HIGH-FREQUENCY WAVES IN A COLLISIONAL PLASMA WITH MAGNETIC FIELD

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ABSTRACT

The damping of high-frequency waves, in a hot plasma in an external uniform magnetic field in the presence of weak Coulomb collisions, is investigated by using the Fokker-Planck equation. The electron-ion collisions play the dominant role; nevertheless, the electron-electron collisions are important for disturbances of finite wave lengths. As far as the electron-ion-collision contribution is concerned, the frictional term exceeds the diffusion term but in electron-electron case, both the frictional and the diffusion contributions are of the same order. The two-body Coulomb collisions have a stabilizing effect on these plasma waves; the magnetic field however does not affect the longitudinal waves but has a tendency to stabilize left-handed polarized wave and to destabilize the right-handed polarized wave.

I. INTRODUCTION

The small amplitude oscillations in a fully ionized plasma in a uniform external megnetic field were studied by Bernstein[1]. He showed that in a collision-free plasma, the self excitation of the waves around thermal equilibrium is not possible.

Comisar[2] and Buti and Jain[3] studied the high-frequency plasma waves in a hot plasma in the absence of any external magnetic field but they took into account the weak Coulomb collisions by using the Fokker-Planck equation of Rosenbluth et al[4]. They found that the electron-ion collisions play more important role in damping the longitudinal as well as the transverse waves; the electron-electron collisions, one has to take into account only if one is interested in finite-wave-length distrubances.

The wave motion in a plasma, where the collisions are too frequent and the applied magnetic field is strong, has been studied by Oppenheim[5] and Liboff[6] using the models known as isotropic Fokker-Planck model and the Liboff-Krook model respectively. Both predicted an infinite number of Larmor resonances; in addition Oppenheim's model described the diffusion process in velocity space. In cold plasma regime and the long wave length magnetohydrodynamic regime, to lowest order, these two models gave the same results.

Following Comisar and Buti and Jain, we consider the effect of an external uniform weak magnetic field on the plasma waves when the collisions are not too frequent which allows us to neglect the many-body collisions. The magnetic field B_0 does not affect the nature of the collisions provided the Larmor radius R_L is much larger than the Debye length λ_D i.e., if the plasma frequency ω_p is much larger than the electron-cyclotron frequency $\Omega = eB_0/mc$. For such small magnetic fields, the radiation is also negligible; so the Fokker-Planck co-efficients remain unaltered and we can use the Fokker-Planck equation of Rosenbluth et al[4]. In this study, we take the contributions of the frictional and the diffusion terms separately both for the electron-electron and the electron-ion collisions; in the former case both contributions are of the same order but in the latter case frictional contribution is much larger than that of diffusion which is comparable to the contributions due to electron-electron collisions. The magnetic field as well as the collisions tend to stabilize the system under consideration.

II. GENERAL THEORY

Let us consider a fully ionized hot plasma in a uniform external magnetic field; the ions in the plasma form a neutralizing background. In equilibrium both the electrons and the ions have Maxwellian distribution of velocities i.e.,

$$f_{oe} = (2\pi v_0^2)^{-3/2} \, e^{-v^2/(2v_0^2)} \tag{1}$$

and

$$f_{oi} = (2\pi V_0^2)^{-3/2} \, e^{-V^2/(2V_0^2)} , \tag{2}$$

where $v_0^2 = KT_e/m$, $V_0^2 = KT_i/M$ and Nf_{oe} and Nf_{oi} are the equilibrium electron and ion distribution functions. For small perturbations, the linearized Fokker-Planck equation for electrons is given by

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} - \frac{Ne}{m}\left[\underset{\sim}{E} \cdot \frac{\partial f_{oe}}{\partial \underset{\sim}{v}} + \frac{1}{c}(\underset{\sim}{v} \times \underset{\sim}{B}_o) \cdot \frac{\partial f}{\partial \underset{\sim}{v}}\right] = \sum_{j=1}^{3} \left(\frac{\partial f}{\partial t}\right)_{cj} , \tag{3}$$

where f is the perturbed electron distribution function and $\underset{\sim}{B}_o$ is the externally applied magnetic field which we shall take along the Z-axis. The collision term $(\partial f/\partial t)_c$ takes care of both the electron-ion and the electron-electron collisions and is represented by[2,3]

$$\left(\frac{\partial f}{\partial t}\right)_{c1} = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\langle \underset{\sim}{\Delta} \rangle_{oi} f\right) + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{v}\, \partial \underset{\sim}{v}} : \left(\langle \underset{\sim}{\Delta} \underset{\sim}{\Delta} \rangle_{oi} f\right) , \tag{4}$$

$$\left(\frac{\partial f}{\partial t}\right)_{c2} = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\langle \underset{\sim}{\Delta} \rangle_{oe} f\right) + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{v}\, \partial \underset{\sim}{v}} : \left(\langle \underset{\sim}{\Delta} \underset{\sim}{\Delta} \rangle_{oe} f\right) \tag{5}$$

and

$$\left(\frac{\partial f}{\partial t}\right)_{c3} = -\frac{\partial}{\partial \underset{\sim}{v}} \cdot \left(\langle \underset{\sim}{\Delta} \rangle f_{oe}\right) + \frac{1}{2} \frac{\partial^2}{\partial \underset{\sim}{v}\, \partial \underset{\sim}{v}} : \left(\langle \underset{\sim}{\Delta} \underset{\sim}{\Delta} \rangle f_{oe}\right) , \tag{6}$$

where

$$\langle \underset{\sim}{\Delta} \rangle_{oi} = \left(1 + \frac{m}{M}\right) \frac{N\Gamma}{2\pi^2} \frac{\partial}{\partial \underset{\sim}{v}} \int d\underset{\sim}{v}'\, f_{oi}(\underset{\sim}{v}') \int \frac{d\underset{\sim}{\xi}}{\xi^2}\, e^{i \underset{\sim}{\xi} \cdot (\underset{\sim}{v} - \underset{\sim}{v}')} , \tag{7}$$

$$\langle \underset{\sim}{\Delta}\underset{\sim}{\Delta} \rangle_{oi,e} = -\frac{N\Gamma}{\pi^2}\frac{\partial^2}{\partial \underset{\sim}{v}\partial \underset{\sim}{v}}\int d\underset{\sim}{v}'\, f_{oi,e}(\underset{\sim}{v}') \int \frac{d\underset{\sim}{\xi}}{\xi^4}\, e^{i\underset{\sim}{\xi}\cdot(\underset{\sim}{v}-\underset{\sim}{v}')} , \tag{8}$$

$$\langle \underset{\sim}{\Delta} \rangle_{oe} = \frac{N\Gamma}{\pi^2}\frac{\partial}{\partial \underset{\sim}{v}}\int d\underset{\sim}{v}'\, f_{oe}(\underset{\sim}{v}') \int \frac{d\underset{\sim}{\xi}}{\xi^2}\, e^{i\underset{\sim}{\xi}\cdot(\underset{\sim}{v}-\underset{\sim}{v}')} , \tag{9}$$

$$\langle \underset{\sim}{\Delta} \rangle = \frac{N\Gamma}{\pi^2}\frac{\partial}{\partial \underset{\sim}{v}}\int d\underset{\sim}{v}'\, f(\underset{\sim}{v}') \int \frac{d\underset{\sim}{\xi}}{\xi^2}\, e^{i\underset{\sim}{\xi}\cdot(\underset{\sim}{v}-\underset{\sim}{v}')} \tag{10}$$

and

$$\langle \underset{\sim}{\Delta}\underset{\sim}{\Delta} \rangle = -\frac{N\Gamma}{\pi^2}\frac{\partial^2}{\partial \underset{\sim}{v}\partial \underset{\sim}{v}}\int d\underset{\sim}{v}'\, f(\underset{\sim}{v}') \int \frac{d\underset{\sim}{\xi}}{\xi^4}\, e^{i\underset{\sim}{\xi}\cdot(\underset{\sim}{v}-\underset{\sim}{v}')} , \tag{11}$$

with $\Gamma = (4\pi e^4/m^2)\,\ln(4\pi N\lambda_D^3)$; $\lambda_D^2 = KT/(4\pi N e^2)$.

If we take the Fourier transforms in space and Laplace transforms in time of all the perturbed quantities, then Eq. (3) goes over to

$$(s + i\underset{\sim}{k}\cdot\underset{\sim}{v})\, f'(\underset{\sim}{k},\underset{\sim}{v},s) - g(\underset{\sim}{k},\underset{\sim}{v}) - \frac{Ne}{m}\underset{\sim}{E}'_k \cdot \frac{\partial f_{oe}}{\partial \underset{\sim}{v}} + \Omega\frac{\partial f'}{\partial \phi} = \sum_{j=1}^{3}\left(\frac{\partial f'}{\partial t}\right)_{cj} \quad ; \quad \mathrm{Re}\, s > 0 , \tag{12}$$

where f' and $\underset{\sim}{E}'_k$ are the Fourier-Laplace transforms of f and $\underset{\sim}{E}$ and $g(\underset{\sim}{k},\underset{\sim}{v})$ is the Fourier transform of the initial perturbation in the distribution function. In writing Eq. (12), we have made use of the cylindrical polar co-ordinates for the velocity i.e., $\underset{\sim}{v} = (v_\perp, \phi, v_z)$. On taking the Fourier-Laplace transforms of Maxwell equations, we obtain

$$\underset{\sim}{E}'_k = \frac{4\pi e}{(s^2+c^2k^2)}\left[ic^2 k \int d\underset{\sim}{v}\, f'(\underset{\sim}{v}) + s\int d\underset{\sim}{v}\, \underset{\sim}{v}\, f'(\underset{\sim}{v})\right]$$

$$+ \frac{1}{(s^2+c^2k^2)}\left[s\underset{\sim}{\varepsilon} + ic\,(\underset{\sim}{k}\times\underset{\sim}{b})\right] , \tag{13}$$

where $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{b}$ correspond to initial perturbations in the electric and the magnetic fields respectively. On eliminating $\underset{\sim}{E}'_k$ from Eqs. (12) and (13), we get

$$(s+i\underset{\sim}{k}\cdot\underset{\sim}{v})f' - g(\underset{\sim}{k},\underset{\sim}{v}) + \Omega\frac{\partial f'}{\partial\phi} - \Lambda\frac{\partial f_{oe}}{\partial\underset{\sim}{v}}\cdot\int d\underset{\sim}{v}\,\underset{\sim}{v}\,f'(\underset{\sim}{v})$$

$$-\frac{ic^2\Lambda}{s}\,\underset{\sim}{k}\cdot\frac{\partial f_{oe}}{\partial\underset{\sim}{v}}\int d\underset{\sim}{v}\, f'(\underset{\sim}{v}) - \frac{Ne}{m}\,(s^2+c^2k^2)^{-1} \tag{14}$$

$$\frac{\partial f_{oe}}{\partial\underset{\sim}{v}}\cdot\left[s\underset{\sim}{\varepsilon} + ic\,(\underset{\sim}{k}\times\underset{\sim}{b})\right] = \sum_{j=1}^{3}\left(\frac{\partial f'}{\partial t}\right)_{cj} ,$$

where $\Lambda = \omega_p^2 s/(s^2+c^2k^2)$; $\omega_p^2 = (4\pi N e^2/m)$.
For wave propagation along the direction of the magnetic field i.e., $\underset{\sim}{k} = k\hat{e}_z$, on introducing the integrating factor exp $[-(s+ikv_z)(\phi-\phi')/\Omega]$, Eq. (14) can be solved to give

$$f'(\underset{\sim}{k},\underset{\sim}{v},s) = \frac{1}{\Omega}\int_{-\infty}^{\phi} d\phi'\,\exp\left[-\frac{1}{\Omega}(s+ikv_z)(\phi-\phi')\right]$$

$$\left[g(\underset{\sim}{k},\underset{\sim}{v}') + \frac{Ne}{m}(s^2+c^2k^2)^{-1}\frac{\partial f_{oe}}{\partial\underset{\sim}{v}'}\cdot\left\{s\underset{\sim}{\varepsilon} + ic\,(\underset{\sim}{k}\times\underset{\sim}{b})\right\} + \right. \tag{15}$$

$$\left. + \frac{ic^2k\Lambda}{s}\frac{\partial f_{oe}}{\partial v_z}\int d\underset{\sim}{v}\, f'(\underset{\sim}{v}) + \Lambda\frac{\partial f_{oe}}{\partial\underset{\sim}{v}'}\cdot\int d\underset{\sim}{v}\,\underset{\sim}{v}\,f'(\underset{\sim}{v}) + \sum_{j=1}^{3}\left\{\frac{\partial f'(\underset{\sim}{v}')}{\partial t}\right\}_{cj}\right]$$

with $\underset{\sim}{v}' = (v_\perp, \phi', v_z)$. On substituting Eq. (2) in Eq. (15) and on performing ϕ' - integration, Eq. (15) reduces to

$$f'(\underset{\sim}{k},\underset{\sim}{v},s) = q(\underset{\sim}{k},\underset{\sim}{v}) - \frac{\Lambda v_z f_{oe}(\underset{\sim}{v})}{v_o^2(s+ikv_z)}\left[\frac{ic^2k}{s}\int d\underset{\sim}{v}\, f'(\underset{\sim}{v})\right.$$

$$\left.+\int d\underset{\sim}{v}\, v_z f'(\underset{\sim}{v})\right] - \frac{\Lambda v_\perp e^{i\phi} f_{oe}(\underset{\sim}{v})}{2v_o^2(s+ikv_z+i\Omega)}\int d\underset{\sim}{v}\,(v_x - iv_y)\, f'(\underset{\sim}{v})$$

$$- \frac{\Lambda v_\perp e^{-i\phi} f_{oe}(\underset{\sim}{v})}{2v_o^2(s+ikv_z-i\Omega)}\int d\underset{\sim}{v}\,(v_x+iv_y)\, f'(\underset{\sim}{v}) \tag{16}$$

$$+ \frac{1}{\Omega}\sum_{j=1}^{3}\int_{-\infty}^{\phi} d\phi'\, e^{-\alpha(\phi-\phi')}\left[\frac{\partial f'(\underset{\sim}{v}')}{\partial t}\right]_{cj},$$

where $\alpha = (s+ikv_z)/\Omega$ and

$$q(\underset{\sim}{k},\underset{\sim}{v}) = \frac{1}{\Omega}\int_{-\infty}^{\phi} d\phi' e^{-\alpha(\phi-\phi')}\left[g(\underset{\sim}{k},\underset{\sim}{v}') + \frac{Ne}{m}(s^2+c^2k^2)\right. \tag{17}$$

$$\left.\frac{\partial f_{oe}(\underset{\sim}{v}')}{\partial \underset{\sim}{v}'}\cdot\left\{s\underset{\sim}{\varepsilon}+ic(\underset{\sim}{k}\times\underset{\sim}{b})\right\}\right].$$

In order to simplify further, we shall take the Fourier transforms in velocity space and if we define

$$F(\underset{\sim}{k},\underset{\sim}{\sigma},s) = \int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, f'(\underset{\sim}{k},\underset{\sim}{v},s), \tag{18}$$

then

$$\int d\underset{\sim}{v}\, f'(\underset{\sim}{v}) = F(\underset{\sim}{k},0,s), \qquad \int d\underset{\sim}{v}\,\underset{\sim}{v}\, f'(\underset{\sim}{v}) = i\left(\frac{\partial F}{\partial \underset{\sim}{\sigma}}\right)_{\underset{\sim}{\sigma}=0}$$

and Eq. (16) becomes

$$F(\underset{\sim}{k},\underset{\sim}{\sigma},s) = Q(\underset{\sim}{k},\underset{\sim}{\sigma}) + \left[\frac{c^2k}{s}F(\underset{\sim}{k},0,s) + \left(\frac{\partial F}{\partial \sigma_z}\right)_{\underset{\sim}{\sigma}=0}\right]I_1(\underset{\sim}{\sigma})$$

$$+ \left(\frac{\partial F}{\partial \sigma_x} + i\frac{\partial F}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} I_2(\underset{\sim}{\sigma}) + \left(\frac{\partial F}{\partial \sigma_x} - i\frac{\partial F}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} I_3(\underset{\sim}{\sigma}) \tag{19}$$

$$+ \sum_{j=1}^{3}\left(\frac{\partial F'}{\partial t}\right)_{cj},$$

where

$$Q(\underset{\sim}{k},\underset{\sim}{\sigma}) = \int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, q(\underset{\sim}{k},\underset{\sim}{v}) \quad , \tag{20}$$

$$I_1(\underset{\sim}{\sigma}) = -\frac{i\Lambda}{v_0^2}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, f_{0e}(\underset{\sim}{v})\frac{v_z}{(s+ikv_z)} \quad , \tag{21}$$

$$I_2(\underset{\sim}{\sigma}) = -\frac{i\Lambda}{2(v_0^2)}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, f_{0e}(\underset{\sim}{v})\frac{v_\perp e^{-i\phi}}{(s+ikv_z-i\Omega)} , \tag{22}$$

$$I_3(\underset{\sim}{\sigma}) = -\frac{i\Lambda}{2v_0^2}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, f_{0e}(\underset{\sim}{v})\frac{v_\perp e^{i\phi}}{(s+ikv_z+i\Omega)} \tag{23}$$

and

$$\left(\frac{\partial F'}{\partial t}\right)_{cj} = \frac{1}{\Omega}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\int_{-\infty}^{\phi} d\phi' e^{-\alpha(\phi-\phi')}\left[\frac{\partial f'(v')}{\partial t}\right]_{cj} . \tag{24}$$

III. DISPERSION RELATION

Let us introduce the Coulomb mean free path $L = v_0^4/(N\Gamma)$ and if we assume that the collisions are infrequent, then the effective collision frequency $\nu_c = v_0/L$ is much smaller than ω_p; so to the lowest order in ν_c , Eq. (19) yields

$$F(\underset{\sim}{k},\underset{\sim}{\sigma},s) = F_0(\underset{\sim}{k},\underset{\sim}{\sigma},s) + \sum_{j=1}^{3}\left[\left(\frac{\partial F'}{\partial t}\right)_{cj}\right]_{F=F_0} , \tag{25}$$

where

$$F_0 = Q(\underset{\sim}{k},\underset{\sim}{\sigma}) + I_1(\underset{\sim}{\sigma})\left[\frac{c^2 k}{s} F(\underset{\sim}{k},0,s) + \left(\frac{\partial F}{\partial \sigma_z}\right)_{\underset{\sim}{\sigma}=0}\right]$$

$$+ I_2(\underset{\sim}{\sigma})\left(\frac{\partial F}{\partial \sigma_x} + i\frac{\partial F}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} + I_3(\underset{\sim}{\sigma})\left(\frac{\partial F}{\partial \sigma_x} - i\frac{\partial F}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0}, \qquad (26)$$

represents F in a collision free plasma. To obtain the dispersion relation, we shall study Eq. (25) for the longitudinal and the transverse oscillations separately.

(a) Longitudinal Oscillations.

In this case

$$\Lambda\left[\frac{c^2}{s}\underset{\sim}{k} F(\underset{\sim}{k},0,s) + \left(\frac{\partial F}{\partial \underset{\sim}{\sigma}}\right)_{\underset{\sim}{\sigma}=0}\right] = \frac{\omega_p^2}{k^2}\underset{\sim}{k} F(\underset{\sim}{k},0,s) \qquad (27)$$

and

$$Q(\underset{\sim}{k},\underset{\sim}{\sigma}) = \frac{1}{\Omega}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}} \int_{-\infty}^{\phi} d\phi'\, e^{-\alpha(\phi-\phi')}\, g(\underset{\sim}{k},\underset{\sim}{v}')$$

$$\equiv Q_L(\underset{\sim}{k},\underset{\sim}{\sigma}) \qquad \text{say} \quad , \qquad (28)$$

which when substituted in Eq. (25) immediately gives

$$F(\underset{\sim}{k},0,s) = Q_L(\underset{\sim}{k},0) - \frac{i\omega_p^2}{k v_0^2} F(\underset{\sim}{k},0,s)\int d\underset{\sim}{v}\, \frac{v_z f_{0e}(\underset{\sim}{v})}{(s+ikv_z)}$$

$$+ \sum_{j=1}^{3}\left\{\left[\left(\frac{\partial F'}{\partial t}\right)_{cj}\right]_{F=Q_L(\underset{\sim}{\sigma}=0)} + \left[\left(\frac{\partial F'}{\partial t}\right)_{cj}\right]_{F=F_L}\right\}, \qquad (29)$$

where

$$F_L = -\frac{i\omega_p^2}{k v_0^2} F(\underset{\sim}{k},0,s)\int d\underset{\sim}{v}\, \frac{v_z f_{0e}(\underset{\sim}{v})}{(s+ikv_z)} \quad . \qquad (30)$$

From Eqs.(29) and (30), it is apparent that the dispersion relation would be independent of Ω and hence the longitudinal oscillations are unaffected by the external magnetic field.

(b) Transverse Oscillations.

In this case $F(\underset{\sim}{k},0,s)$ as well as $(\partial F/\partial\sigma_z)_{\underset{\sim}{\sigma}=0}$ are zero: so Eq. (26) reduces to

$$F_0 = Q(\underset{\sim}{k},\underset{\sim}{\sigma}) + I_2(\underset{\sim}{\sigma})\left(\frac{\partial F}{\partial\sigma_x} + i\frac{\partial F}{\partial\sigma_y}\right)_{\underset{\sim}{\sigma}=0} + I_3(\underset{\sim}{\sigma})\left(\frac{\partial F}{\partial\sigma_x} - i\frac{\partial F}{\partial\sigma_y}\right)_{\underset{\sim}{\sigma}=0}, \tag{27}$$

which for the right-handed polarized wave further simplifies to give

$$F_0 = Q(\underset{\sim}{k},\underset{\sim}{\sigma}) + I_2(\underset{\sim}{\sigma})\left(\frac{\partial F}{\partial\sigma_x} + i\frac{\partial F}{\partial\sigma_y}\right)_{\underset{\sim}{\sigma}=0}. \tag{28}$$

Now if we use Eq. (28) in Eq. (24), we get (see appendix),

$$\left[\left(\frac{\partial F'}{\partial t}\right)_{cj}\right]_{F=F_0} = \nu_c v_0^3\left[Q_j + \int d\underset{\sim}{\xi}\, K_j(\underset{\sim}{\sigma},\underset{\sim}{\xi})\left(\frac{\partial F}{\partial\sigma_x} + i\frac{\partial F}{\partial\sigma_y}\right)_{\underset{\sim}{\sigma}=0}\right], \tag{29}$$

where

$$Q_1 = \frac{1}{2\Omega\pi^2}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\int_{-\infty}^{\phi} d\phi'\, e^{-\alpha(\phi-\phi')}\int\frac{d\underset{\sim}{\xi}}{\xi^2}$$

$$\left[-i\mu\, e^{-\xi^2 v_0^2/2}\,\underset{\sim}{\xi}\cdot\frac{\partial}{\partial\underset{\sim}{v}} + \frac{e^{-\xi^2 v_0^2/2}}{\xi^2}\,\underset{\sim}{\xi}\,\underset{\sim}{\xi} : \frac{\partial^2}{\partial\underset{\sim}{v}\,\partial\underset{\sim}{v}}\right]\left[q(\underset{\sim}{v}')\, e^{i\underset{\sim}{\xi}\cdot\underset{\sim}{v}'}\right], \tag{30}$$

$$Q_3 = \frac{1}{2\Omega\pi^2}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\, f_{0e}(\underset{\sim}{v})\int_{-\infty}^{\phi} d\phi'\, e^{-\alpha(\phi-\phi')}$$

$$\int\frac{d\underset{\sim}{\xi}}{\xi^2}\, e^{i\underset{\sim}{\xi}\cdot\underset{\sim}{v}'}\, Q(\underset{\sim}{k},\underset{\sim}{\xi})\left[\xi^2 - \frac{1}{v_0^2} + \frac{1}{\xi^2 v_0^4}(\underset{\sim}{\xi}\cdot\underset{\sim}{v}')^2\right], \tag{31}$$

with $\mu = (1 + m/M)$. Q_2 is obtained from Q_1 by putting $\mu = 2$ and $\underset{\sim}{V_0} = \underset{\sim}{v_0}$. The kernel K_J in Eq. (29) consists of two parts i.e., $K_J = K_J^f + K_J^d$, where the superscripts f and d refer to the frictional and the diffusion parts respectively and are given by

$$K_1^f(\underset{\sim}{\xi}) = -\frac{\Lambda \mu e^{-\xi_1^2 V_0^2/2}}{4\pi^2 \Omega v_0^2 \xi_1^2} \int_0^\infty dt\, e^{i\Omega t} \int d\underset{\sim}{V}\, V_\perp e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{V}}$$

$$\int_{-\infty}^{\phi} d\phi' f_{0e}(\underset{\sim}{V}') \exp\left[i\underset{\sim}{\xi}\cdot\underset{\sim}{V}' - i\phi' - \alpha(\Omega t + \phi - \phi')\right] \tag{32}$$

$$\left[i\xi_1^2 - \frac{(\underset{\sim}{V}'\cdot\underset{\sim}{\xi})}{v_0^2} - ik\xi_z t + \frac{e^{i\phi'}}{V_\perp}(\xi_x - i\xi_y)\right],$$

$$K_1^d(\underset{\sim}{\xi}) = -\frac{i\Lambda e^{-\xi^2 V_0^2/2}}{4\Omega\pi^2 v_0^2 \xi_1^4} \int_0^\infty dt\, e^{i\Omega t} \int d\underset{\sim}{V}\, V_\perp e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{V}} \int_{-\infty}^{\phi} d\phi' f_{0e}(\underset{\sim}{V}')$$

$$\exp\left[i\underset{\sim}{\xi}\cdot\underset{\sim}{V}' - i\phi' - \alpha(\Omega t + \phi - \phi')\right]\left[\frac{\xi_\perp^2}{V_\perp^2} - \frac{\xi_1^2}{v_0^2} - \right. \tag{33}$$

$$\left. - \frac{2\xi_\perp^2}{V_\perp^2} e^{i(\phi'-\theta)} \cos(\phi'-\theta) + \left\{i\xi_1^2 - \frac{\underset{\sim}{V}'\cdot\underset{\sim}{\xi}}{v_0^2} - ikt\xi_z + \frac{\xi_\perp}{V_\perp} e^{i(\phi'-\theta)}\right\}^2\right],$$

$$K_3^f(\underset{\sim}{\xi}) = \frac{I_2(\underset{\sim}{\xi})}{\Omega\pi^2\xi_1^2} \int d\underset{\sim}{V}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{V}} f_{0e}(\underset{\sim}{V}) \int_{-\infty}^{\phi} d\phi'$$

$$\exp\left[i\underset{\sim}{\xi}\cdot\underset{\sim}{V}' - \alpha(\phi - \phi')\right]\left[\xi_1^2 + \frac{i}{v_0^2}(\underset{\sim}{V}'\cdot\underset{\sim}{\xi})\right] \tag{34}$$

and

$$K_3^d(\underset{\sim}{\xi}) = -\frac{I_2(\underset{\sim}{\xi})}{2\Omega\pi^2\xi_1^4} \int d\underset{\sim}{V}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{V}} f_{0e}(\underset{\sim}{V}) \int_{-\infty}^{\phi} d\phi' \exp\left[i\underset{\sim}{\xi}\cdot\underset{\sim}{V}' - \alpha(\phi - \phi')\right]$$

$$\left[\xi_1^4 + \frac{\xi_1^2}{v_0^2}\left\{1 + 2i(\underset{\sim}{V}'\cdot\underset{\sim}{\xi})\right\} - \frac{1}{v_0^4}(\underset{\sim}{V}'\cdot\underset{\sim}{\xi})^2\right]. \tag{35}$$

Once again K_2 is obtained by putting $\mu = 2$ and $\underset{\sim}{V}_0 = \underset{\sim}{v}_0$ in K_1 . In writing Eq. (33), we have made use of the cylindrical co-ordinates for $\underset{\sim}{\xi}$ i.e., $\underset{\sim}{\xi} = (\xi_\perp, \theta, \xi_z)$. $I_2(\underset{\sim}{\xi})$ is as given in Eq. (22). From Eq. (25), (28) and (29), we get

$$F(\underset{\sim}{k}, \underset{\sim}{\sigma}, s) = Q(\underset{\sim}{k}, \underset{\sim}{\sigma}) + I_2(\underset{\sim}{\sigma}) \left(\frac{\partial F}{\partial \sigma_x} + i \frac{\partial F}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0}$$

$$+ \nu_c v_0^3 \sum_{j=1}^{3} \left[Q_j + \int d\underset{\sim}{\xi}\, K_j(\underset{\sim}{\xi}, \underset{\sim}{\sigma}) \left(\frac{\partial F}{\partial \sigma_x} + i \frac{\partial F}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} \right], \quad (36)$$

which can also be put into the form

$$\left[\left(\frac{\partial F}{\partial \sigma_x}\right) + i \left(\frac{\partial F}{\partial \sigma_y}\right)\right]_{\underset{\sim}{\sigma}=0} = \frac{\Phi(k, s)}{\Psi(k, s)}, \quad (37)$$

where

$$\Phi(k, s) = \left(\frac{\partial Q}{\partial \sigma_x} + i \frac{\partial Q}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} + \nu_c v_0^3 \sum_{j=1}^{3} \left(\frac{\partial Q_j}{\partial \sigma_x} + i \frac{\partial Q_j}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} \quad (38)$$

and

$$\Psi(k, s) = 1 - \left(\frac{\partial I_2}{\partial \sigma_x} + i \frac{\partial I_2}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} - \nu_c v_0^3 \sum_{j=1}^{3} \int d\underset{\sim}{\xi}\,.$$

$$\left(\frac{\partial K_j}{\partial \sigma_x} + i \frac{\partial K_j}{\partial \sigma_y}\right)_{\underset{\sim}{\sigma}=0} . \quad (39)$$

We notice that $\Phi(k,s)$ depends only on the initial conditions of the system and if we consider the case when $\Phi(k,s)$ is analytic in the complex s-plane then to determine the current density of the system, we have to take into account the zeros of $\Psi(k,s)$ which are given by

$$1 = \left(\frac{\partial I_2}{\partial \sigma_x} + i \frac{\partial I_2}{\partial \sigma_y} \right)_{\underset{\sim}{\sigma}=0} + \nu_c v_0^3 \sum_{j=1}^{3} P_j \quad , \tag{40}$$

where

$$P_j = \int d\underset{\sim}{\xi} \left(\frac{\partial K_j}{\partial \sigma_x} + i \frac{\partial K_j}{\partial \sigma_y} \right)_{\underset{\sim}{\sigma}=0} \quad . \tag{41}$$

The subscript 1 correspond to electron-ion collisions whereas 2 and 3 represent electron-electron collisions. Eq. (40) gives the required dispersion relation.

IV. COLLISION CONTRIBUTION

On making use of Eq. (35), from Eq. (41), we obtain

$$\begin{aligned} P_3^d = & \frac{-i\Lambda}{8\sqrt{2}\,\pi^{7/2}\,\Omega\, v_0^3} \int d\underset{\sim}{\xi}\, \frac{\xi_\perp}{\xi^4} \exp\left(-i\theta - \frac{\xi_\perp^2 v_0^2}{2}\right) \\ & \int_0^\infty dt \exp\left[-A_1 \Omega t - \frac{v_0^2}{2}\left(kt+\xi_z\right)^2\right] \int d\underset{\sim}{v}\, v_\perp\, e^{-v^2/(2v_0^2)} \\ & \int_{-\infty}^{\phi} d\phi' \exp\left[i\phi - i\underset{\sim}{\xi}\cdot\underset{\sim}{v}' - \alpha(\phi-\phi')\right]\left[\xi^4 + \frac{\xi^2}{v_0^2} \right. \\ & \left. + \frac{2i\xi^2}{v_0^2}\left(\underset{\sim}{v}'\cdot\underset{\sim}{\xi}\right) - \frac{1}{v_0^4}\left(\underset{\sim}{v}'\cdot\underset{\sim}{\xi}\right)^2\right] \quad , \end{aligned} \tag{42}$$

where $A_1 = (s - i\Omega)/\Omega$. In writing Eq. (42), we have already substituted for $I_2(\xi_\perp)$ the following expression:

$$I_2(\xi_\perp) = -\frac{\Lambda}{2}\,\xi_{1\perp}\, e^{-i\theta}\, e^{-\xi_1^2 v_0^2/2}\int_0^\infty dt\, \exp\left[-A_1\Omega t - \frac{v_0^2}{2}\left(kt + \xi_z\right)^2\right]; \qquad (43)$$

this is easily obtained from Eq. (22) if we make use of the following relations:

$$e^{\lambda\cos\phi} = \sum_{n=-\infty}^{\infty} I_n(\lambda)\, e^{in\phi} \qquad (44)$$

and[7]

$$\int_0^\infty dy\, e^{-p^2y^2}\, y^{\nu+1}\, J_\nu(by) = \frac{b^\nu e^{-b^2/(4p^2)}}{(2p^2)^{\nu+1}} \quad . \qquad (45)$$

J_ν and I_ν are the Bessel functions of the first kind and are related by[7]

$$I_n(iz) = e^{in\pi/2}\, J_n(z) \quad . \qquad (46)$$

On putting $(\phi - \phi') = \beta$ in Eq. (42) and on performing v_z and ϕ integration, after some simplifications we obtain

$$P_3^d = \frac{\Lambda}{4\pi^2 \Omega v_0^2} \int_0^\infty d\beta\, e^{-A_1\beta} \int_0^\infty dt\, e^{-A_1\Omega t} \int d\underset{\sim}{\xi} \frac{\xi_\perp}{\xi^4}$$

$$\exp\left[-\frac{v_0^2}{2}\left\{\xi_\perp^2 + \left(\xi_z - \frac{\beta k}{\Omega}\right)^2 + \left(\xi_z + kt\right)^2\right\}\right] \int_0^\infty dV_\perp V_\perp^2$$

$$e^{-V_\perp^2/(2v_0^2)} \left[J_1(\xi_\perp V_\perp)\left\{\xi_\perp^4 + \frac{3\xi_\perp^2}{v_0^2} + \frac{\beta^2 k^2}{\Omega^2}\xi_z^2 + \right.\right. \tag{47}$$

$$+ \frac{2}{\Omega}\xi_z \beta k\left(\xi_\perp^2 + \frac{1}{v_0^2}\right)\Big\} - J_2(\xi_\perp V_\perp)\frac{\xi_\perp V_\perp}{v_0^2}\Big(2\xi_\perp^2 +$$

$$\left. + \frac{3}{v_0^2} + \frac{2}{\Omega}\xi_z \beta k\Big) + \frac{\xi_\perp^2 V_\perp^2}{v_0^4} J_3(\xi_\perp V_\perp)\right] ;$$

in simplifying use has been made of the recurrence relation, namely

$$J_{\nu+1}(z) + J_{\nu-1}(z) = \frac{2\nu}{z} J_\nu(z) . \tag{48}$$

The $\underset{\sim}{\xi}$ -integration can be easily performed by writing $(1/\xi^4)$ as $\int_0^\infty d\nu\,\nu\, e^{-\nu\xi^2}$ and on further making the assumption that $\delta = kv_0/(s-i\Omega)$, is much less than unity, the other integrations can be performed to give

$$P_3^d = \frac{2}{15}\frac{\Lambda(s-i\Omega)^2}{\pi^{1/2} v_0^3}\,\delta^2 . \tag{49}$$

The other constituents of P_j can be evaluated by proceeding on the lines similar to the evaluation of P_3^d and we finally obtain

$$P_3^f = -\frac{\Lambda(s-i\Omega)^{-2}}{3\pi^{1/2} v_0^3}\left(1 - \frac{7}{5}\delta^2\right) , \tag{50}$$

$$P_1^f = \frac{2^{1/2}\Lambda}{3\pi^{1/2}v_0^3}(s-i\Omega)^{-2}\left(1-\frac{12}{5}\delta^2\right) \quad , \tag{51}$$

$$P_1^d = \frac{2(2)^{1/2}\Lambda}{15\pi^{1/2}v_0^3}(s-i\Omega)^{-2}\delta^2 \quad , \tag{52}$$

$$P_2^f = \frac{\Lambda}{3\pi^{1/2}v_0^3}(s-i\Omega)^{-2}\left(1-\frac{21}{5}\delta^2\right) \tag{53}$$

and

$$P_2^d = \frac{2\Lambda}{5\pi^{1/2}v_0^3}(s-i\Omega)^{-2}\delta^2 \quad . \tag{54}$$

It is interesting to note that though the frictional term is always much greater than the corresponding diffusion term but as far as the electron-electron contribution (P_2+P_3) is concerned, two contributions are of the same order. The total electron-ion and the electron-electron collisions contributions are

$$P_1 = \frac{2\Lambda}{3(2\pi)^{1/2}v_0^3}(s-i\Omega)^{-2}\left(1-2\delta^2\right) \tag{55}$$

and

$$P_2+P_3 = -\frac{2\Lambda}{5\pi^{1/2}v_0^3}(s-i\Omega)^{-2}\delta^2 \quad . \tag{56}$$

V. DISCUSSION

If we substitute Eqs. (43), (55) and (56) in Eq. (40), we get the dispersion relation in the form

$$1 = -\frac{\omega_p^2 s\,(s-i\Omega)^{-1}}{(s^2+c^2k^2)}\left[1-\delta^2+3\delta^4-\frac{\nu_c}{\pi^{1/2}(s-i\Omega)}\times\right.$$

$$\left.\times\left\{\frac{2^{1/2}}{3}\left(1-2\delta^2\right)-\frac{2}{5}\delta^2\right\}\right] , \qquad (57)$$

which on putting $s=-(i\omega+\gamma)$, $\gamma=\gamma_e+\gamma_i$ for $|\gamma|<<|\omega|$ gives the following relations:

$$\omega^2 = c^2k^2+\frac{\omega_p^2\,\omega}{(\omega+\Omega)}\left[1+\frac{k^2v_0^2}{(\omega+\Omega)^2}+\frac{3k^4v_0^4}{(\omega+\Omega)^4}\right] , \qquad (58)$$

$$\gamma_i = \frac{\nu_c}{3(2\pi)^{1/2}}\,\frac{\omega_p^2}{(\omega+\Omega)^2}\left[1+\frac{2k^2v_0^2}{(\omega+\Omega)^2}\right]\left[1-\frac{\Omega\,\omega_p^2}{2\omega(\omega+\Omega)^2}\left\{1+\frac{k^2v_0^2}{(\omega+\Omega)^2}\right\}\right]^{-1} \qquad (59)$$

and

$$\gamma_e = \frac{\nu_c}{5\pi^{1/2}}\,\frac{\omega_p^2}{(\omega+\Omega)^2}\,\frac{k^2v_0^2}{(\omega+\Omega)^2}\left[1-\frac{\Omega\,\omega_p^2}{2\omega(\omega+\Omega)^2}\left\{1+\frac{k^2v_0^2}{(\omega+\Omega)^2}\right\}\right]^{-1} . \qquad (60)$$

In the absence of interparticle collisions, both the electron-ion (γ_i) and the electron-electron (γ_e) parts of γ vanish. Eq. (58) which gives the characteristic frequency is the same as obtained by Bernstein[1]; so we shall not discuss it here. Moreover, when $\Omega\rightarrow 0$, Eqs. (58) - (60) reduce to the ones given in ref.[8](3). On replacing Ω by $(-\Omega)$, we get the corresponding results for the left-handed polarized wave.

It is worth pointing out that the results obtained here are valid only when the magnetic field is weak; under such circumstances, Eqs. (59) and (60) can be rewritten as (to lowest order in Ω)

$$\gamma_i = \frac{\nu_c \omega_p^2}{3(2\pi)^{1/2}\omega^2}\left[1 \mp \frac{\Omega}{\omega}\left(2 - \frac{\omega_p^2}{2\omega^2}\right) + \frac{k^2 v_i^2}{\omega^2}\left(2 \mp \frac{7\Omega\omega_p^2}{\omega^3}\right)\right] \tag{61}$$

and

$$\gamma_e = \frac{\nu_c \omega_p^2}{5\pi^{1/2}\omega^2}\ \frac{k^2 v_i^2}{\omega^2}\left[1 \mp \frac{4\Omega}{\omega} \pm \frac{\Omega\omega_p^2}{2\omega^3}\right] \tag{62}$$

where the upper and the lower signs correspond to the right-handed and the left-handed polarized waves respectively and ω is given by Eq. (58). The requirement, that (Ω/ω_p) be much less than unity, demands that for densities of the order of 10^{16} particles per cm^3, the magnetic field B_0 be much less than $3x10^5$ gauss; this is plausible in many physical systems of interest.

VI. CONCLUSIONS

The two-body collisions have a stabilizing effect on the high-frequency plasma waves when the wave propagation is along the direction of the uniform external magnetic field. However, the magnetic field itself, though stabilizes the left-handed polarized wave, has a tendency to destabilize the right-handed polarized wave. To the lowest order in collision frequency the electron-ion collisions play a dominant role in plasma diffusion; however, the electron-electron collisions are important for disturbances of finite wave lengths.

In strong magnetic field $(\Omega > \omega_p)$, the collisions take place in a somewhat different manner and the theory outlined here does not hold good and needs modification. In fact the Fokker-Planck co-efficients will have to be reevaluated. This study is under investigation and will be reported shortly.

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APPENDIX

From Eq. (24), we have

$$\left(\frac{\partial F'}{\partial t}\right)_{c1} = \frac{1}{\Omega}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}} \int_{-\infty}^{\phi} d\phi'\, e^{-\alpha(\phi-\phi')} \left[\frac{\partial f'(\underset{\sim}{v}')}{\partial t}\right]_{c1} . \tag{A1}$$

Let us consider first the frictional contribution; this is given by

$$\left(\frac{\partial F'}{\partial t}\right)_{c1}^{f} = \frac{-1}{\Omega}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}} \int_{-\infty}^{\phi} d\phi'\, e^{-\alpha(\phi-\phi')} \frac{\partial}{\partial \underset{\sim}{v}'}\cdot\left[\langle\underset{\sim}{\Delta}\rangle_{oi} f'(\underset{\sim}{v}')\right] . \tag{A2}$$

On substituting for $\langle\underset{\sim}{\Delta}\rangle_{oi}$ from Eq. (7) and on using the relation

$$\int d\underset{\sim}{v}\; f_{oi}(\underset{\sim}{v})\, e^{-i\underset{\sim}{\xi}\cdot\underset{\sim}{v}} = e^{-\xi^2 V_0^2/2} ,$$

Eq. (A2) can be written as

$$\left(\frac{\partial F'}{\partial t}\right)_{c1}^{f} = -\frac{iN\Gamma\mu}{2\Omega\pi^2}\int d\underset{\sim}{v}\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}} \int d\phi'\, e^{-\alpha(\phi-\phi')} \tag{A3}$$
$$\frac{\partial}{\partial \underset{\sim}{v}'}\cdot\left[f'(\underset{\sim}{v}')\int d\underset{\sim}{\xi}\, \frac{\underset{\sim}{\xi}}{\xi^2} \exp\left(i\underset{\sim}{\xi}\cdot\underset{\sim}{v}' - \frac{\xi^2 V_0^2}{2}\right)\right] .$$

Now if we use Eq. (16) to zeroth order for $f'(\underset{\sim}{v}')$ for the right-handed polarized wave, Eq. (A3) after some simplifications goes over to

$$\left[\left(\frac{\partial F'}{\partial t}\right)^{f}_{c1}\right]_{F=F_0} = -\left\{\frac{N\Gamma\mu\Lambda}{4\pi^2\Omega v_0^2}\int d\underset{\sim}{v}\, v_\perp\, e^{-i\underset{\sim}{\sigma}\cdot\underset{\sim}{v}}\right.$$

$$\int_{-\infty}^{\phi} d\phi'\, f_{0e}(\underset{\sim}{v}')\, e^{-\alpha(\phi-\phi')}\int_0^\infty dt\, e^{-\Omega(\alpha-i)t}\; e^{-i\phi'}$$

$$\int\frac{d\underset{\sim}{\xi}_1}{\xi_1^2}\exp\left(i\underset{\sim}{\xi}_1\cdot\underset{\sim}{v}' - \frac{V_0^2\xi_1^2}{2}\right)\left(\frac{\partial F}{\partial\sigma_x} + i\frac{\partial F}{\partial\sigma_y}\right)_{\underset{\sim}{\sigma}=0} \tag{A4}$$

$$\left.\left[i\xi_1^2 - \frac{(\underset{\sim}{v}'\cdot\underset{\sim}{\xi}_1)}{v_0^2} - ikt\,\xi_{1z} + \frac{\xi_{1\perp}}{v_\perp}e^{i(\phi'-\theta)}\right]\right\} + N\Gamma Q_1^f ,$$

where Q_1^f is defined by Eq. (30). Eq. (A4) can be immediately rewritten in the form of Eq. (29). Proceeding on the similar lines, the other elements of $(\partial F'/\partial t)_{cj}$ can be easily evaluated and the results given in the text follow.

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