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INTEGRATION OF CONTROL EQUATIONS AND THE PROBLEM OF SMALL TIME CONSTANTS

by J. F. Andrus

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Huntsville, Ala.

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INTEGRATION OF CONTROL EQUATIONS AND THE PROBLEM OF SMALL TIME CONSTANTS

SUMMARY

A system of first-order, linear differential equations with constant coefficients is transformed into a convenient and considerably simplified system of differential equations, referred to as the canonical equations. Alternatively, a transfer function in the form of a rational function is represented by a like system of canonical equations. The latter equations can be readily solved provided the forcing function E_{in} can be conveniently represented. Furthermore, the canonical equations which depend upon poles (eigenvalues) of large magnitude may be discarded when it is evident, as is often the case, that the contributions of their solutions to the response E_{out} of the system are negligibly small. The elimination of such equations enables one to integrate the system numerically with a larger integration step-size than that which would ordinarily be required.

Often it can not be easily decided which, if any, canonical equations can be discarded and, furthermore, E_{in} can not be simply represented and may, in fact, not even be predictable as in cases in which it is related to E_{out} through a system of nonlinear differential equations. Such cases are treated by assuming E_{in} to be a linear function of time over short time intervals Δt; that is, E_{in} is expressed as

$$E_{in}(t) = E_{in}(T) + \frac{t-T}{\Delta t}\left[E_{in}(T + \Delta t) - E_{in}(T)\right]$$

on the interval

$$T \leq t \leq T + \Delta t.$$

Assuming the values of all significant variables to be known at time T, the solution $E_{out}(T + \Delta t)$ to the canonical equations is expressed analytically in terms of the unknown quantity $E_{in}(T + \Delta t)$. The analytical expression for $E_{out}(T + \Delta t)$ is then substituted into the nonlinear differential equations,

thereby effectively eliminating the variable E_{out} from the problem. The resulting set of nonlinear differential equations may be integrated numerically.

The important advantage, derived from determining E_{in} analytically and then eliminating it, is the very significant increase in the numerical integration step-size frequently made possible, especially when the given system of linear differential equations has eigenvalues of large magnitude (or, equivalently, small time constants).

As a by-product of the theory, one obtains the relationship of the eigenvalues and eigenvectors of a linear system of differential equations to the poles and coefficients of the corresponding transfer function.

Two illustrative examples of problems with small time constants are given, and subroutines for implementing the theory on digital computers are described.

INTRODUCTION

This paper is concerned with the numerical integration of systems of equations that have the following form:

$$\dot{y}_k = \theta_k(t, y_1, y_2, \ldots, y_{n_d}, E_{in}, E_{out}) \quad (k = 1, 2, \ldots, n_d) \tag{1a}$$

$$\dot{E}_{in} = \phi(t, y_1, y_2, \ldots, y_{n_d}, E_{in}, E_{out}) \tag{1b}$$

$$L(E_{out})/L(E_{in}) = p(s)/g(s) \tag{1c}$$

Differentiation is with respect to time t, L(f) signifies the Laplace transform F(s) of the function f(t), and p(s) and g(s) are polynomials in s such that the degree of p is equal to or less than the degree of g.

Equations (1) occur frequently in the simulation of control systems. The subsystem composed of (1a) and (1b) represents a system of nonlinear differential equations such as the equations of motion of a space vehicle, whereas the transfer function p(s)/g(s) in (1c) represents a system of linear differential equations with constant coefficients which define a response function

$E_{out}(t)$ in terms of a forcing function $E_{in}(t)$. (It is common to represent filters, actuators, and some other components of control systems by means of transfer functions.)

Very often control engineers are surprised to find that numerical integration of equations (1) requires a prohibitively small integration step-size even though the dependent variables do not appear to vary rapidly enough to warrant such a small step. In an earlier paper [1] the author discussed a frequent cause of such difficulty and showed how the problem could be treated. The present paper formulates the theory from a point of view more familiar to the engineer and more general in application. Much use has been made of experience gained from the extensive use of techniques previously developed [1]. The theory may be applied to advantage even when a small integration step-size is not required. Furthermore, it is applicable to the case in which the coefficients of the linear equations vary slowly with time.

Other papers dealing with very similar problems are cited in the Reference section [2-12]. Reports [2-4] describe analytical solutions which require elaborate coupling of first- and/or second-order linear differential equations. The method of Steinman [3] has the advantage that it is not seriously affected by time variant coefficients. Certaine [6] has an approach similar to that of the present paper, but leaves much to be desired in effecting practical solutions. The methods given by Vasileva and Volosov [7-11] are promising but have received little use: They treat systems of differential equations which may be nonlinear and which include equations having small parameters as coefficients of the higher derivatives.

The appendix of Andrus [1] contains a method for removing several large poles from a system of linear differential equations when it is known that terms in the solution, which correspond to these large poles, are insignificant.

Another approach which can effect some time savings involves partitioning the differential equations into two or more subsystems and integrating each subsystem with a different integration step-size depending upon the response time of the subsystem.

THE NUMERICAL EFFECT OF SMALL TIME CONSTANTS

Solutions to systems of linear differential equations with constant coefficients contain terms of the form $ae^{\lambda t}$ where a and λ are real or complex numbers. (The constant λ is often called a pole or eigenvalue of the system, and the reciprocal of $|\lambda|$ is known as a time constant when λ is a negative real number.) Many numerical methods of integration are derived under the assumption that the solution can be expressed over an interval from T to T + Δt as a Maclaurin series

$$b_0 + b_1(\Delta t) + b_2(\Delta t)^2 + \ldots$$

truncated after several terms. However, if $|\lambda|$ is large, many terms of the series may be required to represent $ae^{\lambda t}$ accurately. The Maclaurin expansion of $e^{\lambda t}$ is

$$e^{\lambda(t + \Delta t)} = e^{\lambda t}\left[1 + \frac{\lambda \Delta t}{1!} + \frac{(\lambda \Delta t)^2}{2!} + \ldots\right] .$$

Since $|\lambda| \cdot \Delta t$ cannot greatly exceed unity in order for the series to be truncated after several terms and still approximate $e^{\lambda(t+\Delta t)}$, the maximum acceptable integration step-size is approximately equal to $1/|\lambda|_{max}$, where $|\lambda|_{max}$ is the magnitude of the pole of largest absolute value. Even if the numerical method of integration is not based directly upon a Maclaurin expansion, the large magnitudes of higher derivatives of $ae^{\lambda t}$ may still cause difficulty when $|\lambda|$ is large.

Often the coefficient a is so small in magnitude that the term $ae^{\lambda t}$ gives rise only to negligibly small variations in the solution (see Appendix A, Example 2). However, if $|\lambda|$ is large, the higher derivatives of $ae^{\lambda t}$ can still be so large in magnitude that one is forced to integrate with a very small integration step-size.

DEFINING THE RESPONSE OF A TRANSFER FUNCTION BY MEANS OF A SYSTEM OF FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

Suppose that the transfer function $L(E_{out})/L(E_{in})$ has been separated into partial fractions:

$$\frac{L(E_{out})}{L(E_{in})} = c + \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{\gamma_{ij}}{(s-\lambda_i)^{n_i-j+1}} \tag{2}$$

where c, γ_{ij}, and λ_i are constants and γ_{ij} and λ_i may be complex. The quantities $\lambda_i (i = 1, 2, \ldots, n)$ are the poles of the transfer function, and λ_i is a pole of multiplicity n_i. It will be shown that the following system of differential equations has the same transfer function:

$$E_{out} = cE_{in} + \sum_{i=1}^{n} \sum_{j=1}^{n_i} h_{ij} \tag{3a}$$

$$\dot{h}_{i1} - \lambda_i h_{i1} = \gamma_{i1} E_{in} \tag{3b}$$

$$\dot{h}_{ij} - \lambda_i h_{ij} = h_{i,j-1} + (\gamma_{ij} - \gamma_{i,j-1}) E_{in} \qquad (j=2,3,\ldots, n_i) \tag{3c}$$

$$h_{ij}(0) = 0 \qquad (j = 1,2, \ldots, n_i) \tag{3d}$$

where i = 1, 2,..., n.

On taking the Laplace transform of the members of (3b) and (3c), one obtains:

$$L(h_{i1}) = \frac{\gamma_{i1}}{s-\lambda_i} L(E_{in})$$

$$L(h_{ij}) = \frac{1}{s-\lambda_i} L(h_{i,j-1}) + \frac{\gamma_{ij} - \gamma_{i,j-1}}{s - \lambda_i} L(E_{in})$$

for j = 2, 3, ..., n_i. Then

$$\sum_{j=1}^{n_i} L(h_{ij}) = \left\{ \left[\frac{\gamma_{i1}}{s-\lambda_i}\right] + \left[\frac{\gamma_{i1}}{(s-\lambda_i)^2} + \frac{\gamma_{i2}-\gamma_{i1}}{s-\lambda_i}\right] \right.$$

$$+ \left[\frac{\gamma_{i1}}{(s-\lambda_i)^3} + \frac{\gamma_{i2}-\gamma_{i1}}{(s-\lambda_i)^2} + \frac{\gamma_{i3}-\gamma_{i2}}{s-\lambda_i}\right] + \dots + \left[\frac{\gamma_{i1}}{(s-\lambda_i)^k} + \frac{\gamma_{i2}-\gamma_{i1}}{(s-\lambda_i)^{k-1}}\right.$$

$$\left. + \dots + \frac{\gamma_{ik}-\gamma_{i,k-1}}{s-\lambda_i}\right] + \dots + \left[\frac{\gamma_{i1}}{(s-\lambda_i)^{n_i}} + \frac{\gamma_{i2}-\gamma_{i1}}{(s-\lambda_i)^{n_i-1}}\right.$$

$$\left.\left. + \dots + \frac{\gamma_{i\,n_i}-\gamma_{i,n_{i-1}}}{s-\lambda_i}\right] \right\} \quad L(E_{in}).$$

Simplification yields

$$\sum_{j+1}^{n_i} L(h_{ij}) = \left[\frac{\gamma_{i1}}{(s-\lambda_i)^{n_i}} + \frac{\gamma_{i2}}{(s-\lambda_i)^{n_i-1}} + \dots + \frac{\gamma_{in_i}}{s-\lambda_i}\right] \quad L(E_{in}). \tag{4}$$

The transfer function (2) may be easily derived from (3a) and (4).

Since equations (3) have the transfer function (1c), the solution to (3) is the solution defined by the transfer function.

If the transfer function (1c) has only real coefficients, then corresponding to the equations (3b) and (3c), where λ_i is any complex (non-real) pole, there are the redundant equations

$$\dot{\bar{h}}_{i1} - \bar{\lambda}_i \bar{h}_{i1} = \bar{\gamma}_{i1} \cdot E_{in}$$

$$\dot{\bar{h}}_{ij} - \bar{\lambda}_i \bar{h}_{ij} = \bar{h}_{i,j-1} + (\bar{\gamma}_{ij} - \bar{\gamma}_{i,j-1}) E_{in} \qquad (j=2,3,\dots,n_i)$$

also present in (3b) and (3c). Here the symbol $\bar{x}$ stands for the complex conjugate of x.

A more important observation is the following: If $|\lambda_k|$ is large and λ_k is negative (in a stable system there can be no positive poles), it is often true that the terms h_{kj}, corresponding to λ_i, are negligibly small. (The relative magnitudes of the constants γ_{ij} $(i = 1, 2, \dots, n;\ j = 1, 2, \dots, n_i)$ may be the deciding factors in such a case. However, the nature of E_{in} is also important.) If h_{kj} is negligibly small for $j = 1, 2, \ldots, n_i$, then the corresponding differential equations of (3) may be discarded and the remaining equations integrated numerically using a larger integration step-size. However, if the contributions of the terms h_{kj} are significant, the numerical problem cannot be completely eliminated, because in such a case a small step-size is intrinsically necessary to determine the rapid responses of the solution E_{out}. But even in the latter case it is often possible to increase the step-size significantly by assuming some form from E_{in} over short time intervals and integrating equations (3) analytically rather than numerically. The remainder of this paper is concerned with the analytic solution to (3) and the manner in which it may be tied in with the numerical integration of the nonlinear equations of (1). If the solution derived in this paper is employed, one need not concerned himself with eliminating negligible equations; the analytic solution essentially enables one to hop over negligibly small variations in E_{out}.

In passing, we remark that when $|\lambda_i|$ is large, the equation $\dot{h}_{i1} - \lambda_i h_{i1} = \gamma_{i1} E_{in}$ may be approximated in some cases by the equation $h_{i1} = -(\gamma_{i1}/\lambda_i)\, E_{in}$. This and similar approximations are discussed by Cohen [12].

REPRESENTATION OF E_{in} AS A LINEAR FUNCTION OVER SHORT INTERVALS OF TIME

Some form for E_{in} must be assumed before equations (3) can be integrated. The function E_{in} will be approximated over the time interval from T to $T + \Delta t$ by means of the line passing through the two points

$$[T,\ E_{in}(T)], \quad [T + \Delta t,\ E_{in}(T + \Delta t)].$$

The corresponding expression for E_{in} is

$$E_{in}(t) = E_{in}(T) + \frac{t-T}{\Delta t}\left[E_{in}(T + \Delta t) - E_{in}(T)\right]. \tag{5}$$

Of course it would be possible to represent $E_{in}(t)$ in other ways. The linear expression has been chosen in order to keep the integration formulas to be derived as simple as possible.

ANALYTICAL INTEGRATION OF THE LINEAR EQUATIONS

Assume that equations (3) have been integrated up to time T and that values of $E_{in}(T)$ and $E_{in}(T+\Delta t)$ have been given. Approximating $E_{in}(t)$ by means of (5) over the time interval from T to T + Δt, we will derive an expression for E_{out} at T + Δt in terms of $E_{in}(T)$, $E_{in}(T+\Delta t)$, and $h_{ij}(T)$ where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$. Later this expression will be tied in with the numerical integration of equations (1a) and (1b).

Specifically, we will integrate the equations

$$\begin{cases} \dot{h}_{i1} - \lambda_i h_{i1} = \gamma_{i1} E_{in} \\ \dot{h}_{ij} - \lambda_i h_{ij} = h_{i,j-1} + (\gamma_{ij} - \gamma_{i,j-1}) E_{in} \qquad (j = 2, 3, \ldots, n_i) \end{cases} \tag{6}$$

from time T to T + Δt in order to obtain $h_{ij}(T+\Delta t)$ in terms of $E_{in}(T)$, $E_{in}(T+\Delta t)$, $h_{i1}(T)$, $h_{i2}(T), \ldots, h_{in_i}(T)$.

For simplicity a new time variable $\tau = t - T$ is introduced. Let

$$E_{in}[\tau] = E_{in}(T + \tau)$$

and similarly for the functions E_{out} and h_{ij}. In other words, the variable τ will have its origin at t = T. Then

$$E_{in}[\tau] = E_{in}[0] + \frac{\tau}{\Delta t} \{E_{in}[\Delta t] - E_{in}[0]\} . \tag{7}$$

Letting τ be the independent variable and taking the Laplace transforms of (6) and (7), one arrives at the following equations:

$$\begin{cases} sL(h_{i1}) - h_{i1}[0] - \lambda_i L(h_{i1}) = \gamma_{i1} L(E_{in}) \\ sL(h_{ij}) - h_{ij}[0] - \lambda_i L(h_{ij}) = L(h_{i,j-1}) + (\gamma_{ij} - \gamma_{i,j-1}) L(E_{in}) \\ L(E_{in}) = \frac{1}{s} E_{in}[0] + \frac{1}{s^2} \frac{E_{in}[\Delta t] - E_{in}[0]}{\Delta t} \qquad . \qquad (j = 2, 3, \ldots, n_i) \end{cases}$$

Let

$$
\begin{cases}
r_0 = E_{in}[0] \\
r_1 = \dfrac{E_{in}[\Delta t] - E_{in}[0]}{\Delta t} \quad .
\end{cases}
\tag{8}
$$

After eliminating $L(E_{in})$ and employing (8), one obtains

$$
\begin{cases}
L(h_{i1}) = \dfrac{1}{s-\lambda_i} \left\{ h_{i1}[0] + \gamma_{i1} \left(\dfrac{r_0}{s} + \dfrac{r_1}{s^2} \right) \right\} \\
L(h_{ij}) = \dfrac{1}{s-\lambda_i} \left\{ L(h_{i,j-1}) + h_{ij}[0] + (\gamma_{ij} - \gamma_{i,j-1}) \left(\dfrac{r_0}{s} + \dfrac{r_1}{s^2} \right) \right\} .
\end{cases}
$$

$$(j=2, 3, \ldots, n_i)$$

Successive substitutions yields

$$
L(h_{ij}) = \sum_{k=1}^{j} \left\{ \frac{h_{ik}[0]}{(s-\lambda_i)^{j-k+1}} + \frac{r_0(\gamma_{ik} - \gamma_{i,k-1})}{(s-\lambda_i)^{j-k+1} s} + \frac{r_1(\gamma_{ik} - \gamma_{i,k-1})}{(s-\lambda_i)^{j-k+1} s^2} \right\}
$$

for $j = 1, 2, \ldots, n_i$ and $\gamma_{i_0} = 0$.

Assuming $\lambda_i \neq 0$ and separating into partial fractions, one obtains

$$
L(h_{ij}) = \sum_{k=1}^{j} \left\{ \frac{h_{ik}[0]}{(s-\lambda_i)^{j-k+1}} + r_0 (\gamma_{ik} - \gamma_{i,k-1}) \right.
$$

$$
\left[- \sum_{\alpha=1}^{j-k+1} \frac{1}{(-\lambda_i)^{j-k-\alpha+2} (s-\lambda_i)^{\alpha}} + \frac{1}{(-\lambda_i)^{j-k+1} s} \right]
$$

$$
\left. + r_1 (\gamma_{ik} - \gamma_{i,k-1}) \left[\sum_{\alpha=1}^{j-k+1} \frac{j-k-\alpha+2}{(-\lambda_i)^{j-k-\alpha+3} (s-\lambda_i)^{\alpha}} - \frac{j-k+1}{(-\lambda_i)^{j-k+2} s} + \frac{1}{(-\lambda_i)^{j-k+1} s^2} \right] \right\} .
$$

The inverse Laplace transform is

$$h_{ij}(\tau) = \sum_{k=1}^{j} \left\{ \frac{h_{ik}[0] \cdot \tau^{j-k} e^{\lambda_i \tau}}{(j-k)!} \right.$$

$$+ r_0 (\gamma_{ik} - \gamma_{i,k-1}) \left[- \sum_{\alpha=1}^{j-k+1} \frac{\tau^{\alpha-1} e^{\lambda_i \tau}}{(-\lambda_i)^{j-k-\alpha+2} (\alpha-1)!} + \frac{1}{(-\lambda_i)^{j-k+1}} \right]$$

$$\left. + r_1 (\gamma_{ik} - \gamma_{i,k-1}) \left[\sum_{\alpha=1}^{j-k+1} \frac{(j-k-\alpha+2) \tau^{\alpha-1} e^{\lambda_i \tau}}{(-\lambda_i)^{j-k-\alpha+3} (\alpha-1)!} - \frac{j-k+1}{(-\lambda_i)^{j-k+2}} + \frac{\tau}{(-\lambda_i)^{j-k+1}} \right] \right\}$$

for $\lambda_i \neq 0$.

Evaluating h_{ij} at $\tau = \Delta t$ and rearranging, one obtains

$$h_{ij}[\Delta t] = \sum_{\beta=1}^{j} h_{i,j-\beta+1}[0] \cdot \frac{(\Delta t)^{\beta-1} e^{\lambda_i \Delta t}}{(\beta-1)!}$$

$$+ r_0 \cdot \sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \left[- \sum_{\alpha=0}^{\beta-1} \frac{(\Delta t)^{\alpha} e^{\lambda_i \Delta t}}{(-\lambda_i)^{\beta-\alpha} \alpha !} + \frac{1}{(-\lambda_i)^{\beta}} \right]$$

$$+ r_1 \cdot \sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \left[\sum_{\alpha=0}^{\beta-1} \frac{(\beta-\alpha) (\Delta t)^{\alpha} e^{\lambda_i \Delta t}}{(-\lambda_i)^{\beta-\alpha+1} \alpha!} - \frac{\beta}{(-\lambda_i)^{\beta+1}} + \frac{\Delta t}{(-\lambda_i)^{\beta}} \right]$$

where β = j-k+1 and $\lambda_i \neq 0$. Therefore,

$$h_{ij}[\Delta t] = \sum_{\beta=1}^{j} h_{i,j-\beta+1}[0] \cdot a_{i\beta} + r_0 b_{ij0} + r_1 b_{ij1} \tag{9}$$

where

$$\left\{\begin{aligned} a_{i\beta} &= \frac{(\Delta t)^{\beta-1} e^{\lambda_i \Delta t}}{(\beta-1)!} \\ b_{ij0} &= \sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \frac{e^{\lambda_i \Delta t}}{(-\lambda_i)^{\beta}} \left[e^{-\lambda_i \Delta t} - \sum_{\alpha=0}^{\beta-1} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \right] \\ b_{ij1} &= \sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \frac{e^{\lambda_i \Delta t}}{(-\lambda_i)^{\beta+1}} \left\{ -\beta \left[e^{-\lambda_i \Delta t} - \sum_{\alpha=0}^{\beta-1} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \right] \right. \\ &\quad \left. + (-\lambda_i \Delta t) \left[e^{-\lambda_i \Delta t} - \sum_{\alpha=0}^{\beta-2} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \right] \right\} \end{aligned}\right. \tag{10a}$$

where $\lambda_i \neq 0$ and the summation $\sum_{\alpha=0}^{\beta-2}$ is nugatory when $\beta = 1$.

When $|\lambda_i \Delta t|^{n_i}$ is much less than unity, in order to avoid subtraction of nearly equal numbers during the computation of b_{ij0} and b_{ij1}, one should employ the following power series expansions:

$$e^{-\lambda_i \Delta t} - \sum_{\alpha=0}^{\beta-1} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} = \sum_{\alpha=\beta}^{\infty} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!}$$

$$-\beta\left[e^{-\lambda_i\Delta t} - \sum_{\alpha=0}^{\beta-1} \frac{(-\lambda_i\Delta t)^{\alpha}}{\alpha!}\right] + (-\lambda_i\Delta t)\left[e^{-\lambda_i\Delta t} - \sum_{\alpha=0}^{\beta-2} \frac{(-\lambda_i\Delta t)^{\alpha}}{\alpha!}\right]$$

$$= \sum_{\alpha=\beta+1}^{\infty} (\alpha - \beta) \frac{(-\lambda_i\Delta t)^{\alpha}}{\alpha!}$$

Equation (9) also holds for the case in which $\lambda_i = 0$. However, for this case the coefficients (10a) must be computed as follows:

$$\begin{cases} a_{i\beta} = \dfrac{(\Delta t)^{\beta-1}}{(\beta-1)!} \\[2ex] b_{ij0} = \displaystyle\sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \cdot \frac{(\Delta t)^{\beta}}{\beta!} \\[2ex] b_{ij1} = \displaystyle\sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \cdot \frac{(\Delta t)^{\beta+1}}{(\beta+1)!} \ . \end{cases} \tag{10b}$$

Substitution of expressions (8) into equation (9) yields

$$h_{ij}[\Delta t] = \sum_{\beta=1}^{j} h_{i,j-\beta+1}[0] \cdot a_{i\beta} + E_{in}[0] \cdot c_{ij0} + E_{in}[\Delta t] \cdot c_{ij1} \tag{11}$$

where

$$c_{ij0} = b_{ij0} - \frac{b_{ij1}}{\Delta t} \ , \quad c_{ij1} = \frac{b_{ij1}}{\Delta t} \ .$$

According to (3a)

$$E_{out}[\Delta t] = c\, E_{in}[\Delta t] + \sum_{i=1}^{n} \sum_{j=1}^{n_i} h_{ij}[\Delta t] \ . \tag{12}$$

Equations (11) and (12) may be rewritten in terms of T as follows:

$$
\begin{cases}
E_{out}(T+\Delta t) = c\, E_{in}(T+\Delta t) + \sum_{i=1}^{n} \sum_{j=1}^{n_i} h_{ij}(T+\Delta t) \\
h_{ij}(T+\Delta t) = \sum_{\beta=1}^{j} h_{i,j-\beta+1}(T) \cdot a_{i\beta} + E_{in}(T) \cdot c_{ij0} + E_{in}(T+\Delta t) \cdot c_{ij1} .
\end{cases}
\tag{13}
$$

Equations (13) are the desired integration formulas. As long as Δt and the coefficients of the transfer function (2) are constant, the coefficients c, $a_{i\beta}$, c_{ij0}, and c_{ij1} will be constant.

One can compute $\dot{E}_{out}$ and $\ddot{E}_{out}$ from

$$\dot{E}_{out}(t) = c\dot{E}_{in}(t) + \sum_{i=1}^{n} \sum_{j=1}^{n_i} \dot{h}_{ij}$$

$$\ddot{E}_{out}(t) = c\ddot{E}_{in}(t) + \sum_{i=1}^{n} \sum_{j=1}^{n_i} \ddot{h}_{ij}$$

where $\dot{h}_{ij}$ and $\ddot{h}_{ij}$ may be computed from equations (3).

In most problems $n_1 = n_2 = \ldots = n_n = 1$. Then (13) reduces to

$$
\begin{cases}
E_{out}(T+\Delta t) = c\, E_{in}(T+\Delta t) + \sum_{i=1}^{n} h_{i1}(T+\Delta t) \\
h_{i1}(T+\Delta t) = h_{i1}(T) \cdot a_{i1} + E_{in}(T) \cdot c_{i10} + E_{in}(T+\Delta t) \cdot c_{i11}
\end{cases}
$$

where

$$c_{i10} = b_{i10} - \frac{b_{i11}}{\Delta t}, \quad c_{i11} = \frac{b_{i11}}{\Delta t}$$

and where

$$a_{i1} = e^{\lambda_i \Delta t}, \quad b_{i10} = \frac{\gamma_{i1}}{\lambda_i} (e^{\lambda_i \Delta t} - 1), \quad b_{i11} = \frac{\gamma_{i1}}{\lambda_i^2} (e^{\lambda_i \Delta t} - 1 - \lambda_i \Delta t)$$

when $\lambda_i \neq 0$ and

$$a_{i1} = 1, \quad b_{i10} = \gamma_{i1} \Delta t, \quad b_{i11} = \gamma_{i1} \frac{(\Delta t)^2}{2}$$

when $\lambda_i = 0$.

TIE-IN OF THE ANALYTICAL SOLUTION TO THE NUMERICAL INTEGRATION OF THE NONLINEAR EQUATIONS

Formulas (13) may be tied in to the numerical integration of the nonlinear equations

$$\begin{cases} \dot{y}_k = \theta_k(t, y_1, y_2, \ldots, y_{n_d}, E_{in}, E_{out}) \quad (k = 1, 2, \ldots, n_d) \\ \dot{E}_{in} = \phi\,(t, y_1, y_2, \ldots, y_{n_d}, E_{in}, E_{out}) \end{cases}$$

in a rather simple manner. Essentially E_{out} may be eliminated from these equations by substituting the righthand member of the first of equations (13) for E_{out}. Assuming the equations have been integrated up to time T, we may write

$$\dot{y}_k = \theta_k \left(t, y_1, y_2, \ldots, y_{n_d}, E_{in}, cE_{in} + \sum_{i=1}^{n} \sum_{j=1}^{n_i} h_{ij}\right)$$

for $t > T$, where $\Delta t = t - T$ and

$$h_{ij} = h_{i,j-\beta+1}(T) \cdot a_{i\beta} + E_{in}(T) \cdot c_{ij0} + E_{in} \cdot c_{ij1},$$

and similarly for the equation $\dot{E}_{in} = \phi$. Then the differential equations may be integrated numerically in the ordinary manner.

If E_{in} is continuous, the approximate solution obtained by the method proposed in this paper will approach the exact solution as Δt approaches zero.

THE CASE OF LINEAR EQUATIONS HAVING TIME DEPENDENT COEFFICIENTS

If the coefficients of p(s) and g(s) in the transfer function (1c) vary with time, then c, λ_i and γ_{ij} will also be time dependent. This problem may be treated by the methods proposed, provided Δt is chosen sufficiently small so that c, λ_i and γ_{ij} may be assumed to be constant over each Δt interval.

DERIVATION OF THE ANALYTICAL SOLUTION DIRECTLY FROM A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

Suppose that the relationship between E_{in} and E_{out} is expressed by means of the equations

$$\begin{cases} \dot{\underline{q}} = A\underline{q} + E_{in}\underline{b} \\ E_{out} = w\,E_{in} + \underline{u}^T\underline{q} \end{cases} \qquad (14)$$

rather than by a transfer function. Here A symbolizes an m × m matrix of constant elements, the symbols $\underline{q}$, $\underline{b}$, and $\underline{u}$ represent m × 1 vectors of constants,

$\underline{u}^T$ is the transpose of $\underline{u}$, and w is a scalar constant. It will now be shown briefly how equations (3) can be derived from (14) employing classical methods for the solution of linear systems.

It is proven in elementary matrix theory that there exists a nonsingular matrix S such that

$$S^{-1} A S = J$$

where J has the form:

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & & \dots & \dots \\ 0 & 0 & \dots & J_{n'} \end{pmatrix}$$

where the 0's represent rectangular arrays of zero elements and

$$J_i = \begin{pmatrix} \rho_i & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & \rho_i & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \rho_i & \dots & 0 & 0 & 0 \\ \dots & & & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 & \rho_{n'_i} \end{pmatrix}$$

The constants $\rho_1, \rho_2, \dots, \rho_{n'}$ are <u>eigenvalues</u> of A and are not necessarily distinct. The matrix J is known as the <u>Jordan</u> <u>canonical</u> <u>matrix</u>.

Since $A = SJS^{-1}$

$$\begin{cases} S^{-1}\underline{\dot{q}} = J(S^{-1}\underline{q}) + E_{in}\,(S^{-1}\underline{b}) \\ E_{out} = w\,E_{in} + (\underline{u}^T S)\;(S^{-1}\underline{q}) \end{cases}$$

Defining $\underline{p} = S^{-1}\underline{q}$, $\underline{g} = S^{-1}\underline{b}$, and $\underline{v}^T = \underline{u}^T S$, the above equations may be expressed as follows:

$$\begin{cases} \dot{\underline{p}} = J\,\underline{p} + E_{in}\,\underline{g} \\ E_{out} = w\,E_{in} + \underline{v}^T\underline{p} \end{cases} \tag{15}$$

Decompose the vectors $\underline{p}$, $\underline{g}$, and $\underline{v}$ in the manner indicated below:

$$\underline{p}^T = (\underline{p}_1^T, \underline{p}_2^T, \ldots, \underline{p}_{n'}^T) \;,$$

$$\underline{g}^T = (\underline{g}_1^T, \underline{g}_2^T, \ldots, \underline{g}_{n'}^T) \;,$$

$$\underline{v}^T = (\underline{v}_1^T, \underline{v}_2^T, \ldots, \underline{v}_{n'}^T) \;,$$

where $\underline{p}_i$, $\underline{g}_i$, and $\underline{v}_i$ $(i = 1, 2, \ldots, n')$ are $n_i' \times 1$ vectors. Now equations (15) may be broken down as follows:

$$\begin{cases} \dot{\underline{p}}_i = J_i \cdot \underline{p}_i + E_{in}\,\underline{g}_i \qquad (i = 1, 2, \ldots, n') \\ E_{out} = w \cdot E_{in} + \sum_{i=1}^{n'} \underline{v}_i^T\,\underline{p}_i \;. \end{cases}$$

Letting

$$\underline{p}_i^T = (p_{i1}, p_{i2}, \ldots, p_{in_i'}) \;,$$

$$\underline{g}_i^T = (g_{i1}, g_{i2}, \ldots, g_{in_i'}) \;,$$

$$\underline{v}_i^T = (v_{i1}, v_{i2}, \ldots, v_{in_i'}) \;,$$

one may obtain the equations

$$\begin{cases} \dot{p}_{i1} - \rho_i p_{i1} = E_{in} g_{i1} \\ \dot{p}_{ij} - \rho_i p_{ij} = p_{i,j-1} + E_{in} g_{ij} \qquad (j = 2, 3, \ldots, n_i') \end{cases}$$

for $i = 1, 2, \ldots, n'$.

Now let $f_{ij} = v_{ij} p_{ij}$ and multiply the differential equation defining $\dot{p}_{ij}$ through by v_{ij}. Thus

$$\begin{cases} \dot{f}_{i1} - \rho_i f_{i1} = E_{in} (v_{i1} g_{i1}) \\ \dot{f}_{ij} - \rho_i f_{ij} = f_{i,j-1} + E_{in} (v_{ij} g_{ij}) \quad (i = 1, 2, \ldots, n'; j = 2, 3, \ldots, n_i') \\ E_{out} = w E_{in} + \sum_{i=1}^{n'} \sum_{j=1}^{n_i'} f_{ij} \end{cases} \tag{16}$$

Assuming $q(0) = 0$, it is obvious that $f_{ij}(0) = 0$. The analytical solution of (16) may now be obtained in the same manner as the solution to equations (3).

The transfer function of (16) is

$$\frac{L(E_{out})}{L(E_{in})} = w + \sum_{i=1}^{n'} \sum_{j=1}^{n_i'} \frac{\sum_{k=1}^{j} v_{ik} g_{ik}}{(s - \rho_i)^{n_i' - j + 1}}$$

Provided $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, one may identify f_{ij}, ρ_i, w, n', n_i' and $\sum_{k=1}^{j} v_{ik} g_{ik}$ with h_{ij}, λ_i, c, n, n_i, and γ_{ij}, respectively. However, in the rare event that the ρ_i's are not distinct, the identification is somewhat more complicated. (For example, if $\rho_i = \rho_k$ and $\rho_i \neq \rho_j$ for $j \neq i, k$, then max (n_i', n_k') must be identified with n_i and n_k is zero.)

For the case $n_i = 1$, the vectors $\underline{s}_i$, and $\underline{r}^T_i$, are simply right and left eigenvectors of A corresponding to the eigenvalue λ_i such that $\underline{r}^T_{i1}\underline{s}_{i1} = 1$. Therefore, for the usual case in which $n_i = 1$ for $i = 1, 2, \ldots, n$, the coefficients γ_{i1} may be determined from the eigenvectors of A; in fact, γ_{i1} would be equal to $(\underline{u}^T \underline{s}_{i1}) \cdot (\underline{r}^T_{i1} \underline{b})$ provided $\rho_i \neq \rho_k$ for $i \neq k$. If $\underline{b}$ has all zero elements except one -- say the first element b_1 of $\underline{b}$ -- then

$$\gamma_{i1} = b_1 (\underline{u}^T \underline{s}_{i1}) , \tag{17}$$

where the first element of the eigenvector $\underline{r}_{i1}$ is arbitrarily chosen to be unity. Equation (17) also establishes a relationship between the eigenvectors of A and the numerators of the partial fractions of the transfer function (2).

CONCLUSIONS

Equations (13) provide a simple integration formula for determining the response E_{out}, at time $T + \Delta t$, of the transfer function (2) from values of the forcing function E_{in} at times T and $T + \Delta t$. The section entitled "Tie-in of the Analytical Solution to the Numerical Integration of the Nonlinear Equations" shows how equation (13) may be easily tied in with the numerical integration of other differential equations relating E_{in} to E_{out}

The coefficients in (13) depend upon Δt and the coefficients of the transfer function in the manner shown in equations (10a) and (10b). Furthermore, the section entitled "Derivation of the Analytical Solution Directly from a System of Linear Differential Equations" derives the relationship of the coefficients in (13) to the eigenvalues and eigenvectors of the coefficient matrix A of the system (14) of first-order linear differential equations. (Indirectly, this also gives the dependence of the coefficients of the transfer function upon the eigenvalues and eigenvectors of A.) The coefficients of (13) must be recomputed if either Δt or the coefficients of (2), or the alternate equations (14) have changed significantly.

The integration formula (13) very often enables one to integrate numerically a system of control equations such as (1) at a much larger integration step-size than the largest which could be used by conventional methods, especially when the transfer function has poles of large magnitude.

Use may also be made of the first-order differential equations (3) which define E_{out} very concisely in terms of E_{in} (without introducing troublesome high derivatives of a single variable). If no difficulty with poles of large magnitude is present, equations (3) provide a simple means for determining E_{out}. They may be integrated numerically, simultaneously with other differential equations, and they may be solved analytically when E_{in} is available as an explicit function of time. When a pole λ_i of large magnitude is present, it is often possible to eliminate it by simply removing from (3) the equations corresponding to λ_i. However, a careful analysis of the particular problem should be made before any equations are eliminated. For this reason one may find it most convenient to utilize the integration formula (13), which is likely to be a faster method even when poles of large magnitude are not present.

APPENDIX A

EXAMPLES

Example 1

A problem in which the forcing function is given explicitly by

$$E_{in} = \sin\alpha t$$

will now be examined. In this case equation (3b) is

$$\dot{h}_{i1} - \lambda_i h_{i1} = \gamma_{i1} \sin\alpha t.$$

Assuming $h_i(0) = 0$ and integrating, one obtains

$$h_{i1}(t) = \frac{\gamma_{i1}}{\lambda_i^2 + \alpha^2} \left(\alpha e^{\lambda_i t} - \alpha\cos \alpha t - \lambda_i \sin\alpha t\right).$$

If $\lambda_i < 0$,

$$|h_{i1}(t)| \leq \left|\frac{\gamma_{i1}}{\lambda_i}\right| \left(2\left|\frac{\alpha}{\lambda_i}\right| + 1\right). \tag{18}$$

In particular, consider the problem in which $\alpha = 1$ and the transfer function is

$$L(E_{out})/L(E_{in}) = \frac{4s^3 + 233s^2 + 998s + 5440}{2s^4 + 224s^3 + 2444s^2 + 440s + 4000}$$

$$= \frac{(5/4)\sqrt{-1}}{s + 1 + \sqrt{-1}} + \frac{-(5/4)\sqrt{-1}}{s + 1 - \sqrt{-1}} + \frac{1}{s + 10} + \frac{1}{s + 100}.$$

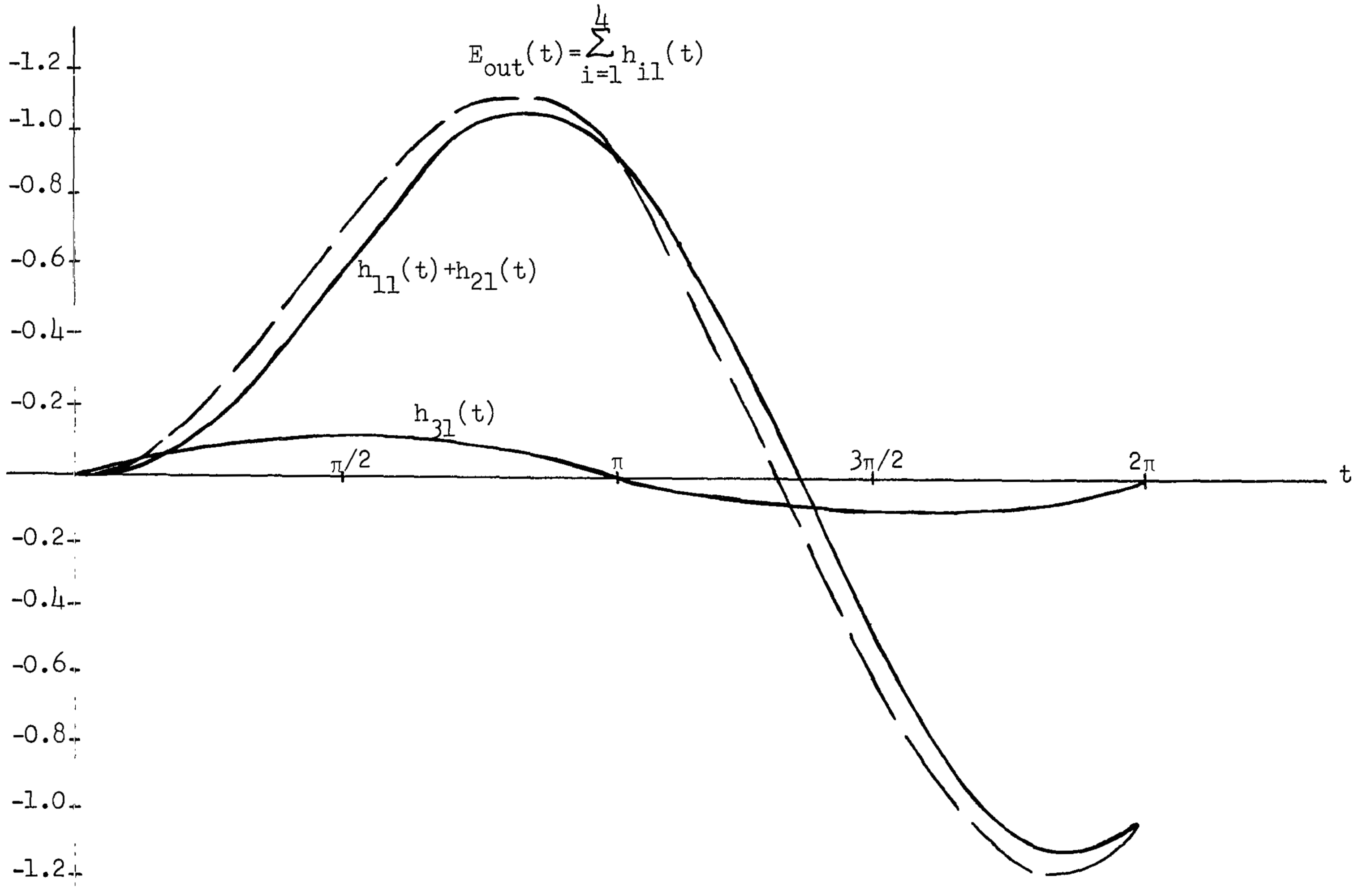
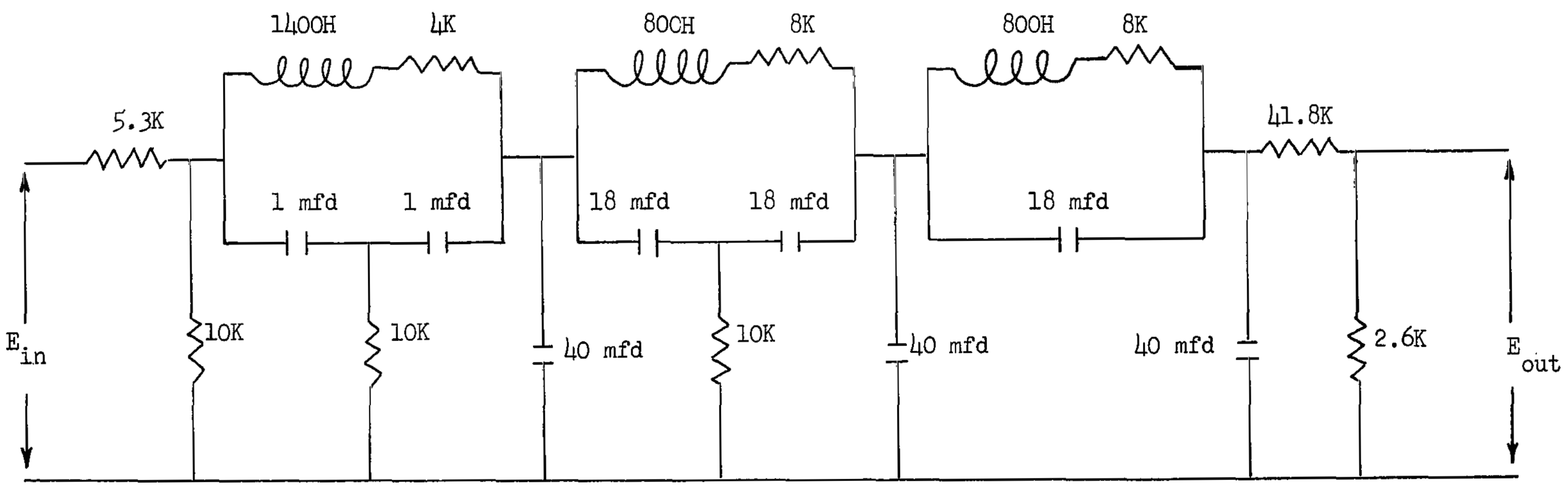


FIGURE A-1. $E_{out}(t)$ OF EXAMPLE 1 AS A FUNCTION OF t



H = Henries, K = Kilo-Ohms, mfd - Micro-Farads

FIGURE A-2. ELECTRICAL NETWORK REPRESENTING A FILTER

Runge-Kutta: Δt = .006 Solid Curve
Δt = .007 Diverged

Analytic Solution: Δt = .02 Solid Curve
Δt = .06 Dotted Curve
Δt = .2 Dashed Curve

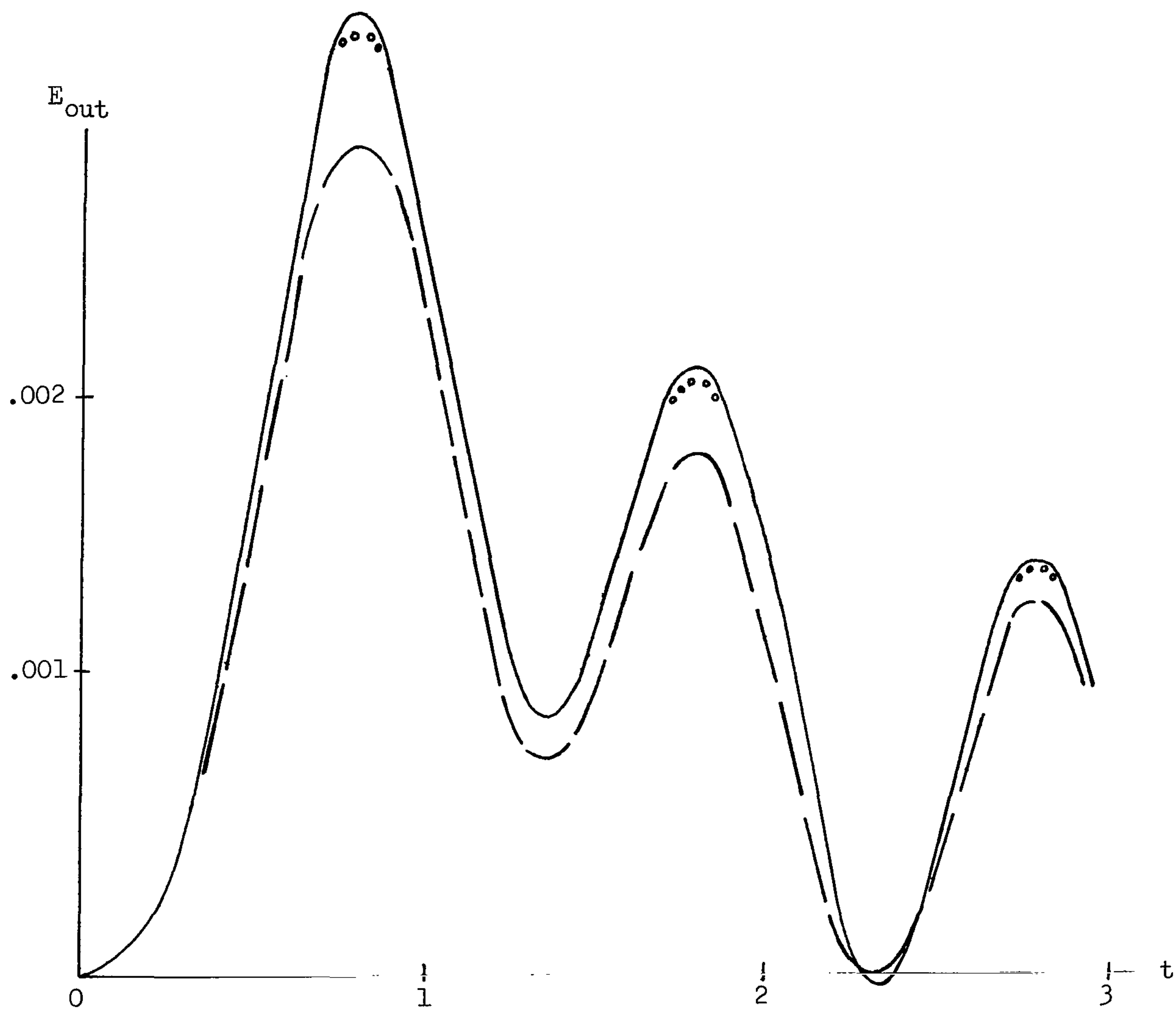


FIGURE A-3. E_{out} OF EXAMPLE 2 AS A FUNCTION OF t

Therefore

$$\lambda_1 = -1 - \sqrt{-1}, \quad \lambda_2 = -1 + \sqrt{-1}, \quad \lambda_3 = -10, \quad \lambda_4 = -100$$

$$\gamma_{11} = (5/4)\sqrt{-1}, \quad \gamma_{21} = -(5/4)\sqrt{-1}, \quad \gamma_{31} = 1, \quad \gamma_{41} = 1 .$$

It follows that

$$h_{11}(t) + h_{21}(t) = \tfrac{1}{2}\left[e^{-t}(2\cos t + \sin t) - 2\cos t + \sin t\right]$$

$$h_{31}(t) = \frac{1}{101}\left(e^{-10t} - \cos t + 10 \sin t\right)$$

$$h_{41}(t) = \frac{1}{10001}\left(e^{-100t} - \cos t + 100 \sin t\right).$$

In Figure A-1, $h_{11}(t) + h_{21}(t)$, $h_{31}(t)$, and $E_{out}(t)$ are plotted as functions of t. Inequality (18) reveals that $|h_{41}| \leq 0.0102$. Therefore, the variable $h_{41}(t)$ is too small in magnitude to be plotted on the same scale, and yet the largeness of $|\lambda_4|$ would make it necessary to use a numerical integration step-size roughly equal to 0.01. A step-size of 0.1 could suffice if the differential equations defining h_{41} were eliminated.

Example 2

The electrical network shown in Figure A-2 is a filter used in an actual control system of a space vehicle. It has been represented by a system of nine differential equations having the form indicated in (14). The response E_{out} was determined by means of the integration formula (13) where E_{in} was defined by

$$E_{in} = \sin(2\pi t).$$

Linear interpolation was used to approximate E_{in} over each Δt time interval. Time increments of $\Delta t = 0.2$ sec, $\Delta t = 0.06$ sec, and $\Delta t = 0.02$ sec were employed.

For comparison the integration was also performed by the fourth order Runge-Kutta method of numerical integration using $\Delta t = 0.006$ sec. The latter technique diverged when an increment of $\Delta t = 0.007$ sec was employed. The results of the comparison are shown in Figure A-3.

APPENDIX B

MATHEMATICAL SUBROUTINES

Subroutine for Decomposition of a Rational Fraction into Partial Fractions

Input Data.

N_R	number of real poles
λ_i $(i = 1, 2, \ldots, N_R)$	distinct real poles
n_i $(i = 1, 2, \ldots, N_R)$	multiplicities of real poles (usually $n_i = 1$)
N_C	one-half the number of complex poles
$\mathrm{Re}(\Lambda_i), \mathrm{Im}(\Lambda_i)$ $(i=1,2,\ldots,N_C)$	real and imaginary parts of complex poles
M	degree of numerator of rational fraction
a_k $(k = 0, 1, \ldots, M)$	coefficients of numerator

Note: A pole is defined to be a root of the denominator of the rational fraction. All complex (non-real) poles are assumed to be distinct. The conjugate of each complex pole given in the input data is also a pole.

The rational fraction must have a numerator of smaller degree than the denominator. The coefficient of the term of highest degree in the denominator is assumed to be unity.

Definition. We define the polynomials

$$p(s) = a_0 s^M + a_1 s^{M-1} + \ldots + a_{M-1} s + a_M$$

$$h_i(s) = \left[\prod_{\substack{k=1\\(k \neq i)}}^{N_R} (s - \lambda_k)^{n_k}\right] \cdot \left[\prod_{k=1}^{N_C} (s^2 + b_k s + c_k)\right] \quad (i=1,2,\ldots,N_R)$$

where

$$b_k = -2\mathrm{Re}(\Lambda_k), \qquad c_k = \left[\mathrm{Re}(\Lambda_k)\right]^2 + \left[\mathrm{Im}(\Lambda_k)\right]^2 .$$

Computation. Compute

$$\gamma_{i1} = \frac{p(\lambda_i)}{h_i(\lambda_i)}$$

$$\gamma_{ij} = \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{ds^{j-1}}\left(\frac{p(s)}{h_i(s)}\right)\right]_{s=\lambda_i} \qquad (j = 2, 3, \ldots, n_i)$$

for i = 1, 2, ..., N_R.

Compute the complex numbers

$$Q_i = \frac{p(\Lambda_i)}{\left[\prod_{k=1}^{N_R} (\Lambda_i - \lambda_k)^{n_k}\right] \cdot \left[\prod_{(k \neq i)}^{N_C} (\Lambda_i^2 + b_k\Lambda_i + c_k)\right]} \qquad (i=1,2,\ldots,N_C)$$

where

$$\Lambda_i = \mathrm{Re}(\Lambda_i) + \sqrt{-1}\,\mathrm{Im}(\Lambda_i) \quad (i = 1, 2, \ldots, N_C) \text{ are complex.}$$

Compute

$$\Gamma_{i1} = \frac{\mathrm{Im}(Q_i)}{\mathrm{Im}(\Lambda_i)}, \qquad \Gamma_{i2} = -\Gamma_{i1} \cdot \left[\mathrm{Re}(\Lambda_i)\right] + \mathrm{Re}(Q_i)$$

for i = 1, 2, ..., N_C.

Output Data.

γ_{ij} $(i = 1, 2, \ldots, N_R;\ j = 1, 2, \ldots, n_i)$

Γ_{1i}, Γ_{2i} $(i = 1, 2, \ldots, N_C)$

coefficients of numerators of partial fractions

Analysis. The procedure described above decomposes the rational fraction

$$\frac{a_0 s^M + a_1 s^{M-1} + \ldots + a_{M-1} s + a_M}{\left[\prod_{k=1}^{N_R} (s - \lambda_k)^{n_k}\right] \left[\prod_{k=1}^{N_C} (s^2 + b_k s + c_k)\right]} \tag{1}$$

into partial fractions:

$$\sum_{k=1}^{N_R} \sum_{j=1}^{n_k} \frac{\gamma_{kj}}{(s - \lambda_k)^{n_k - j + 1}} + \sum_{k=1}^{N_C} \frac{\Gamma_{k1} s + \Gamma_{k2}}{s^2 + b_k s + c_k}$$

It is assumed that the roots of $s^2 + b_k s + c_k$ are the given complex numbers Λ_k and Λ_k conjugate. It is also assumed that the numerator of (1) has lower degree than the denominator, that the real roots $\lambda_1, \lambda_2, \ldots, \lambda_{N_R}$ are distinct, and that there are no repeated complex roots. The method of decomposition is standard.

Subroutine for Computing Coefficients Defining the Response of a Transfer Function

Input Data.

N_R	number of real poles
λ_i $(i = 1, 2, \ldots, N_R)$	distinct real poles

$n_i \quad (i = 1, 2, \ldots, n_i)$	multiplicities of distinct real poles
$\gamma_{ij} (i=1,2,\ldots,N_R; j=1,2,\ldots,n_i)$	numerators of partial fractions corresponding to real poles
N_C	one-half the number of complex poles
$\mathrm{Re}(\Lambda_i), \mathrm{Im}(\Lambda_i) \; (i=1,2,\ldots,N_C)$	real and imaginary parts of complex poles (It is assumed that there are no repeated complex poles.)
$\Gamma_{i1}, \Gamma_{i2} \; (i = 1,2,\ldots,N_C)$	coefficients of numerators of partial fractions corresponding to complex poles
Δt	time interval

Computations. Compute

$$
\begin{array}{l}
i=1,2,\ldots,N_R \\
j=1,2,\ldots,n_i
\end{array}
\left\{
\begin{aligned}
a_{ij} &= \frac{(\Delta t)^{j-1} e^{\lambda_i \Delta t}}{(j-1)!} \\
b_{ij0} &= \sum_{\beta=1}^{j} \frac{\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}}{(-\lambda_i)^{\beta}} \left[1 - e^{\lambda_i \Delta t} \sum_{\alpha=0}^{\beta-1} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \right] \\
b_{ij1} &= \sum_{\beta=1}^{j} \frac{\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}}{(-\lambda_i)^{\beta+1}} \left\{ -\beta \left[1 - e^{\lambda_i \Delta t} \sum_{\alpha=0}^{\beta-1} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \right] \right. \\
&\qquad \left. + (-\lambda_i \Delta t) \left[1 - e^{\lambda_i \Delta t} \sum_{\alpha=0}^{\beta-2} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \right] \right\}
\end{aligned}
\right.
$$

assuming $\lambda_i \neq 0$, where γ_{i0} is defined to be zero. The sum $\sum_{\alpha=0}^{\beta-2}$ is zero when $\beta = 1$.

If $|\lambda_i \cdot \Delta t|^{n_i} < 0.01$, then the term in brackets in the expression for b_{ij0} should be computed by means of the series

$$e^{\lambda_i \Delta} \sum_{\alpha=\beta}^{\infty} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!}$$

and the term in braces is the expression for b_{ij1} should be computed by means of the series

$$e^{\lambda_i \Delta t} \sum_{\alpha=\beta+1}^{\infty} \frac{(-\lambda_i \Delta t)^{\alpha}}{\alpha!} \cdot (\alpha-\beta)$$

When $\lambda_i = 0$, compute

$$\left\{ \begin{aligned} a_{ij} &= \frac{(\Delta t)^{j-1}}{(j-1)!} \\ b_{ij0} &= \sum_{\beta=1}^{j} (\gamma_{i,j-\beta+1} - \gamma_{i,j-\beta}) \quad \frac{(\Delta t)^{\beta}}{\beta!} \\ b_{ij1} &= \sum_{\beta=1}^{j} (\gamma_{i,j-\beta-1} - \gamma_{i,j-\beta}) \quad \frac{(\Delta t)^{\beta+1}}{(\beta+1)!} \end{aligned} \right.$$

where, again, $\gamma_{i0} = 0$.

Also compute

$$c_{ij0} = b_{ij0} - \frac{b_{ij1}}{\Delta t} \quad , \quad c_{ij1} = \frac{b_{ij1}}{\Delta t}$$

for $i = 1, 2, \ldots, N_R$ and $j = 1, 2, \ldots, n_i$.

Compute

$$
\left.
\begin{aligned}
\mathrm{Re}(A_i) &= e^{\mathrm{Re}(\Lambda_i)\cdot \Delta t} \cos \mathrm{Im}(\Lambda_i)\cdot \Delta t \quad \left(\text{Note: } A_i = e^{\Lambda_i \Delta t}\right) \\
\mathrm{Im}(A_i) &= e^{\mathrm{Re}(\Lambda_i)\cdot \Delta t} \sin \left[\mathrm{Im}(\Lambda_i \cdot \Delta t)\right] \\
\mathrm{Re}(\Gamma_i') &= \Gamma_{i1}/2 \\
\mathrm{Im}(\Gamma_i') &= - \frac{\Gamma_{i2} + \Gamma_{i1} \cdot \mathrm{Re}(\Lambda_i)}{2 \cdot \mathrm{Im}(\Lambda_i)} \\
B_{i0} &= \frac{\Gamma_i'}{\Lambda_i} (A_i - 1) \\
B_{i1} &= \frac{\Gamma_i'}{\Lambda_i^2} (A_i - 1 + \Lambda_i \cdot \Delta t) \\
C_{i0} &= B_{i0} - \frac{B_{i1}}{\Delta t}, \qquad C_{i1} = \frac{B_{i1}}{\Delta t}
\end{aligned}
\right\} \quad (i=1,2,\ldots,N_C)
$$

where B_{i0}, B_{i1}, Γ_i', Λ_i, A_i, C_{i0}, C_{i1}, are complex numbers.

If $|A_i \Delta t| < 0.01$, compute

$$A_i - 1 = - A_i \sum_{\alpha=1}^{\infty} \frac{(-\Lambda_i \Delta t)^{\alpha}}{\alpha!}$$

$$A_i - 1 + \Lambda_i \Delta t = - A_i \sum_{\alpha=2}^{\infty} \frac{(-\Lambda_i \Delta t)^{\alpha}}{\alpha!}$$

by series as indicated.

Output Data.

$$\left.\begin{array}{l} \left.\begin{array}{l} a_{ij} \\ c_{ij0} \\ c_{ij1} \end{array}\right\} \quad \begin{array}{l} i = 1, 2, \ldots, N_R \\ j = 1, 2, \ldots, n_i \end{array} \\ \\ \left.\begin{array}{l} \mathrm{Re}(A_i),\ \mathrm{Im}(A_i) \\ \mathrm{Re}(C_{i0}),\ \mathrm{Im}(C_{i0}) \\ \mathrm{Re}(C_{i1}),\ \mathrm{Im}(C_{i1}) \end{array}\right\} \quad i = 1, 2, \ldots, N_C \end{array}\right\} \quad \begin{array}{l} \text{coefficients defining} \\ \text{the response of a} \\ \text{transfer function} \end{array}$$

Analysis. The derivations of the coefficients are contained in the main part of the paper. The coefficients are for use in a subroutine for determining the response of a transfer function of the form

$$\frac{L(E_{out})}{L(E_{in})} = c + \sum_{k=1}^{N_R} \sum_{j=1}^{n_k} \frac{\gamma_{kj}}{(s-\lambda_k)^{n_k-j+1}} + \sum_{k=1}^{N_C} \frac{\Gamma_{k1}\, s + \Gamma_{k2}}{s^2 + b_k s + c_k}$$

Subroutine for Determining the Response of a Transfer Function F(s)

Input Data.

N_R	number of distinct real poles
N_C	one half the number of complex poles
n_i $(i = 1, 2, \ldots, N_R)$	multiplicities of real poles (usually $n_i = 1$)

c (c is usually zero)

$$\left.\begin{array}{l} a_{ij},\ c_{ij0},\ c_{ij1} \quad (i=1,2,\ldots,N_R;\ j=1,2,\ldots,n_i) \\ \mathrm{Re}(A_i),\quad \mathrm{Im}(A_i) \quad (i=1,2,\ldots,N_C) \\ \mathrm{Re}(C_{i0}),\quad \mathrm{Im}(C_{i0}) \quad (i=1,2,\ldots,N_C) \\ \mathrm{Re}(C_{i1}),\quad \mathrm{Im}(C_{i1}) \quad (i=1,2,\ldots,N_C) \end{array}\right\} \text{coefficients}$$

$$\left.\begin{array}{l} E_{in}(T) \\ h_{ij}(T) \quad (i=1,2,\ldots,N_R;\ j=1,2,\ldots,n_i) \\ \mathrm{Re}[H_i(T)],\quad \mathrm{Im}[H_i(T)] \quad (i=1,2,\ldots,N_C) \end{array}\right\} \text{initializing conditions}$$

$E_{in}(t)$ current value of forcing function

Computations.

$$E_{out}(t) = c \cdot E_{in}(t) + \sum_{i=1}^{N_R} \sum_{j=1}^{n_i} h_{ij}(t) + 2 \sum_{i=1}^{N_C} \mathrm{Re}[H_i(t)]$$

where

$$h_{ij}(t) = \sum_{\beta=1}^{j} h_{i,\,j-\beta+1}(T) \cdot a_{i\beta} + E_{in}(T) \cdot c_{ij0} + E_{in}(t) \cdot c_{ij1}$$

$$H_i(t) = H_i(T) \cdot A_i + E_{in}(T) \cdot C_{i0} + E_{in}(t) \cdot C_{i1} \ .$$

Here $H_i(t)$, $H_i(T)$, A_i, C_{i0}, and C_{i1} are all complex.

Note on Handling of Input Data. Since the subroutine may be used in one program to compute the responses of many linear components of a control system, and since the coefficients may change from time to time for any given component, it is probably advisable to place each possible set of input data into an area of common storage. Then, at the time the subroutine is called, the desired set of data can be indicated in some manner.

Even if the form of F(s) is left unchanged, more than one set of coefficients for the component may be required, because the coefficients are dependent upon the time interval $\Delta t = T - t$. And even if the basic integration step-size is constant, some methods of numerical integration will require evaluation of $E_{out}(t)$ at fractional steps. (For fractional steps the computation of $h_{ij}(t)$ and $H_k(t)$ is incidental, and it is not required that these values be stored for future use.)

Output Data.

$E_{out}(t)$	Response at time t	
$h_{ij}(t) \quad (i = 1, 2, \ldots, N_R;\ j=1, 2, \ldots, n_i)$		new initializing conditions
$Re[H_i(t)],\ Im[H_i(t)] \quad (i=1, 2, \ldots, N_C)$		new initializing conditions

Analysis. Consider a transfer function

$$\frac{L(E_{out})}{L(E_{in})} = c + f(s) \tag{1}$$

where c is a constant and f(s) is a rational fraction in which the denominator has larger degree than the degree of the numerator. It is assumed that, before the subroutine has been called, the time histories of E_{in} and E_{out} have been determined up to time T and that the value of $E_{in}(t)$ for some time $t > T$ has been specified. Making the approximation that E_{in} is linear from time T to time t, one may determine approximately, using this subroutine, the corresponding value $E_{out}(t)$ of the response. The derivation of the method employed is contained in the paper, which also explains how this subroutine may be tied in very simply with the numerical integration of differential equations other than those represented by (1).

The initial conditions at time T are expressed by means of the input variables $E_{in}(T)$, $h_{ij}(T)$, and $H_k(T)$. The behavior of f(s) is expressed by means of the coefficients given in the input data. The assumption has been made that none of the complex (non-real) poles of f(s) are repeated.

This subroutine can be used to determine the solution defined by any transfer function (1) with non-repeated complex poles. However, the primary purpose of the subroutine is to avoid the prohibitively small integration step-sizes required by ordinary methods of numerical integration in the presence of poles of large magnitude.

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