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PLASMA STABILITY AND THE BOHR - VAN LEEUWEN THEOREM

by J. Reece Roth

Lewis Research Center

Cleveland, Ohio

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"We can only calculate beforehand, and understand in all observable details, those natural processes in which small errors in the formulation of the premises involve only small errors in the final results. As soon as unstable equilibrium comes into play, this condition is no longer fulfilled."

Hermann von Helmholtz - 1875

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SUMMARY

The characteristics of collisionless, nonrelativistic, magnetically confined plasmas possessing minimum reservoirs of free energy[1] are examined. It is anticipated that such plasmas will be subject to the smallest possible number of instabilities. Postulating a condition of Local Classical Kinetic Equilibrium (LCKE) permits the Bohr - van Leeuwen theorem, familiar in classical solid-state physics, to be applied to the interior region of magnetically confined plasmas. This theorem states that an element of plasma in the interior of such confined plasmas should possess zero net diamagnetism, hence, zero net current transverse to the applied magnetic field. The characteristics of the plasma and magnetic field needed to achieve this result are examined, and a set of sufficient conditions for vanishing transverse current is obtained that is consistent with the minimization of free energy within the plasma element. The most restrictive of these conditions is that the transverse energy density $nmv_{\perp}^2/2$ not vary in a direction transverse to the magnetic field lines. Although this condition may be met in the interior of plasmas, it cannot be met in the sheath region of finite laboratory plasmas.

The deduced characteristics of LCKE plasmas with minimum free energy are compared with results reported in the experimental literature. Examples are found in which the observed characteristics conform to those predicted in this report. In particular, cases are cited in which the observed plasma diamagnetism is less than would ordinarily be expected. Other cases are cited in which the transverse energy density profile remains nearly constant across the plasma interior. Such data are not extensive enough to be considered conclusive, but they do provide support for the viewpoint that any plasma confined for even a short time will approach a minimum-free-energy, LCKE condition in its interior.

A conceptual design is presented of a plasma injection and confinement system, which is aimed at maintaining a condition of minimum free energy during and after plasma buildup. The system consists of a "bumpy torus" confinement geometry together with a suitably modified Penning discharge plasma injection scheme. Experimental results obtained with such a Penning discharge injection scheme are used to show its capabilities for generating an approximately Maxwellian and isotropic plasma within the confinement geometry.

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[1]The term free energy is not used in this report in the precise thermodynamic sense of the Gibbs or Helmholtz free energy; it is intended to refer to those reservoirs of potential energy within the plasma (such as anisotropic or non-Maxwellian velocity distributions) that can drive instabilities which may be detrimental to plasma confinement.

INTRODUCTION

In the field of controlled thermonuclear research, attempts to magnetically confine plasmas have encountered instabilities of many types. The large number of possible instabilities are driven by a relatively small number of reservoirs of free energy within the plasma. Existing approaches to the problem of plasma stability that are based on linearized perturbation theory emphasize a derivation of stability criteria, which specify the characteristics of the plasma and magnetic field required to preserve a hypothetical initial steady state against some particular instability. This steady state may have associated with it large reservoirs of free energy within the plasma. The possible modes of plasma instability are then dealt with one by one by deriving a stability criterion for each instability. Sometimes, two or more of the stability criteria place mutually exclusive restrictions on the plasma and/or magnetic field geometry.

Linearized perturbation theory has been invaluable in documenting the various reservoirs of free energy that can give rise to instabilities, such as anisotropic and non-Maxwellian velocity distributions, for example. However, linearized perturbation theory is open to question, since the resulting stability criteria often assume that large reservoirs of free energy can exist indefinitely without being converted into instabilities by nonlinear phenomena or by some mode of linearized instability other than that under consideration. These linearized theories cannot determine whether nonlinear effects will tap sources of free energy within the plasma and result in a growth of nonlinear instabilities. Such nonlinear effects may result in instabilities that grow much faster than is predicted by the linearized theory.

There are many more possible modes of plasma instability than there are reservoirs of free energy to drive them, even if the nonlinear modes of instability are excluded. This report develops what may be regarded as a flanking approach to the problem of plasma stability and confinement. Rather than attempt to dispose of instabilities one by one by deriving a stability criterion for each, the approach put forward herein insists that the reservoirs of free energy be reduced to their lowest possible value in order that the minimum possible free energy be available to drive instabilities. It is not profitable, however, to insist on thermodynamic equilibrium, which would assure the vanishing of such free energy sources; such a condition is too restrictive to have much practical applicability.

An element of magnetically confined plasma with few reservoirs of free energy may approach the conditions required for application of the Bohr - van Leeuwen theorem, which implies that the current flowing perpendicular to the magnetic field lines is zero. This latter condition can then be used to further specify the plasma characteristics necessary to minimize the reservoirs of free energy, and it may be used to experimentally test whether the reservoirs of free energy in a particular element of plasma are, in fact, minimized.

The reservoirs of free energy within a finite laboratory plasma cannot be reduced completely to zero, of course, since there must always exist an energy density gradient between the plasma and the surrounding vacuum, or wall. By proper attention to the design of the injection and confinement scheme used (as discussed near the end of this report), one can attempt to promote the growth of the plasma interior region, in which no gradients of energy density exist, and in which the reservoirs of free energy are minimized. This interior region may be surrounded by a current-carrying sheath, containing an energy density gradient and possibly other sources of free energy. The discussion herein is based on the postulate that, if the reservoirs of free energy can be restricted to a thin sheath surrounding a large volume of plasma with no available free energy, then instabilities will not arise within the interior, and the plasma as a whole will be subject to fewer and less serious losses than a plasma with reservoirs of free energy throughout its volume. The physical mechanisms underlying this assumption will be discussed below. This report, then, describes a plasma, most of which is in a state of minimum free energy, and considers how such a plasma might be produced.

THE BOHR - VAN LEEUWEN THEOREM

RELATION OF THE BOHR - VAN LEEUWEN THEOREM TO CONVENTIONAL MACROSCOPIC THEORIES OF PLASMA STABILITY

The Momentum Equation

Both the conventional hydromagnetic theory and the discussion of macroscopic plasma stability presented in this report are based on the momentum equation, which describes the macroscopic body forces that act on the plasma as a whole. It is assumed that relativistic effects are not important, that displacement currents in the plasma are negligible, that the effects of collisions among the particles (or viscosity) are negligible, and that the plasma energy density tensor is diagonal, although it may differ in directions parallel and perpendicular to the magnetic field lines. Under these assumptions, the macroscopic momentum equation has been derived in several standard works (refs. 1 to 3) and is given as

$$F \equiv \frac{d}{dt}(nm\underline{v}) = -\nabla \cdot \underline{\underline{p}} + (n_i - n_e)e\,\underline{E} + (\underline{j} \times \underline{B}) + nm\underline{g} \tag{1}$$

where $\underline{v}$ is the macroscopic velocity of the plasma components and $\underline{\underline{p}}$ is the diagonal plasma energy density tensor, given by the sum of the energy density perpendicular and parallel to the magnetic field lines. (All symbols are defined in the appendix.) When the plasma is isotropic in velocity space,

$$\underline{\underline{p}} = p_{\perp} + p_{||} \tag{2}$$

and $\underline{E}$ is the sum of the externally imposed electric field and the internally generated electric field due to charge separation,

$$\underline{E} = \underline{E}_{ex} + \underline{E}_s \tag{3}$$

The term $(n_i - n_e)$ is the net charge density at a point in the plasma, $\underline{j}$ is the total current density flowing within the plasma, and $\underline{B}$ is the net magnetic field. The last term of equation (1) represents the influence of gravitational forces and will be neglected, since its magnitude is much smaller than the other terms under laboratory conditions.

The term on the left side of equation (1) represents the resultant force on the plasma. When this term is nonzero, the plasma is said to be macroscopically unstable and will exhibit gross changes in its configuration in space and/or in time.

The Conventional Magnetostatic and Hydromagnetic Theories

The hydromagnetic theory is usually understood to be based on the momentum equation (eq. (1)), and the assumption is made that $n_i = n_e$, so that no electric fields exist within the plasma. The magnetostatic theory (ref. 4) assumes in addition that the plasma is static in time and that the plasma energy density is scalar, so that equation (1) reduces to

$$\underline{j} \times \underline{B} = \nabla p \tag{4}$$

if gravitational forces are negligible. It should be noted that if $\underline{j} = 0$, equation (4) predicts that p is constant across the plasma. However, neither the hydromagnetic nor the magnetostatic theory provides any motivation for such an assumption. The current is eliminated from this equation by Ampere's law:

$$\nabla \times \underline{B} = \mu_o \underline{j} \tag{5}$$

which yields

$$\frac{1}{\mu_o} (\nabla \times \underline{B}) \times \underline{B} - \nabla p = 0 \tag{6}$$

Making use of the vector identity

$$(\nabla \times \underline{B}) \times \underline{B} = -\frac{1}{2} \nabla B^2 + (\underline{B} \cdot \nabla)\underline{B} \tag{7}$$

and assuming that the magnetic field lines in the plasma are straight, or that the currents flow normal to the applied field in a thin sheath, one has $((\underline{B} \cdot \nabla)\underline{B} = 0)$, and the magnetostatic equation becomes

$$\nabla \left(p + \frac{B^2}{2\mu_o} \right) = 0 \tag{8}$$

which implies that

$$p + \frac{B^2}{2\mu_o} = \text{constant} \tag{9}$$

It is usually inferred (refs. 5 to 8) from equation (9) that magnetically confined plasmas are necessarily diamagnetic, since equation (9) implies that the magnetic field in the vacuum around the plasma must be greater than that in the plasma itself. It should be noted that equation (4) does not require that currents flow throughout the entire volume of a plasma. The analysis reviewed above is identical with that given in many standard works on plasma physics (refs. 4 and 9 to 11). In order to compare this theory with certain experiments (refs. 12 and 13) later in the report, it is desirable to recast equation (9) in a form that makes explicit the contribution to the diamagnetic field of the diamagnetic currents that flow within the plasma.

In most plasma confinement schemes, the net magnetic field in a plasma can be written as the sum of an externally generated vacuum magnetic field B_o, which is curl-free in the plasma volume, and a diamagnetic field b, which is generated by currents flowing in the body of the plasma:

$$\underline{B} = \underline{B}_o + \underline{b} \tag{10}$$

Equation (9) will hold within the body of the plasma and in the surrounding vacuum to yield

$$p + \frac{B^2}{2\mu_o} = \frac{B_o^2}{2\mu_o} \tag{11}$$

If equation (10) is substituted into equation (11),

$$p + \frac{B_o b}{\mu_o} + \frac{b^2}{2\mu_o} = 0 \tag{12}$$

If a ratio of the plasma energy density to the magnetic energy density of the vacuum field is defined as

$$\beta \equiv \frac{2\mu_o p}{B_o^2} = \frac{2\mu_o nkT}{B_o^2} \tag{13}$$

the ratio of the diamagnetic field to the vacuum field can be found by substituting equation (12) into equation (13) to yield

$$\epsilon \equiv \frac{b}{B_o} = -1 + \sqrt{1 - \beta} \tag{14}$$

Equation (14) predicts that the plasma currents always produce a diamagnetic field b and that b is a monotonically increasing function of β.

It is characteristic of both the hydromagnetic and magnetostatic theories that they rule out the possibility of radial electric fields entering into the balance of forces. If significant radial electric fields are present, equation (14) will be an upper bound on the magnitude of the diamagnetic field and, hence, on the sheath currents. It is also characteristic of these theories that they eliminate the current density by substitution, thereby making it implicit in the equations. These theories therefore do not explicitly consider the spatial distribution of current in the plasma or whether it is or may be zero within the plasma volume. Since the Bohr - van Leeuwen theorem predicts that $j_\perp = 0$ under certain conditions, an explicit expression must be derived for the net current flowing in the plasma.

The Bohr - van Leeuwen Theorem

It has just been shown that the diamagnetic field b, which is generated by currents flowing in the plasma, plays an important role in the magnetostatic and hydromagnetic theories. There is a theorem of classical solid-state physics which states that, under very general conditions (to be specified in detail in the following paragraphs), the diamagnetic field b generated by an ensemble of charged particles will approach zero as this ensemble approaches local classical kinetic equilibrium. This theorem has been variously referred to as van Leeuwen's theorem (refs. 14 and 15) and the Bohr - van Leeuwen theorem (ref. 16). This report follows Kittel (ref. 16) and refers to it as the Bohr - van Leeuwen theorem.

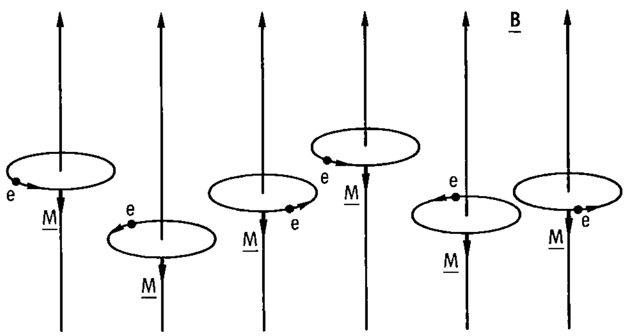


Figure 1. - Diamagnetic contributions of electrons in magnetic field.

This theorem originated early in the 20th century in an attempt to explain the absence of strong diamagnetism when a metallic conductor, with its associated electron gas, is placed in a magnetic field. The absence of strong diamagnetism in such circumstances was held to be very puzzling, since the magnetic moments of the gyrating electrons would seem to line up to oppose the applied field, as shown in figure 1.

Niels Bohr showed, by a physical argument, that the large net diamagnetic contribution of the gyrating electrons is balanced exactly and in detail by the hopping motion of those electrons that bounce off of the perfectly reflecting inner walls of the metal.[2] This mechanism explained the absence of diamagnetism in metals but relied on the special nature of the metallic boundary.

The work of Niels Bohr was generalized and put on a sound theoretical basis by H. J. van Leeuwen (ref. 17). She showed that the net diamagnetism of a collection of charged particles in a magnetic field is zero provided:

(1) Quantum-mechanical effects are negligible

(2) None of the fields or distribution functions are time dependent

(3) The velocity distribution function is Maxwellian along a radius vector in velocity space

(4) The distribution function (and the partition function) of the charged particles in

[2]From 1911 thesis of Niels Bohr which was not available to the author. It is summarized in ref. 15, pp. 95 and 97.

phase space are independent of the value of the magnetic field in which the particles are confined.

These four conditions are referred to collectively as Local Classical Kinetic Equilibrium (LCKE) throughout the remainder of this report, by analogy with the assumption of Local Thermodynamic Equilibrium (LTE) useful in gas dynamics and the theory of stellar structure. LCKE is a state of a system such that randomizing processes have acted to minimize the reservoirs of free energy.

The distinction between local classical kinetic equilibrium and thermodynamic equilibrium is four-fold. Thermodynamic equilibrium requires that there be no energy fluxes in the medium. A medium in LCKE, however, may contain a net flux of radiant energy. A second distinction is that LCKE explicitly excludes quantum-mechanical effects. A third distinction is that LCKE applies locally to a small element of plasma and not to the plasma as a whole. A fourth distinction is that LCKE does not require equipartition of energy as a necessary condition. In principle, if no collisions or other randomizing processes were active in an element of plasma, it would be possible for the plasma element to be in LCKE and yet possess an escape cone in velocity space. For laboratory plasmas, however, the presence of an escape cone will lead to the violation of the second condition of LCKE unless effective isotropy is maintained by steady-state injection of particles, in such a way that the injected particles exactly replace those lost by scattering into the escape cone. Thus, when collisions or other randomizing processes are active in a plasma, LCKE will also imply isotropy in velocity space, if particles are not continuously injected from the outside.

A brief proof of the Bohr - van Leeuwen theorem is as follows: the magnetic moment of a system in LCKE is given by (ref. 14)

$$\underline{M} = \frac{1}{\beta^*} \frac{\partial \ln Z}{\partial B} \tag{15}$$

where β^* is related to the kinetic temperature T by

$$\beta^* \equiv \frac{1}{kT} \tag{16}$$

and Z is the partition function in a phase space of elementary volume h^N expressed by

$$Z(B, T) = \frac{1}{h^N} \int_\Omega \int_V \exp\left[-\beta^* H(\underline{p}, \underline{x})\right] dx_k dp_k \tag{17}$$

In equation (17), Ω and V are the volumes of momentum and physical space, N is the number of degrees of freedom, and H is the Hamiltonian of the gas. When the magnetic field is constant in time, it is conservative, so the Hamiltonian is independent of the magnetic field; when the limits of integration in phase space are also independent of the magnetic field, it follows that $Z(B,T) = Z(T)$, and equation (15) then completes the proof that the magnetic moment $\underline{M}$ is zero. A rigorous and detailed proof of this theorem may be found in references 14 to 17. The requirement that quantum-mechanical effects be negligible was a very restrictive one in the field of solid-state physics. The small diamagnetism actually observed in normal solid materials is attributable to quantum-mechanical effects. The limitation of this theorem to classical physics has therefore resulted in its playing a minor role in the field of solid-state physics, in which it originated.

Application of the Bohr - van Leeuwen Theorem to Plasmas

It is of interest to investigate the conditions under which the Bohr - van Leeuwen theorem can be applied to magnetically confined plasmas. The first requirement of LCKE is easily satisfied, since plasmas of interest in controlled fusion research have high enough energies so that quantum-mechanical effects are negligible. The second requirement, that none of the fields or distribution functions be time dependent, is not satisfied in many experimental situations, but it may be approximated by suitably designed steady-state experiments or experiments in which the plasma characteristics and/or the magnetic field changes slowly compared with the plasma equilibration time. The third requirement of the Bohr - van Leeuwen theorem, that the plasma be Maxwellian along a radius vector in velocity space, is not satisfied in many experimental situations, particularly in neutral injection experiments. The fourth requirement, that the distribution function given by equation (17) be independent of magnetic field, is perhaps the most restrictive of these conditions in plasma physics applications. This requirement excludes experiments in which the magnetic fields change rapidly by comparison with the plasma equilibration time, such as pinch experiments. In such experiments, the Hamiltonian is not constant nor is it independent of magnetic field.

If the plasma is collisionless and if other randomizing processes are not active, the Bohr - van Leeuwen theorem applies to the case in which an escape cone of half-angle

$$\theta_o = \sin^{-1}\sqrt{\frac{B}{B_{max}}} \tag{18}$$

exists in velocity space and the particle motion is adiabatic (refs. 18 and 19). This is the case because it is sufficient for the application of the Bohr - van Leeuwen theorem that the Hamiltonian be independent of the magnetic field and of time.

A state of local classical kinetic equilibrium in the plasma will also be a state of minimum free energy if, in addition to LCKE, the velocity distribution is isotropic in velocity space. (It should be recalled that, in this report, the term minimum free energy implies only a minimizing of those sources of free energy that can give rise to plasma instabilities.) Effective isotropy is not enough to constitute a state of minimum free energy in the presence of collisions, since a plasma cannot maintain effective isotropy without outside help, in the form of particles injected to replace those knocked into the escape cone. True isotropy of the velocity distribution is desirable from the stability viewpoint because an element of plasma in LCKE with an isotropic velocity distribution will be (locally) in a state of maximum disorder, from which it is not possible to produce ordered motion of the type associated with plasma instabilities.

The Bohr - van Leeuwen Theorem and the Literature of Plasma Physics

It is of interest to survey the existing literature of plasma physics with the above discussion as background, and to attempt to ascertain why this theorem has not been more widely applied in the field of plasma physics. Only a few texts on plasma physics discuss the Bohr - van Leeuwen theorem at all (e.g., refs. 20 to 23). All these references base their discussion of the Bohr - van Leeuwen theorem on Niels Bohr's physical model, in which perfectly reflecting walls are necessary to provide the currents that cancel the net contribution from the interior of an element of plasma, and result in zero net diamagnetism for the plasma element. References 20 to 23 strongly suggest that an element of plasma must be surrounded with a perfectly reflecting wall in order for the Bohr - van Leeuwen theorem to be applicable and for the net current to be zero inside it. In fact, an element of the plasma interior may have $\underline{j} = 0$, if it is in LCKE, without being surrounded by a perfectly reflecting wall. All these authors seem unaware that the presence of a perfectly reflecting wall is not a necessary condition for the application of this theorem and that currents arising from density and temperature gradients within the plasma can cancel any net diamagnetism arising from uncompensated currents at the boundary of the plasma element. The impression conveyed by the literature, that perfectly reflecting walls are a necessary condition for the application of the Bohr - van Leeuwen theorem and for zero net current, has undoubtedly done much to limit the application of this theorem in the field of plasma physics.

A consequence of the Bohr - van Leeuwen theorem is that $\underline{j}_\perp = 0$ under the conditions already specified. It has been known for some time (ref. 24) and is widely appreciated in the literature (e.g., refs. 21 to 23) that a plasma in thermodynamic equilibrium will have no net true currents, that result from the motion of net charge from one point to another, since the mass motions that give rise to true currents are zero in thermodynamic equilibrium. The Bohr - van Leeuwen theorem is of interest precisely because LCKE is a less stringent requirement on the plasma than the requirement of thermodynamic equilibrium, and also because the Bohr - van Leeuwen theorem relates to total plasma currents, not just true currents. The total plasma current includes, for example, magnetization currents, which may be nonzero even when the individual guiding centers are stationary.

Perhaps the past limitations in the application of this theorem have come about because it has been assumed to apply to few laboratory plasmas: LCKE is not frequently found, and plasmas often exhibit evidence of internal currents. The present study in effect reverses this approach and starts with the postulate that LCKE is desirable for stability, or that internal currents are undesirable, then proceeds to examine situations under which LCKE may exist. A basic assumption of this approach is that the stability achieved for the main body of the plasma will not be offset by instability in the sheath region, which joins the plasma to the vacuum or wall, and cannot possess LCKE.

An extensive body of literature on plasma stability documents the instability-producing reservoirs of free energy (e.g., refs. 25 to 32). Most of the existing literature on the subject is open to the general type of objection summarized by Helmholtz at the beginning of this report, and discussed in the INTRODUCTION. A perturbation approach is applied to a basically unstable physical system, and the nonlinear effects that take over before the unstable perturbations reach infinite amplitudes are not considered. These approaches also do not consider the possibility that such instabilities may be converted by nonlinear effects into steady-state plasma turbulence, which may not have a detrimental effect on plasma confinement.

The aforementioned references assume that the plasma in question is collisionless (as is also done herein) and that no initial, static electric fields exist in the plasma. In addition, many of these references were obliged in the course of their analysis to make other restrictive assumptions. These assumptions include $\beta << 1$ (refs. 30 and 31), that the adiabatic invariant $mv_{\perp}^2/2B$ is a constant of the motion (refs. 25 and 29), and that no energy gradients exist in the plasma (ref. 30). These assumptions result in the elimination of nonlinear terms from the differential equations that describe the propagation of the disturbance in the plasma.

PLASMA CURRENTS

Assumptions of Analysis

The Bohr - van Leeuwen theorem implies that $\underline{j} = 0$ in a plasma that has attained local classical kinetic equilibrium. In order to relate the condition $\underline{j} = 0$ to the properties of a plasma in LCKE, a general expression for the currents flowing in a plasma is derived in this section. The currents are expressed in terms of the electric and magnetic fields that exist within the plasma volume and as a function of the spatial distribution of plasma density and kinetic temperature. Similar derivations are presented in the existing literature, examples of which are contained in references 25, 33, and 34.

In the derivation of the plasma currents, relativistic effects, the displacement current, and collisions among the particles are neglected. It is also assumed that the plasma is statically confined so that there is no net loss of particles from the plasma volume. Velocity components and vector quantities are decomposed into components parallel and perpendicular to the magnetic field lines.

Sources of Plasma Currents

It is convenient to regard the currents flowing in a statically confined plasma as arising from four sources: electric field currents caused by the presence of an electric field in the plasma $\underline{j}_E$; curvature currents caused by curvature of the magnetic field lines $\underline{j}_c$; gradient currents caused by gradients in the magnetic field $\underline{j}_g$; and magnetization currents arising from vorticity of the plasma magnetization $\underline{j}_m$.

The electric field currents occur when an electric field is present in a plasma. These currents arise from the well-known fact that charged particles in crossed electric and magnetic fields perform a sideways drift in the direction defined by $\underline{E} \times \underline{B}$. Since this drift velocity is independent of both the mass and charge of the particles, the magnitude of the electric field current is proportional to the net difference between positive and negative charge carriers (ref. 33) and is expressed by

$$\underline{j}_E = (n_i - n_e) |e| \frac{\underline{E} \times \underline{B}}{B^2} \tag{19}$$

where the subscripts i and e refer to ions and electrons, respectively. If Maxwell's equation without displacement currents,

$$(n_i - n_e) |e| = \epsilon_o \nabla \cdot \underline{E}_s \tag{20}$$

is used, equation (19) may then be written as

$$\underline{j}_E = -\frac{\epsilon_o(\nabla \cdot \underline{E}_s)}{B^2}(\underline{B} \times \underline{E}) \tag{21}$$

Equation (21) is then the required expression for the electric field drift currents.

A second source of current flow is the drift currents arising from the curvature of the magnetic field lines along which the gyrating particles move (ref. 33). This curvature current is given by

$$\underline{j}_c = \frac{nmv_{||}^2}{B^4} \underline{B} \times (\underline{B} \cdot \nabla)\underline{B} \tag{22}$$

which may be rewritten using the vector identities

$$(\underline{B} \cdot \nabla)\underline{B} \equiv \frac{1}{2} \nabla B^2 - \underline{B} \times (\nabla \times \underline{B}) \tag{23}$$

and

$$\underline{B} \times \left[\underline{B} \times (\nabla \times \underline{B})\right] = \left[\underline{B} \cdot (\nabla \times \underline{B})\right]\underline{B} - B^2(\nabla \times \underline{B}) \tag{24}$$

If equation (24) is used to simplify the equation resulting when equation (23) is substituted into equation (22), the curvature currents are then given by

$$\underline{j}_c = \frac{nmv_{||}^2}{2B^4}(\underline{B} \times \nabla B^2) - \frac{nmv_{||}^2}{B^4}\left[\underline{B} \cdot (\nabla \times \underline{B})\right]\underline{B} + \frac{nmv_{||}^2}{B^2}(\nabla \times \underline{B}) \tag{25}$$

A third cause of current flow in plasmas is the drift currents that arise from magnetic field gradients. These gradient currents are given by (refs. 33 and 34)

$$\underline{j}_g = \frac{nmv_{\perp}^2}{4B^4}(\underline{B} \times \nabla B^2) \tag{26}$$

The final source of time-independent plasma currents is magnetization current. This current is equal to the curl of the magnetic moment per unit volume $\underline{M}$ (ref. 33), which is

$$\underline{j}_m = \nabla \times \underline{M} \tag{27}$$

It is usually argued (refs. 33 and 34) that the magnetic moment per unit volume is just n times the equivalent current caused by a single charge gyrating in a circle of radius $\rho = mv_\perp/eB$. This current gives rise to a diamagnetic contribution to the magnetic field that is antiparallel to $\underline{B}$, as shown in figure 2. The total magnetic moment per unit volume is then given by

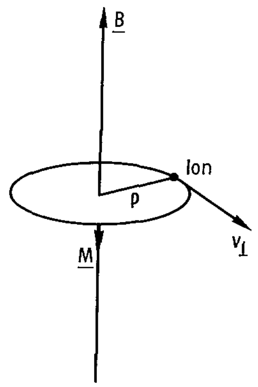


Figure 2. - Characteristic variables of positive ion gyrating in magnetic field.

$$\underline{M} = n\underline{M}' = -niA\,\frac{\underline{B}}{B} \tag{28}$$

The current associated with a particle making $\omega/2\pi$ gyrations per second around a circle of radius ρ is given by

$$i = \frac{e\omega}{2\pi} = \frac{ev_\perp}{2\pi\rho} \tag{29}$$

and the area enclosed by the gyrating charge is

$$A = \pi\rho^2 \tag{30}$$

The magnetic moment per unit volume may then be found by substituting equations (29) and (30) into equation (28), which yields

$$\underline{M} = -\left(\frac{nev_\perp\rho}{2}\right)\frac{\underline{B}}{B} = -\frac{nmv_\perp^2}{2B^2}\,\underline{B} \tag{31}$$

The magnetization current may then be written as

$$\underline{j}_m = \nabla \times \underline{M} = -\nabla \times \frac{nmv_\perp^2}{2B^2}\,\underline{B} \tag{32}$$

This expression for the magnetization current is found in references 33 and 34. Equation (32) does not take into account the effects of an electric field in the plasma normal to the magnetic field.

Net Plasma Currents

The results of the preceding section may be summarized to obtain the net current flowing in a plasma. If strong electric fields are present, the total current is given by

$$\underline{j} = \underline{j_E} + \underline{j_c} + \underline{j_g} + \underline{j_m} \tag{33}$$

where these currents are given, respectively, by equations (21), (25), (26), and (32). Substituting these equations into equation (33) yields

$$\underline{j} = -\frac{nmv_\perp^2}{2B^2}(\nabla \times \underline{B}) + \frac{nmv_{||}^2}{B^2}(\nabla \times \underline{B}) - \frac{nmv_{||}^2}{B^4}\left[\underline{B} \cdot (\nabla \times \underline{B})\right]\underline{B}$$

$$+ \underline{B} \times \nabla\left[\frac{nmv_\perp^2}{2B^2}\right] - \frac{\epsilon_o(\nabla \cdot \underline{E_s})}{B^2}(\underline{B} \times \underline{E}) + \frac{nmv_{||}^2}{2B^4}(\underline{B} \times \nabla B^2) + \frac{nmv_\perp^2}{4B^4}(\underline{B} \times \nabla B^2) \tag{34}$$

Equation (34) may be simplified by using

$$\nabla \times \underline{B} = \mu_o \underline{j} \tag{35}$$

Equation (34) may then be written as

$$\underline{j}\left(1 + \frac{\mu_o nmv_\perp^2}{2B^2} - \frac{\mu_o nmv_{||}^2}{B^2}\right) + \frac{\mu_o nmv_{||}^2}{B^2}\underline{j_{||}}$$

$$= \underline{B} \times \left[\nabla_\perp\left(\frac{nmv_\perp^2}{2B^2}\right) - \frac{\epsilon_o(\nabla \cdot \underline{E_s})}{B^2}\underline{E_\perp} + \frac{nm\left(2v_{||}^2 + v_\perp^2\right)}{4B^4}\nabla_\perp B^2\right] \tag{36}$$

An examination of the right side of equation (36) shows that the currents generated within the plasma are perpendicular to $\underline{B}$. The $\underline{j_{||}}$ currents, therefore, must arise from a net flow of particles along the field lines.

Since the present analysis is concerned only with currents that close on themselves within the plasma, it will be assumed that $\underline{j_{||}} = 0$. The total net plasma current $\underline{j}$ will therefore be equal to the net current flowing perpendicular to the field lines $\underline{j_\perp}$ for the purposes of this report. A parameter γ is defined as

$$\gamma \equiv 1 + \frac{\mu_o nmv_\perp^2}{2B^2} - \frac{\mu_o nmv_{||}^2}{B^2} = 1 + \frac{\mu_o nmv_\perp^2}{2B^2}\left(1 - \frac{2v_{||}^2}{v_\perp^2}\right) \tag{37}$$

so that equation (36) may be written as

$$\underline{j} = \underline{j_\perp} = \frac{\underline{B}}{\gamma} \times \left[\nabla_\perp \left(\frac{nmv_\perp^2}{2B^2}\right) - \frac{\epsilon_o(\nabla \cdot \underline{E_s})}{B^2}\underline{E_\perp} + \frac{nm\left(2v_{||}^2 + v_\perp^2\right)}{4B^4}\nabla_\perp B^2\right] \tag{38}$$

If an isotropy index δ is defined as

$$\delta \equiv 1 - \frac{2v_{||}^2}{v_\perp^2} \tag{39}$$

equation (37) can then be written as

$$\gamma = 1 + \delta \frac{\mu_o nmv_\perp^2}{2B^2} \tag{40}$$

so that the parameter γ is unity whenever the plasma velocity distribution is isotropic in velocity space, and approaches unity as the ratio of plasma to magnetic energy density becomes much less than unity. Equation (38) for the total plasma currents is equal to

$$\underline{j} = \underline{j_\perp} = \frac{\underline{B}}{\gamma B^2} \times \left[\frac{1}{2}\nabla_\perp nmv_\perp^2 - \frac{\delta nmv_\perp^2}{2}\frac{\nabla_\perp B}{B} - \epsilon_o(\nabla \cdot \underline{E_s})\underline{E_\perp}\right] \tag{41}$$

If the electric field in the plasma is negligible, equation (41) may be written as

$$\underline{j} = \frac{nmv_{\perp}^2}{2\gamma B^2}\, \underline{B} \times \left(\frac{\nabla nmv_{\perp}^2}{nmv_{\perp}^2} - \delta\, \frac{\nabla B}{B}\right) \qquad E_{\perp} = 0 \tag{42}$$

where the parameter γ is given by equation (40). If the isotropy index δ is not a function of position transverse to the magnetic field lines, equation (42) may be written as

$$\underline{j} = \frac{nmv_{\perp}^2}{2\gamma B^2}\, \underline{B} \times \nabla_{\perp} \ln \left(\frac{nmv_{\perp}^2}{B^{\delta}}\right) \qquad E_{\perp} = 0 \tag{43}$$

The parameter δ will not be a function of position for particles in a uniform field region between magnetic mirrors.

It is of interest to calculate how the isotropy index δ depends on mirror ratio for particles trapped between magnetic mirrors. It may be shown that, when the only anisotropy in velocity space is a loss cone of half angle θ_o about the magnetic field direction, the root-mean-square value of $v_{\perp}^2$ is given by

$$v_{\perp}^2 = v^2 \left(1 - \frac{1}{3} \cos^2 \theta_o\right) \tag{44}$$

and $v_{||}^2$ is given by

$$v_{||}^2 = \frac{1}{3} v^2 \cos^2 \theta_o \tag{45}$$

When the magnetic field varies slowly in space or time (refs. 18 and 19), the escape cone angle θ_o is related to the local mirror ratio by the relation

$$\sin^2 \theta_o = R \equiv \frac{B}{B_{max}} \tag{46}$$

When the particle is moving between magnetic mirrors in a magnetic field $B = B_{min}$, equation (46) becomes

$$\sin^2 \theta_o = R_m = \frac{B_{min}}{B_{max}} \tag{47}$$

and the isotropy index δ may be written in terms of the mirror ratio. Substituting equations (44), (45), and (46) into equation (39) yields

$$\delta = \frac{3R}{2 + R} \tag{48}$$

where δ may be a function of position along the field lines.

CHARACTERISTICS OF PLASMAS SATISFYING THE BOHR - VAN LEEUWEN THEOREM

General Current-Free Plasmas

A necessary condition for application of the Bohr - van Leeuwen theorem is that the plasma element in question be in local classical kinetic equilibrium. It is of interest to assume that a state of LCKE exists and to investigate the properties of the plasma and magnetic field that can give rise to zero net currents, as the Bohr - van Leeuwen theorem predicts.

Equations (37) and (38) reveal that a singular indeterminancy in $\underline{j_\perp}$ may occur if the parameter γ is zero. If a parameter

$$\beta_\perp \equiv \frac{\mu_o n m v_\perp^2}{2B^2} \tag{49}$$

is defined, it can be seen from equation (37) that γ will be zero only if

$$\frac{2v_{||}^2}{v_\perp^2} = 1 + \frac{1}{\beta_\perp} \tag{50}$$

Equation (50) could hold only in a plasma beam or in something resembling a linear pinch geometry. The requirement that γ be nonzero is satisfied for all devices of interest in the present discussion.

In general, a sufficient condition for $\underline{j_\perp} = 0$ is that the terms within the brackets of equation (41) sum to zero, such that

$$\frac{1}{2}\nabla_{\perp} nmv_{\perp}^2 - \delta \frac{nmv_{\perp}^2}{2} \frac{\nabla_{\perp} B}{B} - \epsilon_o(\nabla \cdot \underline{E}_s)\underline{E}_{\perp} = 0 \tag{51}$$

There are three possible ways in which the terms of equation (51) could counterbalance one another in pairs. Two of these require that the radial electric field in the plasma be so strong that the energy density of the electrostatic field is comparable with the energy density of the plasma or of the magnetic field. Since such strong radial electric fields apparently have not been observed, a balance between the electrostatic term and either of the other two can probably be ruled out. This leaves only the possibility that the first two terms in equation (51) might balance. If the isotropy index δ is not a function of position normal to the field lines, and if the electric field term is negligible, equation (51) then requires that

$$\frac{nmv_{\perp}^2}{2B^{\delta}} = \text{constant} \tag{52}$$

in a direction normal to the field lines. Equations (39) and (48) imply that, in magnetic mirror machines, $1/2\ v_{\perp}^2 \geq v_{||}^2$ and $0 \leq \delta \leq 1.0$. If equation (52) were satisfied, a plasma would behave paramagnetically in a direction normal to the field lines and, in the presence of collisions, it would migrate across the field lines toward regions of strong magnetic field. Satisfaction of equation (52) is a sufficient condition for $j_{\perp} = 0$ if $\underline{E}_{\perp} = 0$ and if δ is not a function of position transverse to the magnetic field lines. Equation (52) cannot be satisfied in the sheath of laboratory plasmas, since the plasma energy density in the sheath typically varies by several orders of magnitude over a distance in which the magnetic field strength varies by no more than a factor of 2. Equation (51) cannot be satisfied in the sheath unless electric fields exist that are much larger than any direct measurement has yet revealed. Currents must therefore flow in the sheath.

There is no evidence that the electrostatic energy density is comparable to the plasma energy density in the interior of confined plasmas. Equation (51) may be satisfied in plasma interiors either by satisfying equation (52), with $\underline{E}_{\perp} = 0$, or by having all three terms of equation (51) separately equal to zero. The radial variation of the magnetic field is usually quite small in the interior of laboratory plasmas and, moreover, $\delta \leq 1.0$. The counterbalancing of terms giving rise to equation (52) implies, therefore, very shallow gradients of energy density, so that the conditions implied by equation (52) will be quite close to, and nearly indistinguishable from, the conditions required to have the three terms of equation (51) separately equal to zero.

Minimum Free-Energy Plasmas

From the preceding discussion, it follows that the most interesting condition under which equation (51) is satisfied and $\underline{j_\perp} = 0$ is that each term of equation (51) separately vanish in the plasma element in question. These conditions can be related to the minimization of free energy in the plasma in the following way: if an element of plasma is in LCKE, the conditions of table I hold:

TABLE I. - CONDITIONS DEFINING LOCAL CLASSICAL KINETIC EQUILIBRIUM (LCKE)

(1) Quantum-mechanical effects are negligible in the plasma.
(2) None of the fields or distribution functions are time dependent.
(3) The distribution function is Maxwellian along a radius vector in velocity space.
(4) The partition function of the charged particles is independent of the value of the magnetic field at the position of the particle.

In the section Application of the Bohr - van Leeuwen Theorem to Plasmas, it was remarked that, if the plasma element in question was isotropic in velocity space and if $\delta = 0$, then the conditions of table I implied that the plasma element was locally in a state of maximum disorder and minimum free energy. If the terms of equation (51) do not counterbalance one another, the Bohr - van Leeuwen theorem states that the condition of minimum free energy described by table I implies, through equation (51), the conditions within a plasma element listed in table II.

TABLE II. - SUFFICIENT PROPERTIES OF A PLASMA ELEMENT WITH $\underline{j_\perp} = 0$

(1) $\underline{E_\perp} = 0$.
(2) Plasma isotropic in velocity space $\delta = 0$, and/or $\nabla_\perp B^2 = 0$.
(3) Plasma energy density uniform across field lines, $nmv_\perp^2 = \text{constant}$.

A state of LCKE cannot be rigorously inferred if $\underline{j_\perp} = 0$, since this might result from the counterbalancing of terms in equation (51). As the plasma equilibrates, however, it will tend to become isotropic in velocity space, δ will approach zero, and equation (52) will assure that the plasma element will approach the conditions of table II. Unless some outside agent maintains isotropy and/or maintains a very high electrostatic energy density (comparable to the plasma energy density), the equilibration of a plasma element can be expected to move it in the direction of satisfying the conditions of table II. The conditions of table II, with $\delta = 0$, are consistent with one's intuitive feeling for the characteristics of a plasma element with no reservoirs of free energy. The conditions of tables I and II are of interest to the apparatus designer, since they describe the characteristics a plasma element should have if it is to possess no reservoirs of free energy capable of being converted into macroscopic or microscopic instabilities.

It should be noted that the three conditions of table II refer only to the plasma properties normal to the field lines. The Bohr - van Leeuwen theorem says nothing about the conditions that must hold along the field lines.

Summary of Bohr - van Leeuwen Theorem

It has been seen that, if an element of plasma is magnetically confined and is in local classical kinetic equilibrium, satisfaction of the Bohr - van Leeuwen theorem implies the conditions enumerated in table II. If $\delta = 0$, such a plasma element will be in a state of minimum free energy. It appears highly probable that laboratory plasmas, if confined long enough for their interior to equilibrate and reach LCKE, will consist of two regions: an interior region in LCKE, in which the diamagnetic currents are zero and the conditions of table II hold, and a sheath region, in which the energy density of the plasma falls off to zero, in which the conditions of table II do not hold, and in which currents may flow.

The approach of a plasma to the state of minimum free energy described in tables I and II is discussed in reference 35. Unless the terms of equation (51) counterbalance one another in the sheath, a current will flow in the sheath. This current is attributable to the energy density gradient, a significant reservoir of free energy. Analysis has shown that instabilities resulting from this source of free energy can be stabilized, at least in part, by radially increasing magnetic fields (ref. 36).

If it is assumed that the net losses caused by instabilities are reduced by maximizing the fraction of the confined plasma in LCKE, the confining fields and injection system should be such as to promote $\underline{j_\perp} = 0$ throughout the interior of the plasma. It is then the task of the apparatus designer to formulate an injection and a confinement scheme that will produce a plasma in LCKE, which satisfies the conditions of table II, and which will make the volume of the interior region much larger than the sheath region. If the sheath region, with its currents and instability-producing sources of free energy, is much smaller in volume than the interior region, then hopefully the sheath region will not significantly affect the stability or confinement of the plasma as a whole. It is also possible that simplifications of stability analyses might result if the instabilities and reservoirs of free energy are limited to a thin sheath.

In most standard works on plasma physics, a plasma is treated as if its entire volume were necessarily a diamagnetic medium, which attempts at all times to migrate from regions of high to low magnetic field strength. It is one of the consequences of the Bohr - van Leeuwen theorem that this intuitive picture requires modification, at least for plasma interiors that have reached LCKE. Elements of plasma in LCKE have $\underline{j}_{\perp} = 0$ and are neither diamagnetic nor paramagnetic. Such elements of plasma therefore will not tend to move toward or away from regions of high magnetic field. A simple axisymmetric magnetic bottle will therefore be as effective in confining a plasma element with $\underline{j}_{\perp} = 0$ as a minimum-B or any other configuration. While absolute stability (in the sense of a ball at the bottom of a cup) is not assured, one can be at least certain that, in the absence of collisions, only an outside force or agency can move the plasma element from the point at which it was created. The effect of the magnetic field configuration on the sheath region where $\underline{j} \neq 0$ must be evaluated for specific cases.

The criteria in table II, deduced from the Bohr - van Leeuwen theorem, refer only to conditions normal to the magnetic field lines. In order to derive conditions that should hold along the magnetic field lines, the forces acting along the field lines must be investigated. This is accomplished in the following section.

FORCE-FREE PLASMAS

THE GENERAL CASE

Plasma characteristics necessary for even momentary confinement may also be deduced from the requirement that the net body force on an element of plasma be zero. In the following section, these characteristics are derived and are shown to be compatible with the results obtained by applying the Bohr - van Leeuwen theorem to a plasma element in LCKE. This approach also specifies the plasma and magnetic field characteristics that should hold in a direction parallel to the magnetic field lines.

In order to study the conditions under which $\underline{F} = 0$, an expression for the divergence of the pressure tensor appearing in equation (1) must be obtained (ref. 33), such that

$$\nabla \cdot \underline{\underline{p}} = \sum_{i} \frac{\partial}{\partial x_j} p_{ij} \tag{53}$$

where

$$p_{ij} = p_{\perp}\delta_{ij} - \frac{\left(p_{\perp} - p_{||}\right)}{B^2} B_i B_j \tag{54}$$

in which δ_{ij} is the Kronecker delta function, and $p_\perp$ and $p_{||}$ are given by

$$p_\perp = \frac{1}{2} nmv_\perp^2 \tag{55}$$

$$p_{||} = nmv_{||}^2 \tag{56}$$

If equation (54) is substituted into equation (53), the divergence of the pressure tensor may be written as

$$\nabla \cdot \underline{\underline{p}} = \nabla p_\perp - \underline{B}\left[(\underline{B} \cdot \nabla)\left(\frac{p_\perp - p_{||}}{B^2}\right) + \frac{p_\perp - p_{||}}{B^2}(\nabla \cdot \underline{B})\right] - \frac{p_\perp - p_{||}}{B^2}(\underline{B} \cdot \nabla)\underline{B} \tag{57}$$

This equation can be simplified by realizing that $\underline{B}(\underline{B} \cdot \nabla) = B^2\nabla_{||}$, $\nabla = \nabla_{||} + \nabla_\perp$, and by using equations (7), (39), and (49), to obtain

$$\nabla \cdot \underline{\underline{p}} = \nabla_\perp p_\perp + \nabla_{||} p_{||} + \frac{\beta_\perp \delta}{2\mu_o} \nabla_{||} B^2 - \frac{\beta_\perp \delta}{2\mu_o} \nabla_\perp B^2 - \delta\beta_\perp(\underline{j} \times \underline{B}) \tag{58}$$

Substituting equation (58) into equation (1), neglecting gravitational forces, and decomposing the forces into those perpendicular and parallel to the field lines yield

$$\underline{F_\perp} = -\nabla_\perp p_\perp + \frac{\beta_\perp \delta}{2\mu_o} \nabla_\perp B^2 + \epsilon_o(\nabla \cdot \underline{E_s})\underline{E_\perp} + (1 + \beta_\perp \delta)(\underline{j} \times \underline{B}) \tag{59}$$

$$\underline{F_{||}} = -\nabla_{||} p_{||} - \frac{\beta_\perp \delta}{2\mu_o} \nabla_{||} B^2 + \epsilon_o(\nabla \cdot \underline{E_s})\underline{E_{||}} \tag{60}$$

A comparison of equation (59) with equation (51) shows that if $\underline{j_\perp} = 0$, as was the case in a plasma in LCKE, then $\underline{F_\perp}$ also vanishes. Thus, as might be expected, the LCKE plasma has no net body force acting on it in a direction perpendicular to the magnetic field.

Substituting equation (38) into equation (59) gives the body forces acting to pull the plasma across the field lines:

$$\underline{F_\perp} = -\nabla_\perp p_\perp + \epsilon_o(\nabla \cdot \underline{E_s})\underline{E_\perp} + \frac{\beta_\perp \delta}{2\mu_o} \nabla_\perp B^2$$

$$+ \frac{B^2(1 + \beta_\perp \delta)}{\gamma} \left\{ \nabla_\perp \left[\frac{nmv_\perp^2}{2B^2} \right] - \frac{\epsilon_o(\nabla \cdot \underline{E_s})\underline{E_\perp}}{B^2} + \frac{nm\left(2v_{||}^2 + v_\perp^2\right)}{4B^4} \nabla_\perp B^2 \right\} \quad (61)$$

The terms of equation (61) may be rewritten to give

$$\underline{F_\perp} = \left[\frac{(1 + \beta_\perp \delta)}{\gamma} - 1\right] \nabla_\perp p_\perp - \left[\frac{(1 + \beta_\perp \delta)}{\gamma} - 1\right] \epsilon_o(\nabla \cdot \underline{E_s})\underline{E_\perp}$$

$$+ \frac{\beta_\perp(1 + \beta_\perp \delta)}{2\mu_o \gamma} \left[\frac{\gamma \delta}{1 + \beta_\perp \delta} + \frac{2v_{||}^2}{v_\perp^2} - 1 \right] \nabla_\perp B^2 = 0 \quad (62)$$

Thus, the net body force acting normal to the field lines is identically equal to zero (in the absence of collisions), regardless of whether a current is flowing in the plasma element in question.

ELECTRIC-FIELD-FREE PLASMAS

Since the particles are very mobile along the magnetic field lines, it is especially desirable that $\underline{F_{||}} = 0$. If electric field forces counterbalanced the forces caused by gradients of magnetic field or of plasma energy density, it would be necessary that the energy density of the electric field be comparable to the plasma and/or magnetic field energy density. The magnitude of the electric fields would then be approximately

$$E_{||} \sim \sqrt{\frac{2nmv_{||}^2}{\epsilon_o}} \quad (63)$$

Such electric fields are quite strong, several orders of magnitude larger than any electric fields as yet experimentally observed. The contribution of $\underline{E_{||}}$ to $\underline{F_{||}}$ is, therefore, ignored in the subsequent discussion.

It is of some interest to find out whether the gradients of $p_{||}$ and B^2 can compensate one another in equation (60) and thereby make $\underline{F_{||}} = 0$ without the necessity of having $\nabla_{||}B^2 = 0$ and $\nabla_{||}p_{||} = 0$ separately.

If $\underline{E_{||}} = 0$, equation (60) may be written

$$\underline{F_{||}} = -p_{||}\left[\frac{\nabla_{||}p_{||}}{p_{||}} + \frac{\delta}{1 - \delta}\frac{\nabla_{||}B}{B}\right] \tag{64}$$

where δ is written in terms of the local mirror ratio of the particle. Using equations (46) and (48) gives

$$\frac{\delta}{1 - \delta} = \frac{3R}{2(1 - R)} = \frac{3B}{2(B_{max} - B)} \tag{65}$$

and substituting equation (65) into equation (64) yields

$$\underline{F_{||}} = -p_{||}\nabla_{||} \ln\left[\frac{p_{||}}{(B_{max} - B)^{3/2}}\right] \tag{66}$$

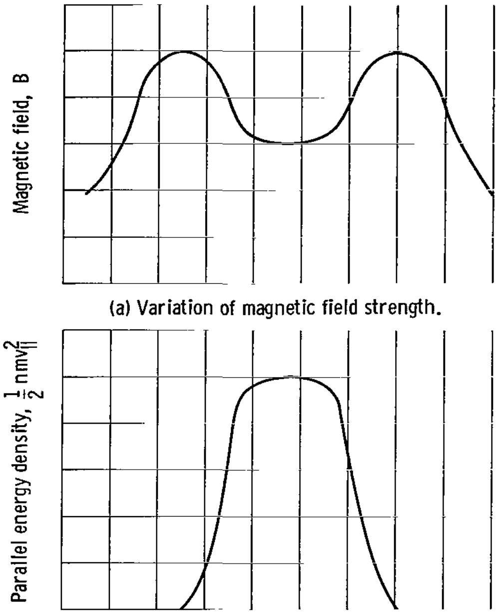
which implies that an exact counterbalancing of the $\nabla_{||}p_{||}$ and the $\nabla_{||}B^2$ term occurs if

$$\frac{p_{||}}{(B_{max} - B)^{3/2}} = \text{constant} = \frac{p_{||o}}{(B_{max} - B_{min})^{3/2}} \tag{67}$$

where o denotes values on the axis and at the midplane of the magnetic bottle. The parallel energy density along a magnetic field line may then be written as

$$p_{||} = p_{||o}\left(\frac{1 - R}{1 - R_m}\right)^{3/2} \tag{68}$$

At the throat of the magnetic bottle, where $B = B_{max}$, the parallel energy density goes to zero. The variation of B and of $p_{||}$ along the axis of a typical mirror machine is shown schematically in figure 3.



(b) Parallel energy density.

Figure 3. - Schematic drawings of magnetic field strength and parallel energy density along axis of mirror machine.

Equation (68) shows that is is possible to have substantial gradients in plasma and magnetic energy density, which counterbalance to give no net body force on the plasma element along the magnetic field lines. Such a situation, however, probably represents a significant reservoir of free energy which may result in instability of the plasma. In order to minimize these reservoirs of free energy and reduce the potential for instability, it is obviously desirable to have each of the three terms of equation (60) separately equal to zero. This, then, implies the plasma characteristics listed in table III. These three conditions supplement the conditions given in table II (p. 20) derived from the Bohr - van Leeuwen theorem, which hold in a direction normal to the magnetic field lines.

TABLE III. - SUFFICIENT CONDITIONS FOR $\underline{F}_{||} = 0$

(1) $\underline{E}_{||} = 0$
and
(2) Plasma is isotropic in velocity space ($\delta = 0$) <u>or</u> $\nabla_{||} B^2 = 0$
and
(3) $p_{||}$ is constant along magnetic field lines

CONTACT BETWEEN THEORY AND EXPERIMENT

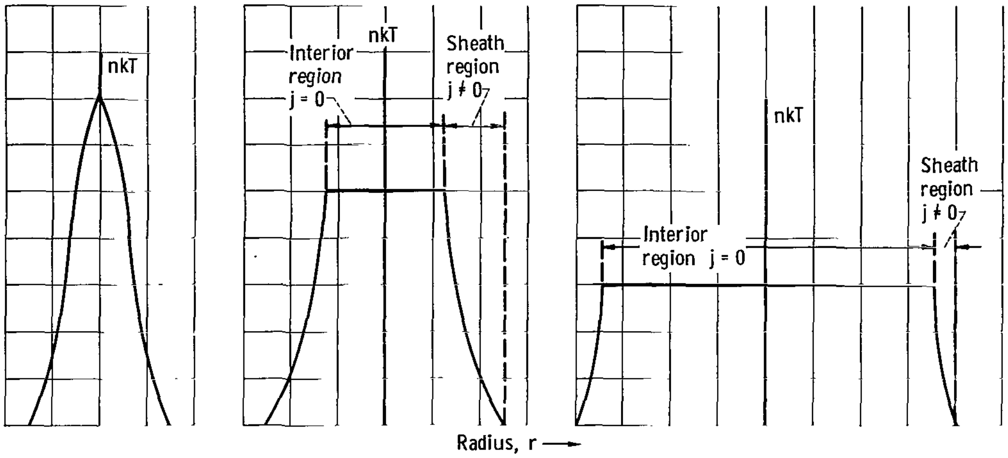
APPROACH TO EQUILIBRIUM OF CONFINED PLASMAS

If one adopts the view that reservoirs of free energy within a confined plasma are self-depleting through the instabilities to which they give rise, it follows that an element of plasma that appears to be stably confined after several multiples of the self-collision or equilibration time will have approached a condition of minimum free energy. This approach to equilibrium has been treated analytically by Abraham (ref. 35), for example. If such a condition is identified with LCKE but not necessarily with thermodynamic equilibrium, the plasma element will have $j_{\perp} = 0$, as a result of the Bohr - van Leeuwen theorem. From the vanishing of the perpendicular current, the relations among the field and plasma energy density gradients in equation (51) follow. The conditions set forth in table II obviously satisfy equation (51). Moreover, consideration of the probable magnitudes of the terms in equation (51) indicates that, as the interior of the plasma approaches LCKE, the gradient transverse to the magnetic field lines to be expected in the plasma energy density $nmv_{\perp}^2/2$ should be much less than that of either the plasma density n or the plasma energy $mv_{\perp}^2/2$ considered individually. Some physical and experimental consequences of this approach are discussed in the following section.

The establishment of LCKE may be limited to the interior of the confined plasma, and departures from LCKE seem necessary at the plasma boundary. The viewpoint adopted herein is that the localization of the free energy and the potential instabilities within a thin sheath should, by reducing the volume and quantity of plasma directly affected, reduce the loss rates of particles and/or energy. It would be desirable if this qualitative argument could be supported by an appropriately designed experiment.

As the plasma characteristics depart further and further from those listed in tables I to III, the reservoirs of free energy become larger, and it is reasonable to postulate that the microscopic and/or macroscopic instabilities will become progressively more troublesome. The successful confinement of a plasma can be looked on as a race between instability losses on the one hand and the randomizing processes that lead to stable confinement on the other. In practically all current plasma confinement schemes, the instabilities predominate, and the plasma is lost before it can equilibrate and become stabilized.

If one could inject an isotropic Maxwellian plasma into a containment device in such a way that the sheath region occupied only a small fraction of the plasma volume, the interior of the plasma would presumably equilibrate, reach LCKE, and remain quiescent. The experimenter could then undertake measurements of its properties and attempt to raise its energy and/or density. The plasma sheath probably would not be current free, since the perpendicular energy density $nmv_{\perp}^2/2$ is not constant across the sheath, and the electric fields in the sheath probably are not strong enough to counterbalance this energy



(a) Time, t = 0. (b) Time, t > 0. (c) Time, t → ∞.

Figure 4. - Development of initially peaked energy density distribution as time and equilibration processes proceed.

density gradient. It is reasonable to suppose that an initially quiescent plasma, which satisfied the conditions of tables I to III in its interior, will diffuse radially outward through collisions or some other collective mechanism in an attempt to establish a uniform energy density throughout all of space. This radial loss is potentially serious, but it may be of manageable proportions if radial cross-field diffusion processes are slow relative to the loss processes along the magnetic field lines.

If a plasma is injected into a confining device at time $t = 0$ with a radial energy density profile such as that shown in figure 4(a), the energy density profile would be expected to assume the configurations shown in figures 4(b) and (c) as equilibration proceeds, and the plasma interior achieves LCKE. The final steady-state thickness of the sheath region would probably be determined by the magnitude and rapidity of the radial diffusional losses across the sheath.

It appears reasonable to relate plasma instabilities to plasma turbulence in the following way: if a plasma is initially created with large reservoirs of free energy (or a high degree of ordered motion), these reservoirs of free energy will drive disturbances whose amplitudes will grow in time. During this initial process of instability growth, the energy will travel from small-scale motions with small amplitudes to larger and larger scale motions. If the amplitude of these instabilities does not grow so large that the plasma is lost from confinement, a steady-state spectrum of plasma fluctuations, or turbulence, will be established. This turbulence spectrum, once established, has the desirable feature that the available free energy will cascade from large-scale low-frequency motions to small-scale high-frequency motions, and will eventually be degraded to the thermal motions of the ions and electrons that compose the plasma. A steady-state spectrum of plasma turbulence can be regarded as a safety valve, or a stabilizing influence, which converts undesirable large-amplitude motions of the plasma into an increased temperature of the plasma constituents. Unfortunately, very few existing fusion research devices operate in a regime where one can reasonably expect to set up plasma turbulence. The initial instabilities are usually entirely suppressed (and this is considered desirable!) or result in loss of the plasma when they do occur.

To avoid loss of a plasma during the initial transient, when a spectrum of steady-state turbulence is being established, the amplitudes of the initial instabilities should be minimized. In view of the nonlinear nature of the phenomena involved (particularly the amplitude of the instabilities), it is very difficult to make a quantitative statement about the maximum tolerable amount of free energy in a confined element of plasma. Such a discussion is beyond the scope of this report. The problem might be most easily approached experimentally. It is clear, however, in which direction one should go - injection and confinement schemes should be designed to reduce the initial reservoirs of free energy to an absolute minimum.

It is desirable to discuss some of the assumptions and physical mechanisms underlying the aforementioned statements. It has been assumed that the region of flat energy density profile within the plasma grows at the expense of the sheath rather than the opposite. It has also been assumed that the presence of a large interior region of minimum free energy results in fewer net instability losses from the plasma than would exist if the sheath extended across the plasma diameter, and thus provided reservoirs of free energy throughout the plasma volume. It is necessary to remember that whenever the radial number density profile has been measured in a plasma, the profile is almost invariably radially decreasing. If collisions are the chief mechanism of equilibration, the plasma would be expected to equilibrate fastest near the axis, where the density and collision frequency are greatest. As time proceeds, the equilibrated region should spread radially outward, in the manner previously discussed in connection with figures 4(a) to (c). Therefore, if a plasma achieves LCKE at all, it should achieve it in the interior, where the equilibration process is most rapid. It appears reasonable to postulate that the number of particles lost from a plasma, because of instabilities, per unit volume and per unit time, is proportional to some positive power of the plasma number density. If the plasma interior is in LCKE, no instability losses should occur from the interior. The number of particles lost due to instabilities in the outer sheath region should be much smaller than it would be because of instabilities in the interior region, if reservoirs of free energy existed in the interior, because of the much lower number density of particles in the sheath. It also appears reasonable that the total free energy of a plasma can be minimized by restricting the reservoirs of free energy to a thin outer sheath. This restriction could come about because the total number of particles in the sheath may be made much smaller than in the plasma interior. Some experimental evidence that tends to confirm the assumptions made in this argument are described in the following section.

PREDICTED CHARACTERISTICS OF STABLY CONFINED PLASMAS

The viewpoint that the interior of stably confined plasmas must approach a condition of LCKE leads to the prediction of at least two previously unrecognized characteristics of the interior of such plasmas. These characteristics are the zero net diamagnetism of the interior of such plasmas, and the uniformity of the plasma energy density transverse to the field lines in the interior of such plasmas.

The vanishing of the diamagnetism of the interior of a confined plasma, or its reduction to values small by comparison with those predicted by the magnetostatic theory, permits the inference that the plasma interior has approached LCKE and that the reservoirs of free energy, which are available to drive instabilities, are correspondingly small. The opposite inference may be made with complete rigor: if substantial transverse currents and/or transverse plasma energy density gradients are detected throughout the plasma volume, the plasma interior is not in LCKE and possesses substantial reservoirs of free energy. These reservoirs of free energy are then available to drive undesirable plasma instabilities.

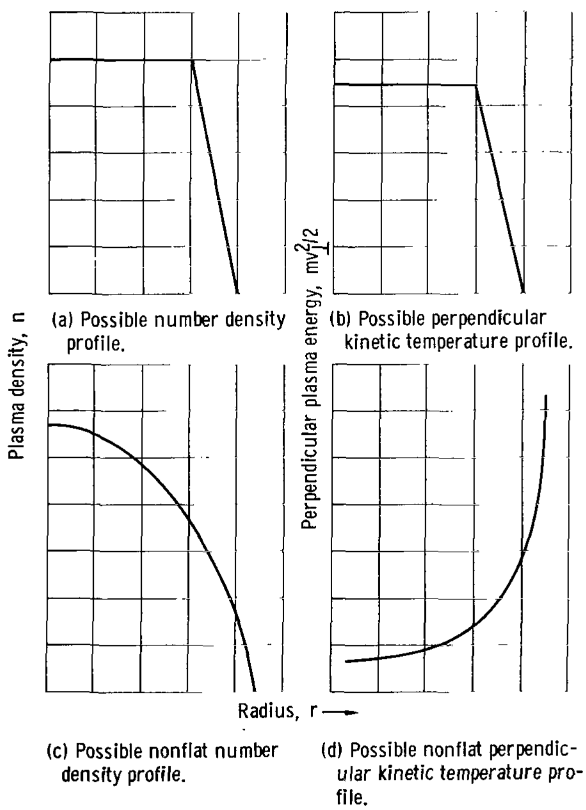


Figure 5. - Possible density and energy profiles that yield constant energy density profile.

The uniformity of the plasma energy density normal to the field lines in a plasma interior is predicted by equation (51) for circumstances where the plasma interior approaches LCKE. Some departure from absolute uniformity may be expected as a result of the presence of a nonvanishing electrostatic energy density and/or a transverse magnetic field gradient coupled with an anisotropy in the velocity distribution $\delta \neq 0$. However, if the plasma density n is to vary by an order of magnitude in the radial direction, the greatest part of this variation must be compensated by a corresponding radial variation in particle energy, or $mv_\perp^2/2$. Measurements of plasma density and temperature variations that result in a nearly constant plasma energy density profile would provide impressive support for the contention that plasma interiors will tend to approach LCKE by themselves.

In experimental tests for a flat energy

density profile, a region of constant energy density across the plasma diameter and a sheath region in which the energy density falls off to zero would be expected. This is illustrated schematically in figures 4(b) and (c). The constant energy density profile could arise from constant profiles of density and temperature, as shown schematically in figures 5(a) and (b). A constant energy density profile could also arise from a radially decreasing density profile and a radially increasing temperature profile, so compensated that their product is constant across the diameter.

The latter type of profile is illustrated in figures 5(c) and (d). Observation of density and temperature profiles like those in figures 5(c) and (d) would be a particularly convincing test of the theory described herein, since a radially increasing temperature would not be expected on any other grounds, unless it were a peculiarity of the injection scheme used. That two such profiles as figures 5(c) and (d) should have a constant product in the plasma interior is unexpected and, to the extent that it is unexpected, is a crucial test of the degree to which the plasma is in LCKE.

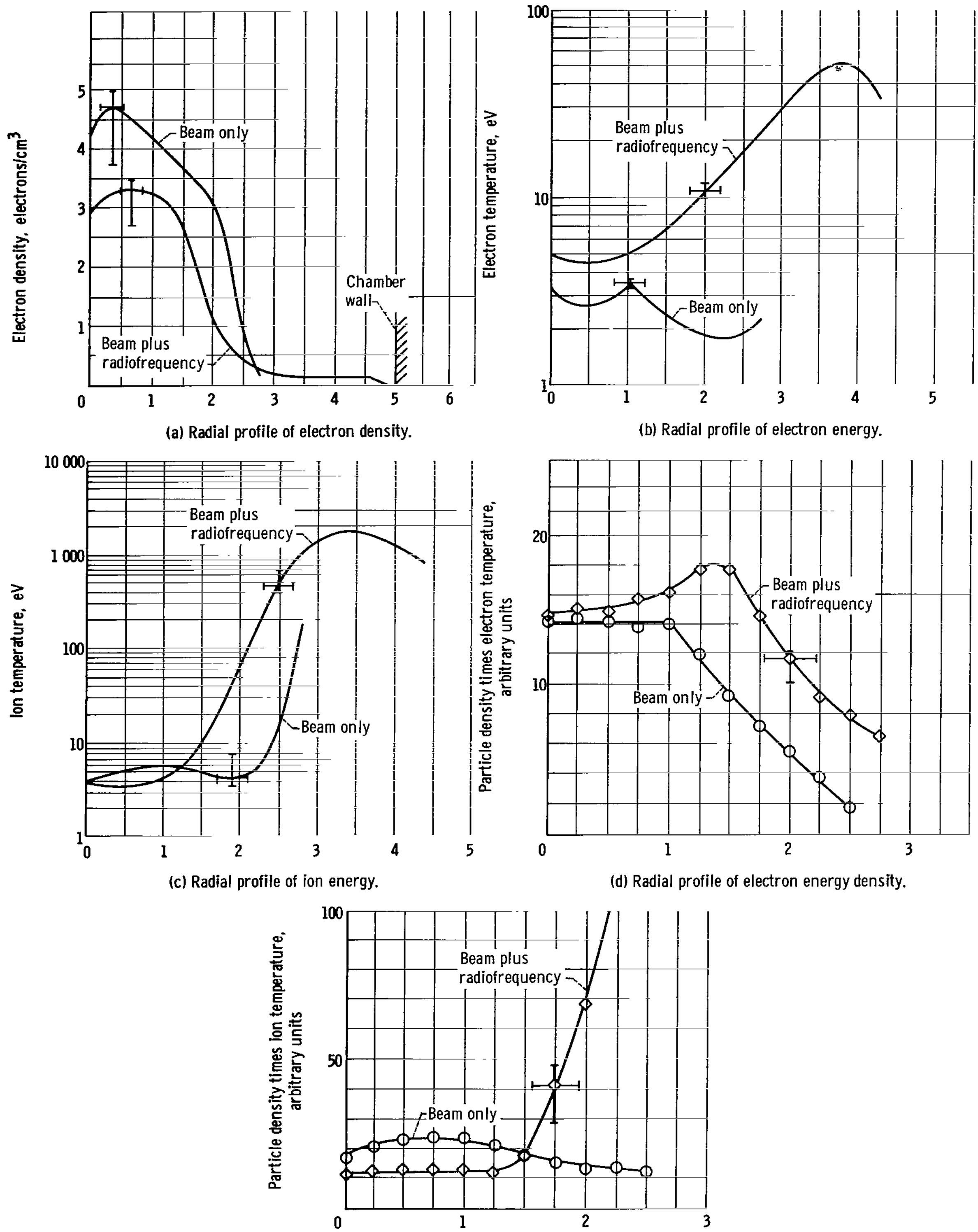
REPORTED PLASMA CHARACTERISTICS

Experimental Evidence of a Constant Radial Energy Density Profile

It might be expected that a decade or more of intensive research in plasma physics and controlled fusion would have produced a large quantity of density and temperature profiles suitable for testing the consequences of the Bohr - van Leeuwen theorem. Surprisingly, an extensive search of the literature revealed practically no such data.

One of the very few simultaneous measurements of radial density and temperature profiles to come to the attention of the author are those made by Krawec (ref. 37). Figure 6(a) shows Krawec's radial electron density profile. In figures 6(b) and (c) are his measurements of the electron and ion radial temperature profiles, and figures 6(d) and (e) show his radial energy density profiles for electrons and ions, respectively. It was assumed that electron and ion densities were equal in computing these latter profiles. The measurements were made on a steady-state plasma from a Phillips Ionization Gage (PIG) source (beam only) and also when a steady-state plasma was subject to ion cyclotron resonance heating at a position between the PIG source and the point at which the measurements were taken (beam plus radiofrequency). The radial temperature and energy density profiles both show an increase, presumably because of the absorption of radio frequency power in the outskirts of the plasma.

The plasma studied in reference 37 was confined between magnetic mirrors, and the conditions of LCKE were probably only approximately met. Nonetheless, figures 6(d) and (e) show a nearly flat energy density profile across 40 percent of the plasma diameter,



(a) Radial profile of electron density.

(b) Radial profile of electron energy.

(c) Radial profile of ion energy.

(d) Radial profile of electron energy density.

(e) Radial profile of ion energy density.

Figure 6. - Radial density and energy profiles from reference 37 with and without addition of radiofrequency power to plasma.

which resulted (in the case of radiofrequency heating of the plasma) from the product of a radially decreasing density and a radially increasing temperature profile. The data presented in figure 6 are typical of a wide variety of data taken under different operating conditions (private communication with Roman Krawec). These data cannot be regarded as conclusive proof that the plasma in question was in a state of LCKE and satisfied the requirements for application of the Bohr - van Leeuwen theorem. These results are in accord with the predictions of the present theory, however.

Other measurements of plasma energy density profiles have been made. Measurements of the radial profile of ion saturation current (proportional to $n(1/2\ mv^2)^{1/2}$) have been made in the model C stellarator, and both flat (ref. 38) and radially decreasing (ref. 39) profiles of ion saturation current have been observed, depending on the operating conditions. The flat profiles were constant over about 80 percent of the discharge diameter. Independent measurements of the density and temperature profiles across the diameter of the model C stellarator have recently been made, and it was found that, under a wide range of conditions, these profiles were flat across most of the diameter, similar to that shown schematically in figures 5(a) and (b) (private communication with A. Haberstich). The resulting flat energy density profile is accord with the aforementioned theory of a plasma in LCKE.

The radial energy density profile of a hot-electron plasma was measured by Ard, England, et al. (ref. 40) by observing the change in the diamagnetic signal caused by a ball dropped along a diameter of the plasma. It was found that the energy density was approximately constant over 50 percent of the plasma diameter. Preliminary experiments on the EPA hot electron plasma at Oak Ridge can be interpreted to indicate that it may have a constant energy density profile over about 75 percent of its diameter (based on preliminary data supplied in private communication by A. C. England). Some observations on the DECA I experiment (ref. 41) suggest the possibility of a constant radial energy density. These investigators observed a radially decreasing density, and an X-ray flux from a radially inserted probe that indicated a radially increasing particle energy.

Experimental Evidence for the Absence of Diamagnetic Currents in Plasmas

A series of measurements have been made on a steady-state, hot-electron plasma at Oak Ridge (refs. 12 and 13). The responsibility for the interpretation of the data given below lies with the present author, and is not necessarily that of the investigators responsible for these experiments. In reference 12, an experiment is described in which $\beta \approx 0.3$ and $B_0 \approx 3000$ gauss. For such a plasma, equation (14) would predict a diamagnetic field ratio of

$$\epsilon \approx -1 + \sqrt{1 - 0.3} = -0.16$$

which would lead one to expect a diamagnetic field of $b \approx -0.16 \times 3000 \approx -480$ gauss in this experiment. In fact, no steady-state diamagnetic field greater than -60 gauss was observed by a gaussmeter inserted along the plasma axis. Not only were the results of this experimental measurement in quantitative disagreement with the magnetostatic theory, but the experimental measurements showed qualitative disagreement, in that the probe measurements revealed a paramagnetic field along some regions of the plasma axis.

Reference 13 contains an account of a similar measurement made in a different apparatus. In this experiment, the plasma was confined in a magnetic field of ≈3000 gauss, and had a β of approximately 0.10. Under these conditions, equation (14) predicts a diamagnetic ratio of $\epsilon \approx -0.05$ and a diamagnetic field of $b \approx -150$ gauss. The experimental observations showed both steady-state diamagnetic and paramagnetic contributions at different points along the plasma axis and a diamagnetic contribution whose magnitude was nowhere greater than 10 gauss. These steady-state measurements should not be confused with the unsteady measurement of flux change through a diamagnetic loop made when the plasma was turned off.

The most probable interpretation of these experiments, in the view of this author, is that the currents that generated the diamagnetic field flowed largely in the outermost boundary of the plasma, and the remoteness of these currents from the axis (combined with the short axial length of the plasma) resulted in the anomalously low diamagnetic field on the axis. The bulk of the plasma between the axis and the boundary sheath would be in LCKE, or near it, and the currents in this region would therefore be zero or small and result in no or little enhancement of the diamagnetic field on the axis. In fact, a calculation of the current densities in the plasma (ref. 13) showed current densities in the outer regions of the plasma that were more than twice those near the axis. The difference may have been even greater than this, since this calculation did not produce a unique distribution of currents in the plasma. The finite axial length of the plasma could not have been responsible for more than a factor of 2 reduction of the diamagnetic field predicted by the magnetostatic theory. Although the confinement of these hot-electron plasmas was such that an escape cone was present, the reduction in diamagnetism may be taken to indicate that an approximation to LCKE existed in the interior of this plasma. The relation given in equation (51) may have been satisfied throughout most of the plasma, although the more restrictive conditions of table II (p. 20) could not be. The hot electrons were observed to be Maxwellian in energy.

A similar hot-electron experiment has been reported by Tanaka, et al. (ref. 42) of the Plasma Physics Institute at Nagoya, Japan. These experimenters reported results in good agreement with those of Oak Ridge (refs. 12 and 13). The diamagnetic field was measured along the axis of a hot-electron plasma, and it was found to be paramagnetic along some regions of the axis and diamagnetic along others. The magnitude of this field was small. The diamagnetic signal was measured while a scraper probe was inserted radially inward at the plasma midplane. It was found that the currents which produced the diamagnetic signal all flowed in an annular volume between 6 and 9 centimeters in radius. The production of X-rays was also monitored as the probe was inserted, and it was found that the particles responsible for the production of X-rays moved in an annulus from 7 to 10 centimeters in radius.

These observations may be explained in two ways: (1) the plasma had a radially increasing temperature profile, with currents flowing in a thin sheath between 6 and 9 centimeters in radius, and a constant energy density profile, or (2) the entire plasma is confined to an annular volume of inner radius 6 centimeters and outer radius 10 centimeters, with a hollow center. It was not possible to distinguish between these two possibilities with the data taken.

In the plasma sheath, where the plasma energy density must fall off to zero, equation (51) implies that any approach to LCKE must produce radial electric fields. Such radial electric fields have been observed by Perkins and Post (ref. 43) in their experiments, and the doppler shift resulting from an $\underline{E} \times \underline{B}$ particle drift has been measured by Drawin and Fumelli (ref. 44). In neither case is the radial electric field large enough to make the electrostatic energy density comparable to the plasma energy density. It is possible, however, that these fields represent an extension of LCKE conditions part of the way into the sheath. Such a conclusion could be confirmed only by a detailed experimental investigation of the radial variation of the electric field and a comparison with the requirements of equation (51).

APPLICATIONS TO THE DESIGN OF FUSION RESEARCH APPARATUS

Criteria for the Design of Magnetic Fields That Promote Plasma Stability and Confinement

In the design of a plasma experiment, the considerations explored herein suggest that the apparatus should be designed in a way that promotes local classical kinetic equilibrium and meets the conditions of table I (p. 20). If this is done and if the plasma is in LCKE, the Bohr - van Leeuwen theorem assures that $\underline{j_\perp} = 0$. If it is also possible to satisfy the even more restrictive conditions of table II (p. 20), the sources of free energy that may drive instabilities should be at a minimum. The requirements of tables I to III should be satisfied both during and after the initial period of plasma buildup. As pointed out in the section APPROACH TO EQUILIBRIUM OF CONFINED PLASMAS, such conditions may be impossible to maintain at the plasma boundary. A system that promotes and maintains such conditions throughout the bulk of the plasma may be the best approximation possible.

The requirement that the velocity distribution of the plasma element be isotropic, or that there be no magnetic field gradients normal to or along the field lines, is best satisfied by utilizing a geometry in which the field lines close on themselves. The exact configuration of the field may depend strongly on the instability processes in the sheath. In order to maintain a Maxwellian velocity distribution, the magnetic field strength and dimensions must be such that nonadiabatic particle losses do not play a significant role in depleting the Maxwellian tail. To summarize, the magnetic field configuration should satisfy the requirements listed in table IV.

TABLE IV. - DESIGN CRITERIA FOR MAGNETIC FIELD CONFIGURATIONS THAT PROMOTE THE MINIMIZATION OF PLASMA FREE ENERGY

(1) The magnetic field lines should close on themselves
and
(2) The magnetic field should not give rise to any escape cone or anisotropy in the velocity distribution of the plasma
or
(3) The magnetic field should have minimum gradients in the plasma, consistent with condition (1)
and
(4) The magnetic field strength and geometry should be such that nonadiabatic effects do not significantly deplete the high-energy Maxwellian tail

Design Criteria for Plasma Injection Schemes That Promote Plasma Stability and Confinement

The plasma injection process should be such that conditions as close as possible to LCKE are maintained throughout the interior of the plasma during and after its buildup. The requirements of such an injection system, consistent with the conditions of tables I to III are summarized in table V.

The conditions of table V should result in the production of a plasma with a minimum of free energy available to drive instabilities. The fourth requirement listed, that the electric field be zero, while consistent with the conditions of tables II and III, is not strictly needed to satisfy the requirements of table I (p. 20). It is clear that the conditions listed in table V are quite difficult of achievement and are not satisfied by most current plasma injection schemes. In particular, neutral injection schemes seem necessarily to result in very non-Maxwellian and anisotropic plasmas.

TABLE V. - DESIGN CRITERIA FOR PLASMA INJECTION SCHEMES THAT

PROMOTE PLASMA STABILITY AND CONFINEMENT

(1) The injected plasma should be Maxwellian along a radius vector in velocity space
(2) The injected plasma should be isotropic in velocity space
(3) The plasma should be injected or created as uniformly as possible throughout the containment volume, <u>or</u> the rate of injection or creation should be slow enough that no appreciable gradients in energy density are produced during buildup
(4) Electric fields should be zero in the plasma

A POSSIBLE SOLUTION TO PLASMA CONTAINMENT AND INJECTION REQUIREMENTS

Magnetic Field Geometry

Many magnetic field geometries are possible that satisfy the criteria of table IV. This section puts forward a particular concept that is compatible with the design constraints in table IV and also possesses the advantages of relative technical simplicity and ease of access for experimental measurements.

This concept is the bumpy torus configuration, which consists of a number of magnetic field coils whose centerlines are disposed around the major diameter of a torus, as shown in figure 7. This configuration was suggested by Gibson, Jordan, and Lauer (refs. 45 to 47) who have done a great deal of valuable work in analytically studying the behavior of particles trapped in such a device (refs. 45 and 47), and in experimentally

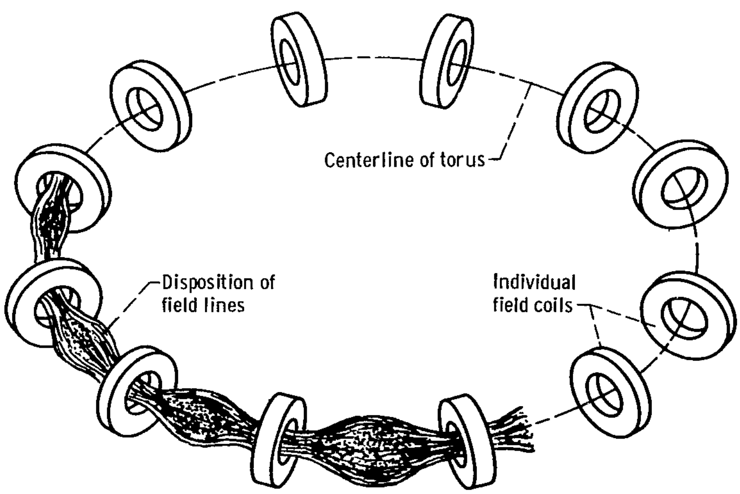


Figure 7. - Schematic drawing of bumpy torus configuration showing 12 coils disposed around toroidal axis.

demonstrating that particles could be adiabatically trapped in such a device (ref. 46). Their studies show no difficulty of principle with this geometry, provided that a suitable injection scheme can be found. Unfortunately, the actual construction of such a device was never carried through to completion and final testing.

A similar device has been built and operated at Kyoto University, Kyoto, Japan and is called the "Heliotron" (ref. 48). The geometry of the device is similar to that of figure 7, with a major diameter of the torus of 2 meters and an inside diameter of the coils of about 20 centimeters. The coils are not operated in the steady state, but are driven by a condenser bank. The plasma in this device is created by preionization and ohmic heating, and is then compressed.

The bumpy torus configuration in figure 11 satisfies most of the requirements of table IV, since the magnetic field lines close on themselves, and the plasma may, in principle, be isotropic. If the plasma is created in an isotropic manner, the velocity distribution should remain so, since particles outside the escape cone in velocity space will be trapped between mirrors, and the escape cone will be filled by particles traveling around the major circumference of the torus.

The configuration does not satisfy the alternate requirement in table IV that the magnetic field possess no gradients in the plasma; both parallel and perpendicular gradients will exist. This requirement is probably not too critical, however, since it arose from the special case in which all gradients were made to vanish as an alternative to isotropy of the velocity distribution. The bumpy torus is superior to a torus with a uniform magnetic field along its axis on practical grounds, since the former permits easy experimental and visual access to the plasma volume and a better access for vacuum pumping of the experimental volume. In addition, if superconducting magnets are used, it is easier to enclose individual coils in dewars than to enclose a uniformly wound toroidal winding. The bumpy torus is probably superior on stability grounds, since the particles trapped between magnetic mirrors could serve to anchor the toroidal plasma ring against macroscopic drift instabilities. The stability properties of the bumpy torus configuration have been investigated by Gardner (ref. 49).

Plasma Injection System

After a confinement device that satisfies the criteria of table IV is obtained, an injection scheme must be developed that satisfies the requirements of table V. None of the conventional injection schemes are suitable for the bumpy torus geometry and also satisfy the requirements of table V. Those injection schemes that come closest to satisfying these constraints (such as injection of plasma bunches across the field lines) are transient rather than steady-state devices and would certainly exhibit severe spatial gradients.

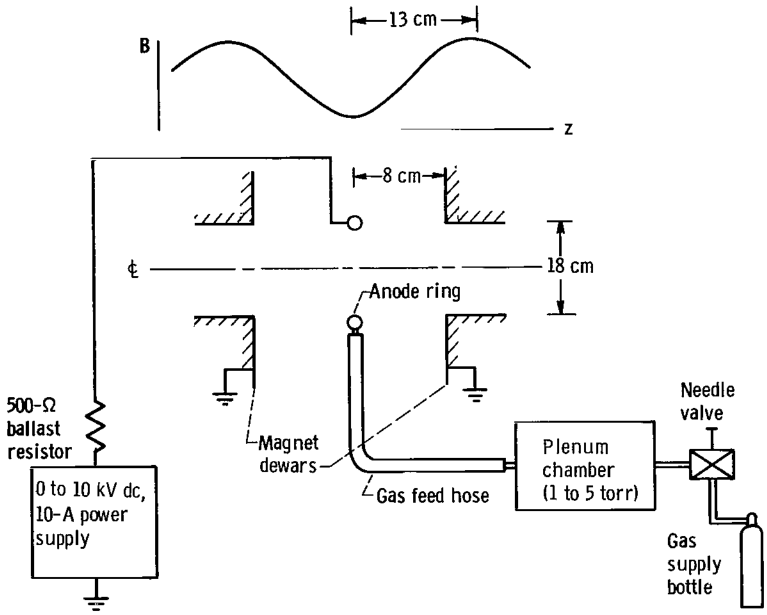


Figure 8. - Electrical and gas flow systems of modified Penning discharge.

An injection scheme has been developed at Lewis Research Center that operates in the steady state, which approximately satisfies the requirements of table V, and which is also well-adapted to the geometry of the bumpy torus. A complete and detailed discussion of this injection scheme is beyond the scope of this report; only these features relevant to the criteria of table V are discussed. The physical arrangement of the source is discussed in detail in reference 50.

This injection scheme is a modification of the conventional Penning discharge (ref. 51), which is shown schematically in figure 8. The anode is a circular hoop located at the midplane of a magnetic bottle and is operated at a positive potential of 1 to 10 kilovolts. The grounded superconducting magnetic dewars (ref. 52), located at the magnetic field maxima, form the cathodes of the discharge. The anode ring and the dewars have an inside diameter of 15 centimeters. An isometric drawing of the discharge configuration is shown in figure 9.

The gas required for operation of the discharge is admitted to the vacuum system through a needle valve and enters a 2-liter plenum chamber, which is maintained at pressures from 1 to 5 torr. The gas inlet line from the plenum chamber is connected to the hollow stainless-steel tubing of which the anode ring is constructed. As is shown in figure 9, the hollow interior of the anode ring serves as a manifold to distribute the incoming gas to 18 holes, which spray the gas radially inward to the discharge. The neutral gas pressure on the axis of the discharge is no more than 5×10^{-5} torr at a gas flow rate adequate to sustain the discharge in the low- and high-power modes of operation. In this geometry, the magnetic mirrors on either side of the anode provide magnetic

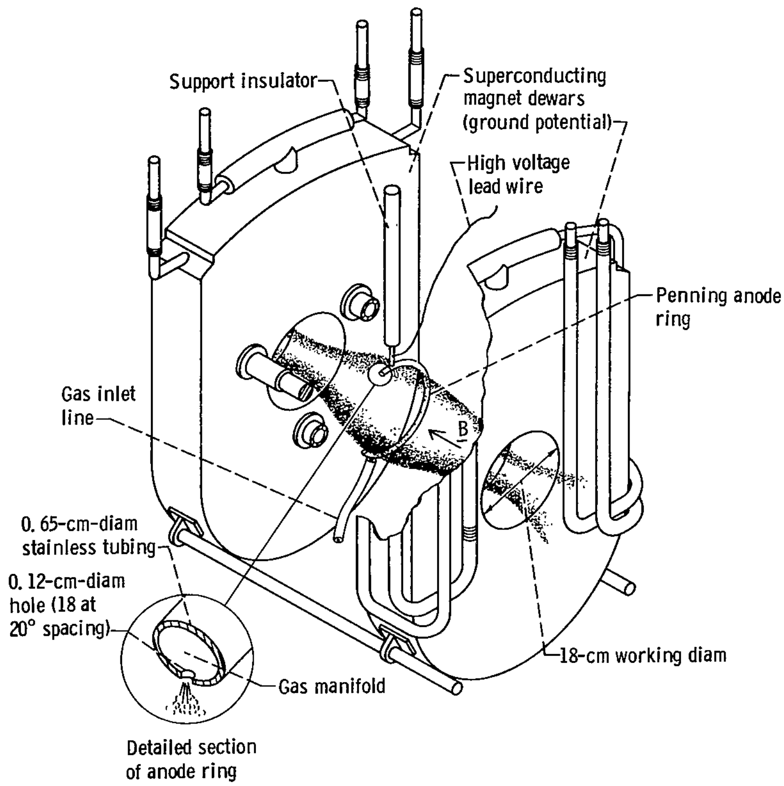


Figure 9. - Isometric cutaway drawing of discharge operating configuration.

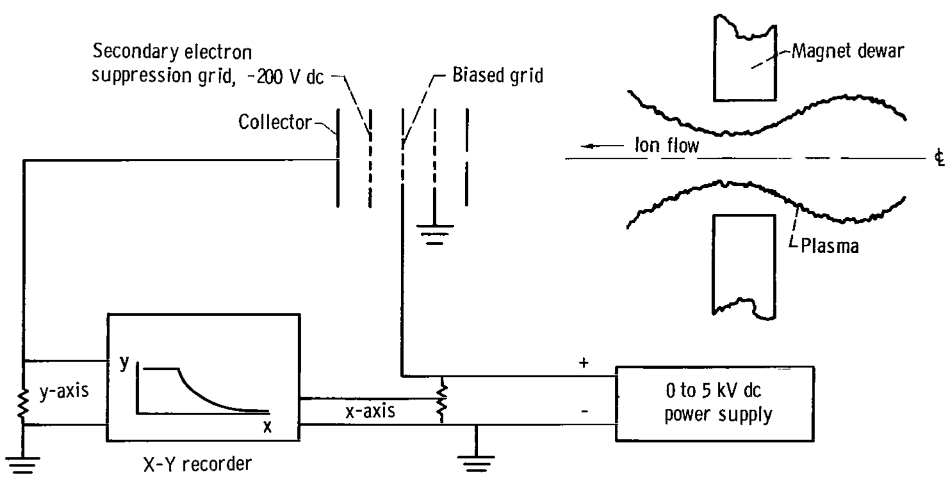


Figure 10. - Apparatus for measuring integrated ion energy spectrum.

forces that aid in trapping the ions produced in the discharge.

A multigrid probe was placed on the axis of the discharge, approximately 30 centimeters beyond the magnetic field maximum, to measure the integrated energy distribution of the ions escaping from the discharge. A schematic of the multigrid probe circuit is shown in figure 10. The collector current is fed into the y-axis of an x-y recorder, and the retarding potential placed on the intermediate grid is connected to the x-axis. An x-y recording of the integrated energy distribution function is then obtained for particles with velocities along the discharge axis. If the ions were Maxwellian on leaving the discharge and isotropic over a hemisphere in velocity space, the x-y recording should have the appearance of an error function. If the ions were Maxwellian and confined to a narrow cone about velocities parallel to the magnetic field lines, the integrated velocity distribution would be exponential.

A typical x-y recording is shown in figure 11. The integrated distribution function is approximately constant up to a value of 132 electron volts, which represents the floating plasma potential of the discharge. Beyond this potential, the integrated energy spectrum drops off with increasing potential on the repeller grid. A computer program was devised to calculate the characteristics of a best fitting integrated distribution function, for both an error-function and exponential dropoff. The solid curve in figure 11 represents the integrated distribution function appropriate to a Maxwellian velocity distribution of deuterium ions of kinetic temperature T_i of 765 electron volts per ion. The observed

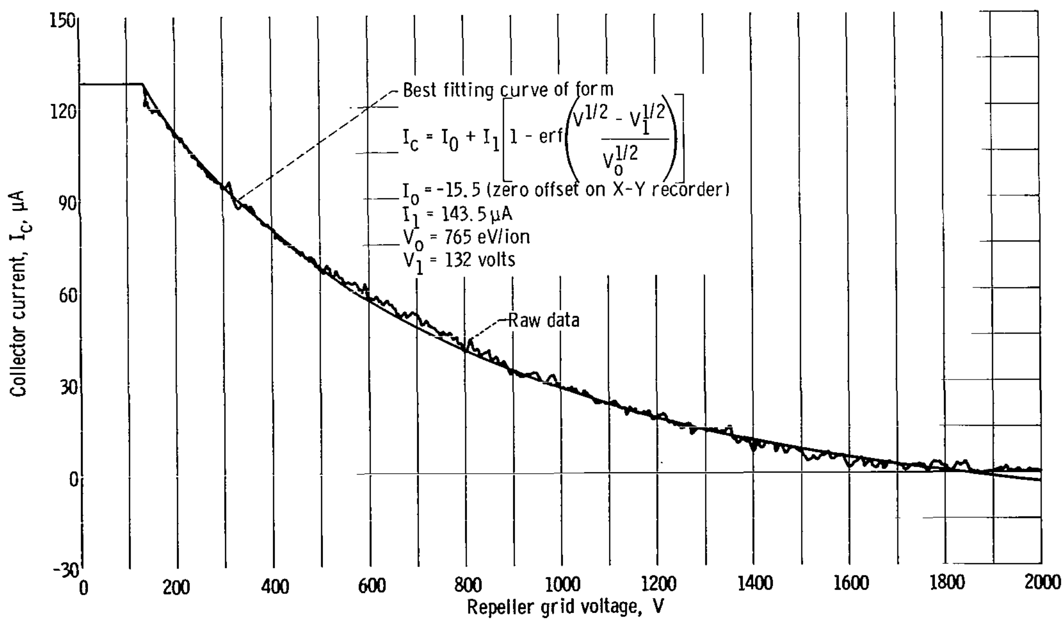


Figure 11. - X-Y recording of integrated ion energy distribution from discharge. Irregular line represents raw data; solid curve represents best fitting error function with floating potential of 132 volts and ion energy of 765 electron volts per deuterium ion. Run W-29 with positive deuterium ions.

integrated distribution function is within a root-mean-square error of 0.86 percent (of the collector current at 132 V) of the best fitting error function. This is expected in this case, since the ions in the discharge were trapped under nonadiabatic (refs. 19 and 20) conditions which would result in an escape cone in velocity space much larger than under adiabatic conditions. This would have enabled the ions leaving the discharge to occupy most of a hemisphere in velocity space and to give an error function dropoff.

Because significant ion trapping was observed in this discharge (ref. 50), the excellent fit of the observed integrated distribution function to that of a Maxwellian leads to the conclusion that the plasma approximates the first two conditions listed in table V (p. 37). If this injection system were applied to the bumpy torus configuration, the plasma should be approximately isotropic in velocity space, since the particles lost to the escape cone will then circulate around the major circumference of the torus. This injection scheme can easily be adapted to the bumpy torus configuration and, in conjunction with it, should be capable of generating and confining an approximately LCKE plasma. Although the bulk of the plasma should approach LCKE and be relatively free of instabilities, the possible losses due to instabilities in a non-LCKE sheath remain an open question.

An experiment based on the bumpy torus with a Penning injection scheme should not be viewed only as a test of the design criteria based on LCKE and the Bohr - van Leeuwen theorem. The device could succeed (i.e., yield hot, dense plasmas) even if some assumptions or design criteria are incorrect or inapplicable. From the considerations discussed herein, it appears that such a device would have a much better chance of achieving the desired results than many other systems.

CONCLUDING REMARKS

Linearized stability analyses select a particular plasma instability and derive a stability criterion that will prevent the growth of that particular instability. These stability criteria usually leave a reservoir of free energy in the plasma, and the requirements of two or more of these criteria as to plasma and magnetic field geometry, etc. are sometimes mutually conflicting.

The approach presented in this report does not attempt to eliminate the large number of possible instabilities one at a time but instead places emphasis on the minimization of the reservoirs of free energy in the plasma, which are fewer in number than the possible modes of plasma instability. The conditions defining local classical kinetic equilibrium (table I, p. 20) are sufficient for application of the Bohr - van Leeuwen theorem. This theorem then yields the restrictions on the plasma properties given by equation (51). A further restriction, aimed at minimization of free energy, gives the special case satisfying equation (51) listed in table II (p. 20). The Bohr - van Leeuwen theorem does not place any restrictions on conditions along the magnetic field lines. The conditions of table III (p. 26) were derived by requiring that zero net forces act along the magnetic field lines, consistent with the minimization of free energy. Tables I to III taken together describe a plasma with a minimum of free energy available to drive instabilities. The only requirement that cannot be met by finite laboratory plasmas is that no plasma energy density gradient exist normal to the field lines. By proper attention to the design of the injection and confinement scheme used, however, it should be possible to create and maintain a plasma whose energy density gradient (and reservoirs of free energy) are restricted to a thin sheath. Such a plasma sheath would surround a large interior region with a flat energy density profile, and minimum free energy. The design criteria in tables IV and V (pp. 36 and 37) should promote the growth of this stable interior region.

The Bohr - van Leeuwen theorem is invoked to show that a constant energy density profile, coupled with zero net diamagnetic currents in the interior region, may be used as an experimental test of whether a plasma has approached a state of LCKE in its interior. The larger the diamagnetic currents, however, and the steeper the energy density gradient in an element of plasma, the larger will be the reservoirs of free energy and the more difficulty can be expected from various instabilities.

The available experimental data on steady-state plasmas have been reviewed and are consistent with a flat energy density profile in the plasma interior. In two cases, the diamagnetism was measured in the plasma interior. These measurements were consistent with a much lower current density in the interior region than in the outer sheath region of the plasma in question. The interior of the plasmas from which these data were taken may be approaching LCKE. This does not necessarily imply that such plasmas represent optimum production and confinement schemes, but merely that the portion of plasma that remained to be measured is near LCKE in its interior.

One injection and confinement system that offers a possible means of minimizing the reservoirs of free energy within a confined plasma is the bumpy torus magnetic field configuration combined with a Penning discharge injection scheme.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, November 17, 1966,
129-02-03-05-22.

APPENDIX - SYMBOLS

A area, m^2

B magnetic field, T

b perturbed magnetic field, T

E electric field, V/m

e electronic charge, C

F force, N

g acceleration of gravity, m/sec^2

H Hamiltonian of system

h^N elementary volume of phase space (eq. (17))

i current, A

j current density, A/m^2

k Boltzmann constant

M total magnetic moment per unit volume

M' instantaneous magnetic moment of a particle

m particle mass, kg

N number of degrees of freedom

n density, $particles/m^3$

p kinetic pressure, also canonical momentum

R local mirror ratio

T kinetic temperature

t time, sec

V particle energy, eV

v velocity, m/sec

x generalized coordinate in physical space

Z partition function, eq. (17)

β ratio of material to magnetic energy density, eq. (13)

β^* normalization constant defined by eq. (16)

γ defined by eq. (37)

δ isotropy index defined by eq. (39)

ϵ defined by eq. (14)

θ_o escape cone angle, eq. (18)

μ_o permeability of free space

ρ gyroradius, m

Ω volume of momentum space (eq. (17))

ω gyrofrequency, sec^{-1}

Subscripts:

c relating to magnetic field curvature

E relating to electric field

e electron

ex external

g relating to magnetic field gradients

i ion

m relating to plasma magnetization

max maximum

min minimum

o steady, unperturbed

s internal

$\perp$ perpendicular to magnetic field lines

$||$ parallel to magnetic field lines

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