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## SUMMARY

The nonhomogeneous heat-conduction equation has been examined subsequent to its transformation into a discrete spectrum eigenvalue form. Theorems have been derived which relate the average source temperature to variations in the thermal conductivity and/or source over any part of the system. These theorems are applicable to purely conducting systems composed of simply or multiply connected sources and nonsources having mathematical interface boundary conditions (continuity of temperature and heat flux). It is expected that these theorems will find practical application for conduction systems, much like the perturbation formulae of nuclear reactor physics. The theorems are general, however, and are not restricted in application to heat conduction systems. The application of these methods to any system behavior that can be described by the heat-conduction equation requires only a physical interpretation of the eigenvalue.

## INTRODUCTION

Frequently, in parametric survey and design feasibility studies, one is interested in some parameter that characterizes the overall performance rather than the detailed behavior of the system in question. Examples of such parameters would be the effective multiplication constant for nuclear reactors or the total heat-transfer rate for heat exchangers. This report describes the development and application of theorems which may be used to relate changes in the parameters that characterize an exclusively steady-state heat-conduction system (geometry, conductivities, and source strengths) to the average source temperature in all source regions. Taking this source temperature as the characteristic index of the system performance, several areas of practical application are possible.

The theorems derived herein are used most effectively in analyzing the effects of parameter variations in systems that have complicated geometries. Depending on the geometrical complexity, both analytical and numerical methods have been widely used to study the behavior of heat conduction in solids. For simple geometrical configurations, the geometry is often approximated by a sphere, cylinder, or slab, and a direct analytical solution is obtained. For geometrically complex systems, however, the relaxed physical interpretation resulting from approximating the actual geometry as a sphere, cylinder, or slab could lead to significant error. Consequently, for complex geometries, recourse is usually made to some numerical method. These numerical methods require representation of an actual continuous system by the finite difference analog of the conduction equation in discrete space. The finite difference equations are usually solved by successive overrelaxation or some similar iterative technique. The result obtained from such a numerical calculation is unique to the particular geometry, material, and source distribution assumed. In order to assess the effects of varying some parameter (material, geometry, etc.), a separate computer calculation is required for each variation. Such studies very often require a considerable amount of computer time. The theorems developed herein require only a single reference calculation of the temperature distribution and appropriate volume integrals (which involve no iteration). These theorems thus permit certain types of parameter survey studies to be performed by hand; hence, a considerable savings in computer time is realized.

It should be noted that, although the theorems as presented herein are shown to apply to heat-conduction systems, this is in no way a restriction. They are mathematically general and may be applied to any system behavior that can be described by the heat-conduction equation. Their derivation was motivated by a desire to explore the physical significance of the eigenvalue of a Sturm-Liouville equation which has a solution that is identical to the heat-conduction equation. Since the Sturm-Liouville equation involves both the eigenvalue and the eigenfunction (instead of just a function), it is possible to obtain additional information about a system when both equations have identical solutions. This additional information is obtained by performing classical operations on the eigenvalue equation and by physically interpreting the resulting expressions.

## FORMULATION OF THE EIGENVALUE PROBLEM

Consider a bounded heat-conduction system D composed of fixed heat generating (source) regions and heat dissipating regions (in which sources are vanishingly small). The system may or may not be at steady state. Assuming linearity of the conduction equation, the temperature at each point in this system may be described by

$$\nabla \cdot K(\vec{r}) \nabla t(\vec{r}, \theta) + \dddot{q}(\vec{r}) = \rho(\vec{r}) C_p(\vec{r}) \frac{\partial t(\vec{r}, \theta)}{\partial \theta} \tag{1}$$

where

| | |
|---|---|
| $\rho$ | density |
| $C_p$ | specific heat |
| $K$ | thermal conductivity |
| $\vec{r}$ | argument denoting position |
| $\nabla$ | gradient operator |
| $t$ | temperature |
| $\theta$ | argument denoting time |
| $\dddot{q}$ | volumetric source strength |

The temperature distribution defined by equation (1) at any particular time $\hat{\theta}$ may also be envisioned as a steady-state eigenfunction solution of the suitably formed three-dimensional (space only) Sturm-Liouville equation, which is

$$-\nabla \cdot K(\vec{r}) \nabla t(\vec{r}, \hat{\theta}) = \bar{\lambda}(\hat{\theta}) \Upsilon(\vec{r}, \hat{\theta}) t(\vec{r}, \hat{\theta}) \tag{2}$$

where the complementary function, $\Upsilon(\vec{r}, \hat{\theta})$, is appropriately chosen. The only restriction placed on equation (2) is that at least one eigenfunction must be identical to the temperature distribution obtained from equation (1) for any selected value of $\hat{\theta}$. The feasibility of such a formulation is guaranteed by the flexibility allowed in specifying $\Upsilon(\vec{r}, \hat{\theta})$. Integration of equation (2) over the system volume gives

$$-\int_D \nabla \cdot K(r) \nabla t(\vec{r}, \hat{\theta}) dV = \bar{\lambda}(\hat{\theta}) \int_D \Upsilon(\vec{r}, \hat{\theta}) t(\vec{r}, \hat{\theta}) dV \tag{3}$$

Since the heat flux $K(\vec{r}) \nabla t(\vec{r}, \hat{\theta})$ is continuous for all $\hat{\theta}$ (except possibly zero), the left side of equation (3) may be transformed by Gauss's theorem to surface integrals of the heat flux over all parts of the system and is recognized as the total heat loss rate from all parts the system at time $\hat{\theta}$. Thus,

$$-\sum_{\text{all } \xi} \int_{\xi} \nabla \cdot K(\vec{r}) \nabla t(\vec{r}, \hat{\theta}) dV = -\sum_{\text{all } \xi} \int_{\xi} K(\vec{r}) \nabla t(\vec{r}, \hat{\theta}) \cdot \overrightarrow{dS}$$

$$= \text{system heat loss rate at time } \hat{\theta}$$

$$= \bar{\lambda}(\hat{\theta}) \int_{D} \Upsilon(\vec{r}, \hat{\theta}) t(\vec{r}, \hat{\theta}) dV \tag{4}$$

where $\xi$ denotes some part of the system, and the sum of the $\xi$ equals D. Normally, only the boundaries of the system contribute to the divergence integrals. The summation has been used to allow for possible discontinuities in the heat flux at $\hat{\theta} = 0$. The eigenvalue is now defined as the ratio of the sum of the heat loss rates to heat production rate for the entire system at time $\hat{\theta}$:

$$\bar{\lambda}(\hat{\theta}) \equiv \frac{-\sum_{\text{all } \xi} \int_{\xi} K(\vec{r}) \nabla t(\vec{r}, \hat{\theta}) \cdot \overrightarrow{dS}}{\int_{D} q'''(\vec{r}) dV} = \frac{\text{Losses/sec}}{\text{Productions/sec}} \tag{5}$$

This definition fixes the integral of $\Upsilon(\vec{r}, \hat{\theta}) t(\vec{r}, \hat{\theta})$ as

$$\int_{D} \Upsilon(\vec{r}, \hat{\theta}) t(\vec{r}, \hat{\theta}) dV = \int_{D} q'''(\vec{r}) dV = \text{total system heat source} \tag{6}$$

but places no further constraint on $\Upsilon(\vec{r}, \hat{\theta})$, so that the formulation of equation (2) is still possible. The form of $\Upsilon(\vec{r}, \hat{\theta})$ is not known for all values of $\hat{\theta}$, however, it can be seen that, for the steady-state ($\hat{\theta} = \infty$), $\bar{\lambda}$ is 1.0 and the functional form of $\Upsilon(\vec{r}, \infty)$ must be

$$\Upsilon(r, \infty) = q(r)/t(r, \infty) \tag{7}$$

This guarentees the equivalence of equations (1) and (2) for the steady state but is not generally valid at all values of $\hat{\theta}$ since each point may be out of balance by a different amount. The steady-state form of this complementary function, given by equation (7), thus transforms the nonhomogeneous steady-state conduction equation into a homogeneous eigenvalue form with known linear operators.

## VIRTUAL IMBALANCE IN THE EIGENSYSTEM

The concept of $\bar{\lambda}$, introduced in the preceding section as a measure of the time-dependent ratio of net losses (Btu/hr) to productions (also Btu/hr), permits a physical interpretation of the heat equilibrium of the system. If $\bar{\lambda}$ is greater than 1.0, the system is cooling down, and if $\bar{\lambda}$ is less than 1.0, it is heating up. For $\bar{\lambda}$ equal to 1.0, the system is at equilibrium. The definition of $\bar{\lambda}$ as an eigenvalue also enables the development of a method for determining the degree of virtual imbalance for the equivalent eigensystem when the physical properties have experienced some alteration. The virtual imbalance referred to herein is the change in the eigenvalue of the equivalent eigenvalue equation (eq. (2)) when the operators are perturbed. It is not a function of time since the eigenvalue equation is not time dependent but is associated with a particular time. Examples of such alterations would be a change in material at a particular location, the introduction of additional sources into the system, or changes in dimensions of subregions of the system. These changes may all be effected by changing the spatial distribution of the thermal conductivity and/or source functions. The following derivation is concerned with the virtual imbalance in the eigensystem, designated as $\delta\bar{\lambda}$, associated with alterations in the conductivity in any region of a geometrically arbitrary system. The approach used is similar to the derivation of first-order eigenvalue changes by the theory of small perturbations. It is shown in detail to emphasize the inherent differences from perturbation theory that arise from the requirement that the eigenfunction must also be the solution to the nonhomogeneous conduction equation.

For simplicity, the system is assumed to be at the steady state. For this condition, the $\hat{\theta}$ argument is dropped, and equation (1) becomes

$$-\nabla \cdot K(\vec{r})\nabla t(\vec{r}) = \dot{q}'''(\vec{r}) \tag{8}$$

For this case, the eigenvalue of the eigentemperature distribution defined by equations (5) and (7) is 1.0 and is denoted by $\bar{\lambda}_0$. Equation (2) is then

$$-\nabla \cdot K(\vec{r})\nabla t_0(\vec{r}) = \bar{\lambda}_0 \Upsilon(r) t_0(\vec{r}) \tag{9}$$

To facilitate the manipulations that follow, this is rewritten in operator form

$$IT_0 = \lambda_0 J T_0 \tag{10}$$

where

$$I \equiv -\nabla \cdot K(r)\nabla \tag{11}$$

$$J \equiv \Upsilon(\vec{r}) \tag{12}$$

$$T_0 \equiv t_0(r) \qquad \text{(zero subscript denotes the unperturbed steady state)} \tag{13}$$

Next, a function is considered that will be called the adjoint eigentemperature. In the discussion that follows, this will be designated by $T_0^*$. This function is defined by

$$I^* T_0^* = \bar{\lambda}_0 J^* T_0^* \tag{14}$$

and has the same boundary conditions as the solution of equation (10).

If the system experiences a change in thermal conductivity in some subregion R contained in D, the loss operator I of region R will change by some amount $\delta I$. The eigenvalue will change by some amount $\delta\bar{\lambda}$ and the eigentemperature distribution by $\delta T$. In order that the source remain unchanged, J must be altered by an amount $\delta J$ such that

$$\int_D (J + \delta J)(T_0 + \delta T)dV = \int_D q'''(r)dV \tag{15}$$

The defining equation for the perturbed system is then

$$(I + \delta I)(T_0 + \delta T) = (\bar{\lambda}_0 + \delta\bar{\lambda})(J + \delta J)(T_0 + \delta T) \tag{16}$$

Multiplication of the perturbed equation (eq. (16)) on the left by the unperturbed adjoint eigentemperature $T_0^*$, and the unperturbed adjoint equation (eq. (14)) by the perturbed temperature $(T_0 + \delta T)$ gives

$$T_0^*(I + \delta I)(T_0 + \delta T) = (\bar{\lambda}_0 + \delta\bar{\lambda})T_0^*(J + \delta J)(T_0 + \delta T) \tag{17}$$

and

$$T_0^* I^*(T_0 + \delta T) = \bar{\lambda}_0(T_0 + \delta T)J^* T_0^* \tag{18}$$

Subtracting equation (18) from equation (17), integrating the result over the system volume, and making use of the identity,

$$\int f\varphi g \, dV = \int g\varphi^* f \, dV \tag{19}$$

(where f and g are any real functions satisfying the same homogeneous boundary conditions, and $\varphi$ is a linear operator) gives, for $\delta\bar{\lambda}$, exactly

$$\delta\bar{\lambda} = \frac{\int_R T_0^* \delta I(T_0 + \delta T)dV - (\bar{\lambda}_0 + \delta\bar{\lambda}) \int_Q T_0^* \delta J(T_0 + \delta T)dV}{\int_Q T_0^* J(T_0 + \delta T)dV} \tag{20}$$

where Q denotes the source region. A significant point here and one worthy of special emphasis is the exact property of equation (20). In the conventional perturbation theory development of eigenvalue perturbations, recourse is made to ignoring products of differential magnitude thus obtaining an approximation for $\delta\bar{\lambda}$ and restricting application of the theory to small perturbations exclusively. No such procedure, however, is required to obtain equation (20); thus, the range of applicability is unlimited for certain types of problems, as will be seen. There are several approximate forms of equation (20) that may be obtained by using $T_0$ in place of $(T_0 + \delta T)$ in one or more of the integrals. If, however, in the interest of preserving the exact nature of the development, this limit is taken in all integrals, the second integral in the numerator vanishes by equations (15) and (6). The resulting expression is

$$\delta\Lambda \equiv \frac{\int_R T_0^* \delta I T_0 \, dV}{\int_Q T_0^* J T_0 \, dV} = \lim_{\delta T \to 0} (\delta\bar{\lambda}) \tag{21}$$

This may be evaluated in general (for zero or nonzero boundaries) using equation (58). The admissability of the limiting operation is a point that may be arbitrary for the analytic version of equation (21); this was not explored. However, for numerical representations of the integrals (the primary source of application of the theorems to be derived is numerical) in equation (21), this operation is correct. It should be noted here that this limit is not the same as the limit that would be obtained using the definition of $\bar{\lambda}$ given by equation (5) at the instant of the perturbation In taking the limit, another deviation from perturbation theory had been introduced. The definition of $\bar{\lambda}$ is the same initially (before the perturbation) whether the weighting function is present or not as can

be seen by comparing equations (5) and (24). However, the limits of $\delta\bar{\lambda}$ as $\delta T$ approaches zero are not. Equation (21) then describes the virtual imbalance in the eigensystem that would result from a change in conductivity in some or all parts of the system.

It is of interest to derive equation (21) by using variational formulation in order to demonstrate that identical results may be obtained with an alternate approach. In operator form,

$$IT_0 = \bar{\lambda}_0 J T_0 \tag{22}$$

and

$$I^* T_0^* = \bar{\lambda}_0 J^* T_0^* \quad \text{(same eigenvalue)} \tag{23}$$

so that a variational principle (Rayleigh quotient) for $\bar{\lambda}_0$ is (ref. 1)

$$\bar{\lambda}_0 = \frac{\int_D T_0^* I T_0 \, dV}{\int_D T_0^* J T_0 \, dV} = \frac{\int_D T_0 I^* T_0^* \, dV}{\int_D T_0 J^* T_0^* \, dV} \tag{24}$$

Varying equation (24) gives, after trivial manipulation,

$$\delta\bar{\lambda} = \frac{\int_R T_0^* \delta I T_0 \, dV - \bar{\lambda}_0 \int_D T_0^* \delta J T_0 \, dV}{\int_Q T_0^* J T_0 \, dV} \tag{25}$$

Again, if the limit is taken as $\delta T$ approaches zero, no variation in $J$ is required. Thus, $\delta J$ is zero and equation (25) becomes

$$\delta\Lambda = \frac{\int_R T_0^* \delta I T_0 \, dV}{\int_Q T_0^* J T_0 \, dV} \tag{26}$$

which is identical to equation (21). The perturbed eigentemperature ($T_0 + \delta T$) is of no interest since it has no known physical significance. This arises from the fact that J is considered linear (independent of T) throughout the perturbation and loses contact with the physical problem for any temperature other than the one for which it was specifically chosen. It should be observed that, if the perturbation is made over most of Q(i.e., if $R \simeq Q$), $\delta\Lambda$ may be approximated by relatively crude estimates of the temperature distribution, the error in the $\delta\Lambda$ so calculated being much less than that of the temperatures used for evaluation (ref. 1). This is because when $R \simeq Q$, equation (26) is a stationary form for $\delta\Lambda$; that is, $\delta\Lambda$ in this case may be taken as the eigenvalue of the equation

$$\delta I T_0 = \delta\Lambda J T_0 \tag{27}$$

the Rayleigh quotient of which is formed by premultiplication by $T_0^*$ and integration over the volume giving equation (26). This may afford some computational convenience when evaluating equation (26) for system changes of this type.

It should be noted that the limiting form $\delta\Lambda$ does not involve J or $\delta J$, so that the functional form of $\Upsilon$ is not required to evaluate $\delta\Lambda$. Hence, the perturbation can be made in a system that was not necessarily at the steady state initially. This was assumed for clarity of presentation but is not a requirement. Hence, the zero subscript in all of the preceding derivation may refer to an initial distribution that has not reached the steady state.

## TEMPERATURE PREDICTION - THEOREMS I AND II

A utilitarian aspect of the eigenvalue approach to heat conduction is the prediction of the change in average source temperature resulting from a change in conductivity and/or source anywhere in the system. In the following discussion, the J coefficient operator is not fixed throughout the perturbation derivation as was done previously. Instead, the time-dependent nature of $\bar{\lambda}$ is considered. In the previous discussion, the operator J was taken to be fixed and associated with a particular temperature distribution and time; thus the perturbed eigentemperature was meaningless. For the purposes of the development that follows, this operator is thought of as varying continuously at all times following the perturbation in such a manner as to ensure that the real temperature and eigentemperature remain identical.

Consider the steady-state form of $\bar{\lambda}_0$ described by equation (24). The ratio $\bar{\lambda}_0$ is understood to be an eigenvalue in this equation. If the time dependence of the temperature and the J operator is recognized, the eigenvalue variation will also be time dependent. The eigenvalue variation is given in operator notation by equation (20), which, for the purpose of this derivation, will be considered to incorporate a time dependence in T, J, and $\bar{\lambda}$. Following a conductivity perturbation, $\delta\lambda$ will assume some transient functional behavior. In the limit as $\theta$ approaches infinity, $\delta\bar{\lambda}$ must become zero since the system tends to regain the steady-state condition ($\bar{\lambda}$ approaches 1.0). This means that $\delta J$ ultimately takes on some nontrivial form in view of the requirement of a continuously varying J operator just discussed. Denoting the final temperature distribution taken on by $T + \delta T$ as $T_p$, the perturbed steady-state distribution (eq. (20)), becomes, in the limit as $\delta\bar{\lambda}$ goes to zero,

$$\delta\bar{\lambda} = \frac{\int_R T_0^* \delta I T_p \, dV - (\lambda_0 + \delta\lambda) \int_Q T_0^* \delta J T_p \, dV}{\int_Q T_0^* J T_p \, dV} = 0 \tag{28}$$

so that

$$\frac{\int_R T_0^* \delta I T_p \, dV}{\int_Q T_0^* J T_p \, dV} = \frac{\bar{\lambda}_0 \int_Q T_0^* \delta J T_p \, dV}{\int_Q T_0^* J T_p \, dV} \tag{29}$$

The adjoint and real eigentemperatures are identical, since equation (10) is self adjoint (ref. 2). The self adjointness of the homogeneous Sturm-Liouville equation only holds for homogeneous boundary conditions of the form $\alpha \partial T_0 / \partial \vec{r} + \beta T_0 = 0$. For the present, these conditions are assumed, that is, zero temperature, zero heat flux, or zero external sink temperature. The treatment of nonhomogeneous boundary conditions is left for subsequent discussion. Since the system is initially at steady state, $\bar{\lambda}_0$ is 1.0. Replacing $T_p$ by $T_0$ in the denominator only of equation (29) (the denominator is arbitrary) and dropping the adjoint distinction give

$$\frac{\int_R T_0 \delta I T_p \, dV}{\int_Q T_0 J T_0 \, dV} = \frac{\int_Q T_0 \delta J T_p \, dV}{\int_Q T_0 J T_0 \, dV} \tag{30}$$

Multiplying the integrands of equation (15) by $T_0$ gives

$$T_0 q''' = T_0(J + \delta J)(T_0 + \delta T) = T_0(J + \delta J)T_p \tag{31}$$

or

$$T_0 \delta J T_p = T_0 q''' - T_0 J T_p \tag{32}$$

From equations (8), (9), (12), and (13) with $\bar{\lambda}_0$ equal to 1.0,

$$T_0 J T_0 = q''' T_0 \tag{33}$$

Substituting equation (33) into equation (32) gives

$$T_0 \delta J T_p = q'''(T_0 - T_p) \tag{34}$$

The $\delta I$ operator will be seen in a subsequent section to involve only the temperature gradient. Hence, $T_p$ may be replaced by $T_0$ exactly in the numerator of the left side of equation (30) if the integrand is multiplied by the ratio of the unperturbed to perturbed conductivities in region R. This is only valid, however, when the heat flow to all points in the perturbed region remains constant (from Fourier's law, the temperature gradient at a point is inversely proportional to the conductivity for constant heat input). If the heat source to the perturbed points changes as a result of the perturbation, a different correction may be required. This is discussed in appendix A. The former correction will be retained herein to simplify the derivation.

Making this replacement and substituting equations (33) and (34) into equation (30) give

$$\frac{\int_R T_0 \delta I\left(\frac{K_0}{K_p}\right) T_0 \, dV}{\int_Q q''' T_0 \, dV} = \frac{\int_Q q'''(T_0 - T_p) dV}{\int_Q q''' T_0 \, dV} \tag{35}$$

Comparing equations (35) and (21) it is noted that, if the correction $K_p/K_0$ is taken outside of the integral as would be possible if the conductivities were constants over R, equation (35) could be written

$$\delta\Lambda = \frac{K_p}{K_0} \frac{\int_Q q'''(T_0 - T_p)dV}{\int_Q q''' T_0 \, dV} \tag{36}$$

where $\delta\Lambda$ is given by equation (26). Also, if the source strength is spatially constant, the numerator and denominator of equation (36) may be divided by the source volume and rearrange to obtain the fractional change in the average temperature of the source due to a conductivity perturbation anywhere in the system.

Theorem I

$$\frac{\delta\overline{T}}{\overline{T}} = \left(\frac{K_0}{K_p}\right)\delta\Lambda \tag{37}$$

This theorem relates the fractional temperature change in the source region to the limiting form of the eigenvalue change that is caused by an alteration in the loss operator of the equivalent eigensystem. The conductivity ratio has been assumed constant over R in this derivation so that equation (37) could be shown simply. Making this assumption is not necessary, however, because, for any numerical model, conductivities must be treated numerically as discontinous step functions. Even in practice, most systems are considered to be composites of materials each having its own conductivity. If the conductivity is temperature dependent, equation (1) is nonlinear; this case is not amenable to the eigenvalue approach, since the operators must be assumed linear. If the source is not spatially constant, a different type of average change is obtained, namely, the change in the source weighted temperatures. The limiting $\delta\Lambda$ is evaluated using equation (57).

An interesting corollary implied by equation (37) is that the relative change in the source temperature resulting from two different changes in the conductivity in a specific region is inversely related to the conductivity ratio and directly related to the ratio of the conductivity changes:

Corollary to Theorem I

$$\frac{\delta\bar{T}_1}{\delta\bar{T}_2} = \frac{\left(K_p\right)_2}{\left(K_p\right)_1}\frac{\delta K_1}{\delta K_2} \tag{38}$$

This is derived by dividing equation (37) for one change (subscript 1) by equation (37) for a second change (subscript 2) in the same region. Then, by equation (57) the temperature distribution dependence may be removed by canceling.

Equation (38) enables comparisons of the relative effects on source temperature of using several different materials in a specific region of the system with no knowledge of the unperturbed system geometry, materials, or source strength (only the unperturbed conductivity in the region of interest is required).

In all of the foregoing, the initial state of the system was assumed to be steady. This is not an essential assumption. The $\bar{\lambda}_0$ could have been carried throughout the development thus giving the fractional change in source temperature based on the average source temperature at the time of the perturbation (which could have been changing with time) and the steady-state temperature finally achieved. This has little practical significance, however, and will not be given detailed consideration.

## Source Effects

To derive an expression for the effect of changes in the source distribution on the unperturbed average source temperature, manipulations shown in equations (28) to (36) are performed. This gives, when it is remembered that the integration is over the whole system instead of with the limits shown in equation (28),

$$\delta\bar{\lambda} = \int_D T_0^* \delta J(T_0 + \delta T)dV = 0 \tag{39}$$

This expression reflects both the change in source distribution because of the perturbation and the change in $J$ in the unperturbed (original) source that is required to offset the temperature change that occurs as a result of the perturbation. Distinguishing these regions by their integral limits, equation (39) may be written as

$$\int_R T_0 \delta J(T_0 + \delta T)dV = -\int_D T_0 \delta J(T_0 + \delta T)dV \tag{40}$$

or

$$\int_R T_0 \delta q''' \, dV = - \int_D q'''(T_p - T_0) dV \tag{41}$$

The last step is similar to manipulations performed in the previous derivation. Retention of the minus sign depends on how $\delta q$ is defined. On the left of equation (40), use has been made of the fact that, prior to the perturbation, J was related to q by

$$JT_0 = q''' \tag{42}$$

If the perturbed region contained no source originally, J was zero and, after the perturbation,

$$\delta J(T_0 + \delta T) = \delta q''' \tag{43}$$

If the sources are spatially constant, manipulation of equation (41) and replacing $T_0$ with $t_0$ give

Theorem II

$$\frac{\delta \overline{T}}{\overline{T}} = \frac{\delta q'''}{q'''} \frac{\int_R t_0 \, dV}{\int_Q t_0 \, dV} \tag{44}$$

The $\delta q'''$ is shown outside of the integral with the numerical model in mind. If the change were not spatially constant, it would have to be represented by a set of constant values over small regions. The total effect would then be found by applying this equation to each small region. This is also true for theorem I. Theorem II relates the fractional change in source temperature (the source being defined as all regions containing sources before the perturbation) to the addition or deletion of a source in any region R.

The corollary to theorem II is formed as was the corollary to theorem I. This is

Corollary to theorem II

$$\frac{\delta \overline{T}_1}{\delta \overline{T}_2} = \frac{(\delta q''')_1}{(\delta q''')_2} \tag{45}$$

As with the corollary to theorem I, this result is not obvious. It allows a direct comparison of the average temperature increases in the source region of a system, which would result from the addition of sources of varying strength in a given location, with no knowledge of the system properties or any temperature redistribution that may result from the introduction of an extraneous source (only the unperturbed source in the region of interest is required).

## EVALUATION OF $\delta\Lambda$ FOR CONDUCTIVITY OR SOURCE CHANGES

In order to emphasize some of the physical restrictions that are encountered in the application of equation (21), its algebraic equivalent must be shown in detail.

The denominator of equation (21) presents no problem since, by equations (8), (12), and (13), it is just

$$\text{Denominator} = \int_Q t_0(\vec{r})\, q'''(\vec{r})\, dV \tag{46}$$

The integrand of the numerator is derived as follows:

$$I = -\nabla \cdot K\nabla \tag{47}$$

$$I + \delta I = -\nabla \cdot (K + \delta K)\nabla \tag{48}$$

By identity,

$$\nabla \cdot S\vec{V} = \nabla S \cdot \vec{V} + S\nabla \cdot \vec{V} \tag{49}$$

where $S$ is any scalar and $\vec{V}$ is any vector. Using equation (49), equation (48) becomes

$$I + \delta I = -\left[\nabla(K + \delta K) \cdot \nabla + (K + \delta K)\nabla^2\right] \tag{50}$$

Also, equation (47) can be written as

$$I = -(\nabla K \cdot \nabla + K\nabla^2) \tag{51}$$

Subtracting equation (51) from equation (50) gives

$$\delta I = - (\nabla \delta K \cdot \nabla + \delta K \nabla^2) \tag{52}$$

Substituting equations (52) and (46) into equation (21) and changing $T_0^*$ and $T_0$ to their equivalent $t_0$ give

$$\delta \Lambda = \frac{- \int_R \left( t_0 \nabla \delta K \cdot \nabla t_0 + t_0 \delta K \nabla^2 t_0 \right) dV}{\int_Q \dddot{q} t_0 \, dV} \tag{53}$$

Making use of the same identity as before yields

$$\nabla \cdot (t_0 \delta K \nabla t_0) = \nabla t_0 \delta K \cdot \nabla t_0 + t_0 \delta K \nabla^2 t_0$$

$$= \delta K \left| \nabla t_0 \right|^2 + t_0 \delta K \nabla^2 t_0 + t_0 \nabla \delta K \cdot \nabla t_0$$

Solving for $-(t_0 \nabla \delta K \cdot \nabla t_0)$ results in

$$-(t_0 \nabla \delta K \cdot \nabla t_0) = -\nabla \cdot (t_0 \delta K \nabla t_0) + \delta K \left| \nabla t_0 \right|^2 + t_0 \delta K \nabla^2 t_0 \tag{54}$$

Substituting equation (54) into equation (53) gives

$$\delta \Lambda = \frac{- \int_R \left[ \nabla \cdot (t_0 \delta K \nabla t_0) \right] dV + \delta K \int_R \left| \nabla t_0 \right|^2 dV}{\int_Q \dddot{q} t_0 \, dV} \tag{55}$$

At this point it is recalled that the boundaries of the system were assumed to be at zero temperature. Then, by Gauss's theorem, the first integral in the numerator of equation (55) can be transformed to a surface integral over the surface S of D to yield

$$\int_R - \left[ \nabla \cdot (t_0 \delta K \nabla t_0) \right] dV = \int_D - \left[ \nabla \cdot (t_0 \delta K \nabla t_0) \right] dV = - \int_S t_0 \delta K \nabla t_0 \cdot d\vec{S} = 0 \tag{56}$$

Hence, for the zero surface temperature condition,

$$\delta\Lambda = \frac{\int_R \delta K |\nabla t_0|^2 \, dV}{\int_Q \dddot{q} t_0 \, dV} \tag{57}$$

If the boundary temperatures are nonzero, the application of equation (57) in equation (37) requires that the temperature be some constant at the physical boundaries of the system. It can be seen that, for this case, the gradient is unaltered and the denominator is increased by the constant surface temperature. The fractional temperature change in the source $\delta\overline{T}/\overline{T}$ is affected identically by the addition of a constant to the temperatures involved. Hence, this is only a normalization effect and equation (57) remains effectively valid for use in equation (37) to predict fractional average source temperature changes.

The $\delta\Lambda$ computed using equation (57) does not represent the virtual imbalance of an eigensystem with nonzero boundaries. If the boundary temperature is some constant, however, $t_0$ in equation (57) is taken to be the solution to the homogeneous boundary value problem formed by subtracting the boundary value from the temperature (ref. 3). All operations required to derive equation (57) may be performed exactly as shown on this temperature excess function (which has been normalized to zero at the boundaries), and equation (57) will again be obtained. Hence, to use equation (57) for the nonhomogeneous boundary condition problem (prescribed nonzero constant), it is noted that the addition or subtraction of a constant to the numerator results in no change. However, the denominator must be decreased appropriately. The exact expression for the nonzero boundary case is then

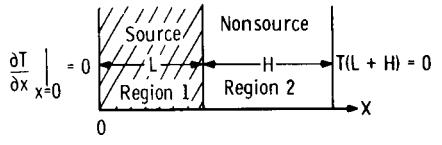
$$\delta\Lambda = \frac{\int_R \delta K |\nabla t_0|^2 \, dV}{\int_Q \dddot{q} t_0 \, dV - t_0(S) \int_Q \dddot{q} \, dV} \tag{58}$$

This correction is never used, however, since it is not required for the $\delta\Lambda$ that appears in equation (37). This is because the boundary temperature must also be included in the denominator of the change in the average source temperature. This is apparent in the simple example calculations that follow. The $\delta\overline{T}/\overline{T}$ is readily seen to be independent of the boundary temperature by adding a constant to the temperature in the conventional solution. Equation (44) does, however, require slight modification when used with a constant nonzero boundary problem. For this case, the numerator should be corrected to give

$$\frac{\delta\overline{T}}{\overline{T}} = \frac{\int_R \delta q''' t_0 \, dV - t_0(S) \int_R \delta q''' \, dV}{\int_Q q''' t_0 \, dV} \tag{59}$$

## EXAMPLE CALCULATIONS

In order to demonstrate the mechanics of the temperature prediction technique (theorem I) proposed herein, a sample calculation is performed for an elementary geometry. The average temperature change in the source region for a simple one-dimensional system is calculated for conductivity perturbations in source and nonsource regions. Theorem II is demonstrated by using a source perturbation in the same geometry. These calculations are elementary examples of the application of the $\delta\Lambda$ method and serve the dual purpose of illustrating the method in detail and providing test cases necessary to complete the exposition. They are trivial examples in the sense that the results are obvious and may be obtained more directly. They are not suggested as typical applications, but are included only to help clarify the concept. To examine theorem I, the infinite slab at steady state (see sketch) is considered.




The conductivity in region 1 is to be perturbed. First, $\delta\overline{T}/\overline{T}$ will be computed using theorem I. Denoting regions by subscripts and perturbed and unperturbed states by superscripts, a conductivity perturbation in region 1 yields

$$\delta\Lambda = \frac{\delta K \int_{(1)} \left| \nabla T_1^0 \right|^2 dV}{\int_{(1)} q''' T_1^0 \, dV} = \frac{\int_0^L \left| \frac{\partial T_1}{\partial x} \right|^2 dx \left( K_1^0 - K_1^p \right)}{\int_0^L q''' T_1^0 \, dx} \tag{60}$$

The temperature distributions assuming continuity of heat flux and temperature at $x = L$, symmetry at $x = 0$, and zero at $x = L + H$ are

$$T_1^0 = \frac{q'''}{2K_1^0}(L^2 - x^2) + \frac{q'''LH}{K_2^0} \tag{61}$$

$$T_2^0 = \frac{q'''}{K_2^0}(L^2 - xL) + \frac{q'''LH}{K_2^0} \tag{62}$$

The gradient in region 1 is

$$\frac{\partial T_1^0}{\partial x} = -\frac{q'''x}{K_1^0}; \quad \left|\frac{\partial T_1^0}{\partial x}\right|^2 = \frac{q'''^2x^2}{\left(K_1^0\right)^2} \tag{63}$$

which gives

$$\delta\Lambda = \frac{\left(K_1^0 - K_1^p\right)\dfrac{q'''^2}{\left(K_1^0\right)^2}\displaystyle\int_0^L x^2\,dx}{\displaystyle\int_0^L \frac{q'''^2L^2}{2K_1^0}\,dx - \int_0^L \left(\frac{q'''^2}{2K_1^0}\right)x^2\,dx + \frac{q^2LH}{K_2^0}\int_0^L dx} = \frac{\delta K_1}{K_1^0}\left(\frac{1}{1 + \dfrac{3HK_1^0}{LK_2^0}}\right) \tag{64}$$

and the fractional change in the source temperature is, by equation (37),

$$\frac{\delta\overline{T}}{\overline{T}_0} = \frac{K_1^0}{K_1^p}\delta\Lambda = \frac{\delta K_1}{K_1^p}\left(\frac{1}{1 + \dfrac{3HK_1^0}{LK_2^0}}\right) \tag{65}$$

This is compared with the result obtained conventionally as follows: $\delta\overline{T}/\overline{T}$ in the source computed using equation (61) gives

$$\overline{T}_0 = \frac{\int_0^L \frac{q'''}{2K_1^0}\left(L^2 - x^2 + \frac{2LHK_1^0}{K_2^0}\right)dx}{\int_0^L dx} = \frac{q'''L^2}{3K_1^0} + \frac{q'''LH}{K_2^0} \tag{66}$$

$$\overline{T}_p = \frac{\int_0^L \frac{q'''}{2K_1^p}\left(L^2 - x^2 + \frac{2LHK_1^p}{K_2^0}\right)dx}{\int_0^L dx} = \frac{q'''L^2}{3K_1^p} + \frac{q'''LH}{K_2^0} \tag{67}$$

and

$$\frac{\delta\overline{T}}{\overline{T}} = 1.0 - \frac{\frac{q'''L^2}{3K_1^p} + \frac{q'''LH}{K_2^0}}{\frac{q'''L^2}{3K_1^0} + \frac{q'''LH}{K_2^0}} = \frac{\delta K_1}{K_1^p}\left(\frac{1}{1 + \frac{3HK_1^0}{LK_2^0}}\right) \tag{68}$$

which is identical to the result obtained in equation (65). It is noted that, for a conductivity perturbation in the nonsource region, both methods give

$$\frac{\delta\overline{T}}{\overline{T}} = \frac{\delta K_2}{K_2^p}\left(\frac{1}{1 + \frac{LK_2^0}{3HK_1^0}}\right) \tag{69}$$

In both of these calculations, the $K_0/K_p$ correction was used since the heat flow to the perturbed regions was not affected because of the symmetry of the perturbation.

To minimize notation, theorem II may be readily examined by assuming that the conductivities are the same in all regions. Let the perturbation consist of adding a source to region 2 having the same unit strength as in region 1 before the perturbation. The change in source strength in region 2 is then

$$\delta q''' = q'''_{p_2} - q'''_{0_2} = q'''_1 - 0 = q'''_1 \equiv q''' \tag{70}$$

Theorem II gives, for the fractional change in $\overline{T}_1$,

$$\frac{\delta \overline{T}}{\overline{T}} = \frac{\delta q''' \int_R T_0 \, dV}{\int_Q q''' T_0 \, dV} = \frac{\int_L^{L+H} T_2^0 \, dx}{\int_0^L T_1^0 \, dx} \tag{71}$$

$$\frac{\delta \overline{T}}{\overline{T}} = \frac{3H^2}{2L^2 + 6LH} \tag{72}$$

After the perturbation, $q'''$ will be in both regions. Since K was assumed the same in both regions, we have a homogeneous slab of thickness L + H. The temperature distribution in the perturbed case is then

$$T_p = \frac{q'''}{2K}\left[(L + H)^2 - x^2\right] \tag{73}$$

and the average over region 1 is

$$\overline{T}_p = \frac{q'''}{2K}\left[(L + H)^2 - \frac{L^2}{3}\right] \tag{74}$$

Using equation (61), the unperturbed average is

$$\overline{T}_0 = \frac{q'''}{2K}\left(\frac{2}{3}L^2 + 2LH\right) \tag{75}$$

giving a fractional change of

$$\frac{\delta\overline{T}}{\overline{T}} = \frac{3H^2}{2L^2 + 6LH}$$

which agrees with equation (72).

# APPLICATIONS

It is not the intent of this report to elaborate on the application of the proposed theorems since the a priori practicality of any concept is difficult to assess depending strongly, as it does, on individual ingenuity. It is worthwhile, however, to devote some discussion to possible areas of application for the sake of completeness.

## Survey Calculations

As suggested in the INTRODUCTION, this method is most powerful as a practical tool when applied to conduction problems having complex geometries. Suppose, for example, that a design survey is to be performed for some heat-conduction device that, out of necessity, has a complicated geometrical configuration. It is also assumed that the geometry is of sufficient complexity to preclude any hand calculational approach and that the principal interest is in the source temperature. In this situation, the required calculations must be performed numerically. The disadvantage here is that, as some parameter is varied, say the size of a particular part of the system, several machine calculations must be performed in order to establish a trend. This could be expensive depending on the complexities of the system and the machine code used to analyze it. The eigenvalue method, however, would require only one calculation of the temperature distribution for a reference case and an additional calculation of the volume integrals of the gradients squared (which would require negligible machine time compared to the temperature calculation). With this information, several types of parameter variation computations could be performed by hand using theorem I.

## Nonlinearities

Effects of nonlinearities are also amenable to hand calculation using theorem I. Suppose that, because of extreme temperature variations, the thermal conductivity in

the above problem varies strongly with position. To approximate the average source temperature, the temperature distribution is first calculated assuming temperature independent conductivities, and then the deviations in the source temperature due to the nonlinearity in K are summed.

## Equivalence of Perturbations

Another possible application of theorem I is the calculation of changes in one region that are required to offset changes that are made in another region in order for the source temperature to remain constant.

## Comparison Calculations

If the relative effect of several different materials in one specific region is of interest, the corollary to theorem I may be applied directly. In fact, it requires no detailed knowledge of the system (geometry, source strength, etc.); only the unperturbed conductivity in the region of interest is required.

## Source Interactions

Consider a large source such as an electrical bus bar embedded in a thick insulator. A second source of the same unit strength is to be inserted in the same insulator. The effect on the first source (average temperature change) is seen from theorem II to be equal to the ratio of the unperturbed temperature integral over the region that would be occupied by the second source to this integral over the first source. Hence, if the unperturbed temperature distribution is known, the optimum position (and shape) of the additional source may be selected intelligently so as to minimize the interaction. With some ingenuity, several applications of this kind are possible.

## Perturbations

For problems of complicated geometry, the classical application of perturbation theory suggests that a simpler problem be solved which closely resembles the problem in question. The solution of the complicated problem is then obtained by perturbing the simpler problem. The eigenvalue methods of theorems I and II may also be applied in

this manner but have the added advantage of imposing no constraint on the magnitude of the perturbation (for problems in which the perturbation does not affect the heat sources), which is an inherent weakness of perturbation theory.

To summarize the discussion of practicality, it may be said that the practical aspects of these methods are in essence all founded on the ability to anticipate the effect of a change in any conduction system using only the unperturbed temperature distribution. For geometrically complicated problems, especially those that must be solved numerically, the savings in computer time could be significant.

## DISCUSSION OF THE EIGENVALUE METHOD

It is of interest to note a few of the mathematical properties of the eigenvalue approach to heat conduction. Fortunately, the complementary function never requires evaluation and acts only as a mathematical expedient. In the derivation of theorem I, in effect $(T_0 + \delta T)$ is replaced by $T_0$ exactly and not approximately as in the first-order theory of small perturbations. The significance of this is that theorem I is valid for changes of any magnitude for problems in which the heat source to the perturbed points is constant, whereas first-order perturbation theory is valid only for small perturbations. Thus, theorem I is a more powerful calculational device for problems of this type than perturbation methods are for eigenvalue problems. For problems having variable heat source to the perturbed points, the method can be used as an approximation similar to first order perturbation theory. This is discussed in appendix A.

In addition to this, for $R \simeq Q$, the eigenvalue method described herein retains the capacity of yielding an approximation for $\delta\Lambda$ that will generally be much more accurate than the temperature distributions used to evaluate it because of its stationary property. Hence, crude estimates of the temperature may be used to obtain relatively accurate values of $\delta\Lambda$.

## CONCLUDING REMARKS

The solution of the nonhomogeneous heat-conduction equation for an arbitrary geometry has been viewed as an eigenfunction over the same geometric space having homogeneous boundary conditions. An approach similar to first-order perturbation theory was used to examine changes in the eigenvalue that would be caused by changes in the conductivity or source strength distributions. As a result of this, theorems have been derived that relate the average source temperature to changes in thermal conductivity or sources at any or all points in the thermal conduction system. It is expected that these theorems

may find practical application for conduction systems much like the perturbation formulae of nuclear reactor physics. They are especially suited to geometrically complex systems that must be analyzed numerically in that they enable certain types of parameter variation studies to be performed by hand using the computer solution for a single reference case. These theorems were developed out of a recognition of the physical significance of the eigenvalue of an equivalent Sturm-Liouville solution. Their extended application to other physical systems, which can be described by the heat-conduction equation, requires only a physical interpretation of $\delta\Lambda$.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, December 2, 1966,
122-29-05-08-22.

# APPENDIX A

## REPLACEMENT OF THE PERTURBED GRADIENT BY THE UNPERTURBED GRADIENT IN EQUATION (30)

Conduction calculations performed numerically yield the temperature at a set of node points. In order to obtain the gradient at a point (the gradient is required to evaluate the $\delta\Lambda$ expression), a seven-point-difference description might be used in rectangular coordinates, for example, where

$$\nabla T\Big|_i \equiv \sum_j \left(T_{i+1} - T_i\right)_j \Delta_j^{-1} - \sum_j \left(T_{i-1} - T_i\right)_j \Delta_j^{-1}; \qquad j = x, y, z \tag{A1}$$

where the unit vectors have been omitted for simplicity and $\Delta$ denotes the appropriate mesh interval. Hence, the gradient is seen to be numerically a simple function of the temperature differences between i and its immediate neighbors. For simplicity, only one of these temperature differences is discussed. A correction to this $\Delta T$ that will account for a change in the conductivity of the material separating the two reference points is examined. The function required to evaluate the correction might possibly be obtained numerically along with the reference calculation of the temperature distribution. This is not suggested, however, since a simpler method may be possible depending on the nature of the problem and the numerical procedure used to solve it. It is shown herein primarily to give some insight into the behavior of the gradient correction and to provide qualitative bounds on the range of validity of the temperature prediction method when used as an approximation (without correction).

From elementary, direct-current theory, the voltage drop (which is analogous to the temperature difference) across any resistance R is given by

$$\Delta V = \left(\frac{E_T}{R_T + R}\right) R \tag{A2}$$

where $E_T$ is the Thévenin equivalent electromotive force and $R_T$ is the Thévenin equivalent resistance of the system measured across the points in question in the absence of R and any source.

The ratio of perturbed to unperturbed voltage drops and, thus, temperature differences by analogy, across the same points, is then

$$\frac{\Delta T_p}{\Delta T_0} = \frac{\mathcal{R}_p}{\mathcal{R}_0} \frac{(\mathcal{R}_T + \mathcal{R}_0)}{(\mathcal{R}_T + \mathcal{R}_p)} = \frac{\left(\frac{\mathcal{R}_T}{\mathcal{R}_0} + 1\right)}{\left(\frac{\mathcal{R}_T}{\mathcal{R}_p} + 1\right)} \tag{A3}$$

where the $\mathcal{R}$ values refer to thermal resistances

$$\mathcal{R} \equiv \frac{l}{\mathscr{A} K} \tag{A4}$$

in which $l$ is the separation distance between the points and $\mathscr{A}$ is the area of heat transfer.

Qualitatively, a perturbation can be characterized by the volume over which it applies and the amount by which the conductivity has been changed. Some insight into the behavior of the correction for limiting combinations of these characteristics can be obtained using (A3) if the problem is such that the following may be assumed:

(a) perturbation over a small volume $\Rightarrow \mathcal{R}_T >>> \mathcal{R}_0$ or $\mathcal{R}_p$

(b) small perturbation in $K \Rightarrow \mathcal{R}_0 \approx \mathcal{R}_p$

(c) perturbation over a large volume $\Rightarrow \mathcal{R}_T <<< \mathcal{R}_0$ or $\mathcal{R}_p$

(d) large change in $K \Rightarrow \mathcal{R}_p <<< \mathcal{R}_0$ or the reverse.

Considering all combinations of the above, we have

| Combination | $\Delta T_p/\Delta T_0$ |
|---|---|
| Large volume, small change in K | $\approx 1.0$ |
| Large volume, large change in K | $\approx 1.0$ |
| Small volume, small change in K | $\approx 1.0$ |
| Small volume, large change in K | $\approx \mathcal{R}_p/\mathcal{R}_0 = K_0/K_p$ |

If the Thévenin equivalent resistances can be computed, an exact correction is possible. If these are not available, however, the above considerations should provide some feel for the validity of the result.

# APPENDIX B

## SYMBOLS

| | |
|---|---|
| $\mathscr{A}$ | area of heat transfer |
| D | limit of integration denoting whole system |
| $E_T'$ | Thévenin equivalent electromotive force |
| f, g | any real functions |
| I | loss operator |
| i | subscript denoting node point in numerical conduction model |
| J | operator form of $\Upsilon$ |
| j | summation index denoting coordinate axes in rectangular coordinates |
| K | thermal conductivity |
| L, H | region dimension in example calculation |
| $l$ | distance between two node points |
| p | superscript or subscript denoting perturbed state |
| Q | limit of integration denoting source region |
| $q'''$ | volumetric source strength |
| R | limit of integration denoting perturbed region |
| $R_T'$ | Thévenin equivalent resistance |
| $\mathscr{R}$ | resistance to heat flow |
| $\vec{r}$ | vector denoting position |
| S | surface limit of integration denoting surface of whole system |
| T | operator form of t |
| $\overline{T}$ | average source temperature |
| t | temperature |
| x | space variable in example calculation |
| $\alpha, \beta$ | arbitrary constants |
| $\delta$ | variation operator |
| $\delta\overline{\lambda}$ | perturbation in $\overline{\lambda}$ when $\overline{\lambda}$ is taken to be an eigenvalue |
| $\delta\Lambda$ | limiting of $\delta\overline{\lambda}$ |
| $\overline{\lambda}$ | ratio of losses to sources for whole system |
| $\xi$ | limit of integration denoting some part of system |
| $\Upsilon$ | complimentary function |
| $\varphi$ | arbitrary linear operator |
| * | superscript denoting adjoint function |
| $\nabla$ | gradient operator |
| 0 | superscript or subscript denoting unperturbed or steady state |
| (1), (2) | limits of integration in example calculation |

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