NASA TECHNICAL NOTE

NASA TN D-3939

NASA TN D-3939

N 67-23303

(ACCESSION NUMBER)

52

(PAGES)

(NASA CR OR TMX OR AD NUMBER)

1

(THRU)

(CODE)

08

(CATEGORY)

FACILITY FORM 602

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1967

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SUMMARY

This report describes a means of extending the area of application of digital computers beyond the numerical data processing stage and reducing the need for human participation in the formulation of certain types of computer problems. By the use of tensor calculus and a computer language designed to facilitate the use of symbolic mathematical computation, a method has been devised whereby a digital computer can be used to do non-numeric work: that is, symbolic algebraic manipulation and differentiation.

To illustrate the techniques involved, a digital computer has been used to derive the equations of motion of a point mass in a general orthogonal curvilinear coordinate system. Since this operation involves a formulation in terms of first- and second-order differential coefficients, it provides a good demonstration of a computer's capability to do non-numeric work and to assist in the formulation process which normally precedes the numerical data processing stage. Moreover, this particular problem serves to illustrate the advantages of the mathematical techniques employed. With the program prepared for this purpose the computer will derive the equations of motion in any coordinate system requested by the user. Results are presented for the following coordinate systems: cylindrical polar, spherical polar, oblate spheroidal and prolate spheroidal.

INTRODUCTION

Research undertaken with the object of promoting man-computer interaction has directed attention to the use of computers for non-numeric operations. In particular, the possibility of using digital computers to derive the equations of motion and of mathematical physics in a general curvilinear coordinate system has been explored. Traditionally, these functions were considered to be the exclusive preserve of the scientist. Nevertheless, as is shown in this report, digital computers can participate in the performance of such tasks. These computers have been used almost exclusively in the past and continue to be used extensively for numerical analyses of all kinds. In fact, excessive preoccupation with their arithmetic capabilities has tended to obscure their potential for non-numeric operations. Nevertheless, the extensive logic and storage capabilities of these computers, combined with the evolution of new computer languages, enable them to be used for a wide range of non-numeric operations. The author is aware of only two previous attempts to use computers in this manner: (a) Reference 1 describes an interesting technique whereby a digital computer uses the method of Lagrange to derive equations of motion. The technique, as described, was not completely satisfactory in that part of the operation has to be performed manually. (b) Reference 2 describes how an IBM 7094 computer, equipped with a Formac compiler, was used to obtain the Christoffel symbols of the first and second kind for 12 orthogonal curvilinear coordinate systems.

If the extensive logic and storage capabilities of these computers are to be used to full advantage, a departure from conventional techniques of formulation may be necessary. For example, when conventional methods are used, the form which the equations of motion and of mathematical physics assume depends on the coordinate system used to describe the problem. This dependence, which is due to the practice of expressing vectors in terms of their physical components, can be removed by the simple expedient of expressing all vectors in terms of their tensor components.

As a consequence of the geometrical simplification inherent in the tensor method, the operations involved in formulating problems in unfamiliar curvilinear coordinate systems can be reduced to routine computer operations. It is this aspect of the tensor method which is so attractive for the types of computer applications contemplated. It is the purpose of this report to use the tensor method to show that digital computers can be used to do non-numeric work. With this object in mind, a computer program was written to demonstrate the effectiveness of the proposed technique. This program, in the Formac computer language, was used to derive the equations of motion of a point mass in a variety of curvilinear coordinate systems. To derive the equations of motion of a particle by this method, the user need only know the coordinate transformation equations relating the curvilinear coordinates to an orthogonal Cartesian triad. When this program is used and the coordinate transformation equations are supplied as input, the computer will derive the equations of motion. The equations of motion obtained will be relative to the curvilinear coordinate system specified by the coordinate transformation equations used as input. The computer presents the results in Fortran language. However, for the convenience of readers, the Fortran statements are translated to conventional mathematical symbolism.

NOMENCLATURE

$\bar{A}$	vector
$\mathscr{A}$	physical components of $\bar{A}$
$A^i(x)$	contravariant vector components in the x coordinate system
$A_j(x)$	covariant vector components in the x coordinate system
$A^{ij}(x)$	components of a contravariant bivector in the x coordinate system
$A_{ij}(x)$	components of a covariant bivector in the x coordinate system
$A^i_j(x)$	components of a mixed bivector in the x coordinate system

$A^{i}_{,j}$	covariant derivative of a contravariant vector
$A_{i,j}$	covariant derivative of a covariant vector
$\overline{a}_i(x)$	system of base vectors in the x coordinate system
$\hat{a}_i(x)$	system of unit vectors in the directions of $\overline{a}_i(x)$
$\overline{a}^i(x)$	system of base vectors reciprocal to $\overline{a}_i(x)$
$\hat{a}^i(x)$	system of unit base vectors in the directions of $\overline{a}^i(x)$
$B^i(y)$	contravariant vector components in the y coordinate system
$B_j(y)$	covariant vector components in the y coordinate system
$B^{ij}(y)$	components of a contravariant bivector in the y coordinate system
$B_{ij}(y)$	components of a covariant bivector in the y coordinate system
$B^i_j(y)$	components of a mixed bivector in the y coordinate system
$\overline{b}_j(y)$	system of base vectors in the y coordinate system
$\overline{b}^j(y)$	system of base vectors reciprocal to $\overline{b}_j(y)$
$F_i(y)$	covariant component of gravitational force vector in the y coordinate system
$F_j(x)$	covariant component of gravitational force vector in the x coordinate system
g_{ij}	$\overline{a}_i \cdot \overline{a}_j$
g^{ij}	$\overline{a}^i \cdot \overline{a}^j$
I^{ij}	components of the inertia tensor in the x coordinate system
$\overline{I}^{ij}$	components of the inertia tensor in the y coordinate system
M	mass of space vehicle
$\overline{r}$	position vector
r	scalar magnitude of $\overline{r}$
$\overline{T}$	thrust vector
T^i	contravariant component of the thrust vector
T_i	covariant component of the thrust vector

$U^j(x)$	contravariant component of velocity in the x coordinate system
$V^i(y)$	contravariant component of velocity in the y coordinate system
x^i	system coordinates
$x^i(y^1,y^2,y^3)$	functional form of the transformation from the y coordinate system to the x coordinate system
y^i	system coordinates
$y^i(x^1,x^2,x^3)$	functional form of the transformation from the x coordinate system to the y coordinate system
z	displacement along the axis of a cylinder
$[ij,k]$	Christoffel symbol of the first kind
$\left\{ {i \atop jk} \right\}$	Christoffel symbol of the second kind
α^i_j	constant coefficients
α_i	scalar magnitude of $\bar{a}_i$
β^i	scalar magnitude of $\bar{a}^i$
δ^i_j	Kronecker delta
θ	polar angle
τ	physical component of the thrust vector
φ	gravitational potential function
$\nabla\varphi$	gradient of gravitational potential function
ψ	angular displacement in longitude or azimuth

Superscripts

α,i,j,k,l	indices of contravariance

Subscripts

i,j,k,l	indices of covariance

ANALYSIS

Transformation Laws

Vector components.- To facilitate the computer processing of vectors and bivectors, all such entities should be expressed in terms of their tensor components and a corresponding set of base vectors, rather than in terms of their physical components and a set of unit base vectors. When referred to a general curvilinear coordinate system, a vector $\bar{A}$ may be expressed in the following alternative forms:

$$\bar{A} = A^i\bar{a}_i = A_j\bar{a}^j \tag{1}$$

If, in some expression, a certain index occurs twice, this means that the expression is to be summed with respect to that index for all admissible values of the index, that is,

$$A^i\bar{a}_i = \sum_{i=1}^{n} A^i\bar{a}_i$$

$$A_j\bar{a}^j = \sum_{j=1}^{n} A_j\bar{a}^j$$

where A^i, A_j are the tensor components of the vector $\bar{A}$, and $\bar{a}_i$, $\bar{a}^j$ are the corresponding systems of base vectors. In accordance with established convention, contravariant components will be denoted by superscripts and covariant components by subscripts. It is necessary to keep in mind the distinction between contravariance and covariance because if general coordinate transformations are contemplated, the transformation law for the components of a contravariant vector denoted by superscripts differs from that for a covariant vector denoted by subscripts. It must be emphasized, however, that the covariance or contravariance of tensor components is not an intrinsic property of the entity under consideration. The distinction is due to the way in which the entity is related to its environment, the coordinate system, to which it is referred. For a coordinate transformation from a coordinate system x to a coordinate system y given by

$$y^i = y^i(x^1,x^2,x^3) \tag{2}$$

the transformation law for the components of a contravariant vector A^i is (see appendix A and ref. 3):

$$B^j(y) = \frac{\partial y^j}{\partial x^i} A^i(x) \tag{3}$$

where $A^i(x)$ are the contravariant components in the x coordinate system and $B^j(y)$ are the components when referred to the y coordinate system. For the same transformation of coordinates, other vectors, such as the gradient of a scalar point function, obey a different transformation law. These are the covariant vectors denoted by subscripts. Assuming that the coordinate transformation is reversible and one-to-one, the appropriate transformation law for these vector components is (see appendix A)

$$B_j(y) = \frac{\partial x^i}{\partial y^j} A_i(x) \tag{4}$$

where $A_i(x)$ are the covariant components in the x coordinate frame and $B_j(y)$ are the covariant components when referred to the y coordinate frame. As the following argument shows, the distinction between these two transformation laws vanishes when the transformation is orthogonal Cartesian. Let x^i be the components of a position vector $\bar{r}$ when referred to the x coordinate system which is orthogonal Cartesian. Likewise, let y^j be components of the same vector when referred to another orthogonal Cartesian system. In this case the transformation of coordinates is given by

$$y^i = \alpha^i_j x^j \tag{5}$$

where the α^i_j are constants. The position vector $\bar{r}$ is invariant with respect to coordinate transformations. Hence, the square of the vector is also invariant. Therefore,

$$x^j x^j = y^i y^i = \alpha^i_j \alpha^i_k x^j x^k = \delta^j_k x^j x^k$$

therefore

$$\alpha^i_j \alpha^i_k = \delta^j_k \tag{6}$$

where δ^j_k is the Kronecker delta, that is, (see ref. 4)

$$\delta^j_k = \begin{matrix} 1 & \text{for} & j = k \\ 0 & \text{for} & j \neq k \end{matrix}$$

Equation (6) is the orthogonality condition which may be used to solve equation (5) for x^j. If both sides of equation (5) are multiplied by α^i_k,

$$\alpha^i_j \alpha^i_k x^j = \alpha^i_k y^i$$

and

$$\delta^j_k x^j = x^k = \alpha^i_k y^i$$

therefore,

$$x^j = \alpha^i_j y^i \tag{7}$$

From equation (5), it is seen that

$$\frac{\partial y^i}{\partial x^j} = \alpha^i_j \tag{8}$$

and from equation (7)

$$\frac{\partial x^j}{\partial y^i} = \alpha^i_j \tag{9}$$

It follows from equations (8) and (9) that

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial x^j}{\partial y^i} \tag{10}$$

As a consequence of equation (10), the distinction between contravariant and covariant vectors disappears when coordinate transformations are confined to orthogonal Cartesian systems. This also explains why there is no preoccupation with these vectors in the study of ordinary vector analysis.

Base vectors.- Subscripts assigned to a system of base vectors indicate that they are covariant in character and obey the covariant transformation law. See equation (4) and appendix A. Therefore, if $\bar{a}_i(x)$ are a system of base vectors in the x coordinate system and $\bar{b}_j(y)$ are the corresponding base vectors in the y coordinate system, then

$$\bar{b}_j(y) = \frac{\partial x^i}{\partial y^j}\,\bar{a}_i(x) \tag{11}$$

In this connection it should be noted that to every system of base vector $\bar{a}_i$, there exists a reciprocal system of base vectors $\bar{a}^i$ with the following property:

$$\bar{a}^j \cdot \bar{a}_i = \delta^j_i = \bar{a}_i \cdot \bar{a}^j \tag{12}$$

A superscript is assigned to the reciprocal base vectors to indicate their contravariant character, and to emphasize the fact that they obey the contravariant transformation law. (See eq. (3) and appendix A.) Hence, if $\bar{a}^i(x)$ are the reciprocal base vectors in the x coordinate system and $\bar{b}^j(y)$ are the corresponding base vectors in the y coordinate system, then

$$\bar{b}^j(y) = \frac{\partial y^j}{\partial x^i}\,\bar{a}^i(x) \tag{13}$$

In a curvilinear coordinate system the base vectors are, in general, not unit vectors, but are functions of the coordinates; that is,

$$\bar{a}_i = \bar{a}_i(x^1,x^2,x^3) \tag{14}$$

$$\bar{a}^j = \bar{a}^j(x^1,x^2,x^3) \tag{15}$$

The base vectors $\bar{a}_i$ may be obtained as follows: let $d\bar{r}$ be the differential of a position vector $\bar{r}$ and let dx^i be the corresponding differentials of the coordinates. Then by substituting $d\bar{r}$ for $\bar{A}$, and dx^i for A^i in equation (1), we have

$$d\bar{r} = dx^i \bar{a}_i \tag{16}$$

From equation (16) the base vectors $\bar{a}_i$ are given by

$$\bar{a}_i = \frac{\partial \bar{r}}{\partial x^i} \tag{17}$$

In an orthogonal Cartesian frame of reference, the base vectors $\bar{a}_i$ constitute a triad of mutually orthogonal unit vectors. However, in problem formulation, it is usually convenient to use a more general curvilinear coordinate system. When this is done, the magnitudes of the base vectors generally differ from unity.

Vector Derivatives and the Christoffel Symbols

The scalar product of any two base vectors $\bar{a}_i$ and $\bar{a}_j$ may be defined as follows:

$$\bar{a}_i \cdot \bar{a}_j = \bar{a}_j \cdot \bar{a}_i = g_{ij} \tag{18}$$

Likewise, the scalar product of the reciprocal base vectors $\bar{a}^i$ and $\bar{a}^j$ may be defined as

$$\bar{a}^i \cdot \bar{a}^j = \bar{a}^j \cdot \bar{a}^i = g^{ij} \tag{19}$$

The symmetry of g_{ij} and g^{ij} follows from the nature of the scalar product. Certain combinations of the partial derivatives of these scalar products with respect to the system coordinates are useful in obtaining the derivative of a vector, in formulating the equations of motion, or writing the equations of mathematical physics in a general curvilinear coordinate system. The definitions that follow are ascribed to Christoffel and are called Christoffel symbols (see ref. 5). There are two of these symbols, the first of which is defined as

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{20}$$

The Christoffel symbol of the second kind is

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}\,[ij,l] \tag{21}$$

<u>Derivatives of a contravariant vector</u>.- The utility of the Christoffel symbols is immediately apparent when an attempt is made to find the partial

derivative of a base vector, or its reciprocal, with respect to any system coordinate. Any vector $\overline{A}$ may be expressed in the forms given in equation (1). Furthermore, since the base vectors are, in general, functions of the coordinates, it follows that the derivative of $\overline{A}$ with respect to any coordinate must involve the Christoffel symbols. From equation (1), the partial derivative of the contravariant form of the vector $\overline{A}$ with respect to the coordinate x^k is given by

$$\frac{\partial \overline{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k}\overline{a}_i + A^i \frac{\partial \overline{a}_i}{\partial x^k} \tag{22}$$

Since $\overline{a}_i \cdot \overline{a}_j = g_{ij}$, then

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \overline{a}_i}{\partial x^k} \cdot \overline{a}_j + \overline{a}_i \cdot \frac{\partial \overline{a}_j}{\partial x^k} \tag{23}$$

Likewise,

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial \overline{a}_j}{\partial x^i} \cdot \overline{a}_k + \overline{a}_j \cdot \frac{\partial \overline{a}_k}{\partial x^i} \tag{24}$$

and

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial \overline{a}_i}{\partial x^j} \cdot \overline{a}_k + \overline{a}_i \cdot \frac{\partial \overline{a}_k}{\partial x^j} \tag{25}$$

Since

$$\overline{a}_i = \frac{\partial \overline{r}}{\partial x^i}$$

it follows that

$$\frac{\partial \overline{a}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial \overline{r}}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial \overline{r}}{\partial x^j}\right) = \frac{\partial \overline{a}_j}{\partial x^i} \tag{26}$$

From equations (23) through (26)

$$\frac{\partial \overline{a}_i}{\partial x^j} \cdot \overline{a}_k = [ij,k] \tag{27}$$

Therefore if equation (12) is used, the rate of change of the base vector $\overline{a}_i$ with respect to x_j assumes the form

$$\frac{\partial \overline{a}_i}{\partial x^j} = [ij,k]\overline{a}^k \tag{28}$$

Equation (28) gives the required rate of change of the base vector $\overline{a}_i$ with respect to a system coordinate, in terms of the Christoffel symbol of the first kind and the reciprocal base vectors. A more convenient form is obtained if both sides of equation (28) are multiplied scalarly by the reciprocal base vector $\bar{a}^l$ to yield

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}^l = [ij,k]\bar{a}^k \cdot \bar{a}^l \tag{29}$$

From equation (19), it is seen that

$$\bar{a}^k \cdot \bar{a}^l = g^{kl}$$

therefore

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}^l = [ij,k]g^{kl} \tag{30}$$

In terms of the defining formula (21), equation (30) may be rewritten as follows:

$$\frac{\partial \bar{a}_i}{\partial x^j} \cdot \bar{a}^l = \begin{Bmatrix} l \\ ij \end{Bmatrix} \tag{31}$$

Therefore,

$$\frac{\partial \bar{a}_i}{\partial x^j} = \begin{Bmatrix} l \\ ij \end{Bmatrix} \bar{a}_l \tag{32}$$

By substitution of equation (32) in equation (22) the partial derivative of a vector $\bar{A}$ with respect to the system coordinate x^k is

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k} \bar{a}_i + A^i \begin{Bmatrix} l \\ ik \end{Bmatrix} \bar{a}_l \tag{33}$$

The indices i and l in the second term on the right side of equation (33) are dummy[1] indices, and may therefore be replaced by any other convenient indices, except k. To have a common base vector $\bar{a}_i$, equation (33) may be rewritten as follows

$$\frac{\partial \bar{A}}{\partial x^k} = \left(\frac{\partial A^i}{\partial x^k} + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^j \right) \bar{a}_i = A^i_{,k} \bar{a}_i \tag{34}$$

Furthermore, since

$$\frac{\partial \bar{A}}{\partial x^k} \frac{dx^k}{dt} = \frac{d\bar{A}}{dt}$$

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[1]As already indicated, a repeated index implies summation with respect to that index. Since the summation index can be changed at will, it is usually referred to as a dummy index. Of course, the range of admissible values of the index must be preserved.

and

$$\frac{\partial A^i}{\partial x^k}\frac{dx^k}{dt} = \frac{dA^i}{dt}$$

the intrinsic derivative, or the total derivative with respect to time of the contravariant form of the vector $\bar{A}$, may be obtained from equation (34) in the following form:

$$\frac{d\bar{A}}{dt} = \left(\frac{dA^i}{dt} + \left\{ {i \atop jk} \right\} A^j \frac{dx^k}{dt}\right)\bar{a}_i = A^i_{,k}\frac{dx^k}{dt}\bar{a}_i \tag{35}$$

where $A^i_{,k}$ is the covariant derivative of the contravariant vector A^i with respect to x^k.

The notation $A^i_{,j}$ suggests that the covariant derivative of a contravariant vector is not a simple covariant or contravariant vector. As the notation implies, $A^i_{,j}$ is a mixed tensor, with one index of contravariance and one index of covariance (see appendix A). If a single-valued, reversible functional transformation of the form given in equation (2) is assumed, the transformation law for this type of entity is

$$B^i_k(y) = \frac{\partial y^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^k}A^\alpha_\beta(x)$$

where $A^\alpha_\beta(x)$ are the components in the x coordinate system and $B^i_k(y)$ are the corresponding components in the y coordinate system.

In an orthogonal Cartesian reference frame

$$g_{ij} = \bar{a}_i \cdot \bar{a}_j = \delta^i_j = \bar{a}^i \cdot \bar{a}^j = g^{ij}$$

Therefore, since all these scalar products are constants, it follows that the Christoffel symbols vanish. In this case, the covariant derivative of a contravariant vector reduces the sum of the partial derivatives of its physical components along a set of fixed axes

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k}\bar{a}_i \qquad i = 1,2,3$$

Likewise, the intrinsic derivative of a vector reduces to the ordinary time rates of change of the physical components along a set of fixed axes.

For a general space of three dimensions, equation (35) assumes the form

$$\frac{d\bar{A}}{dt} = \left(\frac{dA^1}{dt} + f^1\right)\bar{a}_1 + \left(\frac{dA^2}{dt} + f^2\right)\bar{a}_2 + \left(\frac{dA^3}{dt} + f^3\right)\bar{a}_3 \tag{36}$$

where

$$f^1 = \left[A^1\left(\begin{Bmatrix}1\\11\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}1\\12\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}1\\13\end{Bmatrix}\frac{dx^3}{dt}\right) + A^2\left(\begin{Bmatrix}1\\21\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}1\\22\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}1\\23\end{Bmatrix}\frac{dx^3}{dt}\right)\right.$$

$$\left. + A^3\left(\begin{Bmatrix}1\\31\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}1\\32\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}1\\33\end{Bmatrix}\frac{dx^3}{dt}\right)\right] \tag{37}$$

$$f^2 = \left[A_1\left(\begin{Bmatrix}2\\11\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}2\\12\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}2\\13\end{Bmatrix}\frac{dx^3}{dt}\right) + A_2\left(\begin{Bmatrix}2\\21\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}2\\22\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}2\\23\end{Bmatrix}\frac{dx^3}{dt}\right)\right.$$

$$\left. + A^3\left(\begin{Bmatrix}2\\31\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}2\\32\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}2\\33\end{Bmatrix}\frac{dx^3}{dt}\right)\right] \tag{38}$$

$$f^3 = \left[A^1\left(\begin{Bmatrix}3\\11\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}3\\12\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}3\\13\end{Bmatrix}\frac{dx^3}{dt}\right) + A^2\left(\begin{Bmatrix}3\\21\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}3\\22\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}3\\23\end{Bmatrix}\frac{dx^3}{dt}\right)\right.$$

$$\left. + A^3\left(\begin{Bmatrix}3\\31\end{Bmatrix}\frac{dx^1}{dt} + \begin{Bmatrix}3\\32\end{Bmatrix}\frac{dx^2}{dt} + \begin{Bmatrix}3\\33\end{Bmatrix}\frac{dx^3}{dt}\right)\right] \tag{39}$$

The intrinsic derivative of a contravariant vector in a space of three dimensions contains 27 Christoffel symbols. However, because of the symmetry of the Christoffel symbols,

$$\begin{Bmatrix}k\\ij\end{Bmatrix} = \begin{Bmatrix}k\\ji\end{Bmatrix} \tag{40}$$

and the number of independent Christoffel symbols reduces to 18.

Derivatives of a covariant vector.- The second alternative from equation (1) may be used to express the vector $\bar{A}$ in terms of its covariant tensor components and reciprocal base vectors; that is,

$$\bar{A} = A_i\bar{a}^i \tag{41}$$

In this case, the partial derivative of the vector $\bar{A}$ with respect to x^k is given by

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A_i}{\partial x^k}\bar{a}^i + A_i\frac{\partial \bar{a}^i}{\partial x^k} \tag{42}$$

From equation (12)

$$\frac{\partial \bar{a}^i}{\partial x^k} \cdot \bar{a}_j + \bar{a}^i \cdot \frac{\partial \bar{a}_j}{\partial x^k} = 0$$

therefore

$$\frac{\partial \bar{a}^i}{\partial x^k} \cdot \bar{a}_j = -\bar{a}^i \cdot \frac{\partial \bar{a}_j}{\partial x^k} \tag{43}$$

Substituting equation (32) in equation (43) gives

$$\frac{\partial \bar{a}^i}{\partial x^k} \cdot \bar{a}_j = -\bar{a}^i \cdot \begin{Bmatrix} l \\ jk \end{Bmatrix} \bar{a}_l = -\begin{Bmatrix} i \\ jk \end{Bmatrix}$$

therefore

$$\frac{\partial \bar{a}^i}{\partial x^k} = -\begin{Bmatrix} i \\ jk \end{Bmatrix} \bar{a}^j \tag{44}$$

Substituting equation (44) in equation (42) gives the partial derivative of the vector $\bar{A}$ with respect to x^k in the following form

$$\frac{\partial \bar{A}}{\partial x^k} = \frac{\partial A_i}{\partial x^k} \bar{a}^i - \begin{Bmatrix} i \\ jk \end{Bmatrix} A_i \bar{a}^j \tag{45}$$

The indices i and j in the second term on the right side of equation (45) are dummies, and may therefore be replaced by any other indices, except k. In terms of the base vectors $\bar{a}^i$, equation (45) may be rewritten as follows

$$\frac{\partial \bar{A}}{\partial x^k} = \left(\frac{\partial A_i}{\partial x^k} - \begin{Bmatrix} j \\ ik \end{Bmatrix} A_j\right) \bar{a}^i = A_{i,k} \bar{a}^i \tag{46}$$

where $A_{i,k}$ defines the covariant derivative of the covariant vector A_i with respect to x^k.

It may be noted that the covariant derivative of a covariant vector is not a vector. As the notation implies, $A_{i,j}$ is a doubly covariant tensor, that is, a tensor with two indices of covariance. If a single-valued, reversible functional transformation, of the form given in equation (2) is again assumed, the transformation law for entities of this kind is

$$B_{ij}(y) = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j} A_{\alpha\beta}(x)$$

where $A_{\alpha\beta}(x)$ are the components in the x coordinate system, and $B_{ij}(y)$ are the corresponding components in the y coordinate system. It may be mentioned in passing that moment of inertia, which is a second order tensor, has a transformation law of this form. (See appendix A.)

It appears, therefore, that the operation of covariant differentiation of a vector or tensor increases the covariance by one index. That is, the x^j covariant derivative of the contravariant vector A^i is $A^i_{,j}$, which is a mixed tensor, with one index of contravariance and one index of covariance. The x^j covariant derivative of the covariant vector A_i is $A_{i,j}$. This is a doubly covariant tensor or a tensor with two indices of covariance. The intrinsic derivative of the covariant form of the vector $\bar{A}$ is obtained from equation (46) in the following form

$$\frac{d\bar{A}}{dt} = \left(\frac{dA_i}{dt} - \left\{ {j \atop ik} \right\} A_j \frac{dx^k}{dt}\right) \bar{a}^i = A_{i,k} \frac{dx^k}{dt} \bar{a}^i \tag{47}$$

For a general space of three dimensions equation (47) assumes the form

$$\frac{d\bar{A}}{dt} = \left(\frac{dA_1}{dt} - f_1\right) \bar{a}^1 + \left(\frac{dA_2}{dt} - f_2\right) \bar{a}^2 + \left(\frac{dA_3}{dt} - f_3\right) \bar{a}^3 \tag{48}$$

where

$$f_1 = \left[A_1\left(\left\{ {1 \atop 11} \right\} \frac{dx^1}{dt} + \left\{ {1 \atop 12} \right\} \frac{dx^2}{dt} + \left\{ {1 \atop 13} \right\} \frac{dx^3}{dt}\right) + A_2\left(\left\{ {2 \atop 11} \right\} \frac{dx^1}{dt} + \left\{ {2 \atop 12} \right\} \frac{dx^2}{dt} + \left\{ {2 \atop 13} \right\} \frac{dx^3}{dt}\right)\right.$$

$$\left. + A_3\left(\left\{ {3 \atop 11} \right\} \frac{dx^1}{dt} + \left\{ {3 \atop 12} \right\} \frac{dx^2}{dt} + \left\{ {3 \atop 13} \right\} \frac{dx^3}{dt}\right)\right] \tag{49}$$

$$f_2 = \left[A_1\left(\left\{ {1 \atop 21} \right\} \frac{dx^1}{dt} + \left\{ {1 \atop 22} \right\} \frac{dx^2}{dt} + \left\{ {1 \atop 23} \right\} \frac{dx^3}{dt}\right) + A_2\left(\left\{ {2 \atop 21} \right\} \frac{dx^1}{dt} + \left\{ {2 \atop 22} \right\} \frac{dx^2}{dt} + \left\{ {2 \atop 23} \right\} \frac{dx^3}{dt}\right)\right.$$

$$\left. + A_3\left(\left\{ {3 \atop 21} \right\} \frac{dx^1}{dt} + \left\{ {3 \atop 22} \right\} \frac{dx^2}{dt} + \left\{ {3 \atop 23} \right\} \frac{dx^3}{dt}\right)\right] \tag{50}$$

$$f_3 = \left[A_1\left(\left\{ {1 \atop 31} \right\} \frac{dx^1}{dt} + \left\{ {1 \atop 32} \right\} \frac{dx^2}{dt} + \left\{ {1 \atop 33} \right\} \frac{dx^3}{dt}\right) + A_2\left(\left\{ {2 \atop 31} \right\} \frac{dx^1}{dt} + \left\{ {2 \atop 32} \right\} \frac{dx^2}{dt} + \left\{ {2 \atop 33} \right\} \frac{dx^3}{dt}\right)\right.$$

$$\left. + A_3\left(\left\{ {3 \atop 31} \right\} \frac{dx^1}{dt} + \left\{ {3 \atop 32} \right\} \frac{dx^2}{dt} + \left\{ {3 \atop 33} \right\} \frac{dx^3}{dt}\right)\right] \tag{51}$$

As in the case of the intrinsic derivative of the contravariant vector, the intrinsic derivative of the covariant form of the vector $\bar{A}$ is seen to contain 27 Christoffel symbols. However, because of the symmetry implied by equation (40), the number of independent Christoffel symbols is again reduced to 18.

Special Coordinate Systems

The large number of terms appearing in equations (35) and (47) is due to the generality of these equations which are applicable to any space of three dimensions. Fortunately, for the three-dimensional spaces most commonly used, both of these equations reduce to a more manageable form.

For example, if base vectors of unit length are denoted by $\hat{a}_i$ or $\hat{a}^i$, then in a cylindrical polar coordinate system,

$$\left.\begin{aligned} \bar{a}_1 &= \hat{a}_1 & g_{11} &= 1 \\ \bar{a}_2 &= x^1\hat{a}_2 & g_{22} &= (x^1)^2 \\ \bar{a}_3 &= \hat{a}_3 & g_{33} &= 1 \end{aligned}\right\} \tag{52}$$

and

$$\left.\begin{aligned} \bar{a}^1 &= \hat{a}^1 & g^{11} &= 1 \\ \bar{a}^2 &= \frac{1}{x^1}\hat{a}^2 & g^{22} &= \frac{1}{(x^1)^2} \\ \bar{a}^3 &= \hat{a}^3 & g^{33} &= 1 \end{aligned}\right\} \tag{53}$$

As a consequence of equations (52) and (53) there are only two independent, nonzero Christoffel symbols in a cylindrical polar coordinate system. These are

$$\left.\begin{aligned} \left\{ {1 \atop 22} \right\} &= -x^1 \\ \left\{ {2 \atop 12} \right\} = \left\{ {2 \atop 21} \right\} &= \frac{1}{x^1} \end{aligned}\right\} \tag{54}$$

Hence, a contravariant vector referred to this coordinate system has a time rate of change as follows:

$$\frac{d\bar{A}}{dt} = \left(\frac{dA^1}{dt} + A^2\left\{ {1 \atop 22} \right\} \frac{dx^2}{dt}\right)\bar{a}_1 + \left[\frac{dA^2}{dt} + \left\{ {2 \atop 12} \right\}\left(A^1\frac{dx^2}{dt} + A^2\frac{dx^1}{dt}\right)\right]\bar{a}_2 + \frac{dA^3}{dt}\bar{a}_3 \tag{55}$$

Likewise, the time rate of change of a covariant vector referred to this coordinate system is given by

$$\frac{d\bar{A}}{dt} = \left(\frac{dA_1}{dt} - A_2\left\{ {2 \atop 12} \right\}\frac{dx^2}{dt}\right)\bar{a}^1 + \left(\frac{dA_2}{dt} - A_1\left\{ {1 \atop 22} \right\}\frac{dx^2}{dt} - A_2\left\{ {2 \atop 21} \right\}\frac{dx^1}{dt}\right)\bar{a}^2 + \frac{dA_3}{dt}\bar{a}^3 \tag{56}$$

In spherical polar coordinates,

$$\left.\begin{aligned} \bar{a}_1 &= \hat{a} & g_{11} &= 1 \\ \bar{a}_2 &= x^1\hat{a}_2 & g_{22} &= (x^1)^2 \\ \bar{a}_3 &= x^1 \sin x^2 \hat{a}_3 & g_{33} &= (x^1 \sin x^2)^2 \end{aligned}\right\} \tag{57}$$

and

$$\left.\begin{aligned} \bar{a}^1 &= \hat{a}^1 & g^{11} &= 1 \\ \bar{a}^2 &= \frac{1}{x^1}\,\hat{a}^2 & g^{22} &= \frac{1}{(x^1)^2} \\ \bar{a}^3 &= \frac{1}{x^1 \sin x^2}\,\hat{a}^3 & g^{33} &= \frac{1}{(x^1 \sin x^2)^2} \end{aligned}\right\} \tag{58}$$

In this case there are six independent, nonzero Christoffel symbols. These are

$$\left.\begin{aligned} \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= -x^1 & \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} &= -\sin x^2 \cos x^2 \\ \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} &= \frac{1}{x^1} & \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} &= \frac{1}{x^1} \\ \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} &= -x^1 \sin^2 x^2 & \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} &= \cot x^2 \end{aligned}\right\} \tag{59}$$

When the Christoffel symbols are substituted in equation (35), the time rate of change of a contravariant vector referred to a spherical coordinate system assumes the following form:

$$\begin{aligned} \frac{d\bar{A}}{dt} = {} & \left(\frac{dA^1}{dt} + A^2 \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{dx^2}{dt} + A^3 \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} \frac{dx^3}{dt}\right) \bar{a}_1 \\ & + \left[\frac{dA^2}{dt} + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \left(A^1 \frac{dx^2}{dt} + A^2 \frac{dx^1}{dt}\right) + A^3 \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} \frac{dx^3}{dt}\right] \bar{a}_2 \\ & + \left[\frac{dA^3}{dt} + \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} \left(A^1 \frac{dx^3}{dt} + A^3 \frac{dx^1}{dt}\right) + \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} \left(A^2 \frac{dx^3}{dt} + A^3 \frac{dx^2}{dt}\right)\right] \bar{a}_3 \end{aligned} \tag{60}$$

The corresponding rate of change of a covariant vector is obtained by substitution from equation (59) in equation (47). In this case,

$$\frac{d\bar{A}}{dt} = \left(\frac{dA_1}{dt} - A_2 \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \frac{dx^2}{dt} - A_3 \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} \frac{dx^3}{dt}\right) \bar{a}^1$$

$$+ \left(\frac{dA_2}{dt} - A_1 \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} \frac{dx^2}{dt} - A_2 \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} \frac{dx^1}{dt} - A_3 \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} \frac{dx^3}{dt}\right) \bar{a}^2$$

$$+ \left[\frac{dA_3}{dt} - A_1 \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} \frac{dx^3}{dt} - A_2 \begin{Bmatrix} 2 \\ 33 \end{Bmatrix} \frac{dx^3}{dt} - A_3 \left(\begin{Bmatrix} 3 \\ 31 \end{Bmatrix} \frac{dx^1}{dt} + \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} \frac{dx^2}{dt}\right)\right] \bar{a}^3 \tag{61}$$

Alternative Derivation of the Christoffel Symbols

In equations (20) and (21) the Christoffel symbols have been defined in terms of the scalar product of two of the base vectors These symbols can also be derived from the equations of coordinate transformation by the following method, which is more suitable for the applications contemplated in this report.

In a rectangular Cartesian coordinate reference frame, with coordinates denoted by y^i, an element of arc of length $d\bar{s}$ may be expressed in the following form

$$d\bar{s} = \hat{a}_\alpha \, dy^\alpha = \hat{a}_\beta \, dy^\beta$$

therefore

$$ds^2 = \hat{a}_\alpha \cdot \hat{a}_\beta \, dy^\alpha \, dy^\beta = \delta^\alpha_\beta \, dy^\alpha \, dy^\beta = dy^\alpha \, dy^\alpha$$

Consider a curvilinear coordinate system with coordinates denoted by x^i, and assume that the x and y coordinates are related by a set of transformation equations as follows

$$y^i = y^i(x^1, x^2, x^3) \tag{62}$$

The element of arc $d\bar{s}$ in the x coordinate system assumes the form

$$d\bar{s} = \bar{a}_i \, dx^i = \bar{a}_j \, dx^j$$

therefore

$$ds^2 = (\bar{a}_i \, dx^i) \cdot (\bar{a}_j \, dx^j) = g_{ij} \, dx^i \, dx^j = dy^\alpha \, dy^\alpha$$

$$dy^\alpha \, dy^\alpha = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j} \, dx^i \, dx^j = g_{ij} \, dx^i \, dx^j$$

and

$$g_{ij} = \left(\frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j}\right) \tag{63}$$

If the transformation equation (62) is reversible and one-to-one, then

$$x^i = x^i(y^1, y^2, y^3) \tag{64}$$

By substitution from equation (63) in equation (20), the Christoffel symbol of the first kind assumes the following form:

$$[ij,k] = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} \frac{\partial y^\alpha}{\partial x^k} \tag{65}$$

Likewise, substituting equation (63) in equation (21) gives for the Christoffel symbol of the second kind

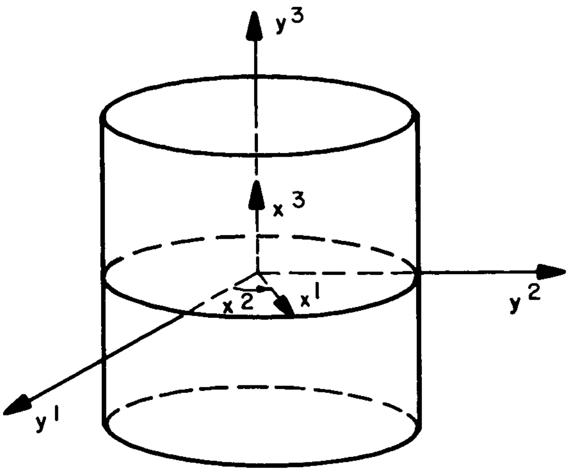
$$\left\{ {i \atop jk} \right\} = \frac{\partial^2 y^\alpha}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial y^\alpha} \tag{66}$$

COMPUTER APPLICATIONS

It is interesting to note that the preceding derivations differ from the conventional approach only in the method of expressing a vector in terms of its components and associated base vectors. Conventionally, a vector is expressed in terms of its physical components and a corresponding set of unit base vectors. The proposed method, which is the method of tensor analysis, uses an almost identical formulation. The important difference being that in this case the components are not, in general, the physical components. Instead, they are the tensor components which obey transformation laws corresponding to their variance. The transformation laws for contravariant and covariant vectors are given by equations (3) and (4), respectively. When the base vectors define an orthogonal Cartesian reference frame, the physical components and the tensor components are equal. It follows that the tensor method reduces to the conventional method for problems formulated in orthogonal Cartesian reference systems.

As a consequence of the geometrical simplification inherent in the tensor method, the operations involved in obtaining derivatives and formulating the equations of mathematical physics in unfamiliar curvilinear coordinate systems are routine operations. It is this feature of the tensor method which makes it so attractive for computer applications. Because of their logic and storage capabilities, digital computers are well suited to such routine operations if they are properly programmed.

If the Christoffel symbols are stored for all coordinate systems that are likely to be used, the rates of change of any vector can be obtained by a straightforward application of equation (35) or (47). Alternatively, if the functional forms given by equations (62) and (64) are known, the Christoffel symbols may be obtained from equations (65) and (66) without using storage space. This technique may be illustrated by using the transformation from an orthogonal Cartesian reference frame to a curvilinear coordinate system in which x^i are cylindrical polar coordinates. If the curvilinear coordinate system is cylindrical polar, the Cartesian coordinates y^i are related to the curvilinear coordinates by the following transformation equations (see sketch (a)):



Sketch (a)

$$\left.\begin{aligned} y^1 &= x^1 \cos x^2 \\ y^2 &= x^1 \sin x^2 \\ y^3 &= x^3 \end{aligned}\right\} \qquad (67)$$

The inverse transformation is given by

$$\left.\begin{aligned} x^1 &= \sqrt{(y^1)^2 + (y^2)^2} \\ x^2 &= \tan^{-1}\left(\frac{y^2}{y^1}\right) \\ x^3 &= y^3 \end{aligned}\right\} \qquad (68)$$

By substitution from equations (67) and (68) in equations (65) and (66), all the Christoffel symbols may be obtained. For the special case considered there are only 2 independent, nonzero Christoffel symbols out of a total of 18. Of course, in practical applications, the operation of obtaining Christoffel symbols from formulas (65) and (66) and the transformation equations would be performed by a computer programmed for non-numeric operations.

By way of illustration, equation (66) will be used to obtain the nonzero Christoffel symbols of the second kind. With the exception of the dummy index, the superscripts appearing on the right side of the equation (66) must correspond to those appearing in the Christoffel symbol. For example

$$\left\{\begin{matrix} 1 \\ 22 \end{matrix}\right\} = \frac{\partial^2 y^\alpha}{\partial x^2 \partial x^2} \frac{\partial x^1}{\partial y^\alpha}$$

Carrying out the summation implied by the dummy index α gives

$$\left\{\begin{matrix} 1 \\ 22 \end{matrix}\right\} = \frac{\partial^2 y^1}{\partial x^2 \partial x^2} \frac{\partial x^1}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^2 \partial x^2} \frac{\partial x^1}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^2 \partial x^2} \frac{\partial x^1}{\partial y^3}$$

Substitution for the partial differential coefficients from the functional relationships given by equations (67) and (68) gives

$$\left\{\begin{matrix} 1 \\ 22 \end{matrix}\right\} = [-x^1 \cos x^2(\cos x^2) - x^1 \sin x^2(\sin x^2)] = -x^1$$

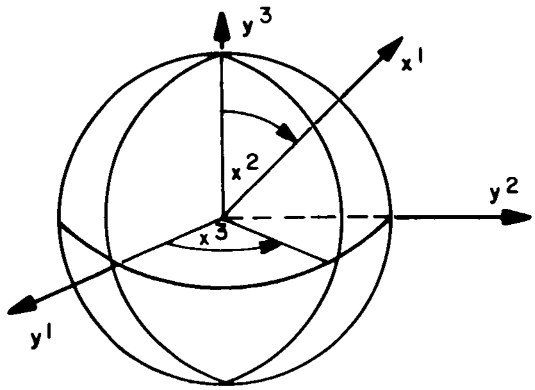
Likewise,

$$\left\{\begin{matrix} 2 \\ 12 \end{matrix}\right\} = \frac{\partial^2 y^1}{\partial x^1 \partial x^2} \frac{\partial x^2}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^1 \partial x^2} \frac{\partial x^2}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^1 \partial x^2} \frac{\partial x^2}{\partial y^3}$$

therefore

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \left[-\sin x^2\left(\frac{-\sin x^2}{x^1}\right) + \cos x^2\left(\frac{\cos x^2}{x^1}\right)\right] = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \frac{1}{x^1}$$

The procedure for determining the Christoffel symbols for a spherical polar coordinate system is the same as that used for a cylindrical polar coordinate system. In this case, however, the terms of equation (66) have to be obtained from a different set of coordinate transformation equations. The Cartesian coordinates y^i are related to the spherical polar coordinates x^i by the following transformation equations. (See sketch (b).)



Sketch (b)

$$\left.\begin{aligned} y^1 &= x^1 \sin x^2 \cos x^3 \\ y^2 &= x^1 \sin x^2 \sin x^3 \\ y^3 &= x^1 \cos x^2 \end{aligned}\right\} \qquad (69)$$

The inverse transformation is given by

$$\left.\begin{aligned} x^1 &= \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \\ x^2 &= \tan^{-1}\left(\frac{\sqrt{(y^1)^2 + (y^2)^2}}{y^3}\right) \\ x^3 &= \tan^{-1}\left(\frac{y^2}{y^1}\right) \end{aligned}\right\} \qquad (70)$$

By substitution from equations (69) and (70) in equation (66), all the Christoffel symbols may be obtained. For the special case being considered there are 6 nonzero Christoffel symbols out of a total of 18. Of course, as indicated previously, the operation of obtaining Christoffel symbols from formula (66) and the use of the transformation equations would be performed by a computer programmed for this kind of operation. By way of illustration equation (66) will again be used to obtain the nonzero Christoffel symbols. By substitution from equations (69) and (70) in the expanded form of equation (66) it is found that

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \left(\frac{\partial^2 y^1}{\partial x^2 \partial x^2}\frac{\partial x^1}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^2 \partial x^2}\frac{\partial x^1}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^2 \partial x^2}\frac{\partial x^1}{\partial y^3}\right) = -x^1$$

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = \left(\frac{\partial^2 y^1}{\partial x^3 \partial x^3}\frac{\partial x^2}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^3 \partial x^3}\frac{\partial x^2}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^3 \partial x^3}\frac{\partial x^2}{\partial y^3}\right) = -\sin x^2 \cos x^2$$

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \left(\frac{\partial^2 y^1}{\partial x^1 \partial x^2}\frac{\partial x^2}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^1 \partial x^2}\frac{\partial x^2}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^1 \partial x^2}\frac{\partial x^2}{\partial y^3}\right) = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \frac{1}{x^1}$$

$$\left\{ {3 \atop 13} \right\} = \left(\frac{\partial^2 y^1}{\partial x^1 \partial x^3} \frac{\partial x^3}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^1 \partial x^3} \frac{\partial x^3}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^1 \partial x^3} \frac{\partial x^3}{\partial y^3} \right) = \left\{ {3 \atop 31} \right\} = \frac{1}{x^1}$$

$$\left\{ {1 \atop 33} \right\} = \left(\frac{\partial^2 y^1}{\partial x^3 \partial x^3} \frac{\partial x^1}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^3 \partial x^3} \frac{\partial x^1}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^3 \partial x^3} \frac{\partial x^1}{\partial y^3} \right) = -x^1 \sin^2 x^2$$

$$\left\{ {3 \atop 23} \right\} = \left(\frac{\partial^2 y^1}{\partial x^2 \partial x^3} \frac{\partial x^3}{\partial y^1} + \frac{\partial^2 y^2}{\partial x^2 \partial x^3} \frac{\partial x^3}{\partial y^2} + \frac{\partial^2 y^3}{\partial x^2 \partial x^3} \frac{\partial x^3}{\partial y^3} \right) = \left\{ {3 \atop 32} \right\} = \cot x^2$$

These results are seen to agree with those obtained in equation (59).

The Velocity Vector

Two methods of obtaining the Christoffel symbols have been indicated: one using the methods of vector calculus and the other using known differential coefficients from the coordinate transformation equations. Since the latter method is more adaptable to digital logic, equations (65) and (66) rather than equations (27) and (31) are used to determine the Christoffel symbols. Given the Christoffel symbols, it is seen that there are two forms for the intrinsic or absolute derivative of a vector. Equation (35) gives the intrinsic derivative in terms of the contravariant vector components; and equation (47) gives the same in terms of the covariant components. Either of these equations may be used. However, to avoid the necessity of transforming covariant components into contravariant components, and vice versa, by the methods of appendix A, it is better to match the formula to the variance of the vectors. In the course of the analysis, it will become evident what the variance of the vectors is. For example, the variance of the differential elements can be determined as follows: the differential elements dy^i in the y coordinate system are related to the elements dx^j in the x coordinate system by the following equation:

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j \qquad (71)$$

By comparing this equation with equation (3), it is seen that the differential elements are the components of a contravariant vector. Likewise, equation (71) shows that the components of velocity in the y coordinate system are related to those in the x coordinate system by the equation

$$\frac{dy^i}{dt} = \frac{\partial y^i}{\partial x^j} \frac{dx^j}{dt}$$

that is,

$$V^i(y) = \frac{\partial y^i}{\partial x^j} U^j(x) \qquad (72)$$

where $V^i(y)$ are the velocity components in the y coordinate system, and $U^j(x)$ are the velocity components in the x coordinate system. Comparison of equation (72) with equation (3) shows that the components of the velocity vector also obey the contravariant transformation law. To obtain the velocity vector from equation (35), the position vector $\overline{r}$ is substituted for the vector $\overline{A}$; that is,

$$\overline{A} = A^i \overline{a}_i = \overline{r} \tag{73}$$

Hence, in a cylindrical polar coordinate system

$$A^1 = x^1 , \qquad A^2 = 0 , \qquad A_3 = x^3 \tag{74}$$

By substitution of these values in equation (55), the velocity vector is obtained as follows

$$\overline{V} = \frac{dx^1}{dt} \overline{a}_1 + \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} x^1 \frac{dx^2}{dt} \overline{a}_2 + \frac{dx^3}{dt} \overline{a}_3$$

When the appropriate value of the Christoffel symbol is substituted from equation (54), the tensor components of the velocity vector are given by

$$\overline{V} = \frac{dx^1}{dt} \overline{a}_1 + \frac{dx^2}{dt} \overline{a}_2 + \frac{dx^3}{dt} \overline{a}_3$$

that is,

$$\overline{V} = \frac{dx^i}{dt} \overline{a}_i \tag{75}$$

In order to reduce equation (75) to the conventional form, where the physical components of velocity are associated with a set of unit base vectors, equation (B1) may be used to express the base vectors in unitary form. In this form the velocity $\overline{V}$ is given by

$$\overline{V} = \frac{dx^1}{dt} \hat{a}_1 + \left(x^1 \frac{dx^2}{dt} \right) \hat{a}_2 + \frac{dx^3}{dt} \hat{a}_3 \tag{76}$$

If the coordinate x^1 is identified with the radial distance r, the coordinate x^2 with the polar angle θ, and the coordinate x^3 with the axial displacement z, the equation for the velocity in a cylindrical polar coordinate system assumes the familiar form

$$\overline{V} = \frac{dr}{dt} \hat{a}_1 + \left(r \frac{d\theta}{dt} \right) \hat{a}_2 + \frac{dz}{dt} \hat{a}_3 \tag{77}$$

where $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ are a triad of mutually orthogonal unit vectors in the directions of increasing r, θ, and z, respectively.

In a spherical polar coordinate system, the vector $\bar{r}$ has the following components.

$$A^1 = x^1 , \quad A^2 = A^3 = 0$$

When these values are substituted in equation (60), the velocity vector in this coordinate system is given by

$$\bar{V} = \frac{dx^1}{dt} \bar{a}_1 + \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} x^1 \frac{dx^2}{dt} \bar{a}_2 + \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} x^1 \frac{dx^3}{dt} \bar{a}_3$$

Again, by substitution of the Christoffel symbols from equation (59), the velocity vector may be expressed in terms of its tensor components and a corresponding set of base vectors as follows:

$$\bar{V} = \frac{dx^1}{dt} \bar{a}_1 + \frac{dx^2}{dt} \bar{a}_2 + \frac{dx^3}{dt} \bar{a}_3$$

that is,

$$\bar{V} = \frac{dx^i}{dt} \bar{a}_i \tag{78}$$

From a comparison of equation (75) with (78), it is seen that when expressed in terms of its tensor components, the velocity vector has the same form in both coordinate systems. This is true, in general, since by definition

$$\bar{V} = \frac{d\bar{r}}{dt} = \frac{\partial \bar{r}}{\partial x^i} \frac{dx^i}{dt} \tag{79}$$

And substitution from equation (17) in equation (79) gives

$$\bar{V} = \frac{dx^i}{dt} \bar{a}_i$$

Of course, the physical components of velocity are different, as can be seen when the methods of appendix B are used to reduce the base vectors to unitary form. By substitution from equation (57) in equation (78), the velocity vector may be expressed in terms of its physical components and a set of unit base vectors as follows:

$$\bar{V} = \frac{dx^1}{dt} \hat{a}_1 + \left(x^1 \frac{dx^2}{dt}\right) \hat{a}_2 + \left(x^1 \sin x^2 \frac{dx^3}{dt}\right) \bar{a}_3 \tag{80}$$

When the coordinate x^1 is identified with the radial distance r, the coordinate x^2 with the polar angle θ, and the coordinate x^3 with the azimuth angle ψ, the equation for $\bar{V}$ assumes the more familiar form

$$\bar{V} = \frac{dr}{dt} \hat{a}_1 + \left(r \frac{d\theta}{dt}\right) \hat{a}_2 + \left(r \sin \theta \frac{d\psi}{dt}\right) \hat{a}_3 \tag{81}$$

where $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ are a triad of mutually orthogonal unit vectors in the directions of increasing r, θ, and ψ, respectively.

The Acceleration Vector

If the acceleration vector were required to formulate the equations of motion of a particle, the velocity vector $\overline{V}$ would be substituted for the vector $\overline{A}$ in the equation for the intrinsic derivative. As shown in equation (72), the velocity vector assumes the contravariant form; therefore, if the tensor components of velocity are substituted for the components of $\overline{A}$ in equation (35),

$$\overline{A} = \overline{V} = \frac{dx^i}{dt}\,\overline{a}_i \tag{82}$$

Hence, in a general curvilinear coordinate system, the acceleration vector is given by

$$\frac{d\overline{V}}{dt} = \left(\frac{dV^i}{dt} + \begin{Bmatrix} i \\ jk \end{Bmatrix} V^j \frac{dx^k}{dt}\right)\overline{a}_i \tag{83}$$

By substitution from equation (82) in equation (83), the acceleration vector may be written in the following alternative form

$$\frac{d\overline{V}}{dt} = \left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\,\frac{dx^k}{dt}\right)\overline{a}_i \tag{84}$$

This equation gives the acceleration in any coordinate system, provided the Christoffel symbols are appropriate to the coordinate system chosen to describe the problem.

In a three-dimensional cylindrical polar coordinate system, equation (83) reduces to the form given by equation (55), when the vector $\overline{V}$ is substituted for the vector $\overline{A}$. Likewise, in a three-dimensional spherical polar coordinate system, equation (83) reduces to the form given by equation (60), when the vector $\overline{V}$ is substituted for the vector $\overline{A}$. If equation (60) is used to obtain the acceleration vector, the tensor components of velocity, rather than the physical components, must always be used. The tensor components of velocity are given by equation (78). These are

$$A^1 = \frac{dx^1}{dt}, \qquad A^2 = \frac{dx^2}{dt}, \qquad A^3 = \frac{dx^3}{dt} \tag{85}$$

Substituting these values in equation (60) gives the acceleration in terms of spherical polar coordinates

$$\frac{d\bar{V}}{dt} = \left[\left(\frac{d^2x^1}{dt^2} + \begin{Bmatrix}1\\22\end{Bmatrix} \frac{dx^2}{dt}\frac{dx^2}{dt} + \begin{Bmatrix}1\\33\end{Bmatrix} \frac{dx^3}{dt}\frac{dx^3}{dt}\right)\bar{a}_1\right.$$

$$+ \left(\frac{d^2x^2}{dt^2} + 2\begin{Bmatrix}2\\12\end{Bmatrix} \frac{dx^1}{dt}\frac{dx^2}{dt} + \begin{Bmatrix}2\\33\end{Bmatrix} \frac{dx^3}{dt}\frac{dx^3}{dt}\right)\bar{a}_2$$

$$\left. + \left(\frac{d^2x^3}{dt^2} + 2\begin{Bmatrix}3\\13\end{Bmatrix} \frac{dx^1}{dt}\frac{dx^3}{dt} + 2\begin{Bmatrix}3\\23\end{Bmatrix} \frac{dx^2}{dt}\frac{dx^3}{dt}\right)\bar{a}_3\right] \tag{86}$$

By substitution of the Christoffel symbols from equation (59) in equation (86), the acceleration vector may be expressed in terms of its tensor components and associated base vectors as follows:

$$\frac{d\bar{V}}{dt} = \left[\frac{d^2x^1}{dt^2} - x^1\left(\frac{dx^2}{dt}\right)^2 - x^1 \sin^2 x^2\left(\frac{dx^3}{dt}\right)^2\right]\bar{a}_1$$

$$+ \left[\frac{d^2x^2}{dt^2} + \frac{2}{x^1}\frac{dx^1}{dt}\frac{dx^2}{dt} - \sin x^2 \cos x^2\left(\frac{dx^3}{dt}\right)^2\right]\bar{a}_2$$

$$+ \left(\frac{d^2x^3}{dt^2} + \frac{2}{x^1}\frac{dx^1}{dt}\frac{dx^3}{dt} + 2\cot x^2 \frac{dx^2}{dt}\frac{dx^3}{dt}\right)\bar{a}_3 \tag{87}$$

The corresponding physical components of the acceleration vector may be obtained from the base vectors expressed in terms of unit vectors in accordance with equation (57). When appropriate substitutions are made, equation (87) gives

$$\frac{d\bar{V}}{dt} = \left[\frac{d^2x^1}{dt^2} - x^1\left(\frac{dx^2}{dt}\right)^2 - x^1\left(\sin x^2 \frac{dx^3}{dt}\right)^2\right]\hat{a}_1$$

$$+ \left[x^1\frac{d^2x^2}{dt^2} + 2\frac{dx^1}{dt}\frac{dx^2}{dt} - x^1 \sin x^2 \cos x^2\left(\frac{dx^3}{dt}\right)^2\right]\hat{a}_2$$

$$+ \left(x^1 \sin x^2 \frac{d^2x^3}{dt^2} + 2\sin x^2 \frac{dx^1}{dt}\frac{dx^3}{dt} + 2x^1 \cos x^2 \frac{dx^2}{dt}\frac{dx^3}{dt}\right)\hat{a}_3$$

Equations of Motion in a General Curvilinear Coordinate System

In using tensor methods to derive equations of motion, it is again important to remember that the acceleration and force vectors must always be expressed in terms of their tensor components rather than their physical components. Hence, the two sides of every equation must balance with respect to their covariant or contravariant properties before applying Newton's second law of motion. In this connection it is worth noting that, although the acceleration vector is expressed in contravariant form in equation (84), the force vector may appear in the form of a covariant tensor. The force vector assumes the covariant form when it appears as the gradient of a scalar point function. This occurs in the equations of motion of a space vehicle which, in addition to the thrust force, is subject to gravitational forces. If the gravitational forces are expressed in the form of the gradient of a gravitational potential function, the force vector is

$$\overline{F} = \nabla\varphi + \overline{T} \tag{88}$$

where φ is the gravitational potential function, which may include the influence of oblateness and extraterrestrial gravitational forces, and $\overline{T}$ is the thrust vector. (See refs. 6 and 7.)

The tensor form of the gradient of a scalar point function assumes the following form

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^i}\,\overline{a}^i \tag{89}$$

The use of the reciprocal base vector $\overline{a}^i$ in equation (89) is justified by the following considerations: the components of the gradient of the gravitational potential function in the y coordinate system are related to those in the x coordinate system by the following equation:

$$\frac{\partial\varphi}{\partial y^i} = \frac{\partial\varphi}{\partial x^j}\,\frac{\partial x^j}{\partial y^i}$$

or

$$F_i(y) = \frac{\partial x^j}{\partial y^i}\,F_j(x) \tag{90}$$

where

$$F_i(y) = \frac{\partial\varphi}{\partial y^i}$$

$$F_j(x) = \frac{\partial\varphi}{\partial x^j}$$

The transformation of the components of the gradient vector from the x coordinate system to the y coordinate system (eq. (90)) obeys the covariant transformation law as defined in equation (4).

The equation of motion of a point mass which is subject to gravitational and thrust forces is obtained by combining equations (84) and (88).

$$M\left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right)\bar{a}_i = \nabla\varphi + \bar{T} \tag{91}$$

where M is the mass.

It is seen that the acceleration components represented by the left-hand side of this equation are all contravariant. The thrust vector, on the other hand, is usually given in terms of its physical components, and as already indicated in equation (90), the gravitational forces assume the form of covariant vectors. To have a force system compatible with the accelerations, it is necessary to convert all the force terms to the contravariant form. The potential gradient function may be converted to contravariant form with the aid of equation (A22). From equations (89) and (A22)

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^j}\bar{a}^j = g^{ij}\frac{\partial\varphi}{\partial x^j}\bar{a}_i \tag{92}$$

The thrust vector may be expressed in the following alternative forms

$$\bar{T} = T^i\bar{a}_i = \tau^i\hat{a}_i$$

where T^i are the contravariant tensor components of the thrust vector, and τ^i are the corresponding physical components. The physical components of the thrust vector are related to the contravariant tensor components by equation (B1)

$$T^i = \frac{1}{\sqrt{g_{(ii)}}}\tau^i \tag{93}$$

By substitution from equations (92) and (93) in equation (91), the equation of motion assumes the following form

$$M\left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right)\bar{a}_i = \left(g^{ij}\frac{\partial\varphi}{\partial x^j} + \frac{\tau^i}{\sqrt{g_{(ii)}}}\right)\bar{a}_i$$

Therefore

$$M\left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right) = g^{ij}\frac{\partial\varphi}{\partial x^j} + \frac{\tau^i}{\sqrt{g_{(ii)}}} \tag{94}$$

When the expression for the gravitational forces is expanded in a general three-dimensional coordinate system, equation (94) becomes

$$M\left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right) = g^{i1}\frac{\partial\varphi}{\partial x^1} + g^{i2}\frac{\partial\varphi}{\partial x^2} + g^{i3}\frac{\partial\varphi}{\partial x^3} + \frac{\tau^i}{\sqrt{g_{(ii)}}} \tag{95}$$

However, in a rectangular coordinate system

$$g^{ij} = 0 \quad \text{for} \quad i \neq j$$

and

$$g^{(ii)} = 1/g_{(ii)}$$

where the parentheses imply suspension of the summation convention.

Substituting these values in equation (95) gives for orthogonal systems

$$M\left(\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right) = \frac{1}{g_{(ii)}}\frac{\partial\varphi}{\partial x^i} + \frac{\tau^i}{\sqrt{g_{(ii)}}} \tag{96}$$

Equation (96) may be rewritten as follows:

$$M\left(g_{(ii)}\frac{d^2x^i}{dt^2} + g_{(ii)}\begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^k}{dt}\right) = \frac{\partial\varphi}{\partial x^i} + \sqrt{g_{(ii)}}\,\tau^i$$

or in the alternative form

$$M\left(g_{(ii)}\frac{d^2x^i}{dt^2} + [jk,i]\frac{dx^j}{dt}\frac{dx^k}{dt}\right) = \frac{\partial\varphi}{\partial x^i} + \sqrt{g_{(ii)}}\,\tau^i \tag{97}$$

From the point of view of non-numeric computer operations, it is more expedient to eliminate the Christoffel symbols and the metric tensors from equation (97). These symbols are related to the coordinate transformation equations by equations (65) and (63). Substituting from equations (63) and (65) in equation (97) gives

$$M\left[\left(\frac{\partial y^\alpha}{\partial x^{(i)}}\frac{\partial y^\alpha}{\partial x^{(i)}}\right)\frac{d^2x^i}{dt^2} + \left(\frac{\partial^2 y^\alpha}{\partial x^j\partial x^k}\frac{\partial y^\alpha}{\partial x^i}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}\right] = \frac{\partial\varphi}{\partial x^i} + \sqrt{\frac{\partial y^\alpha}{\partial x^{(i)}}\frac{\partial y^\alpha}{\partial x^{(i)}}}\,\tau^i \tag{98}$$

As indicated previously, a repeated index implies summation with respect to that index. An exception to this rule occurs when repeated indices are enclosed in parentheses. Parentheses around an index imply that the summation convention is to be suspended for that index. This means that for each value of the index i, equation (98) must be summed on α, j, and k. For example, when equation (98) is summed on α, it appears as follows:

$$\begin{aligned}
&M\left[\left(\frac{\partial y^1}{\partial x^{(i)}}\frac{\partial y^1}{\partial x^{(i)}} + \frac{\partial y^2}{\partial x^{(i)}}\frac{\partial y^2}{\partial x^{(i)}} + \frac{\partial y^3}{\partial x^{(i)}}\frac{\partial y^3}{\partial x^{(i)}}\right)\frac{d^2x^i}{dt^2}\right. \\
&\left. + \left(\frac{\partial^2 y^1}{\partial x^j\partial x^k}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^j\partial x^k}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^j\partial x^k}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}\right] \\
&\qquad = \frac{\partial\varphi}{\partial x^i} + \sqrt{\frac{\partial y^1}{\partial x^{(i)}}\frac{\partial y^1}{\partial x^{(i)}} + \frac{\partial y^2}{\partial x^{(i)}}\frac{\partial y^2}{\partial x^{(i)}} + \frac{\partial y^3}{\partial x^{(i)}}\frac{\partial y^3}{\partial x^{(i)}}}\,\tau^i
\end{aligned} \tag{99}$$

The left side of this equation must also be summed on j and k. When each of these indices is permitted to take the values 1, 2, 3, in turn, equation (99) assumes the following form:

$$
\begin{aligned}
M\Bigg[& \left(\frac{\partial y^1}{\partial x^{(i)}}\frac{\partial y^1}{\partial x^{(i)}} + \frac{\partial y^2}{\partial x^{(i)}}\frac{\partial y^2}{\partial x^{(i)}} + \frac{\partial y^3}{\partial x^{(i)}}\frac{\partial y^3}{\partial x^{(i)}}\right)\frac{d^2x^i}{dt^2} \\
& + \left(\frac{\partial^2 y^1}{\partial x^1\partial x^1}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^1\partial x^1}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^1\partial x^1}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^1}{dt}\frac{dx^1}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^1\partial x^2}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^1\partial x^2}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^1\partial x^2}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^1}{dt}\frac{dx^2}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^1\partial x^3}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^1\partial x^3}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^1\partial x^3}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^1}{dt}\frac{dx^3}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^2\partial x^1}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^2\partial x^1}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^2\partial x^1}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^2}{dt}\frac{dx^1}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^2\partial x^2}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^2\partial x^2}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^2\partial x^2}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^2}{dt}\frac{dx^2}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^2\partial x^3}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^2\partial x^3}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^2\partial x^3}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^2}{dt}\frac{dx^3}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^3\partial x^1}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^3\partial x^1}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^3\partial x^1}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^3}{dt}\frac{dx^1}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^3\partial x^2}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^3\partial x^2}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^3\partial x^2}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^3}{dt}\frac{dx^2}{dt} \\
& + \left(\frac{\partial^2 y^1}{\partial x^3\partial x^3}\frac{\partial y^1}{\partial x^i} + \frac{\partial^2 y^2}{\partial x^3\partial x^3}\frac{\partial y^2}{\partial x^i} + \frac{\partial^2 y^3}{\partial x^3\partial x^3}\frac{\partial y^3}{\partial x^i}\right)\frac{dx^3}{dt}\frac{dx^3}{dt}\Bigg] \\
& = \frac{\partial \varphi}{\partial x^{(i)}} + \sqrt{\frac{\partial y^1}{\partial x^{(i)}}\frac{\partial y^1}{\partial x^{(i)}} + \frac{\partial y^2}{\partial x^{(i)}}\frac{\partial y^2}{\partial x^{(i)}} + \frac{\partial y^3}{\partial x^{(i)}}\frac{\partial y^3}{\partial x^{(i)}}}\;\tau^i \qquad (100)
\end{aligned}
$$

The form of this equation is well suited to routine non-numeric computer operations. The large number of terms appearing in equation (100) is due to the generality of this equation, which is applicable to any space of three dimensions. Moreover, since this equation is applicable to any space of three dimensions, it may be permanently stored in the computer. Hence, to obtain the equations of motion in any system of coordinates, the only information required is the special form of equation (2) relating that system of coordinates to the orthogonal Cartesian coordinates y^i.

Computer Derivations of Equations of Motion in Special Systems of Coordinates

Spherical polar coordinates.- Consider a transformation of coordinates specifying the relation between the spherical polar coordinates x^i and the orthogonal Cartesian coordinates y^i. In this case, equation (2) becomes (see sketch (b)):

$$y^1 = x^1 \sin x^2 \cos x^3$$

$$y^2 = x^1 \sin x^2 \sin x^3$$

$$y^3 = x^1 \cos x^2$$

These coordinate transformation equations were supplied as input to an IBM 7094 computer, which was programmed for non-numeric operations.

When the computer was instructed to perform the operations involved in equation (100), the following output was obtained in Fortran language:

```
FOR I = 1,
The expression input for  Y(I) is given below.
X(1)*FMCSIN(X(2))*FMCCOS(X(3))$

FOR I = 2,
The expression input for  Y(I) is given below.
X(1)*FMCSIN(X(2))*FMCSIN(X(3))$

FOR I = 3,
The expression input for  Y(I) is given below.
X(1)*FMCCOS(X(2))$


FOR I = 1
The equation for  I = 1 is given below.
M*(P(1)-R(2)**2.0*X(1)-R(3)**2.0*X(1)*FMCSIN(X(2))**2.0)$
=DPHI(1)+TAU(1)$

FOR I = 2
The equation for  I = 2 is given below.
M*(P(2)*X(1)**2.0+R(1)*R(2)*X(1)*2.0-R(3)**2.0*X(1)**2.0*FMCSIN(X(2))*FMCCOS
(X(2)))$=DPHI(2)+TAU(2)*X(1)$

FOR I = 3
The equation for  I = 3 is given below.
M*(P(3)*X(1)**2.0*FMCSIN(X(2))**2.0+R(1)*R(3)*X(1)*FMCSIN(X(2))**2.0*2.0+R(2)
*R(3)*X(1)**2.0*FMCSIN(X(2))*FMCCOS(X(2))*2.0)$
=DPHI(3)+TAU(3)*X(1)*FMCSIN(X(2))$
```

Time on	Comp/load time, min	Executive time, min
950.09	0.78	0.41

In interpreting these Fortran statements, it must be remembered that:

$$R(i) = \frac{dx^i}{dt}$$

$$P(i) = \frac{d^2x^i}{dt^2}$$

In terms of conventional mathematical symbolism, these equations assume the following form:

$$M\left[\frac{d^2x^1}{dt^2} - x^1\left(\frac{dx^2}{dt}\right)^2 - x^1\left(\sin x^2 \frac{dx^3}{dt}\right)^2\right] = \frac{\partial\varphi}{\partial x^1} + \tau^1$$

$$M\left[(x^1)^2 \frac{d^2x^2}{dt^2} + 2x^1 \frac{dx^1}{dt}\frac{dx^2}{dt} - (x^1)^2 \sin x^2 \cos x^2\left(\frac{dx^3}{dt}\right)^2\right] = \frac{\partial\varphi}{\partial x^2} + x^1\tau^2$$

$$M\left[\left(x^1 \sin x^2\right)^2 \frac{d^2x^3}{dt^2} + 2x^1 \sin^2 x^2 \frac{dx^1}{dt}\frac{dx^3}{dt} + 2(x^1)^2 \sin x^2 \cos x^2 \frac{dx^2}{dt}\frac{dx^3}{dt}\right]$$

$$= \frac{\partial\varphi}{\partial x^3} + x^1 \sin x^2\tau^3$$

Because of its generality, equation (100) is applicable in all coordinate systems. Therefore, to obtain the equations of motion in any other coordinate system, all that is required is to supply the computer with the appropriate coordinate transformation equations.

Cylindrical polar coordinates.- As a further illustration of the procedure involved, consider the equations of motion in a cylindrical polar system of coordinates. In this case, the coordinate transformation equations are (see sketch (a)):

$$y^1 = x^1 \cos x^2$$

$$y^2 = x^1 \sin x^2$$

$$y^3 = x^3$$

When these coordinate transformation equations were used to evaluate the terms of equation (100), the following computer output was obtained.

```
FOR I = 1
The expression input for  Y(I) is given below.
X(1)*FMCCOS(X(2))$

FOR I = 2
The expression input for  Y(I) is given below.
X(1)*FMCSIN(X(2))$

FOR I = 3
The expression input for  Y(I) is given below.
X(3)$


The equation for  I = 1 is given below.
M*(P(1)-R(2)**2.0*X(1))$
=DPHI(1)+TAU(1)$

The equation for  I = 2 is given below.
M*(P(2)*X(1)**2.0+R(1)*R(2)*X(1)*2.0)$
=DPHI(2)+TAU(2)*X(1)$

The equation for  I = 3 is given below.
M*P(3)$
=DPHI(3)+TAU(3)$
```

Time on	Comp/load time, min	Executive time, min
037.78	1.15	0.14

Translating these equations from Fortran language to conventional mathematical symbolism yields the following:

$$M\left(\frac{d^2x^1}{dt^2} - x^1 \frac{dx^2}{dt} \frac{dx^2}{dt}\right) = \frac{\partial\varphi}{\partial x^1} + \tau^1$$

$$M\left[(x^1)^2 \frac{d^2x^2}{dt^2} + 2x^1 \frac{dx^1}{dt} \frac{dx^2}{dt}\right] = \frac{\partial\varphi}{\partial x^2} + x^1\tau^2$$

$$M\left(\frac{d^2x^3}{dt^2}\right) = \frac{\partial\varphi}{\partial x^3} + \tau^3$$

Prolate spheroidal coordinates.- Another interesting system of orthogonal curvilinear coordinates is the prolate spheroidal coordinates. Coordinate surfaces are obtained by rotating a family of confocal ellipses and hyperbolas about their major axes. Rotating these conic sections gives rise to a system of prolate spheroids and hyperboloids of two sheets. A family of planes through the axis of rotation completes the system of orthogonal surfaces. The curvilinear coordinate systems generated by the curves of intersection of these surfaces are useful in certain quantum mechanical problems (ref. 8). The transformation equations relating this system of coordinates to the orthogonal Cartesian system are as follows:

$$y^1 = a \sinh x^1 \sin x^2 \cos x^3$$

$$y^2 = a \sinh x^1 \sin x^2 \sin x^3$$

$$y^3 = a \cosh x^1 \cos x^2$$

To obtain the equations of motion relative to a prolate spheroidal system of coordinates, these transformation equations were substituted for equation (2) in the computer program. Execute time was 1.63 minutes. Omitting the print-out in Fortran language, the equations of motion were obtained as follows:

$$M\left[a^2(\sin^2 x^2 + \sinh^2 x^1) \frac{d^2x^1}{dt^2} + 2a^2 \sin x^2 \cos x^2 \frac{dx^1}{dt} \frac{dx^2}{dt} + a^2 \sinh x^1 \cosh x^1 \frac{dx^1}{dt} \frac{dx^1}{dt} - a^2 \sinh x^1 \cosh x^1 \frac{dx^2}{dt} \frac{dx^2}{dt} - a^2 \sin^2 x^2 \sinh x^1 \cosh x^1 \frac{dx^3}{dt} \frac{dx^3}{dt}\right] = a \sqrt{\sin^2 x^2 + \sinh^2 x^1}\, \tau^1 + \frac{\partial\varphi}{\partial x^1}$$

$$M\left[a^2(\sin^2 x^2 + \sinh^2 x^1) \frac{d^2x^2}{dt^2} - a^2 \sin x^2 \cos x^2 \frac{dx^1}{dt} \frac{dx^1}{dt} + 2a^2 \sinh x^1 \cosh x^1 \frac{dx^1}{dt} \frac{dx^2}{dt} + a^2 \sin x^2 \cos x^2 \frac{dx^2}{dt} \frac{dx^2}{dt} - a^2 \sin x^2 \cos x^2 \sinh^2 x^1 \frac{dx^3}{dt} \frac{dx^3}{dt}\right] = a\left(\sqrt{\sin^2 x^2 + \sinh^2 x^1}\right)\tau^2 + \frac{\partial\varphi}{\partial x^2}$$

$$M\left[a^2 \sin^2 x^2 \sinh^2 x^1 \frac{d^2x^3}{dt^2} + 2a^2 \sin^2 x^2 \sinh x^1 \cosh x^1 \frac{dx^1}{dt} \frac{dx^3}{dt} + 2a^2 \sin x^2 \cos x^2 \sinh^2 x^1 \frac{dx^2}{dt} \frac{dx^3}{dt}\right] = a \sin x^2 \sinh x^1\, \tau^3 + \frac{\partial\varphi}{\partial x^3}$$

Oblate spheroidal coordinates.- Confocal ellipses and hyperbolas rotated about their minor axes generate the oblate spheroids and hyperboloids of one sheet (ref. 9). These surfaces, together with a family of planes through the axis of rotation, constitute a family of orthogonal surfaces. The curvilinear coordinate systems generated by the curves of intersection of these surfaces are called oblate spheroidal coordinates. Oblate spheroids are sometimes referred to as planetary ellipsoids, because the Earth and the planet Jupiter are approximately of this form. The transformation equations relating this system of coordinates to the orthogonal Cartesian system are as follows:

$$y^1 = a \cosh x^1 \sin x^2 \cos x^3$$

$$y^2 = a \cosh x^1 \sin x^2 \sin x^3$$

$$y^3 = a \sinh x^1 \cos x^2$$

These transformation equations take the place of equation (2) in the computer. In this case, the time required to execute the program was again 1.63 minutes. Omitting the print-out in Fortran language, the equations of motion relative to a system of oblate spheroidal coordinates were obtained in the following form:

$$M\left[a^2(\sinh^2 x^1 + \cos^2 x^2)\frac{d^2x^1}{dt^2} + a^2(\sinh x^1 \cosh x^1)\frac{dx^1}{dt}\frac{dx^1}{dt}\right.$$

$$- 2a^2 \cos x^2 \sin x^2 \frac{dx^1}{dt}\frac{dx^2}{dt} - a^2 \sinh x^1 \cosh x^1 \frac{dx^2}{dt}\frac{dx^2}{dt}$$

$$\left. - a^2 \cosh x^1 \sinh x^1 \sin^2 x^2 \frac{dx^3}{dt}\frac{dx^3}{dt}\right] = a(\sqrt{\sinh^2 x^1 + \cos^2 x^2})\tau^1 + \frac{\partial\varphi}{\partial x^1}$$

$$M\left[a^2(\sinh^2 x^1 + \cos^2 x^2)\frac{d^2x^2}{dt^2} + a^2 \sin x^2 \cos x^2 \frac{dx^1}{dt}\frac{dx^1}{dt}\right.$$

$$+ 2a^2 \sinh x^1 \cosh x^1 \frac{dx^1}{dt}\frac{dx^2}{dt} - a^2 \sin x^2 \cos x^2 \frac{dx^2}{dt}\frac{dx^2}{dt}$$

$$\left. - a^2 \cosh^2 x^1 \sin x^2 \cos x^2 \frac{dx^3}{dt}\frac{dx^3}{dt}\right] = a\sqrt{\sinh^2 x^1 + \cos^2 x^2}\ \tau^2 + \frac{\partial\varphi}{\partial x^2}$$

$$M\left[a^2 \cosh^2 x^1 \sin^2 x^2 \frac{d^2x^3}{dt^2} + 2a^2 \sinh x^1 \cosh x^1 \sin^2 x^2 \frac{dx^1}{dt}\frac{dx^3}{dt}\right.$$

$$\left. + 2a^2 \cosh^2 x^1 \sin x^2 \cos x^2 \frac{dx^2}{dt}\frac{dx^3}{dt}\right] = a \cosh x^1 \sin x^2 \tau^3 + \frac{\partial\varphi}{\partial x^3}$$

CONCLUSIONS

The extensive logic and storage capabilities of digital computers, combined with the new computer languages, enable them to be used for a wide range of non-numeric operations. Research indicates that these computers can be used more effectively for this purpose if all vector quantities are expressed in terms of their tensor components, rather than in terms of their physical components. Because of the geometrical simplification inherent in the tensor method, the formulation of problems in curvilinear coordinate systems can be reduced to routine computer operations. The results obtained suggest that the exploitation and extension of these techniques should lead to a substantial reduction in the man hours required to formulate and process engineering and scientific problems.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Dec. 8, 1966
125-19-01-37

APPENDIX A

TRANSFORMATION FORMULAS FOR VECTORS AND BIVECTORS

Base Vectors

The transformation laws and, hence, the covariant and contravariant character of the base vectors and their reciprocals may be obtained as follows: Let the differential of a position vector be denoted by $d\bar{r}$. Then if $\bar{a}_i(x)$ are the base vectors in the x coordinate system, and $\bar{b}_j(y)$ the base vectors in the y coordinate system, the differential $d\bar{r}$ may be expressed in the following alternative forms:

$$d\bar{r} = \bar{a}_i(x)dx^i = \bar{b}_j(y)dy^j = \bar{b}_j(y)\frac{\partial y^j}{\partial x^i}dx^i \tag{A1}$$

Therefore

$$\bar{a}_i(x) = \frac{\partial y^j}{\partial x^i}\bar{b}_j(y) \tag{A2}$$

Likewise,

$$\bar{a}_i(x)\frac{\partial x^i}{\partial y^j}dy^j = \bar{b}_j(y)dy^j$$

therefore

$$\bar{b}_j(y) = \frac{\partial x^i}{\partial y^j}\bar{a}_i(x) \tag{A3}$$

It is seen from equations (A2) and (A3) that the base vectors $\bar{a}_i$ and $\bar{b}_j$ obey the covariant transformation law; consequently, the use of subscripts is justified.

Reciprocal Base Vectors

To each system of base vectors $\bar{a}_i$ there exists a reciprocal system of vectors $\bar{a}^j$ with the following property

$$\bar{a}_i \cdot \bar{a}^j = \delta_i^j = \bar{a}^j \cdot \bar{a}_i \tag{A4}$$

where δ_i^j is the Kronecker delta; that is,

$$\delta_i^j = \begin{matrix} 1 & \text{for} & j = i \\ 0 & \text{for} & j \neq i \end{matrix}$$

Scalar multiplication of each side of equation (A2) by $\bar{b}^j(y)$ gives on using (A4)

$$\bar{b}^j(y) \cdot \bar{a}_i(x) = \frac{\partial y^j}{\partial x^i} \tag{A5}$$

Similarly, from equations (A3) and (A4) it is seen that

$$\bar{a}^i(x) \cdot \bar{b}_j(y) = \frac{\partial x^i}{\partial y^j} \tag{A6}$$

Equation (A1) referred to the reciprocal system of base vectors assumes the form

$$d\bar{r} = \bar{a}^i(x)dx_i = \bar{b}^j(y)dy_j \tag{A7}$$

therefore

$$dy_j = \bar{b}_j(y) \cdot \bar{a}^i(x)dx_i$$

$$dy_j = \frac{\partial x^i}{\partial y^j}\, dx_i \tag{A8}$$

and

$$dx_i = \bar{a}_i(x) \cdot \bar{b}^j(y)dy_j$$

therefore

$$dx_i = \frac{\partial y^j}{\partial x^i}\, dy_j \tag{A9}$$

From equations (A7) and (A8)

$$\bar{a}^i(x)dx_i = \bar{b}^j(y)\, \frac{\partial x^i}{\partial y^j}\, dx_i$$

therefore

$$\bar{a}^i(x) = \frac{\partial x^i}{\partial y^j}\, \bar{b}^j(y) \tag{A10}$$

Likewise, from equations (A7) and (A9)

$$\bar{b}^j(y) = \frac{\partial y^j}{\partial x^i}\, \bar{a}^i(x) \tag{A11}$$

From equations (A10) and (A11), it is seen that the reciprocal base vectors obey the contravariant law of transformation; therefore, the superscript notation is justified.

Vector Transformations

Equations (A10) and (A11) may be used to obtain the transformation law for a vector $\bar{A}$, where

$$\bar{A} = A^i\bar{a}_i = A_j\bar{a}^j \tag{A12}$$

If $\bar{A} = A^i(x)\bar{a}_i(x)$ when the vector $\bar{A}$ is referred to the x coordinate system, and if $\bar{A} = B^j(y)\bar{b}_j(y)$ when referred to the y coordinate system, invariance of $\bar{A}$ requires that

$$B^j(y)\bar{b}_j(y) = A^i(x)\bar{a}_i(x) \tag{A13}$$

From equations (A2) and (A13), the appropriate transformation law is obtained as follows:

$$B^j(y) = \frac{\partial y^j}{\partial x^i} A^i(x) \tag{A14}$$

Equation (A14) is the contravariant transformation law for the components of the vector $\bar{A}$. When $\bar{A}$ is referred to the x coordinate system with base vectors $\bar{a}_i(x)$, which obey the covariant transformation law, the components $A^i(x)$ obey the contravariant transformation law; hence, the use of superscripts is justified. If $\bar{A}$ is referred to the reciprocal base system $\bar{a}^i$, then from equation (A12)

$$\bar{A} = A_i\bar{a}^i$$

On a transformation of coordinates from the x coordinate system to the y coordinate system, invariance of $\bar{A}$ requires that

$$A_i(x)\bar{a}^i(x) = B_j(y)\bar{b}^j(y) \tag{A15}$$

From equations (A10) and (A15), the appropriate transformation law is obtained as follows:

$$B_j(y) = \frac{\partial x^i}{\partial y^j} A_i(x) \tag{A16}$$

It is seen that when a vector $\bar{A}$ is referred to a coordinate system with reciprocal base vectors, which obey the contravariant law, the corresponding components of $\bar{A}$ obey the covariant law, and the use of subscripts is therefore justified.

Raising and Lowering of Indices

Lowering indices.- The vector $\bar{A}$ may be expressed in the alternative forms given in equation (A12). Scalar multiplication of each side of equation (A12) by $\bar{a}_j$ gives

$$(\bar{a}_i \cdot \bar{a}_j)A^i = A_j(\bar{a}^j \cdot \bar{a}_j) \tag{A17}$$

By substitution from equations (18) and (A4) in equation (A17), the following result is obtained

$$g_{ij}A^i = A_j \tag{A18}$$

Again by substitution for A_j from equation (A18) in equation (A12)

$$\bar{a}_i = g_{ij}\bar{a}^j \tag{A19}$$

Equation (A18) gives the transformation from the contravariant components to the covariant components of a vector. The corresponding transformation of base vectors is given by equation (A19). These operations are usually referred to as lowering the index.

Raising indices.- Scalar multiplication of each side of equation (A12) by $\bar{a}^i$ gives

$$A^i(\bar{a}_i \cdot \bar{a}^i) = A_j(\bar{a}^j \cdot \bar{a}^i) \tag{A20}$$

Substitution from equations (19) and (A4) in equation (A20) gives

$$A^i = g^{ij}A_j \tag{A21}$$

When this expression for A^i is substituted in the left-hand side of equation (A12), the following result is obtained

$$g^{ij}A_j\bar{a}_i = A_j\bar{a}^j$$

therefore

$$\bar{a}^j = g^{ij}\bar{a}_i \tag{A22}$$

Equation (A21) enables the contravariant components of a vector to be expressed in terms of its covariant components. Equation (A22) gives the corresponding transformation of base vectors. These operations are usually referred to as raising the index.

Bivector Transformations

A second-order tensor is characterized by having two indices. Both indices can be superscripts, in which case the tensor is doubly contravariant. Tensors of this kind are sometimes referred to as the contravariant components of a bivector (ref. 10). When both indices are subscripts, the tensors are doubly covariant, or the components of a covariant bivector. It sometimes happens that one of the indices is a superscript and the other one a subscript. Entities of this kind are called mixed tensors or the components of a mixed bivector.

Contravariant bivectors.- As in the case of vectors or first-order tensors, bivectors are entities whose properties are independent of the reference frames used to describe them. Equations (A13) and (A15) are mathematical expressions of this statement, insofar as it applies to vectors. As might be expected, the invariance of a bivector in going from a coordinate system x with base vectors $\bar{a}_j(x)$, to a coordinate system y with base vectors $\bar{b}_i(y)$, involves the equality of two dyadics. The coefficients of the individual dyads in these dyadics are the components of the bivectors. If in the x coordinate system with base vectors $\bar{a}_j$ the bivector is given by

$$A^{\alpha\beta}(x)\bar{a}_\alpha(x)\bar{a}_\beta(x)$$

and if in the y coordinate system with base vectors $\bar{b}_i$ this bivector assumes the form

$$B^{ij}(y)\bar{b}_i(y)\bar{b}_j(y)$$

invariance requires that

$$B^{ij}(y)\bar{b}_i(y)\bar{b}_j(y) = A^{\alpha\beta}(x)\bar{a}_\alpha(x)\bar{a}_\beta(x) \tag{A23}$$

By substitution from equation (A2) in equation (A23)

$$B^{ij}(y)\bar{b}_i(y)\bar{b}_j(y) = A^{\alpha\beta}(x)\,\frac{\partial y^i}{\partial x^\alpha}\,\bar{b}_i(y)\,\frac{\partial y^j}{\partial x^\beta}\,\bar{b}_j(y) \tag{A24}$$

Therefore, by equating coefficients of like dyads in equation (A24), the required transformation law is obtained as follows:

$$B^{ij}(y) = \frac{\partial y^i}{\partial x^\alpha}\,\frac{\partial y^j}{\partial x^\beta}\,A^{\alpha\beta}(x) \tag{A25}$$

This is the transformation law for the components of a contravariant bivector.

Covariant bivectors.- Since covariant bivectors are characterized by two subscripts, it follows that the formulation of the dyadics will be in terms of the reciprocal base vectors. That is, if $A_{\alpha\beta}(x)$ are the components of the covariant bivector in the x coordinate system, and $B_{ij}(y)$ are the corresponding components in the y coordinate system, invariance of the bivectors requires that

$$B_{ij}(y)\bar{b}^i(y)\bar{b}^j(y) = A_{\alpha\beta}(x)\bar{a}^\alpha(x)\bar{a}^\beta(x) \tag{A26}$$

Substitution from equation (A10) in equation (A26) gives

$$B_{ij}(y)\bar{b}^i(y)\bar{b}^j(y) = A_{\alpha\beta}(x)\,\frac{\partial x^\alpha}{\partial y^i}\,\bar{b}^i(y)\,\frac{\partial x^\beta}{\partial y^j}\,\bar{b}^j(y) \tag{A27}$$

therefore

$$B_{ij}(y) = \frac{\partial x^{\alpha}}{\partial y^{i}} \frac{\partial x^{\beta}}{\partial y^{j}} A_{\alpha\beta}(x) \tag{A28}$$

Equation (A28) is the transformation law for the components of a covariant bivector.

<u>Mixed bivectors</u>.- A mixed bivector has one index of covariance and one index of contravariance. In this case, the bivectors consist of base vectors and reciprocal base vectors. The invariance requirements may be stated as follows:

$$B^{i}_{j}(y)\bar{b}_{i}(y)\bar{b}^{j}(y) = A^{\alpha}_{\beta}(x)\bar{a}_{\alpha}(x)\bar{a}^{\beta}(x) \tag{A29}$$

Substitution from equations (A2) and (A10) in equation (A29) gives

$$B^{i}_{j}(y)\bar{b}_{i}(y)\bar{b}^{j}(y) = A^{\alpha}_{\beta}(x) \frac{\partial y^{i}}{\partial x^{\alpha}} \bar{b}_{i}(y) \frac{\partial x^{\beta}}{\partial y^{j}} \bar{b}^{j}(y)$$

Therefore,

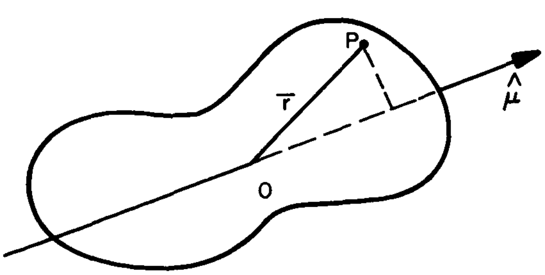
$$B^{i}_{j}(y) = \frac{\partial y^{i}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial y^{j}} A^{\alpha}_{\beta}(x) \tag{A30}$$

The components of mixed bivectors transform according to equation (A30).

Moments and Products of Inertia as the Components of a Bivector

Moments and products of inertia provide a good illustration of the components of a bivector. Moreover, since the moments and products of inertia are the components of a Cartesian bivector, either equation (A25) or (A28) may be used to transform the components. The equality of covariant and contravariant components of Cartesian bivectors follows from equation (10).

Consider the rigid-body shown in sketch (c). Let m be the mass of a particle of the body at the point P, and $\bar{r}$ its position vector relative to the point O, the center of mass of the body. The moment of inertia I of the body about an axis through O, in the direction of the unit vector $\hat{\mu}$ is given by



Sketch (c)

$$I = \Sigma m(\bar{r}x\hat{\mu}) \cdot (\bar{r}x\hat{\mu}) = \hat{\mu} \cdot \bar{\Phi} \cdot \hat{\mu}$$

where Φ is the inertia dyadic which is defined as follows:

$$\Phi = \Sigma m(r^2\bar{I} - \bar{r}\bar{r})$$

and $\overline{I}$ is the identity dyadic; therefore,

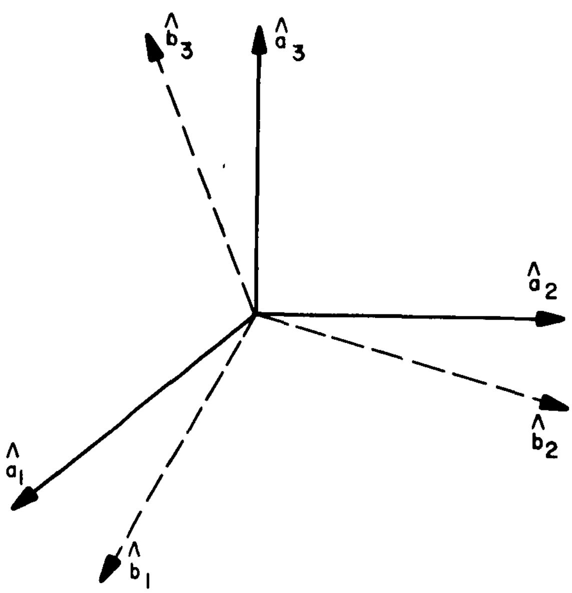
$$\Phi = (I_{x^1x^1}\hat{a}_1\hat{a}_1 - I_{x^1x^2}\hat{a}_1\hat{a}_2 - I_{x^1x^3}\hat{a}_1\hat{a}_3 - I_{x^2x^1}\hat{a}_2\hat{a}_1 + I_{x^2x^2}\hat{a}_2\hat{a}_2 - I_{x^2x^3}\hat{a}_2\hat{a}_3$$

$$- I_{x^3x^1}\hat{a}_3\hat{a}_1 - I_{x^3x^2}\hat{a}_3\hat{a}_2 + I_{x^3x^3}\hat{a}_3\hat{a}_3)$$

The components $I_{x^ix^j}$ are the moments and products of inertia.

In order to take advantage of the tensor notation, the inertia tensor will be denoted by $I^{ij}\hat{a}_i\hat{a}_j$ when it is referred to the x coordinate system, and by $\overline{I}^{ij}\hat{b}_i\hat{b}_j$ when referred to the y coordinate system. See sketch (d). The components of the inertia tensor are related to the components of the inertia dyadic Φ as follows:

$$\left.\begin{aligned} I^{(jj)} &= I_{x^{(j)}x^{(j)}} \\ I^{iy} &= -I_{x^ix^j} \qquad i \neq j \end{aligned}\right\} \tag{A31}$$



Sketch (d)

Consider a transformation of coordinates in which the transformation equations are:

$$\left.\begin{aligned} y^1 &= x^1 \cos\theta - x^3 \sin\theta \\ y^2 &= x^2 \\ y^3 &= x^1 \sin\theta + x^3 \cos\theta \end{aligned}\right\} \tag{A32}$$

These equations represent a rotation about the $\hat{a}_2$ axis, the angle of rotation being θ.

Equation (A25) may be used to find the moments and products of inertia relative to the y coordinate frame. These are:

$$\overline{I}^{ij} = \frac{\partial y^i}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta} I^{\alpha\beta} \tag{A33}$$

When equation (A32) was supplied as input to a digital computer, which was programmed to determine the inertia components $\overline{I}^{ij}$ according to the transformation law (A33), the following output was obtained in Fortran language:

```
FOR I= 1     FOR J=1

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.

    II(1,1)*FMCCOS(THETA)**2.0-II(3,1)*FMCSIN(THETA)*FMCCOS(THETA)-II(1,3)*FMCSIN(THETA)*FMCCOS(THETA)+II(3,3)*FMCSIN(Q00000
    THETA)**2.0$                                                                                                       Q00000

FOR I= 1     FOR J=2

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.
                                                                                                                       Q00000
    II(1,2)*FMCCOS(THETA)-II(3,2)*FMCSIN(THETA)$

FOR I= 1     FOR J=3

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.

    II(1,1)*FMCSIN(THETA)*FMCCOS(THETA)-II(3,1)*FMCSIN(THETA)**2.0+II(1,3)*FMCCOS(THETA)**2.0-II(3,3)*FMCSIN(THETA)*  Q00000
    FMCCOS(THETA)$                                                                                                     Q00000

FOR I= 2     FOR J=1

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.
                                                                                                                       Q00000
    II(2,1)*FMCCOS(THETA)-II(2,3)*FMCSIN(THETA)$

FOR I= 2     FOR J=2

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.
                                                                                                                       Q00000
    II(2,2)$

FOR I= 2     FOR J=3

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.
                                                                                                                       Q00000
    II(2,1)*FMCSIN(THETA)+II(2,3)*FMCCOS(THETA)$

FOR I= 3     FOR J=1

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.

    II(1,1)*FMCSIN(THETA)*FMCCOS(THETA)+II(3,1)*FMCCOS(THETA)**2.0-II(1,3)*FMCSIN(THETA)**2.0-II(3,3)*FMCSIN(THETA)*  Q00000
    FMCCOS(THETA)$                                                                                                     Q00000

FOR I= 3     FOR J=2

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.
                                                                                                                       Q00000
    II(1,2)*FMCSIN(THETA)+II(3,2)*FMCCOS(THETA)$

FOR I= 3     FOR J=3

THE EXPRESSION OUTPUT FOR IBARIJ IS GIVEN BELOW.

    II(1,1)*FMCSIN(THETA)**2.0+II(3,1)*FMCSIN(THETA)*FMCCOS(THETA)+II(1,3)*FMCSIN(THETA)*FMCCOS(THETA)+II(3,3)*FMCCOS(Q00000
    THETA)**2.0$                                                                                                       Q00000
```

In terms of conventional mathematical symbolism, these equations assume the following form:

$$I_{y_1y_1} = C^2\theta I_{x_1x_1} + C\theta S\theta I_{x_1x_3} + C\theta S\theta I_{x_3x_1} + S^2\theta I_{x_3x_3}$$

$$I_{y_1y_2} = -C\theta I_{x_1x_2} + S\theta I_{x_3x_2} = I_{y_2y_1}$$

$$I_{y_1y_3} = C\theta S\theta I_{x_1x_1} - C^2\theta I_{x_1x_3} + S^2\theta I_{x_3x_1} - S\theta C\theta I_{x_3x_3} = I_{y_3y_1}$$

$$I_{y_2y_2} = I_{x_2x_2}$$

$$I_{y_2y_3} = -S\theta I_{x_2x_1} - C\theta I_{x_2x_3} = I_{y_3y_2}$$

$$I_{y_3y_3} = S^2\theta I_{x_1x_1} - S\theta C\theta I_{x_1x_3} - C\theta S\theta I_{x_3x_1} + C^2\theta I_{x_3x_3}$$

where

$$C\theta = \cos\theta$$

and

$$S\theta = \sin\theta$$

APPENDIX B

PHYSICAL COMPONENTS

The transformation from covariant to contravariant components and vice versa was discussed in appendix A. This appendix is concerned with the transformation from covariant and contravariant components to physical components and vice versa (ref. 11).

It frequently happens that an analysis can be performed and the results obtained, without reference to physical components. However, sometimes a quantity, such as a force, is known only in terms of its physical components. In this case, the transformation from physical components to tensor components must be determined. The appropriate transformations may be obtained as follows: From equation (18)

$$\bar{a}_i \cdot \bar{a}_j = g_{ij}$$

therefore

$$\bar{a}_i \cdot \bar{a}_i = g_{(ii)}$$

Let

$$\bar{a}_i = \alpha_i \hat{a}_i$$

where α_i as a scalar magnitude and $\hat{a}_i$ is a unit vector. With this notation

$$\bar{a}_i \cdot \bar{a}_i = (\alpha_i)^2 = g_{(ii)}$$

Therefore,

$$\alpha_i = \sqrt{g_{(ii)}}$$

that is,

$$\bar{a}_i = \sqrt{g_{(ii)}}\ \hat{a}_i \tag{B1}$$

where the parentheses imply suspension of the summation convention. Hence, if A^i are the contravariant tensor components of a vector $\bar{A}$, and if $\mathcal{A}^i$ are the corresponding physical components, then

$$A^i \bar{a}_i = \left(\sqrt{g_{(ii)}} A^i\right) \hat{a}_i = \mathcal{A}^i \hat{a}_i$$

therefore

$$\mathcal{A}^i = \sqrt{g_{(ii)}}\ A^i \tag{B2}$$

Likewise, from equation (19),

$$\bar{a}^i \cdot \bar{a}^j = g^{ij}$$

Let

$$\bar{a}^i = \beta^i \hat{a}^i$$

where β^i is a scalar magnitude and $\hat{a}^i$ is a unit reciprocal base vector. Therefore,

$$\bar{a}^i \cdot \bar{a}^i = g^{(ii)} = (\beta^i)^2$$

therefore

$$\beta^i = \sqrt{g^{(ii)}}$$

therefore

$$\bar{a}^i = \sqrt{g^{(ii)}}\ \hat{a}^i \tag{B3}$$

Hence, if A_i are the covariant components of a vector $\bar{A}$, and if $\mathcal{A}_i$ are the corresponding physical components

$$A_i\bar{a}^i = \left(\sqrt{g^{(ii)}}\ A_i\right) \hat{a}^i = \mathcal{A}_i\hat{a}^i$$

therefore

$$\mathcal{A}_i = \sqrt{g^{(ii)}}\ A_i$$

Moreover, if the coordinate system is orthogonal, the physical components can be expressed in the following alternative forms

$$\mathcal{A}_i = \sqrt{g^{(ii)}}\ A_i = \frac{1}{\sqrt{g_{(ii)}}}\ A_i \tag{B4}$$

Equation (A18) may be used to show that $A_i = A^i$ in orthogonal coordinate systems. From equations (B4) and (A18)

$$\mathcal{A}_i = \frac{1}{\sqrt{g_{(ii)}}}\ A_i = \frac{1}{\sqrt{g_{(ii)}}}\left[g_{(ii)}A^i\right] = \sqrt{g_{(ii)}}\ A^i = \mathcal{A}^i$$

That is

$$\mathcal{A}_i = \mathcal{A}^i$$

That $\mathcal{A}_i \neq \mathcal{A}^i$ in nonorthogonal coordinate systems may be seen as follows:

$$\mathcal{A}_i = \sqrt{g^{(ii)}}\, A_i = \sqrt{g^{(ii)}}\, g_{ij} A^j$$

Therefore, in this case,

$$\mathcal{A}_i \neq \sqrt{g_{(ii)}}\, A^i = \mathcal{A}^i$$

The fact that $\mathcal{A}^i \neq \mathcal{A}_i$ in nonorthogonal coordinate systems is a consequence of the relation

$$A_i = g_{ij} A^j$$

APPENDIX C

TRANSFORMATION FORMULAS FOR COVARIANT DERIVATIVES

The transformation from covariant tensor components to contravariant tensor components and vice versa, was discussed in appendix A. The relationship between physical components and tensor components was derived in appendix B. In this appendix, it will be shown that the covariant derivative of a covariant vector may be obtained from the covariant derivative of a contravariant vector and vice versa. The method is analogous to the procedure of raising and lowering of indices, which was discussed in appendix A.

From equations (34) and (46)

$$\left(\frac{\partial A^i}{\partial x^k} + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^j\right)\bar{a}_i = \left(\frac{\partial A_i}{\partial x^k} - \begin{Bmatrix} j \\ ik \end{Bmatrix} A_j\right)\bar{a}^i \tag{C1}$$

that is

$$A_{i,k}\bar{a}^i = A_{j,k}\bar{a}^j = A^i_{,k}\bar{a}_i \tag{C2}$$

From equation (A19)

$$g_{ij}\bar{a}^j = \bar{a}_i \tag{C3}$$

Substitution from (C3) in (C2) gives

$$A^i_{,k}(g_{ij}\bar{a}^j) = A_{j,k}\bar{a}^j$$

therefore

$$g_{ij}A^i_{,k} = A_{j,k} \tag{C4}$$

that is

$$g_{ij}\left(\frac{\partial A^i}{\partial x^k} + \begin{Bmatrix} i \\ \beta k \end{Bmatrix} A^\beta\right) = \frac{\partial A_j}{\partial x^k} - \begin{Bmatrix} \alpha \\ jk \end{Bmatrix} A_\alpha \tag{C5}$$

Similarly, from equation (A22)

$$g^{ij}\bar{a}_i = \bar{a}^j \tag{C6}$$

Substitution from equation (C6) in equation (C2) gives

$$A_{j,k}(g^{ij}\bar{a}_i) = A^i_{,k}\bar{a}_i$$

Therefore,

$$A^{i}_{,k} = g^{ij} A_{j,k} \tag{C7}$$

that is,

$$\left(\frac{\partial A^{i}}{\partial x^{k}} + \left\{ \begin{matrix} i \\ \beta k \end{matrix} \right\} A^{\beta}\right) = g^{ij}\left(\frac{\partial A_{j}}{\partial x^{k}} - \left\{ \begin{matrix} \alpha \\ jk \end{matrix} \right\} A_{\alpha}\right) \tag{C8}$$

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NASA-Langley, 1967 —— 8 A-2519

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