at arbitrarily chosen instants. In the time-intervals between these instants, the direction of the thrust is constant with respect to the sun-spacecraft vector and the plane of the initial orbit. Moreover, in these time-intervals the thrust acceleration is assumed to be inversely proportional to the square of the distance to the sun. The actual variation of the thrust acceleration (owing to mass depletion and change in power received from the sun) may be approximated by step-by-step changes in the constant of proportionality.

The results are compared with numerical integrations of the equations of motion. If the ratio of the thrust acceleration and the gravitational acceleration is less than 1/10, the difference between the results given by the approximate formulas and those to numerical integration is, at most, 1%. After 1.5 revolutions around the sun, no signs of growing errors could yet be discovered. The computing time needed to obtain numerical results from the approximate formulas is appreciably less than the time needed to perform numerical integrations of the equations of motion.

It is concluded that the asymptotic series derived in this Report is a useful tool for the design of solar-powered, interplanetary, low-thrust trajectories.

A Two-Variable Asymptotic Solution for Three-Dimensional, Solar-Powered, Low-Thrust Trajectories in the Vicinity of the Ecliptic Plane

I. Introduction

Current breakthroughs in the design of large solar panels have caused much interest in solar-powered, electrically propelled space vehicles. This type of spacecraft is attractive because of the very high specific impulses which are within the reach of advanced electrical propulsion systems. Under contract with Jet Propulsion Laboratory (JPL) a study of solar–electric spacecraft was recently completed at Hughes Aircraft Company (Ref. 1). The Advanced Technical Studies Office at JPL, also, is tentatively planning a study group effort on solar-powered, low-thrust spacecraft. To support this work, suitable trajectory programs are necessary.

For trajectory computations, an integrating program developed by Melbourne and Sauer is available (Ref. 2). Numerical programs like this are relatively slow on the computer, and complicated. Therefore, a program which is simple and fast, and which gives approximate results, would be a useful tool for the design of low-thrust trajectories if the approximate results are accurate enough. Such a program would be analogous to a patched conic program for ballistic interplanetary trajectories.

In this Report, an approximate, analytic, closed-form solution is derived for the variables which describe a three-dimensional, solar-powered, low-thrust trajectory. The answers are given in the form of somewhat lengthy, but essentially simple, algebraic formulas containing sines, cosines and exponentials. Consequently, computer time is appreciably shorter than it is for a more accurate numerical integrating program.

The approximate solution is derived with the two-variable asymptotic expansion technique (Ref. 3). This method can be used for other types of perturbations of Kepler's equations as well, a possibility which has already received the attention of G. A. Flandro, of the Systems Analysis Section, at JPL.

A similar approximate solution can probably be derived for nuclear-powered low-thrust spacecraft, but the mathematical approach would have to differ from that followed in the present work.

The reader who is mainly interested in the final results can find these at the end of Sections IV and V (Eqs. 59–67 and Eq. 73). The approximations made in the analysis may be ascertained by a perusal of Section II and the beginning of Section IV. The accuracy of the approximate formulas is evaluated by a comparison with numerical integrations in Section VII.

II. Derivation and Discussion of the Dimensionless Equations

Consider a space vehicle with a solar–electric propulsion system, under gravitational influence of the sun. The gravitational field is assumed to be identical to the field generated by one point-mass.

In spherical polar coordinates (r, ϕ, ψ) (see Fig. 1), the equations of motion (assuming ψ is small, that is, restricting ourselves to trajectories in the vicinity of the ecliptic plane) are:

$$\frac{d}{dt}\left(\frac{mdr}{dt}\right) = -\frac{GMm}{r^2} + mr(\dot{\phi}^2 + \dot{\psi}^2) + F_r \tag{1}$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = rF_\phi \tag{2}$$

$$\frac{d}{dt}(mr^2\dot{\psi}) = rF_\psi - mr^2\psi\dot{\phi}^2 \tag{3}$$

F_r, F_ϕ, F_ψ are the components of the thrust vector; for the rest, the notation is obvious. For the mass we have

$$\dot{m} = -\frac{F}{c} \tag{4}$$

where $F = (F_r^2 + F_\phi^2 + F_\psi^2)^{1/2}$, and c is the exhaust velocity. Because electrical propulsion systems have a very high exhaust velocity, $\dot{m}$ is small and can be neglected in approximate calculations. Equations (1–3) become

$$\ddot{r} = -\frac{GM}{r^2} + r(\dot{\phi}^2 + \dot{\psi}^2) + \frac{F_r}{m} \tag{5}$$

$$\frac{d}{dt}(r^2\dot{\phi}) = r\frac{F_\phi}{m} \tag{6}$$

$$\frac{d}{dt}(r^2\dot{\psi}) = r\frac{F_\psi}{m} - r^2\psi\dot{\phi}^2 \tag{7}$$

Substituting $r = 1/u$, and using ϕ as the independent variable instead of t, gives for Eqs. (5) and (6)

$$\frac{d^2u}{d\phi^2} + u\left[1 + \left(\frac{d\psi}{d\phi}\right)^2\right] = GMk - \frac{k}{u^2}\frac{F_r}{m} - \frac{k}{u^3}\frac{du}{d\phi}\frac{F_\phi}{m} \tag{8}$$

$$\frac{dk}{d\phi} = -2\frac{k^2}{u^3}\frac{F_\phi}{m} \tag{9}$$

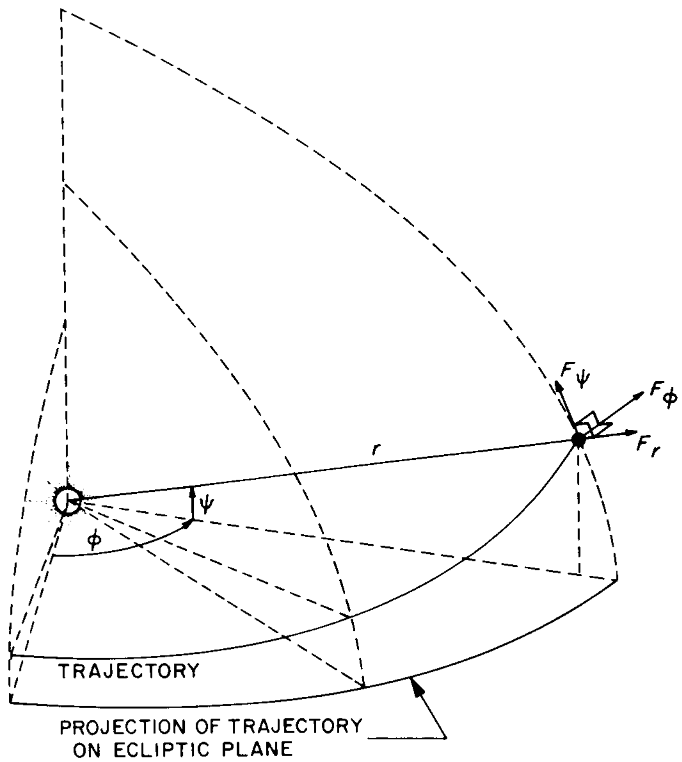

Fig. 1. Definition of the coordinate system (r, ϕ, ψ)

where $k = u^4/\dot{\phi}^2 = 1/H_\phi^2$, and H_ϕ is the ϕ-component of the angular momentum of the spacecraft. Equation (8) is the equation for an oscillator with nonlinear restoring force and damping. Equations of this type have been studied extensively. This is the reason for the change of variables.

Equation (7) transforms into

$$\frac{d^2\psi}{d\phi^2} - \frac{1}{2k}\frac{dk}{d\phi}\frac{d\psi}{d\phi} + \psi = \frac{k}{u^3}\frac{F_\psi}{m} \tag{10}$$

Because the spacecraft receives its power from the sun, the available power is proportional to $1/r^2$. At the present time, it is expected that in reality the thrust force will be approximately proportional to $r^{-1.7}$. It is assumed here that the thrust *acceleration* is proportional to $1/r^2$. Therefore we write

$$\frac{F_\phi}{m} = u^2Q\eta(\phi), \quad \frac{F_r}{m} = u^2Q\zeta(\phi), \quad \frac{F_\psi}{m} = u^2Q\xi(\phi) \tag{11}$$

where

$$\eta^2(0) + \zeta^2(0) + \xi(0)^2 = 1$$

so that the constant of proportionality Q is given by

$$Q = \frac{1}{u^2(0)} \frac{F(0)}{m} \tag{12}$$

Because η, ζ and ξ depend on ϕ, changes in thrust, direction and control setting can be included in the present model. Also, by periodically readjusting η, ζ and ξ, the assumed thrust acceleration can be made to approximate more closely the actual thrust acceleration which is not proportional to $1/r^2$.

The spacecraft is assumed to be oriented toward the sun. Hence the thrust direction is constant with respect to the sun–spacecraft vector (assuming that the exhaust nozzle does not move with respect to the spacecraft). It is also assumed that the thrust direction makes a constant angle with the plane of the initial orbit. If we assume that the thrust acceleration is proportional to $1/r^2$, this means that η, ζ and ξ are constant as long as the spacecraft maintains the same orientation.

In view of these considerations, the actual situation can probably be approximated closely if η, ζ and ξ are specified as follows:

$$\left.\begin{aligned} \eta(\phi) &= \eta_n \\ \zeta(\phi) &= \zeta_n \\ \xi(\phi) &= \xi_n \end{aligned}\right\} \qquad \text{for } \phi_n < \phi < \phi_{n+1}$$

where η_n, ζ_n and ξ_n are constants. The points $\phi = \phi_n$, at which the thrust acceleration components are readjusted, may be chosen arbitrarily. Equations (8–10) then become

$$\frac{d^2u}{d\phi^2} + u\left[1 + \left(\frac{d\psi}{d\phi}\right)^2\right] = GMk - Q\zeta_n k - Q\eta_n \frac{k}{u}\frac{du}{d\phi} \tag{13}$$

$$\frac{dk}{d\phi} = -2Q\eta_n \frac{k^2}{u} \tag{14}$$

$$\frac{d^2\psi}{d\phi^2} - \frac{1}{2k}\frac{dk}{d\phi}\frac{d\psi}{d\phi} + \psi = \frac{k}{u} K\xi_n \tag{15}$$

It is convenient to make these equations dimensionless. The initial distance to the sun r_0 is the only unit of length that is available. The two accelerations GM/r_0^2 and Q/r_0^2 give two time units, namely

$$T_1 = \frac{(r_0^3)^{1/2}}{(GM)^{1/2}} \qquad \text{and} \qquad T_2 = \frac{(r_0^3)^{1/2}}{Q^{1/2}}$$

We have $T_1 = 1/2\pi\,(r_0/a_0)^{3/2} \cdot P$ where a_0 is the semi-major axis, and P the period of the ellipse which the spacecraft described before the thrust was turned on. At least a time T_2 is necessary for the thrust to bring about an appreciable change of orbit.

It will be assumed that the thrust is small compared with the gravitational force, i.e., $T_1 << T_2$. In the beginning of the flight, we have $t << T_2$; the thrust does not yet have a large effect. Eventually, we have $t >> T_2$ and the thrust has changed the orbit drastically.

The occurrence of two time-scales is of great significance. It is the fundamental cause which makes the perturbation problem under consideration (the thrust is regarded as a small perturbation) singular. This means that it will be impossible to approximate the trajectory by an asymptotic series uniformly accurate for all times. The two-variable method has been devised to increase the time-span for which the asymptotic series is a good approximation.

When dimensionless variables $u^* = ur_0$, $t^* = t/T_1$ are introduced, and the asterisks omitted, the equations become

$$\frac{d^2u}{d\phi^2} + u\left[1 + \left(\frac{d\psi}{d\phi}\right)^2\right] = k - \varepsilon k\left(\zeta_n + \eta_n \frac{1}{u}\frac{du}{d\phi}\right) \tag{16}$$

$$\frac{dk}{d\phi} = -2\varepsilon\eta_n \frac{k^2}{u} \tag{17}$$

$$\frac{d^2\psi}{d\phi^2} - \frac{1}{2k}\frac{dk}{d\phi}\frac{d\psi}{d\phi} + \psi = \varepsilon\xi_n \frac{k}{u} \tag{18}$$

where

$$\varepsilon = \frac{Q}{GM} << 1$$

These equations are difficult to solve even if only an approximation to the solution is sought by substituting an asymptotic series. After the substitution, one is still left with nonlinear equations because of the term $(d\psi/d\phi)^2$ in Eq. (16).

However, for elliptic trajectories close to the ecliptic plane, the term $(d\psi/d\phi)^2$ will be small at all times; therefore this term is neglected in Eq. (16) so that the equations become

$$\frac{d^2u}{d\phi^2} + u = k - \varepsilon k \left[\zeta_n + \eta_n \frac{1}{u}\frac{du}{d\phi} \right] \tag{19}$$

$$\frac{dk}{d\phi} = -2\varepsilon\eta_n \frac{k^2}{u} \tag{20}$$

$$\frac{d^2\psi}{d\phi^2} - \frac{1}{2k}\frac{dk}{d\phi}\frac{d\psi}{d\phi} + \psi = \varepsilon\xi_n \frac{k}{u} \tag{21}$$

Equations (19) and (20) can now be solved independently from (21). After k has been found, Eq. (21) can be integrated. The solution of Eqs. (19) and (20) gives the "projection" of the trajectory on the ecliptic plane. Equation (21) gives, essentially, the distance above the ecliptic plane.

First, an approximate solution for Eqs. (19) and (20) will be derived (Sections III and IV). Then Eq. (21) will be integrated (Section V).

III. An Initially Valid Asymptotic Series for the Ecliptic Projection of the Trajectory

A straightforward way to obtain an approximate solution to Eqs. (19) and (20) for small ε is to substitute the following asymptotic series:

$$\left.\begin{aligned} u &= U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots \\ k &= K^{(0)} + \varepsilon K^{(1)} + \varepsilon^2 K^{(2)} + \cdots \end{aligned}\right\} \tag{22}$$

However, this procedure results in a series which can furnish a good approximation to the exact solution only for small times. This will now be demonstrated by studying, as an example, the case in which the spacecraft at the initial instant when the thrust is turned on describes a circle around the Sun in the ecliptic plane, and $\xi_0 = \zeta_0 = 0, \eta_0 = 1$. Equations (19) and (20) specialize to

$$\frac{d^2u}{d\phi^2} + u = k - \varepsilon \frac{k}{u}\frac{du}{d\phi} \tag{23}$$

$$\frac{dk}{d\phi} = -2\varepsilon \frac{k^2}{u} \tag{24}$$

Because Eq. (22) must be a solution for arbitrary (small) values of ε, and different powers of ε are linearly independent, terms with like powers of ε must satisfy Eqs. (23) and (24) by themselves. Before terms with like powers of ε can be collected, the following Taylor series must be substituted for $1/u$:

$$\frac{1}{u} = \frac{1}{U^{(0)}\left[1 + \varepsilon U^{(1)}/U^{(0)} + \varepsilon^2 U^{(2)}/U^{(0)} + O(\varepsilon^3)\right]} = \frac{1}{U^{(0)}}\left[1 - \varepsilon \frac{U^{(1)}}{U^{(0)}} + \varepsilon^2 \left(\frac{U^{(1)2}}{U^{(0)2}} - \frac{U^{(2)}}{U^{(0)}}\right) + O(\varepsilon^3)\right]$$

For the first three terms of Eq. (22), the following linear equations are obtained (collecting terms proportional to ε^0, ε^1 and ε^2, respectively):

$$\left.\begin{aligned} \frac{d^2U^{(0)}}{d\phi^2} + U^{(0)} &= K^{(0)} \\ \frac{dK^{(0)}}{d\phi} &= 0 \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} \frac{d^2U^{(1)}}{d\phi^2} + U^{(1)} &= K^{(1)} - \frac{K^{(0)}}{U^{(0)}}\frac{dU^{(0)}}{d\phi} \\ \frac{dK^{(1)}}{d\phi} &= -2\frac{K^{(0)2}}{U^{(0)}} \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} \frac{d^2U^{(2)}}{d\phi^2} + U^{(2)} &= K^{(2)} - \frac{K^{(1)}}{U^{(0)}}\frac{dU^{(0)}}{d\phi} - \frac{K^{(0)}}{U^{(0)}}\frac{dU^{(1)}}{d\phi} + \frac{K^{(0)}U^{(1)}}{U^{(0)2}}\frac{dU^{(0)}}{d\phi} \\ \frac{dK^{(2)}}{d\phi} &= -4\frac{K^{(0)}K^{(1)}}{U^{(0)}} + 2\frac{K^{(0)2}U^{(1)}}{U^{(0)2}} \end{aligned}\right\} \tag{27}$$

The units were chosen such that, initially, $u = 1$. If we assume the spacecraft is initially in a circular orbit, then $k = 1$ and $du/d\phi = 0$ at $\phi = 0$. When we remember that the initial conditions must be satisfied for arbitrary values of ε, substitution of Eq. (22) gives, at $\phi = 0$:

$$\left.\begin{aligned} &U^{(0)} = K^{(0)} = 1 \\ &U^{(n)} = K^{(n)} = 0 \qquad (n = 1, 2, 3, \cdots) \\ &\frac{dU^{(n)}}{d\phi} = 0 \qquad (n = 0, 1, 2, \cdots) \end{aligned}\right\} \tag{28}$$

Equations (25–27) with initial conditions (Eq. 28) areeasily solved. The final result is

$$\left.\begin{aligned} &u = 1 + \varepsilon\, 2(\sin\phi - \phi) + \varepsilon^2 \{2(1 - \cos\phi) + 2\phi^2 - 3\phi \sin\phi\} + O(\varepsilon^3) \\ &k = 1 - \varepsilon\, 2\phi + \varepsilon^2 \{4(1 - \cos\phi) + 2\phi^2\} + O(\varepsilon^3) \end{aligned}\right\} \tag{29}$$

The appearance of terms proportional to ϕ (secular terms) is worthy of note. For $\phi \cong 1/\varepsilon$, successive terms become of comparable magnitude; the series no longer has the fundamental property of asymptotic series, namely

$$\lim_{\varepsilon \to 0} \frac{U^{(n+1)}}{U^{(n)}} = 0$$

Hence, the applicability of Eq. (29) is restricted to $\phi << 1/\varepsilon$.

A comparison with more exact numerical results obtained with the integrating program of Melbourne and Sauer (Ref. 2) is shown in Figs. 2 and 3. As expected, the initially valid series is not a good approximation for large values of ϕ.

As can be seen from these figures, the series obtained with the two-variable technique has a much larger region of applicability. The two-variable method will be described and applied to the problem under consideration in the following section.

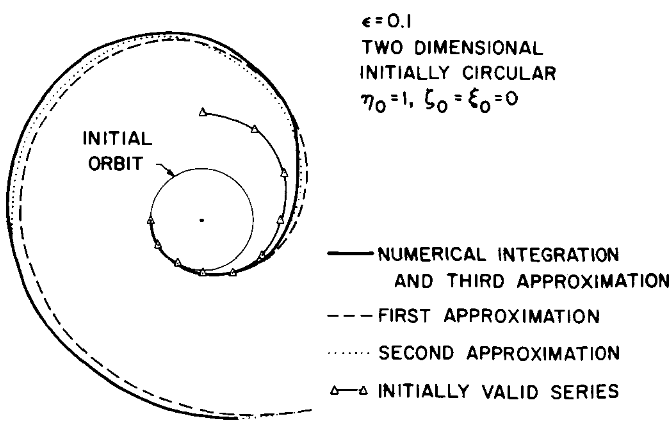

Fig. 2. Comparison of approximate method with numerical integration; initial eccentricity = 0; ε = 0.1

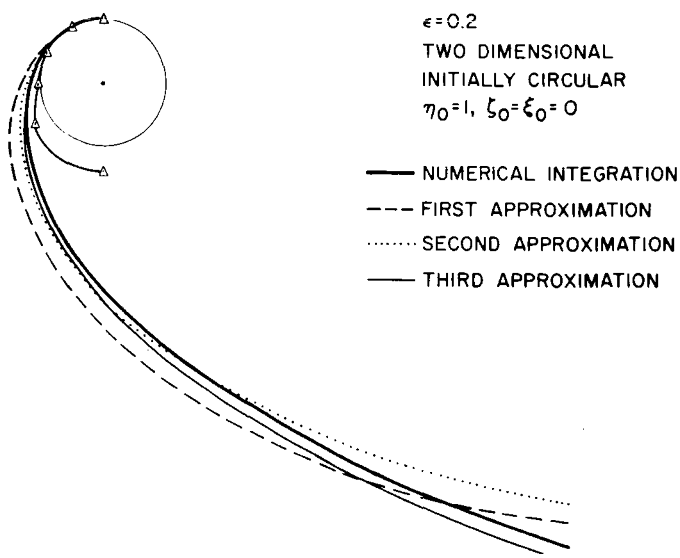

Fig. 3. Comparison of approximate method with numerical integration; initial eccentricity = 0; ε = 0.2

IV. A Two-Variable Asymptotic Series for the Ecliptic Projection of the Trajectory

The term *ecliptic projection* is used in the sense of Fig. 1.

The breakdown of the initially valid series for large values of ϕ demonstrates that the perturbation problem under consideration is singular. (For a good treatment of singular perturbation problems see Ref. 4.) Physically, this breakdown is caused by the fact that the thrust, even as $\varepsilon \to 0$, will bring about large effects if ϕ is large enough.

As shown in Section II, the attracting body brings about changes in the spacecraft's position in a characteristic time T_1; the thrust brings about changes in a characteristic time $T_2 >> T_1$. Prof. J. D. Cole of Caltech proposed* to regard variables that vary appreciably in a time T_1 as functions of a fast variable $\phi^+ = (1 + \omega_2 \varepsilon^2 + \omega_3 \varepsilon^3 + \cdots)\phi$ (the Poincaré-variable); and variables that change with a characteristic time T_2 (like the elements of the osculating conic) as functions of a slow variable $\tilde{\phi} = \varepsilon(1 + a\varepsilon + b\varepsilon^2 + \cdots)\phi$. For many differential equations of the nonlinear oscillator type, this procedure gives a uniformly valid asymptotic series. In the present case, complete uniformity cannot be obtained; but the domain of validity is larger than for the initially valid series.

Before the two-variable method is applied, the problem will be reformulated using an exact solution.

The reader may convince himself that Eqs. (19) and (20) have the following exact solution:

$$\left.\begin{aligned} u &= A_n e^{-\beta_n \phi} \\ k &= \frac{\beta_n}{2\varepsilon\eta_n} A_n e^{-\beta_n \phi} \end{aligned}\right\} \qquad (30)$$

where

$$\beta_n = \frac{1}{2\varepsilon\eta_n}\left[1 - \varepsilon\zeta_n - ((1 - \varepsilon\zeta_n)^2 - 8\varepsilon^2\eta_n^2)^{1/2}\right]$$

This exact solution breaks down when ε is so large that $(1 - \varepsilon\zeta_n)^2 - 8\varepsilon^2\eta_n^2 < 0$. Requiring that u be continuous at the points $\phi = \phi_n$ gives

$$A_{n+1} = A_n e^{(\beta_{n+1} - \beta_n)\phi_{n+1}} \qquad (31)$$

*See discussion by Kevorkian in Ref. 3.

With this choice of A_n the exact solution for k is discontinuous:

$$k(\phi_{n+1} + 0) - k(\phi_{n+1} - 0) = A_n e^{-\beta_n \phi_{n+1}}\left(\frac{\beta_{n+1}}{2\varepsilon\eta_{n+1}} - \frac{\beta_n}{2\varepsilon\eta_n}\right)$$

β_n has the following Taylor series expansion:

$$\begin{aligned} \beta_n = {} & 2\varepsilon\eta_n + 2\varepsilon^2\eta_n\zeta_n + \varepsilon^3(4\eta_n^3 + 2\eta_n\zeta_n^2) \\ & + \varepsilon^4(12\eta_n^3\zeta_n + 2\eta_n\zeta_n^3) + \varepsilon^5(2\eta_n\zeta_n^4 \\ & + 24\eta_n^3\zeta_n^2 + 16\eta_n^5) + O(\varepsilon^6) \end{aligned} \qquad (32)$$

Hence

$$\frac{\beta_{n+1}}{2\varepsilon\eta_{n+1}} - \frac{\beta_n}{2\varepsilon\eta_n} = \varepsilon(\zeta_{n+1} - \zeta_n) + O(\varepsilon^2)$$

It follows that the discontinuity of the exact solution for k at the points $\phi = \phi_n$ is of order ε.

The exact solution gives at an arbitrary point $\phi = \phi_a$

$$\frac{du}{d\phi} = -\beta_n A_n e^{-\beta_n \phi_a}$$

The osculating conic corresponding to the exact solution at $\phi = \phi_a$ is

$$u = \frac{\beta_n}{2\varepsilon\eta_n} A_n e^{-\beta_n\phi_a}\left[1 - \frac{2\varepsilon\eta_n}{\cos\alpha}\sin(\phi - \phi_a + \alpha)\right] \qquad (33)$$

where α is given by

$$\tan\alpha = \frac{1}{2\varepsilon\eta_n} - \frac{1}{\beta_n} = \frac{\zeta_n}{\eta_n} + \varepsilon\eta_n + O(\varepsilon^2)$$

Hence, $\cos\alpha = O(1)$, so that the eccentricity of the osculating conic is $O(\varepsilon)$.

We will restrict ourselves to trajectories which are more general than the exact solution, but have eccentricities of $O(\varepsilon)$. It turns out that this restriction on the eccentricity of the osculating conic is not a severe limitation. A comparison of the present approximate theory with that of numerical results (Section VII) shows that the approximate theory is still accurate for eccentricities higher than 0.5, for $\varepsilon = 0.1$.

To include more general trajectories with eccentricities of $O(\varepsilon)$, an $O(\varepsilon)$ correction must be added to the exact solution. Also, this correction makes it possible to correct the $O(\varepsilon)$ discontinuities in the exact solution for k. We write

$$\left.\begin{aligned} u &= A_n e^{-\beta_n \phi}(1+\varepsilon U) \\ k &= \frac{\beta_n}{2\varepsilon\eta_n} A_n e^{-\beta_n \phi}(1+\varepsilon K) \end{aligned}\right\} (\phi_n < \phi < \phi_{n+1}) \quad (34)$$

where A_n is given by Eq. (31).

Substitution of Eq. (34) in (19) and (20) results in

$$\begin{aligned} \frac{d^2U}{d\phi^2} - 2\beta_n \frac{dU}{d\phi} + (1+\beta_n^2)\,U + \frac{1}{\varepsilon}\left(1+\beta_n^2 - \frac{\beta_n}{2\varepsilon\eta_n}\right) &= \frac{\beta_n}{2\varepsilon\eta_n} K \\ - \frac{\beta_n}{2}\frac{1+\varepsilon K}{1+\varepsilon U}\left[\frac{dU}{d\phi} - \frac{\beta_n}{\varepsilon}(1+\varepsilon U)\right] &- \frac{\beta_n \zeta_n}{2\varepsilon\eta_n}\varepsilon K - \frac{\beta_n \zeta_n}{2\varepsilon\eta_n} \end{aligned} \quad (35)$$

$$\frac{dK}{d\phi} - \beta_n K - \frac{\beta_n}{\varepsilon} = - \frac{\beta_n}{\varepsilon}\frac{(1+\varepsilon K)^2}{1+\varepsilon U} \quad (36)$$

The Taylor series for β_n gives

$$\begin{aligned} \frac{1}{\varepsilon}\left(1+\beta_n^2 - \frac{\beta_n}{2\varepsilon\eta_n}\right) = &- \zeta_n + \varepsilon(2\eta_n^2 - \zeta_n^2) + \varepsilon^2(2\eta_n^2\zeta_n - \zeta_n^3) \\ &+ \varepsilon^3(8\eta_n^4 - \zeta_n^4) + O(\varepsilon^4) \end{aligned} \quad (37)$$

Equations (35) and (36) will now be solved asymptotically with the two-variable method (Ref. 3).

Two new variables are introduced:

$$\left.\begin{aligned} &\text{The fast variable}: \phi^+ = (1 + \omega_2\varepsilon^2 + \omega_3\varepsilon^3 + \cdots)\,\phi \\ &\text{The slow variable}: \tilde{\phi} = \varepsilon(1 + a\varepsilon + b\varepsilon^2 + \cdots)\,\phi \end{aligned}\right\} \quad (38)$$

The constants ω_2, a, b, etc., are chosen so that terms which cause nonuniformity are eliminated. It turns out that in the present case this is not possible, so that there is a certain arbitrariness in the selection of ω_2, a, b, etc., and a completely uniform result cannot be obtained. The problem contains already a slow variable, namely $\beta_n\phi$. It seems natural therefore to define $\tilde{\phi}$ as

$$\tilde{\phi} = \frac{\beta_n}{2\eta_n}\phi = [\varepsilon + \varepsilon^2\zeta_n + \varepsilon^3(2\eta_n^2 + \zeta_n^2) + \varepsilon^4(6\eta_n^2\zeta_n + \zeta_n^3) + \varepsilon^5(8\eta_n^4 + 12\eta_n^2\zeta_n^2 + \zeta_n^4) + O(\varepsilon^6)]\,\phi \quad (39)$$

We also define

$$\phi^+ = \phi \quad (40)$$

These two variables are now regarded as independent, so that Eqs. (35) and (36) become partial differential equations. The rule for differentiation is

$$\frac{d}{d\phi} = \frac{\partial}{\partial\phi^+} + \frac{\beta_n}{2\eta_n}\frac{\partial}{\partial\tilde{\phi}} \quad (41)$$

Derivatives with respect to ϕ^{+} will be indicated by a subscript 1; with respect to $\widetilde{\phi}$, by a subscript 2. Equations (35) and (36) become

$$
\begin{aligned}
&U_{11} + \frac{2\beta_n}{2\eta_n} U_{12} + \frac{\beta_n^2}{4\eta_n^2} U_{22} - 2\beta_n U_1 - \frac{2\beta_n^2}{2\eta_n} U_2 + (1+\beta_n^2)\, U + \frac{1}{\varepsilon}\left(1 + \beta_n^2 - \frac{\beta_n}{2\varepsilon\eta_n}\right) \\
&= \frac{\beta_n}{2\varepsilon\eta_n} K - \frac{\beta_n}{2}\,\frac{1+\varepsilon K}{1+\varepsilon U}\left[U_1 + \frac{\beta_n}{2\eta_n} U_2 - \frac{\beta_n}{\varepsilon}(1+\varepsilon U)\right] - \frac{\beta_n\zeta_n}{2\varepsilon\eta_n}\,\varepsilon K - \frac{\beta_n\zeta_n}{2\varepsilon\eta_n}
\end{aligned}
\tag{42}
$$

$$
K_1 + \frac{\beta_n}{2\eta_n} K_2 - \beta_n K - \frac{\beta_n}{\varepsilon} = -\frac{\beta_n}{\varepsilon}\,\frac{(1+\varepsilon K)^2}{1+\varepsilon U}
\tag{43}
$$

The following asymptotic series are substituted in these equations:

$$
\left.
\begin{aligned}
U(\phi,\widetilde{\phi};\varepsilon) &= U^{(0)}(\phi,\widetilde{\phi}) + \varepsilon U^{(1)}(\phi,\widetilde{\phi}) + \varepsilon^2 U^{(2)}(\phi,\widetilde{\phi}) + \cdots \\
K(\phi,\widetilde{\phi};\varepsilon) &= K^{(0)}(\phi,\widetilde{\phi}) + \varepsilon K^{(1)}(\phi,\widetilde{\phi}) + \varepsilon^2 K^{(2)}(\phi,\widetilde{\phi}) + \cdots
\end{aligned}
\right\}
\tag{44}
$$

Substitution of Eq. (44) in Eqs. (42) and (43), insertion of the Taylor series for β_n, and collection of terms with like powers of ε gives the following equations:

$$
\left.
\begin{aligned}
U_{11}^{(0)} + U^{(0)} &= K^{(0)} \\
K_1^{(0)} &= 0
\end{aligned}
\right\}
\tag{45}
$$

$$
\left.
\begin{aligned}
U_{11}^{(1)} - 2U_{12}^{(0)} - 4\eta_n U_1^{(0)} + U^{(1)} &= K^{(1)} - \eta_n U_1^{(0)} \\
K_1^{(1)} + K_2^{(0)} - 2\eta_n K^{(0)} &= -4\eta_n K^{(0)} + 2\eta_n U^{(0)}
\end{aligned}
\right\}
\tag{46}
$$

$$
\left.
\begin{aligned}
&U_{11}^{(2)} + 2U_{12}^{(1)} + 2\zeta_n U_{12}^{(0)} - 4\eta_n(U_1^{(1)} + U_2^{(0)}) - 4\eta_n\zeta_n U_1^{(0)} + U^{(2)} + 4\eta_n^2 U^{(0)} + U_{22}^{(0)} = \\
&K^{(2)} + 2\eta_n^2 K^{(0)} - \eta_n\zeta_n U_1^{(0)} - \eta_n K^{(0)} U_1^{(0)} + 2\eta_n^2 K^{(0)} + \eta_n U^{(0)} U_1^{(0)} - \eta_n U_2^{(0)} - \eta_n U_1^{(1)} \\
&K_1^{(2)} + K_2^{(1)} + \zeta_n K_2^{(0)} - 2\eta_n K^{(1)} - 2\eta_n\zeta_n K^{(0)} = -2\eta_n(U^{(0)2} - U^{(1)} - 2K^{(0)}U^{(0)} + 2K^{(1)} + K^{(0)2}) \\
&- 2\eta_n\zeta_n(2K^{(0)} - U^{(0)})
\end{aligned}
\right\}
\tag{47}
$$

$$
\left.
\begin{aligned}
&U_{11}^{(3)} + 2U_{12}^{(2)} + 2\zeta_n U_{12}^{(1)} + 2(2\eta_n^2 + \zeta_n^2)\, U_{12}^{(1)} + U_{22}^{(1)} + 2\zeta_n U_{22}^{(0)} - 4\eta_n(U_1^{(2)} + U_2^{(1)} + \zeta_n U_2^{(0)}) \\
&- 4\eta_n\zeta_n(U_1^{(1)} + U_2^{(0)}) + (8\eta_n^3 + 4\eta_n\zeta_n^2)\, U_1^{(0)} + U^{(3)} + 4\eta_n^2 U^{(1)} + 8\eta_n^2\zeta_n U^{(0)} + 8\eta_n^4 - \zeta_n^4 = \\
&K^{(3)} + 2\eta_n^2 K^{(1)} + 4\eta_n^2\zeta_n K^{(0)} - \eta_n U_1^{(2)} - \eta_n U_2^{(1)} - (2\zeta_n + K^{(0)})\,\eta_n U_2^{(0)} + 8\eta_n^4 + 6\eta_n^2\zeta_n^2 - \eta_n(K^{(0)} + \zeta_n)\, U_1^{(1)} \\
&+ 4\eta_n^2\zeta_n K^{(0)} + \eta_n U^{(0)} U_1^{(1)} + \eta_n U^{(0)} U_2^{(0)} + \eta_n U_1^{(0)} \{(\zeta_n + K^{(0)})\, U^{(0)} - K^{(1)} - U^{(0)2} + U^{(1)} - \zeta_n K^{(0)} \\
&- 2\eta_n^2 - \zeta_n^2\} + 2\eta_n^2 K^{(1)} \\
&K_1^{(3)} + K_2^{(2)} + \zeta_n K_2^{(1)} + (2\eta_n^2 + \zeta_n^2)\, K_2^{(0)} - 2\eta_n K^{(2)} - 2\eta_n\zeta_n K^{(1)} - (4\eta_n^3 + 2\eta_n\zeta_n^2)\, K^{(0)} - 12\eta_n^3\zeta_n - 2\eta_n\zeta_n^3 = \\
&- 4\eta_n U^{(0)} U^{(1)} + 2\eta_n U^{(2)} - 4\eta_n K^{(0)} U^{(0)2} + 4\eta_n K^{(0)} U^{(1)} - 2\eta_n\zeta_n U^{(0)2} + 2\eta_n\zeta_n U^{(1)} \\
&+ (4\eta^3 + 2\eta_n\zeta_n^2 + 4\eta_n\zeta_n K^{(0)} + 2\eta_n K^{(0)2})\, U^{(0)} + 4\eta_n U^{(0)} K^{(1)} - (12\eta_n^3\zeta_n + 2\eta_n\zeta_n^3 \\
&+ 8\eta_n^3 K^{(0)} + 4\eta_n\zeta_n^2 K^{(0)} + 2\eta_n\zeta_n K^{(0)2}) - 4\eta_n(\zeta_n + K^{(0)})\, K^{(1)} - 4\eta_n K^{(2)} + 2\eta_n U^{(0)}
\end{aligned}
\right\}
\tag{48}
$$

To find the solution of Eqs. (45–48) for $\phi_n < \phi < \phi_{n+1}$, initial conditions for $U^{(0)}, U^{(1)}, \cdots, K^{(0)}, K^{(1)}, \cdots$ at $\phi = \phi_n$ have to be given. These conditions follow from the requirement that u, $du/d\phi$, and k be continuous at $\phi = \phi_n$. Suppose we have at $\phi = \phi_n - 0$

$$\left.\begin{aligned} u\Big|_{\phi=\phi_n-0} &= A_n e^{-\beta_n\phi_n}(1+\varepsilon B_n) \\ \frac{du}{d\phi}\Bigg|_{\phi=\phi_n-0} &= A_n e^{-\beta_n\phi_n}\varepsilon C_n \\ k\Big|_{\phi=\phi_n-0} &= A_n e^{-\beta_n\phi_n}(1+\varepsilon D_n) \end{aligned}\right\} \tag{49}$$

Continuity of u, k and $du/d\phi$ requires that at $\phi = \phi_n + 0$ (since A_n has been defined such that $A_n e^{-\beta_n\phi_n} = A_{n+1} e^{-\beta_{n+1}\phi_n}$)

$$\left.\begin{aligned} u\Big|_{\phi=\phi_n+0} &= A_{n+1} e^{-\beta_{n+1}\phi_n}(1+\varepsilon B_n) \\ \frac{du}{d\phi}\Bigg|_{\phi=\phi_n+0} &= A_{n+1} e^{-\beta_{n+1}\phi_n}\varepsilon C_n \\ k\Big|_{\phi=\phi_n+0} &= A_{n+1} e^{-\beta_{n+1}\phi_n}(1+\varepsilon D_n) \end{aligned}\right\} \tag{50}$$

From Eqs. (34), (41), and (44) follows

$$\left.\begin{aligned} \frac{du}{d\phi} &= A_n e^{-\beta_n\phi}\,[\varepsilon(U_1^{(0)} - 2\eta_n) + \varepsilon^2(U_1^{(1)} + U_2^{(0)} - 2\eta_n U^{(0)} - 2\eta_n\zeta_n) \\ &\quad + \varepsilon^3(U_1^{(2)} + \zeta_n U_2^{(0)} + U_2^{(1)} - 2\eta_n U^{(1)} - 2\eta_n\zeta_n U^{(0)} - 4\eta_n^3 - 2\eta_n\zeta_n^2) + O(\varepsilon^4)] \\ u &= A_n e^{-\beta_n\phi}\,[1 + \varepsilon U^{(0)} + \varepsilon^2 U^{(1)} + \varepsilon^3 U^{(2)} + O(\varepsilon^4)] \\ k &= A_n e^{-\beta_n\phi}\,[1 + \varepsilon(K^{(0)} + \zeta_n) + \varepsilon^2(K^{(1)} + \zeta_n K^{(0)} + 2\eta_n^2 + \zeta_n^2) \\ &\quad + \varepsilon^3\{K^{(2)} + \zeta_n K^{(1)} + (2\eta_n^2 + \zeta_n^2)K^{(0)} + 6\eta_n^2\zeta_n + \zeta_n^3\} + O(\varepsilon^4)] \end{aligned}\right\} \tag{51}$$

Hence

$$\left.\begin{aligned} B_n &= [U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots]\Big|_{\phi=\phi_n} \\ C_n &= [U_1^{(0)} - 2\eta_n + \varepsilon(U_1^{(1)} + U_2^{(0)} - 2\eta_n U^{(0)} - 2\eta_n\zeta_n) \\ &\quad + \varepsilon^2(U_1^{(2)} + \zeta_n U_2^{(0)} + U_2^{(1)} - 2\eta_n U^{(1)} - 2\eta_n\zeta_n U^{(0)} - 4\eta_n^3 - 2\eta_n\zeta_n^2)]\Big|_{\phi=\phi_n} \\ D_n &= [K^{(0)} + \zeta_n + \varepsilon(K^{(1)} + \zeta_n K^{(0)} + 2\eta_n^2 + \zeta_n^2) \\ &\quad + \varepsilon^2(K^{(2)} + \zeta_n K^{(1)} + 2\eta_n^2 K^{(0)} + \zeta_n^2 K^{(0)} + 6\eta_n^2\zeta_n + \zeta_n^3)]\Big|_{\phi=\phi_n} \end{aligned}\right\} \tag{52}$$

Froms Eqs. (50) and (51) follow the conditions necessary to obtain a unique solution of Eqs. (45–48):

$$\left.\begin{aligned}
&U^{(0)}\Big|_{\phi=\phi_n} = B_n \\
&U^{(1)}\Big|_{\phi=\phi_n} = 0 \\
&U^{(2)}\Big|_{\phi=\phi_n} = 0 \\
&U_1^{(0)}\Big|_{\phi=\phi_n} = 2\eta_n + C_n \\
&U_1^{(1)}\Big|_{\phi=\phi_n} = -\,U_2^{(0)}\Big|_{\phi=\phi_n} + 2\eta_n B_n + 2\eta_n \zeta_n \\
&U_1^{(2)}\Big|_{\phi=\phi_n} = -\,\zeta_n U_2^{(0)}\Big|_{\phi=\phi_n} - U_2^{(1)}\Big|_{\phi=\phi_n} + 2\eta_n \zeta_n B_n + 4\eta_n^3 + 2\eta_n \zeta_n^2 \\
&K^{(0)}\Big|_{\phi=\phi_n} = D_n - \zeta_n \\
&K^{(1)}\Big|_{\phi=\phi_n} = -\,\zeta_n D_n - 2\eta_n^2 \\
&K^{(2)}\Big|_{\phi=\phi_n} = -\,2\eta_n^2 D_n - 2\eta_n^2 \zeta_n
\end{aligned}\right\} \tag{53}$$

With Eq. (53), the solutions of Eqs. (45–48) can be obtained at the cost of much labor but in a straightforward way. To bring out an important feature of the two-variable technique, the application of the so-called "boundedness condition", the derivation of $U^{(0)}$ and $K^{(0)}$ will be given. The derivations of $U^{(1)}$, $U^{(2)}$, $K^{(1)}$, $K^{(2)}$ are quite similar and are not reproduced here, but the final results are given.

Equation (45) gives

$$\left.\begin{aligned}
&U^{(0)}(\phi, \tilde{\phi}) = f^{(0)}(\tilde{\phi})\,[1 + e^{(0)}(\tilde{\phi}) \cos\{\phi - \omega^{(0)}(\tilde{\phi})\}] \\
&K^{(0)}(\phi, \tilde{\phi}) = f^{(0)}(\tilde{\phi})
\end{aligned}\right\} \tag{54}$$

The functions $f^{(0)}(\tilde{\phi}), e^{(0)}(\tilde{\phi})$ and $\omega^{(0)}(\tilde{\phi})$ are still undetermined, but follow from the boundedness condition applied to $U^{(1)}$ and $K^{(1)}$. With Eq. (54), Eq. (46) gives

$$U_{11}^{(1)} + U^{(1)} = 2\frac{df^{(0)} e^{(0)}}{d\tilde{\phi}} \sin(\phi - \omega^{(0)}) - 2f^{(0)} e^{(0)} \frac{d\omega^{(0)}}{d\tilde{\phi}} \cos(\phi - \omega^{(0)}) - \eta_n f^{(0)} e^{(0)} \sin(\phi - \omega^{(0)}) - \frac{df^{(0)}}{d\tilde{\phi}}\,\phi \tag{55}$$

$$K^{(1)} = -\frac{df^{(0)}}{d\tilde{\phi}}\,\phi + 2\eta_n f^{(0)} e^{(0)} \sin(\phi - \omega^{(0)}) + f^{(1)}(\tilde{\phi}) \tag{56}$$

The terms proportional to $\sin(\phi - \omega^{(0)})$ and $\cos(\phi - \omega^{(0)})$ in Eq. (55) generate in the solution for $U^{(1)}$ terms like $\phi\cos(\phi - \omega^{(0)})$ and $\phi\sin(\phi - \omega^{(0)})$; i.e., secular terms which are undesirable appear. Also, $U^{(1)}$ will contain a term proportional to ϕ^2, and $K^{(1)}$ contains a term proportional to ϕ. Because of these secular terms, the validity of the initially valid series is restricted to small values of ϕ. However, with the two-variable method, the undesirable terms can be eliminated by a judicious choice of the as-yet-undetermined functions $f^{(0)}(\tilde{\phi}), e^{(0)}(\tilde{\phi})$ and $\omega^{(0)}(\tilde{\phi})$.

The physical reason for the elimination of the secular terms is that the elements of the osculating conic change under influence of the thrust only and are therefore functions only of the slow variable $\tilde{\phi}$. Hence $U^{(1)}$ and $K^{(1)}$ may not be proportional to ϕ. This is called the "boundedness condition." Its application gives

$$\left.\begin{aligned} &\frac{df^{(0)}}{d\tilde{\phi}} = 0 \\ &2\frac{df^{(0)}e^{(0)}}{d\tilde{\phi}} - \eta_n f^{(0)}e^{(0)} = 0 \\ &\frac{d\omega^{(0)}}{d\tilde{\phi}} = 0 \end{aligned}\right\} \tag{57}$$

Hence

$$\begin{aligned} f^{(0)} &= \text{constant} \\ f^{(0)}e^{(0)} &= \text{constant}\; e^{\frac{1}{2}\eta_n(\tilde{\phi}-\tilde{\phi}_n)} \\ \omega^{(0)} &= \text{constant} \end{aligned}$$

Using Eqs. (53) and (54), we obtain

$$\begin{aligned} K^{(0)} &= D_n - \zeta_n \\ U^{(0)} &= D_n - \zeta_n + e^{\frac{1}{2}\eta_n(\tilde{\phi}-\tilde{\phi}_n)}\left[(B_n + \zeta_n - D_n)\cos(\phi - \phi_n) + (2\eta_n + C_n)\sin(\phi - \phi_n)\right] \end{aligned}$$

In the same way, $U^{(1)}$, $K^{(1)}$, $U^{(2)}$, $K^{(2)}$ can be found. To determine $U^{(2)}$, the boundedness condition has to be applied to $U^{(3)}$. This is why the equation for $U^{(3)}$ has been written out (Eq. 48). The complete results are given in the following pages. The approximate formulas for u and k completely determine the trajectory. The accuracy is evaluated in Section VI.

The time as a function of ϕ can be determined as follows: since

$$(k)^{\frac{1}{2}} = u^2\frac{dt}{d\phi}$$

then

$$t = \int_0^{\phi}\frac{(k)^{\frac{1}{2}}}{u^2}\,d\phi \tag{58}$$

This integral can be integrated either numerically or analytically by replacing $1/u^2\,(k)^{\frac{1}{2}}$ by its asymptotic series. The analytic integration of the series for $1/u^2\,(k)^{\frac{1}{2}}$, although straightforward, is rather cumbersome. However, it turns out that numerical integration of Eq. (58) with a step as large as 30° using the trapezium rule (approximating $1/u^2\,(k)^{\frac{1}{2}}$ by the average of the values at the beginning and the end of the interval) is sufficiently accurate. When actual missions are studied, the thrust acceleration will have to be readjusted periodically at the points $\phi = \phi_n$, as has been explained before. The size of the step for the numerical integration of Eq. (58) can be made to correspond with these periodic readjustments. This procedure would save computer time.

The final results for the ecliptic projection of the trajectory are

$$[\phi_n < \phi < \phi_{n+1}]$$

$$u = A_n e^{-2\eta_n \tilde{\phi}} [1 + \varepsilon U^{(0)} + \varepsilon^2 U^{(1)} + \varepsilon^3 U^{(2)}] \tag{59}$$

$$k = \frac{\beta_n}{2\varepsilon\eta_n} A_n e^{-2\eta_n \tilde{\phi}} [1 + \varepsilon K^{(0)} + \varepsilon^2 K^{(1)} + \varepsilon^3 K^{(2)}] \tag{60}$$

$$t = \int_0^{\phi} \frac{(k)^{1/2}}{u^2} d\phi$$

$$\tilde{\phi} = \frac{\beta_n}{2\eta_n} \phi$$

$$A_n = A_{n-1} e^{(\beta_n - \beta_{n-1})\phi_n}, \qquad A_0 = 1$$

$$\beta_n = \frac{1}{2\varepsilon\eta_n} [1 - \varepsilon\zeta_n - \{(1 - \varepsilon\zeta)^2 - 8\varepsilon^2\eta_n^2\}^{1/2}] \tag{61}$$

$$U^{(0)} = D_n - \zeta_n + e^{1/2\eta_n(\tilde{\phi} - \tilde{\phi}_n)} [(B_n - D_n + \zeta_n) \cos(\phi - \phi_n) + (2\eta_n + C_n) \sin(\phi - \phi_n)] \tag{62}$$

$$\begin{aligned} U^{(1)} = {} & 6\eta_n^2 + 6\eta_n C_n + (B_n - D_n + \zeta_n)^2 + C_n^2 - D_n\zeta_n - e^{\eta_n(\tilde{\phi} - \tilde{\phi}_n)} [(B_n - D_n + \zeta_n)^2 + (2\eta_n + C_n)^2] \\ & + e^{1/2\eta_n(\tilde{\phi} - \tilde{\phi}_n)} \Big[(-2C_n\eta_n + D_n\zeta_n - 2\eta_n^2) \cos(\phi - \phi_n) + \left(\frac{3}{2} B_n\eta_n + \frac{1}{2} D_n\eta_n + \frac{3}{2} \eta_n\zeta_n\right) \sin(\phi - \phi_n) \\ & - \frac{9}{8} \eta_n^2 (2\eta_n + C_n)(\tilde{\phi} - \phi_n) \cos(\phi - \phi_n) + \frac{9}{8} \eta_n^2 (B_n - D_n + \zeta_n)(\tilde{\phi} - \tilde{\phi}_n) \sin(\phi - \phi_n) \Big] \end{aligned} \tag{63}$$

$$\begin{aligned} U^{(2)} = {} & -4B_n\eta_n^2 + 6D_n\eta_n^2 - B_nC_n\eta_n + C_nD_n\eta_n + C_n\eta_n\zeta_n - 2\eta_n^2\zeta_n + (\phi - \tilde{\phi}_n) 2\eta_n [(D_n - \zeta_n) - (D_n - \zeta_n)^3] \\ & + (A) [1 - e^{\eta_n(\tilde{\phi} - \tilde{\phi}_n)}] + \frac{1}{3} \eta_n e^{\eta_n(\tilde{\phi} - \tilde{\phi}_n)} \{[(B_n - D_n + \zeta_n)^2 - (2\eta_n + C_n)^2] \sin 2(\phi - \phi_n) \\ & - 2(2\eta_n + C_n)(B_n - D_n + \zeta_n) \cos 2(\phi - \phi_n)\} \\ & + e^{1/2\eta_n(\tilde{\phi} - \tilde{\phi}_n)} \sin(\phi - \phi_n) \Big\{ \frac{1}{3} B_n^2\eta_n + \frac{5}{3} C_n^2\eta_n + \frac{1}{3} D_n^2\eta_n - \frac{2}{3} B_nD_n\eta_n + \frac{13}{6} B_n\eta_n\zeta_n - \frac{2}{3} D_n\eta_n\zeta_n \\ & + \frac{211}{24} C_n\eta_n^2 + \frac{167}{12} \eta_n^3 + \frac{11}{6} \eta_n\zeta_n^2 + (\phi - \tilde{\phi}_n) \Big[-\frac{81}{32} C_n\eta_n^3 - \frac{45}{16} \eta_n^4 + \frac{9}{8} \eta_n^2\zeta_n (B_n + \zeta_n) \\ & + 3\eta (D_n - \zeta_n)^2 (2\eta_n + C_n) - \eta_n (2\eta_n + C_n) \Big] + (\tilde{\phi} - \tilde{\phi}_n)^2 \frac{81}{128} \eta_n^4 (2\eta_n + C_n) \Big\} \\ & + e^{1/2\eta_n(\tilde{\phi} - \tilde{\phi}_n)} \cos(\phi - \phi_n) \Big\{ \frac{5}{3} C_n\eta_n (B_n - D_n) + \frac{16}{3} B_n\eta_n^2 - \frac{22}{3} D_n\eta_n^2 \\ & - \frac{1}{3} C_n\eta_n\zeta_n + \frac{10}{3} \eta_n^2\zeta_n + (\tilde{\phi} - \tilde{\phi}_n) \Big[-\frac{63}{32} B_n\eta_n^3 - \frac{9}{32} D_n\eta_n^3 - \frac{9}{8} C_n\eta_n^2\zeta_n - \frac{135}{32} \eta_n^3\zeta_n \\ & + 3\eta_n (D_n - \zeta_n)^2 (B_n - D_n + \zeta_n) - \eta_n (B_n - D_n + \zeta_n) \Big] - (\tilde{\phi} - \tilde{\phi}_n)^2 \frac{81}{128} \eta_n^4 (B_n - D_n + \zeta_n) \Big\} \end{aligned} \tag{64}$$

where

$$
\begin{aligned}
(A) = {} & 2B_n^2(D_n - \zeta_n) - 4B_nD_n^2 + 12C_n^2D_n + 2D_n^3 + B_nC_n\eta_n + 10B_nD_n\zeta_n \\
& - 2C_n^2\zeta_n + 13C_nD_n\eta_n - 8D_n^2\zeta_n + 2B_n(\eta_n^2 - 2\zeta_n^2) - 9C_n\eta_n\zeta_n + 2D_n(7\eta_n^2 + 4\zeta_n^2) - 6\eta_n^2\zeta_n - 2\zeta_n^3
\end{aligned}
$$

$$K^{(0)} = D_n - \zeta_n \tag{65}$$

$$
\begin{aligned}
K^{(1)} = {} & (B_n - D_n + \zeta_n)^2 + C_n^2 + 6C_n\eta_n - D_n\zeta_n + 6\eta_n^2 - e^{\eta_n(\bar{\phi} - \bar{\phi}_n)}\left[(B_n - D_n + \zeta_n)^2 + (2\eta_n + C_n)^2\right] \\
& + e^{\frac{1}{2}\eta_n(\bar{\phi} - \bar{\phi}_n)}\left[-2\eta_n(2\eta_n + C_n)\cos(\phi - \phi_n) + 2\eta_n(B_n - D_n + \zeta_n)\sin(\phi - \phi_n)\right]
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
K^{(2)} = {} & -4B_n\eta_n^2 + 6D_n\eta_n^2 - B_nC_n\eta_n + C_nD_n\eta_n + C_n\eta_n\zeta_n - 2\eta_n^2\zeta_n \\
& + (\bar{\phi} - \bar{\phi}_n)2\eta_n\left[(D_n - \zeta_n) - (D_n - \zeta_n)^3\right] + (A)\left[1 - e^{\eta_n(\bar{\phi} - \bar{\phi}_n)}\right] \\
& - \frac{1}{2}\eta_n e^{\eta_n(\bar{\phi} - \bar{\phi}_n)}\left\{\left[(B_n - D_n + \zeta_n)^2 - (2\eta_n + C_n)^2\right]\sin 2(\phi - \phi_n)\right. \\
& \left. - 2(2\eta_n + C_n)(B_n - D_n + \zeta_n)\cos 2(\phi - \phi_n)\right\} \\
& + e^{\frac{1}{2}\eta(\bar{\phi} - \bar{\phi}_n)}\left\{(2B_n\eta_n^2 - 2C_n\eta_n\zeta_n - 6D_n\eta_n^2 - 2\eta_n^2\zeta_n)\cos(\phi - \phi_n)\right. \\
& + (-2B_n\eta_n\zeta_n + C_n\eta_n^2 + 6\eta_n^3 + 2\eta_n\zeta_n^2)\sin(\phi - \phi_n) \\
& \left. - \frac{9}{4}\eta_n^3(B_n - D_n + \zeta_n)(\bar{\phi} - \bar{\phi}_n)\cos(\phi - \phi_n) - \frac{9}{4}\eta_n^3(2\eta_n + C_n)(\phi - \phi_n)\sin(\phi - \phi_n)\right\}
\end{aligned} \tag{67}
$$

Scrutiny of the successive terms in the asymptotic series for U and K obtained in Eqs. (62–67) reveals that this series is not uniformly valid. The reason for this has already been given following Eq. (38). The domain of validity of the two-variable asymptotic series is given by

$$\phi << \left|\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}\right|$$

This domain is larger than the domain of validity of the initially valid series.

V. An Approximate Formula for the Elevation Above the Ecliptic Plane

The elevation above the ecliptic plane is given by Eq. (21). Substitution of

$$\frac{1}{r} = A_n e^{-\beta_n\phi}(1 + \varepsilon U), \qquad k = \frac{A_n\beta_n}{2\varepsilon\eta}e^{-\beta_n\phi}(1 + \varepsilon K)$$

results in

$$\frac{d^2\psi}{d\phi^2} + \left\{\frac{1}{2}\beta_n - \frac{1}{2}\varepsilon(1 + \varepsilon K)^{-1}\frac{dk}{d\phi}\right\}\frac{d\psi}{d\phi} + \psi = \frac{\beta_n\xi_n}{2\eta_n}\frac{1 + \varepsilon K}{1 + \varepsilon U} \tag{68}$$

Since $K_1^{(0)} = 0$, $dK/d\phi = O(\varepsilon)$. Hence to $O(\varepsilon)$ Eq. (68) is equivalent to

$$\frac{d^2\psi}{d\phi^2} + \frac{1}{2}\beta_n\frac{d\psi}{d\phi} + \psi = \frac{\beta_n\xi_n}{2\eta_n} \tag{69}$$

The solution of this equation is

$$\psi = \psi_a = e^{-\frac{1}{4}\beta_n\phi}\left[\left(1 - \frac{\beta_n^2}{16}\right)^{-\frac{1}{2}} \left\{H_n + \frac{\beta_n}{4}\left(G_n - \frac{\beta_n \xi_n}{2\eta_n}\right)\right\} \sin\left(1 - \frac{\beta_n^2}{16}\right)^{\frac{1}{2}} \phi \right.$$
$$\left. + \left(G_n - \frac{\beta_n\xi_n}{2\eta_n}\right) \cos\left(1 - \frac{\beta_n^2}{16}\right)^{\frac{1}{2}} \phi\right] + \frac{\beta_n\xi_n}{2\eta_n} \qquad (70)$$

where G_n and H_n are constants of integration. Substitution of $\psi = \psi_a + \psi_b$ in Eq. (68) gives the following equation for ψ_b:

$$\frac{d^2\psi_b}{d\phi^2} + \left\{\frac{1}{2}\beta_n - \frac{1}{2}\varepsilon(1 + \varepsilon K)^{-1}\frac{dK}{d\phi}\right\}\frac{d\psi_b}{d\phi} + \psi_b$$
$$= \frac{1}{2}\varepsilon(1 + \varepsilon K)^{-1}\frac{dK}{d\phi}\frac{d\psi_a}{d\phi} + \frac{\beta_n\xi_n}{2\eta_n}\left(\frac{1 + \varepsilon K}{1 + \varepsilon U} - 1\right) \qquad (71)$$

If one lets ψ_a take care of the initial conditions, the initial conditions for ψ_b are

$$\psi_b(\phi_n) = \frac{d\psi_b}{d\phi}(\phi_n) = 0$$

Since Eq. (71) is the equation for a harmonic oscillator with small damping, the two-variable method can be used to obtain an approximation for ψ_b. Let

$$\psi_b = \psi^{(0)}(\phi, \tilde{\phi}) + \varepsilon\psi^{(1)}(\phi, \tilde{\phi}) + \varepsilon^2\psi^{(2)}(\phi, \tilde{\phi}) + \cdots$$

The two-variable method gives the following result:

$$\psi^{(0)} = 0$$
$$\psi^{(1)} = \frac{\xi_n}{\eta_n}\sinh\left(\frac{1}{2}\eta_n\tilde{\Phi}\right)[(D_n - \zeta_n)\sin\Phi + (\eta_n + C_n)\cos\Phi] \qquad (72)$$

where

$$\Phi = \phi - \phi_n$$

It does not seem necessary to derive higher approximations.

Summarizing, the following approximate formula for ψ has been obtained:

$$\psi = \frac{\beta_n\xi_n}{2\eta_n} + e^{-\frac{1}{4}\beta_n\Phi}\left[\left(1 - \frac{\beta_n^2}{16}\right)^{-\frac{1}{2}}\left\{H_n + \frac{\beta_n}{4}\left(G_n - \frac{\beta_n\xi_n}{2\eta_n}\right)\right\}\sin\left(1 - \frac{\beta_n^2}{16}\right)^{\frac{1}{2}}\Phi + \left(G_n - \frac{\beta_n\xi_n}{2\eta_n}\right)\cos\left(1 - \frac{\beta_n^2}{16}\right)^{\frac{1}{2}}\Phi\right]$$
$$+ \varepsilon\frac{\xi_n}{\eta_n}\sinh\left(\frac{1}{4}\beta_n\Phi\right)[(D_n - \zeta_n)\sin\Phi + (\eta_n + C_n)\cos\Phi] \qquad (73)$$

where

$$G_n = \psi(\phi_n), \qquad H_n = \frac{d\psi}{d\phi}(\phi_n), \qquad \Phi = \phi - \phi_n$$

VI. A Procedure to Approximate the True Thrust Acceleration

It has been pointed out already that although the power received from the sun is proportional to $1/r^2$, the thrust force in general does not obey the same proportionality. Suppose we have the dimensionless expression

$$F = \varepsilon m_0 u^p \tag{74}$$

where m_0 is the total mass of the spacecraft at time $t = 0$, and the exponent p is left undetermined. Since $dm/dt = -F/c$ the mass at time t is given by

$$m(t) = m_0 \left[1 - \int_0^t \frac{\varepsilon u^p}{c} dt \right] \tag{75}$$

where c is the dimensionless exhaust velocity.

The true thrust acceleration a_F is given by

$$a_F = \varepsilon u^p \left[1 - \varepsilon \int_0^t \frac{u^p}{c} \right]^{-1} \tag{76}$$

For constant propulsion system control setting, the propulsion system will be a function of the solar distance only. With a fair degree of generality one can assume that $c = \gamma u^q$, where γ is a constant.

With $\quad dt = \frac{1}{\phi} d\phi = \frac{(k)^{1/2}}{u^2} d\phi$

one obtains

$$a_F(\phi) = \varepsilon u^p \left[1 - \frac{\varepsilon}{\gamma} \int_0^\phi u^{p-q-2} (k)^{1/2} d\phi \right]^{-1} \tag{77}$$

The integral in Eq. (77) can be evaluated in the same way as the integral in Eq. (58) for the time t. The appropriate size of the step must be determined by comparison with results obtained by numerical integrating, for instance, with the program of Melbourne and Sauer (Ref. 2) which prints out mass and thrust acceleration as a function of time. The difference between the thrust acceleration given by the approximation method and that given by the Melbourne–Sauer program should be less than ½%. The points $\phi = \phi_n$ are made to correspond with the size of the step.

After each step $\phi_{n-1} \to \phi_n$ in the computation of a_F, the constants ξ_n, η_n and ζ_n must be readjusted for the computation in the next interval $\phi_n \to \phi_{n+1}$. Ideally, we have

$$\varepsilon^2 (\xi^2 + \eta^2 + \zeta^2) \equiv a_F^2 \tag{78}$$

The following formula seems a suitable approximation:

$$\xi_n^2 + \eta_n^2 + \zeta_n^2 = \frac{1}{\varepsilon^2} \left\{ a_F(\phi_n) + \frac{1}{2} [a_F(\phi_n) - a_F(\phi_{n-1})] \right\}^2 \tag{79}$$

The size of the step should also be subject to the requirement that the deviation of $\varepsilon (\xi_n^2 + \eta_n^2 + \zeta_n^2)^{1/2}$ from $a_F(\phi_{n+1})$ be less than ½%.

If the thrust direction is changed, the ratio's ξ_{n/η_n} and ζ_{n/η_n} are changed accordingly. For the purpose of preliminary mission design and optimization, it is appropriate to allow changes in propulsion system control setting and thrust direction only at the points $\phi = \phi_n$. This procedure would save computer time while the work of Melbourne and Sauer indicates that the results are not very sensitive to the details of the thrust program.

VII. Comparison of Approximate Methods With Numerical Integrations

For the case that the thrust acceleration is exactly proportional to $1/r^2$, and propulsion system control setting and thrust direction relative to the spacecraft–sun and spacecraft–Canopus vectors are constant, comparisons have been made with numerical integrations. In this case, no readjustments of the assumed thrust acceleration are necessary; the subscript n assumes the value 0 only. For two-dimensional trajectories, the program developed by Melbourne and Sauer (Ref. 2) for the IBM 7094 computer was used. In the three-dimensional case, this program does not allow the user to prescribe the thrust acceleration. Therefore, a separate program called NUTRAL was written for the IBM 1620 computer which integrates the three-dimensional equations of motion using a Runge-Kutta integration technique. NUTRAL stands for *numerical integration of trajectories of spacecraft with low thrust*.

The results of the present asymptotic theory as given by Eqs. (59–67) and Eq. (73) have been programmed for the IBM 1620 computer. The program is called ASTRAL (*asymptotic series for trajectories of spacecraft with low thrust*).

The following cases were computed:

(1) Two-dimensional, initial orbit circular, $\eta_0 = 1$, $\zeta_0 = 0$, $\varepsilon = 0.1$.

(2) Two-dimensional, initial orbit circular, $\eta_0 = 1$, $\zeta_0 = 0$, $\varepsilon = 0.2$.

(3) Two-dimensional, initial orbit elliptic with eccentricity = 0.2, $\eta_0 = 1$, $\zeta_0 = 0$, $\varepsilon = 0.1$.

(4) Two-dimensional, initial orbit circular, $\eta_0 = -1$, $\zeta_0 = 0$, $\varepsilon = 0.1$.

(5) Three-dimensional, initial orbit circular in the ecliptic plane, $\eta_0 = \frac{1}{2}(3)^{1/2}$, $\zeta_0 = 0$, $\xi_0 = \frac{1}{2}$, $\varepsilon = 0.1$.

In ASTRAL, the time-integral Eq. (61) was computed for all these cases with the trapezium-rule, using a step size of 30°.

For cases (1) and (2), a first, second, and third approximation was computed with the present approximate theory using only the first, the first two, and the first three terms, respectively, in the asymptotic series for U and K. The results are shown in Figs. 2 and 3. For $\varepsilon = 0.1$, the successive approximations approach quite rapidly the result given by numerical integration. The third approximation as programmed in ASTRAL is compared with numerical integration in Table 1. It is noteworthy, in the approximate theory, that the error does not increase steadily as ϕ increases; instead, it oscillates. This is a characteristic feature of the two-variable method. After reaching a maximum of a few percent at $\phi = 398.2°$, the error decreases again to about 1% in r, almost nothing in H_ϕ, and less than 1% in t. From Fig. 3, it is clear that for $\varepsilon = 0.2$ the error becomes intolerably large. The present approximate theory seems to be useful only for ε smaller than about 0.15. In practice, it appears that the value of ε will be about 0.06. For such a low value of ε, the error in the approximate theory should be considerably less even than in Table 1.

The present theory assumes (see Section IV) that the eccentricity of the osculating conic is $O(\varepsilon)$. For case (3), the program of Melbourne and Sauer indicates that during the first revolution the eccentricity of the osculating conic reaches a value of 0.53. Despite this high value of the eccentricity, the error in the approximate theory is small, as shown in Fig. 4 and Table 2.

Table 1. Comparison of third approximation with numerical integration; initial eccentricity = 0

ϕ	r		H_ϕ		t	
	Numerical	Approximate	Numerical	Approximate	Numerical	Approximate
108.4	1.1806	1.1831	1.1815	1.1834	1.9000	1.9065
206.1	2.2591	2.2682	1.3939	1.3975	5.4000	5.4315
316.4	4.0323	4.0246	1.7956	1.7894	16.0000	16.1034
398.2	3.8242	3.6844	1.9977	2.0750	27.0000	30.1624
443.6	3.8599	3.8093	2.2209	2.2153	35.4000	35.2515
462.6	4.0703	4.0218	2.2791	2.2757	37.7000	37.5087
Two-dimensional $\varepsilon = 0.1$; $\eta_0 = 1$; $\zeta_0 = 0$						

Table 2. Comparison of third approximation with numerical integration; initial eccentricity = 0.2

ϕ	r		H_ϕ		t	
	Numerical	Approximate	Numerical	Approximate	Numerical	Approximate
108.7	1.4775	1.4831	1.2779	——	2.1000	2.1143
209.5	3.9358	4.1318	1.5833	1.5664	10.1200	10.7721
314.2	4.3242	4.3092	2.0492	1.9943	31.8000	34.6533
393.5	3.8113	3.6370	2.3000	2.2254	41.7000	44.3093
448.7	4.6717	4.4900	2.4666	2.3829	48.6000	50.8540
492.9	6.8712	6.8632	2.6362	2.5500	58.2000	60.4484
521.1	9.7245	10.3347	2.7819	2.6916	70.2000	74.0477
Two-dimensional $\varepsilon = 0.1$; $\eta = 1$; $\zeta = 0$						

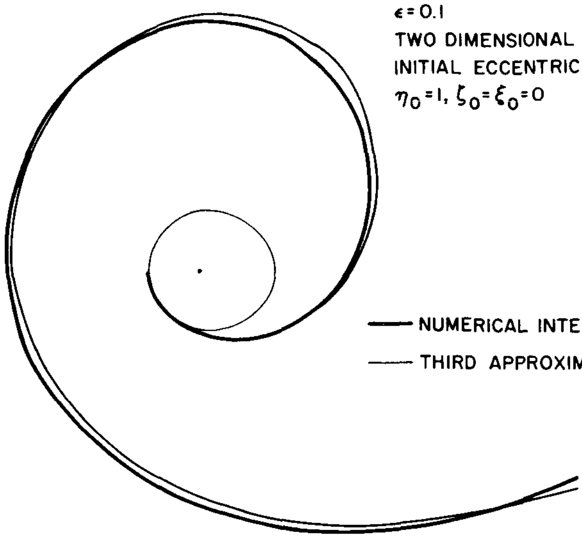

Fig. 4. Comparison of approximate method with numerical integration; initial eccentricity = 0.2; ε = 0.1

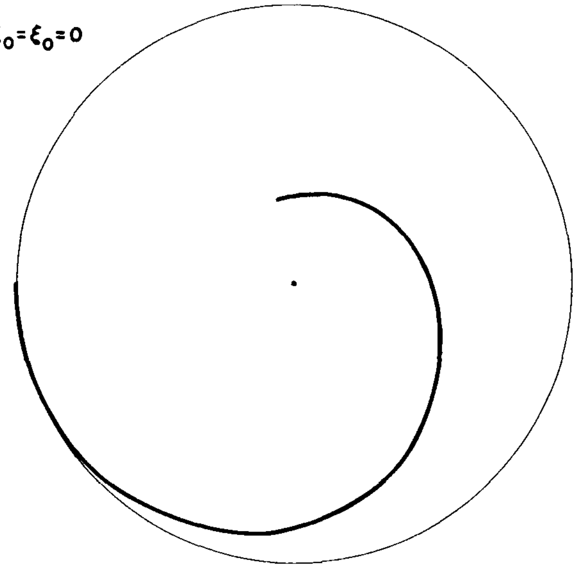

Fig. 5. Comparison of approximate method with numerical integration; initial eccentricity = 0; ε = 0.1, inward

In Fig. 5, an inward trajectory is represented; the approximate theory is as accurate as for the outward trajectory in Fig. 2.

The results for a three-dimensional trajectory, given in Table 3, show that the accuracy is highly satisfactory up to $\phi = 240°$, where the spacecraft is 4° above the ecliptic.

ASTRAL and NUTRAL compute the same quantities on the same computer. The running time of ASTRAL is only about one-tenth that of NUTRAL.

It can be concluded that the accuracy of the approximate formulas developed in the foregoing pages, and the gain in computer time realized by ASTRAL, are sufficient to justify the development and use of ASTRAL as a design tool for three-dimensional, interplanetary, solar-powered, low-thrust trajectories.

Table 3. Comparison of third approximation with numerical integration; three-dimensional; initial eccentricity (in ecliptic plane) = 0

ϕ Degrees	r		H_ϕ		t		ψ	
	Numerical	Approximate	Numerical	Approximate	Numerical	Approximate	Numerical, degrees	Approximate, degrees
60	1.029	1.030	1.087	1.086	1.018	1.024	1.32	1.49
120	1.213	1.212	1.176	1.176	2.143	2.161	3.69	4.17
180	1.698	1.689	1.280	1.281	3.873	3.913	4.77	5.36
240	2.563	2.532	1.420	1.420	7.332	7.357	4.05	4.64
300	3.238	3.193	1.597	1.591	13.408	13.246	2.94	1.72
360	3.169	3.147	1.771	1.764	19.943	19.642	2.52	0.68
Three-dimensional $\varepsilon = 0.1$; $\eta_0 = \frac{1}{2}(3)^{\frac{1}{2}}$; $\zeta = 0$; $\xi_0 = \frac{1}{2}$								

Nomenclature

a_F acceleration due to thrust

A_n constant of integration, defined in Eq. (31)

B_n, C_n, D_n constant of integration, defined in Eq. (52)

F thrust force

GM gravitational acceleration due to the sun at unit distance

G_n constant, defined in Eq. (69)

H_n constant, defined in Eq. (71)

H_ϕ ϕ-component of the angular momentum of the spacecraft

k $k = 1/H_\phi^2$

m mass of spacecraft

m_0 mass of spacecraft at the moment that the thrust turned on

r distance of spacecraft to the sun

t time

u $u = 1/r$

β_n constant, see Eq. (30)

ε ratio of gravitational and thrust acceleration at the instant that the thrust is turned on

ϕ angular position in ecliptic plane; fast variable

$\tilde{\phi}$ slow variable, defined in Eq. (39)

ϕ_n point where thrust acceleration is readjusted

η_n ϕ-component, ξ_n ψ-component, ζ_n r-component of thrust acceleration in interval $\phi_n < \phi < \phi_{n+1}$

ψ angular position relative to ecliptic plane (see Fig. 1)

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