NASA CONTRACTOR REPORT

NASA CR-765

NASA CR-765

N67-23719

(ACCESSION NUMBER) (THRU)

47 1

(PAGES) (CODE)

CR-765 19

(NASA CR OR TMX OR AD NUMBER) (CATEGORY)

FACILITY FORM 602

PERIODIC SOLUTIONS FOR THE RESTRICTED THREE-BODY PROBLEM OF CELESTIAL MECHANICS

by P. Lanzano

Prepared by

NORTH AMERICAN AVIATION, INC.

Downey, Calif.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1967

NASA CR-765

PERIODIC SOLUTIONS FOR THE RESTRICTED THREE-BODY PROBLEM OF CELESTIAL MECHANICS

By P. Lanzano

Prepared under Contract No. NASw-1309 by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

FOREWORD

This document represents the final technical report on Contract NASw-1309 entitled "Periodic Solutions for the Restricted Three-Body Problem of Celestial Mechanics." Dr. Paolo Lanzano has been the principal investigator. This report consists of two sections, corresponding to the two tasks of said contract.

Two technical papers have stemmed from this investigation. Both papers have been authored by the principal investigator and will shortly appear in Icarus--International Journal of the Solar System. One paper is entitled "Contributions to the Elliptic Restricted Three-Body Problem," the other paper is entitled "Stability of a Class of Periodic Orbits in the Restricted Three-Body Problem."

SUMMARY

Certain basic questions pertaining to the Restricted Three-Body Problem of Celestial Mechanics have been considered, advancing the knowledge in the field.

The first section of the report deals with a number of intrinsic properties for the circular problem. It has been possible to express the curvature of the generic orbit in terms of the velocity field and the angle that the orbit makes with the zero-velocity curves, obtainable from the Jacobi integral. An application of this intrinsic representation has been given for a periodic orbit, whereby the period of the orbit can be expressed as a double integral extended to the area bounded by the closed curve. Also, the existence of periodic orbits, which can be considered as an infinitesimal isoenergetic displacement of a given periodic orbit, has been proved by ascertaining periodic solutions to a Mathieu equation.

The second section of the report deals with a series representation of periodic solutions about a Lagrangian Triangular point. Two families of periodic solutions were obtained. The synodic coordinates of the orbits have been expressed as trigonometric series of time, the coefficients of the various harmonics depending on a real parameter related to the initial conditions.

TABLE OF CONTENTS

INTRINSIC PROPERTIES OF PERIODIC SOLUTIONS FOR THE CIRCULAR PROBLEM

This section deals with the second task of the contract. The object has been to study intrinsic properties of periodic solutions and of the associated zero-velocity contours, obtainable from the Jacobi integral. By intrinsic properties, we mean those properties which do not depend on the choice of the reference system, e.g., for a plane curve, the concept of curvature is an intrinsic one, because one would obtain the same invariant quantity associated with a given point of the curve, independently of the chosen reference.

We feel that the objectives of this task have been achieved, within the context of the proposed research program. This report will contain only a short description of the results. A paper furnishing more details on this study and which has been entitled "Stability of a Class of Periodic Orbits in the Restricted Three-Body Problem" has been accepted for publication in Icarus.

After a short introduction, describing the equations of motion and setting forth certain definitions and notations required in the sequel, we plan to discuss:

(a) an analytic representation of the zero-velocity contours,

(b) an intrinsic representation of general orbits which gives rise to an integral application for periodic orbits,

(c) the existence of periodic, infinitesimal displacements from a given periodic orbit.

We feel that the last mentioned subject is a very promising one and that it should be investigated more deeply in connection with a study on stability of orbits.

Introduction

We consider the circular case of the Restricted Three-Body problem and we limit ourselves to the planar motion. In other words, we study the motion of the third body P_3, of infinitesimal mass, in the plane defined by the circular orbits of the two massive bodies P_1, P_2 about the centroid O of the system. In the usual synodic reference $(O; x, y)$, rotating uniformly about the centroid, the equations of motion for the third body take the form

$$\begin{aligned} \ddot{x} - 2\dot{y} &= U_x \\ \ddot{y} + 2\dot{x} &= U_y \end{aligned} \tag{1}$$

Dots, as usual, denote derivatives with respect to time and letters, used as subscripts, represent partial derivatives with respect to those variables. Here we have

$$\begin{aligned} U(x,y) &\equiv \frac{1}{2}(x^2+y^2) + \sum_{1}^{2}{}_j (\mu_j / r_j) \\ &\equiv \frac{1}{2} \sum_{1}^{2}{}_j \; \mu_j \left(r_j^2 + \frac{2}{r_j} \right) - \frac{1}{2}\mu_1 \mu_2 \end{aligned} \tag{2}$$

$$(j = 1, 2)$$

μ_1, μ_2 being the masses of the primaries with

$$\mu_1 + \mu_2 = 1$$

and r_1, r_2 are the distances of P_3 from P_1 and P_2 respectively. The relationship between them is as follows

$$r_2^2 = 1 - 2 r_1 \cos \theta + r_1^2 \qquad (3)$$

where θ is the angle measured from the positive direction of the synodic x-axis to the line joining P_1 with P_3. Let us recall that the coordinates of P_1, P_2 are $(-\mu_2, 0)$ and $(\mu_1, 0)$, respectively so that

$$\text{distance } (P_1 P_2) \equiv 1 .$$

Multiplying the first of equations (1) by $\dot{x}$, the second equation by $\dot{y}$ and adding, we obtain

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x} U_x + \dot{y} U_y$$

which yields the first integral of motion

$$V^2 \equiv \dot{x}^2 + \dot{y}^2 = 2 U(x, y) - \text{constant} \qquad (4)$$

This energy integral, known as the Jacobi integral, relates the magnitude of the synodic velocity vector with the position coordinates of the third body.

Analytic representation of the zero-velocity curves

The zero-velocity curves can be obtained from formula (4) by letting $V = 0$. In other words, they constitute the locus of the positions that the third body can reach, in the synodic plane, with zero-velocity. Using (2) and (4), their equation can be written as follows

$$\sum_{1}^{2}{}_{j} \mu_j \left(r_j^2 + \frac{2}{r_j} \right) = J \qquad (5)$$

where J, the so-called Jacobi constant, results from the agglomeration of the two constant terms, appearing in the previous formulae.

The geometry of this family of curves varies according to the value of the Jacobi constant. For large values of J, the locus consists of three closed curves: two separate oval-shaped curves, each surrounding one of the primaries, and an almost-circular curve, the so-called curtain, surrounding both ovals. Since for the motion of the third body it must be $V \geqslant 0$, it is easily seen that the allowable region of motion for the third body is within each oval and in the domain exterior to the curtain.

With diminishing values of J, the curtain shrinks in size whereas the two ovals become increasingly elongated toward each other, along the x-axis of the synodic reference, until for a critical value of J, depending on the mass ratio of the primaries, both ovals will develop a common point (on the x-axis between the primaries) which is a singular point for the resulting curve. There are now only two components of the locus. For smaller values of J, the two ovals coalesce giving rise to the so-called envelope; this looks like a dumbbell shaped curve, increasing in size with decreasing values of J but still separate from the curtain.

Two other singular points develop at consecutive stages on the x-axis, one on each side of the primaries, whereby contact will occur between envelope and curtain. Subsequently, the envelope of the primaries will coalesce with the curtain, consecutively on each side; thus, we see that the number of components goes from two to one and then back to two. Eventually the latter two components shrink in size to the Lagrangian points. For values of J smaller than this limiting value no real point can exist on the locus.

We plan to obtain a suitable analytic representation for some components of these zero-velocity curves. For this purpose, let us consider the expansion of $1/r_2$ in powers of r_1, by means of Legendre polynomials. From (3) we shall write

$$\frac{1}{r_2} \equiv \left(1 - 2 r_1 \cos\theta + r_1^2 \right)^{-1/2}$$

$$\equiv \sum_0^{\infty}{}_j \; (r_1)^j \; P_j \; (\cos\theta) .$$

Upon substituting this series expansion within (5) and performing appropriate simplifications, we get

$$r_1 = r_0 \left\{ 1 + \left[\frac{1}{2\mu_1} + \mu P_2(\alpha) \right] r_1^3 + \mu \sum_{3}^{\infty} {}_j \, (r_1)^{j+1} P_j(\alpha) \right\} \tag{6}$$

Here we have set

$$\mu = \frac{\mu_2}{\mu_1} \; ; \qquad \alpha = \cos\theta \; ; \qquad r_0 = \frac{2\mu_1}{J - 3\mu_2} .$$

Since the fundamental series expansion we have used is convergent, and therefore valid, only when $r_1 < 1$, it is to be understood that only certain portions of the curve can thus be represented. Also, both μ and r_0 shall be considered small parameters, which means not only that the ratio of the primary masses must be small but also that certain values of the Jacobi constant might have to be excluded.

We propose to find a solution for (6) of the following form

$$r_1 = r_0 \left\{ 1 + r_0^3 \left[\sum_{0}^{\infty} {}_j \, A_j(\alpha) \, r_0^j \right] \right\}$$

which is an infinite series in r_0 and where the coefficients A_j, to be determined, are functions of α. One finds at once

$$A_0(\alpha) = \frac{1}{2\mu_1} + \mu P_2(\alpha)$$

whereas, the remaining unknown functions will be obtained by equating coefficients of equal powers of τ_0, appearing in the left and right-hand sides of the following formula

$$\sum_0^\infty{}_j \, A_j(\alpha)\, \tau_0^j = A_0(\alpha)\left[1 + \tau_0^3 \sum_0^\infty{}_k \, A_k(\alpha)\, \tau_0^k\right]^3$$

$$+ \mu \sum_3^\infty{}_j \, P_j(\alpha)\, \tau_0^{j-2}\left[1 + \tau_0^3 \sum_0^\infty{}_k \, A_k(\alpha)\, \tau_0^k\right]^{j+1}.$$

Let us introduce the following notation

$$\left(\sum_0^\infty{}_j \, A_j\, \tau_0^j\right)^k \equiv \sum_0^\infty{}_j \, A_j^{(k)}\, \tau_0^j$$

whereby $A_n^{(k)}$ represents the coefficient of the n-th power of τ_0 in the expansion of the k-th power (in the sense of Cauchy) of the original series. It is easy to realize that $A_n^{(k)}$ is a polynomial in the A_j's with $j \leq n$.

Making use of this notation, we have been able to establish the following general recurrent relationship

$$A_n = A_0\left(3\, A^{(1)}_{n-3} + 3\, A^{(2)}_{n-6} + A^{(3)}_{n-9}\right)$$

$$+ \mu\left\{P_{n+2} + \sum_3^{n-1}{}_j \, P_j\left[\sum_1{}_k \binom{j+1}{k} A^{(k)}_{n-(j-2)-3k}\right]\right\}. \tag{7}$$

Here both the P's and A's are functions of α, and the last summation extends to that value of k satisfying, for given n and j, whichever of the two inequalities

$$k \leq j+1 \quad ; \quad n-(j-2)-3k \geq 0$$

occurs first. The other symbol in the above formula is a binomial coefficient.

Using (7), it is easy to express the first few coefficients of the series expansion. We list them here:

$$A_1 \equiv \mu P_3 \quad ; \qquad A_2 \equiv \mu P_4$$

$$A_3 \equiv 3 A_0^2 + \mu P_5$$

$$A_4 \equiv \mu \left(P_6 + 7 A_0 P_3 \right)$$

$$A_5 \equiv \mu \left(P_7 + 8 A_0 P_4 + 4 \mu P_3^2 \right)$$

$$A_6 \equiv 12 A_0^3 + \mu \left(P_8 + 9 A_0 P_5 + 9 \mu P_3 P_4 \right)$$

$$A_7 \equiv \mu \Big[P_9 + 10 A_0 P_6 + 45 A_0^2 P_3$$

$$+ 5\mu \left(P_4^2 + 2 P_3 P_5 \right) \Big]$$

$$A_8 \equiv \mu \left[P_{10} + 11 A_0 \left(P_7 + 5 \mu P_3^2 \right) \right.$$

$$\left. + 55 A_0^2 P_4 + 11 \mu \left(P_3 P_6 + P_4 P_5 \right) \right] .$$

It is understood that in the above formulae both A_0 and the P_j's (Legendre polynomials) are functions of α. We have thus achieved an explicit polar representation of the zero-velocity curves. The convergence of this series expansion depends primarily on the magnitude of μ and r_0.

For recent works pertaining to the zero-velocity curves, mention should be made to Kopal (1959) and Lanzano (1960). Kopal deals with these curves in discussing the Roche limit of close binary systems. This investigator's interest in these families of curves has been from the point of view of guidance of space vehicles along collision orbits.

Intrinsic representation of orbits

We plan to exhibit a characterization of the generic orbit of the circular, restricted problem which is independent of the particular coordinate system employed. In geometric parlance, this is referred to as an intrinsic representation of a curve. For this purpose, we begin by eliminating the time variable and by reducing the two equations of motion, which contain derivatives with respect to time, into a single differential equation relating the unknown function $y(x)$ and its derivatives $y'(x)$, $y''(x)$ with respect to x. During this process we must also fix a value for the Jacobi constant: This isoenergetic reduction limits the number of parameters upon which the totality of orbits in consideration depends. In other words, we shall consider orbits corresponding to the same value of the energy constant. As a further step, we shall identify the intrinsic character of the various terms appearing in the resulting differential equation.

We denote by primes the derivatives with respect to x. From (1) we have

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = \dot{x}U_y - \dot{y}U_x - 2\left(\dot{x}^2 + \dot{y}^2\right)$$

and from (4), since $y' = \dot{y}/\dot{x}$ we get

$$\dot{x}^2\left[1 + (y')^2\right] = 2U - J.$$

Consequently, we can write

$$y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

$$= \frac{1}{\dot{x}^2}\left(U_y - U_x y'\right) - \frac{2}{\dot{x}}\left[1 + (y')^2\right].$$

Upon elimination of $\dot{x}$ through previous formulae and after simplification we get

$$\frac{y''}{\left[1+(y')^2\right]^{3/2}} = -\frac{2}{(2U-J)^{1/2}} + \frac{U_y - U_x y'}{(2U-J)\left[1+(y')^2\right]^{1/2}}.$$

The expression appearing in the left side in the previous formula is the curvature of the orbit. Let us recall, for future reference, that the curvature of a plane curve is defined as

$$k \equiv d\beta / ds$$

where β is the angle that the tangent line to the curve makes with any fixed line in the plane of the curve and s is the arclength. Since this fixed line can coincide with a tangent to another point on the curve, the intrinsic character of this concept is therefore evident.

Using the fact that

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + (y')^2$$

we have

$$\left[1 + (y')^2\right]^{-1/2} = \frac{dx}{ds} = \cos\beta$$

$$y'\left[1 + (y')^2\right]^{-1/2} = \frac{dy}{dx}\,\frac{dx}{ds} = \frac{dy}{ds} = \sin\beta\,.$$

We also know that

$$V^2 = 2U - J$$

Therefore, the previous relationship can be written as follows

$$k = -\frac{2}{V} + \frac{1}{V^2}\left(U_y \cos\beta - U_x \sin\beta\right) \quad (8)$$

This is the sought intrinsic representation of the general orbit. Clearly, the last term in (8) can be written by means of the gradient

$$\left(U_x^2 + U_y^2\right)^{1/2}$$

and of the angle γ between the tangent to the orbit and the normal to the zero-velocity curves

$$U(x,y) = \text{constant.}$$

A simple calculation will show that

$$k = -\frac{2}{V} + \frac{1}{V^2}\left(U_x^2 + U_y^2\right)^{1/2} \sin\gamma .$$

Another way of modifying the representation (8) is as follows: differentiating $V^2 = 2U - J$ with respect to s, we easily get

$$V \frac{dV}{ds} = \left(U_x^2 + U_y^2\right)^{1/2} \cos\gamma$$

which can be used to obtain

$$k = -\frac{2}{V} + \frac{1}{V}\frac{dV}{ds} \tan\gamma .$$

Here the curvature of the orbit is expressed in terms of the velocity, the tangential acceleration $V\,dV/ds$ and the angle with the zero-velocity curves. Let us remark, at this point, that these curves stem from the very nature of the problem and ought to be considered intrinsically related to it.

We shall make use of the previously obtained intrinsic representation to investigate a property of periodic orbits. This is an integral property and will imply the application of Gauss theorem pertaining to the equivalence between the integral extended to the area bounded by said curve.

Let C be a periodic orbit, in the synodic plane, having period τ. Suppose, for sake of simplicity, that C is all contained within the permissible domain of motion, without any common boundary with the zero-velocity curve. Also, that

C has no double points (i.e. loops) so that, when described counterclockwise, its tangent line sweeps through an angle of 2π radians. If we multiply (8) by V and integrate with respect to t, between zero and τ, we get

$$\int_0^\tau V \frac{d\beta}{ds}\, dt = -2\tau + \int_0^\tau \frac{1}{V}\left(U_y \cos\beta - U_x \sin\beta\right) dt.$$

The first integral is simply

$$\beta(\tau) - \beta(0)$$

which equals 2π because of the conditions imposed on C. The integral which appears in the right-hand side can be transformed into a line integral, with respect to the arc-length, along the contour of C; its integrand can be easily seen to be

$$-\frac{1}{V^2}\frac{dU}{dq}$$

where q is the exterior normal to the contour. On the other hand, from

$$V^2 = 2U - J$$

we have

$$V\frac{dV}{dq} = \frac{dU}{dq}$$

therefore

$$\frac{1}{V^2}\frac{dU}{dq} = \frac{1}{V}\frac{dV}{dq} = \frac{d}{dq}(\ln V).$$

Using Gauss theorem, we can transform the line integral into a double integral, extended to the area bounded by the curve C, of the function $\Delta^2 (\ln V)$, where

$$\Delta^2 F \equiv F_{xx} + F_{yy}$$

is the Laplace operator. If the functions in consideration do not have any singularities in the domain bounded by C, then the period τ of the orbit can be represented as follows

$$2\tau + 2\pi = -I$$

where

$$I \equiv \iint \Delta^2 \ln V(x, y; J)\, dx\, dy$$

is a double integral extended to the area bounded by C.

Certain modifications are required if the path of the periodic orbits surrounds one or both of the massive bodies. In fact, the function $U(x, y)$ becomes infinite at the positions occupied by these masses. The procedure is to describe a circle of finite radius about each of these points and consider the above integral in the region of regularity of the function, finally to take the limit of the integral as the radius of the circle approaches zero.

As a parenthetical remark, we wish to mention here that the above procedure, which has led to an intrinsic property for periodic solutions, is akin to the one employed in differential geometry for establishing the well known Gauss-Bonnet formula. In both cases, the use of Gauss integral theorem is the key feature. We are dealing here with a surface S upon which a curvilinear coordinate system has been established. Considering a closed curve C on S, the fundamental quantities involved are the geodesic curvature k_g of C and the Gaussian curvature K of the surface in the domain bounded by C. The geodesic curvature of a curve, belonging to a surface, is defined by means of the geodesics tangent to its points in the same manner as the usual curvature is defined for a plane curve by means of its tangent lines.

Using Liouville expression for the geodesic curvature, see Struik (1950) pp. 154-156, the Gauss-Bonnet formula can be written as follows

$$\int_C k_g \, ds = 2\pi - \iint_S K \, dS$$

where the double integral is extended to that portion of the surface S interior to the curve C.

Periodic displacement of a periodic orbit

Let us suppose we know a particular solution $\bar{x}(t), \bar{y}(t)$ of the restricted circular problem. Let us consider the curve defined by

$$\bar{x}(t) + \varepsilon \xi(t) \quad ; \quad \bar{y}(t) + \varepsilon \eta(t)$$

where ε is a small parameter. In order that the latter curve be a solution of the same system of equations (1), with an error of the order higher than ε, it must be

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi\, U_{xx}(\bar{x},\bar{y}) + \eta\, U_{xy}(\bar{x},\bar{y}) \\ \ddot{\eta} + 2\dot{\xi} &= \xi\, U_{xy}(\bar{x},\bar{y}) + \eta\, U_{yy}(\bar{x},\bar{y}) \end{aligned} \tag{9}$$

where the second derivatives are evaluated along the given periodic solution. To the same order of approximation, the Jacobi integral gives rise to a first integral for the differential system (9):

$$\dot{\xi}\,\dot{\bar{x}} + \dot{\eta}\,\dot{\bar{y}} = \xi\, U_x(\bar{x},\bar{y}) + \eta\, U_y(\bar{x},\bar{y}) . \tag{10}$$

Since no arbitrary constant appears in (10), it means that the original value of the Jacobi constant has not been modified by any infinitesimal amount. A solution of (9) satisfying (10) is called an isoenergetic infinitesimal displacement of the original solution.

There exists a one-to-one correspondence between points of the original orbit and its infinitesimal displacement, corresponding points (P, P') belonging to the same value of t. It is convenient to refer the point P' to the intrinsic reference constituted by the tangent and normal to the periodic plane curve at the corresponding point P. Let (p, q) denote the coordinates of P' in this reference frame.

It is the purpose of this study to investigate under which circumstances the infinitesimal displacement $(\bar{x} + \varepsilon\xi;\ \bar{y} + \varepsilon\eta)$ is also a periodic solution. Since both $\bar{x}(t)$ and $\bar{y}(t)$ are periodic functions, the crux of the matter consists in finding periodic solutions for the differential system (9) which has periodic coefficients. The transformation of (9) into a differential system pertaining to (p, q) can be achieved by simple operations of elimination and differentiation, see e.g. Smart (1953) and Lanzano (1961). The new system can be written

$$\ddot{q} + Q(t)\, q = 0$$

$$\frac{d}{dt}\left(\frac{p}{2V}\right) = \frac{q}{V}\, P(t)$$

where

$$P(t) \equiv 1 + \frac{1}{V^2}\left(\dot{\bar{x}}\,\ddot{\bar{y}} - \ddot{\bar{x}}\,\dot{\bar{y}}\right)$$

$$Q(t) \equiv \frac{\ddot{V}}{V} + 2P^2(t) + 2 - U_{xx}(\bar{x}, \bar{y}) - U_{yy}(\bar{x}, \bar{y}).$$

It is to be understood that the velocity V, given by the Jacobi integral, and all other quantities involved, have been evaluated along the given periodic solution.

$Q(t)$ is a periodic function and, in most instances, can be expressed as a trigonometric series

$$Q(t) \equiv Q_0 - 2 \sum_{1}^{\infty}{}_k \, Q_k \cos(2kt)$$

where the coefficients Q_k $(k = 0, 1, \ldots)$ initiate with terms in the k-th power of a small parameter. Neglecting all the coefficients but Q_0 and Q_1, we are therefore led to seeking periodic solutions for the Mathieu equation

$$\ddot{q} + \left[Q_0 - 2\, Q_1 \cos(2t) \right] q = 0 \, . \qquad (11)$$

In other words, we must study how to represent a periodic solution of (11) and at the same time determine the relationship that must hold between the two constants Q_0, Q_1 for the existence of a periodic solution. This relationship will represent an intrinsic characteristic of the original periodic orbit.

The general solution of (11) depends upon two parameters $(C_0; \delta)$ and can be written as follows

$$q(t;\, C_0, \delta) \equiv \sum_{-\infty}^{\infty}{}_k \, C_k(Q_1) \cos\left[(c + 2k)t - \delta\right] \qquad (12)$$

The coefficients C_k of this trigonometric series are infinite power series in Q_1. c is a constant to be determined; obviously, for a periodic solution, c must be a rational number, not necessarily an integer.

We consider Q_1 to be a small parameter; we shall also change the order of the various terms within (12) and rearrange it from a trigonometric series into a power series in Q_1. This latter form will be used to ascertain the coefficients appearing in (12). Let us remark, at this stage, that when $Q_1 = 0$, a particular solution of (11) is given by

$$\cos\left(\sqrt{Q_0}\, t\right)$$

so that $Q_0 = c^2$. This suggests that, when $Q_1 \neq 0$, one should consider a solution of equation (11) of the following form

$$\begin{cases} q(t;1,0) \equiv \cos(ct) + \sum_{1}^{\infty}{}_k\, Q_1^k\, E_k(t) \\ Q_0 \equiv c^2 + \sum_{1}^{\infty}{}_k\, Q_1^k\, F_k \end{cases} \tag{13}$$

Here the E_k's are functions of time and will be ascertained in the sequel; the F_k's are constants which depend on the parameter c. Upon replacing the two series expansions (13) within the Mathieu equation (11), we obtain

$$
\begin{cases}
\ddot{E}_1 + c^2 E_1 = \cos[(c-2)t] + \cos[(c+2)t] \\
\qquad\qquad\qquad - F_1 \cos(ct) & (14) \\
\ddot{E}_k + c^2 E_k = 2 E_{k-1} \cos(2t) - F_k \cos(ct) \\
- \sum_{1}^{k-1}{}_j \; F_j \, E_{k-j} \quad ; \quad (k = 2, 3, \cdots) \; .
\end{cases}
$$

This system of linear differential equations allows the recursive determination of the various functions $E_k(t)$ in terms of the E_r's with $r = 1, \ldots, k-2, k-1$. The F_k's can be determined by imposing the periodicity condition, i.e., that no secular terms shall appear in the solution of (14). It is easy to realize that this can be achieved by equating to zero the coefficient of $\cos(ct)$, after having substituted the lower order approximations within the right-hand side of (14).

It can be seen at once that $E_1(t)$ consists of terms in

$$\cos\left[(c \pm 2)t\right]$$

and that $F_1 = 0$. Proceeding recursively, we find that $E_k(t)$ contains terms in .

$$\cos\left[(c \pm 2r)t\right]$$

with $r = k, k-2, k-4, \ldots$. Only when k is even can we get a term in $\cos(ct)$, so that, in this case, it will be $F_k \neq 0$. On the other hand, if k is an odd number, then $F_k = 0$.

Because of this remark, equation (14) can be rewritten as follows:

$$\ddot{E}_k + c^2 E_k = 2 E_{k-1} \cos(2t) - F_k \cos(ct)$$
$$- F_2 E_{k-2} - F_4 E_{k-4} - \cdots \; ; \quad (k = 2, 3, \ldots)$$

whence we see that, when k is even, F_k is equal to the sum of the two coefficients of

$$\cos\left[(c \pm 2)t\right]$$

appearing in E_{k-1}.

Since a particular solution of the differential equation

$$\ddot{y} + c^2 y = A \cos\left[(c + 2k)t\right]$$

is

$$y = B(c;k) \cos\left[(c+2k)t\right]$$

where

$$B(c;k) \equiv - \frac{A}{4k(c+k)} , \qquad (15)$$

then one can see that the coefficient of

$$\cos\left[(c+2k)t\right]$$

in $E_k(t)$ is given by

$$\prod_{1}^{k}{}_{r} \frac{-1}{4r(c+r)} = \frac{(-1)^k \; \Gamma(c+1)}{2^{2k}\,(k!)\,\Gamma(c+k+1)}$$

where $\Gamma(x)$ is the gamma function. Another remark that can be made by considering formula (15) is that if we change k into $-k$ in the original differential equation, the coefficient of its particular solution becomes $B(-c;k)$.

It is easy to realize that the C_k's are power series in Q_1 starting with Q_1^k. We give below the leading terms:

$$\frac{C_1}{C_0} = -\frac{Q_1}{4(c+1)} - \frac{c^2+4c+7}{128(c+1)^3(c+2)(c-1)}\,Q_1^3$$

$$-\frac{c^6+5c^5+8c^4+8c^3+47c^2-13c+232}{3072(c+1)^5(c-1)^3(c+2)(c+3)(c-2)}\,Q_1^5+\cdots$$

$$\frac{C_2}{C_0} = \frac{Q_1^2}{32(c+1)(c+2)} +$$

$$\frac{c^2+5c+10}{768(c+1)^3(c-1)(c+2)(c+3)}\,Q_1^4+\cdots$$

$$\frac{C_3}{C_0} = - \frac{Q_1^3}{384\,(c+1)\,(c+2)\,(c+3)}$$

$$- \frac{c^2 + 6c + 13}{8192\,(c+1)^3\,(c-1)\,(c+2)\,(c+3)\,(c+4)}\, Q_1^5 + \cdots$$

$$\frac{C_k}{C_0} = \frac{(-1)^k\, Q_1^k\, \Gamma(c+1)}{2^{2k}\,(k!)\,\Gamma(c+k+1)} + \cdots$$

the powers of Q_1 decrease by two units.

Also,

$$Q_0 = c^2 + \frac{Q_1^2}{2(c^2-1)} + \frac{5c^2+7}{32\,(c^2-1)^3\,(c^2-4)}\, Q_1^4$$

$$+ \frac{9c^4 + 58c^2 + 29}{64\,(c^2-1)^5\,(c^2-4)\,(c^2-9)}\, Q_1^6 + \cdots . \tag{16}$$

This is the sought relationship between Q_0, Q_1 for periodic solutions.

If Q_1 is given and c has an assigned value (not an integer), then the corresponding Q_0 is easily obtainable. On the other hand, should Q_0 and Q_1 be the known quantities and c the number to be determined, then, no doubt, (16) is not very convenient for that purpose.

To obtain a more suitable formula that yields the value of c in an explicit fashion, we must have recourse to the Floquet theory pertaining to differential equations with periodic coefficients. The property to be exploited is the following: let $q(t; Q_0, Q_1)$ be that particular solution of (11) with initial conditions

$$q(0; Q_0, Q_1) = 1$$

$$\dot{q}(0; Q_0, Q_1) = 0$$

then, it is known, see e.g., Flügge (1956) p. 208 and Meixner (1954) p. 103 that

$$\cos(\pi c) = q(\pi; Q_0, Q_1) .$$

On considering the expansion

$$q(t; Q_0, Q_1) \equiv \sum_{0}^{\infty}{}_{k}\, Q_1^{k}\, q_k(t; Q_0)$$

as a power series of Q_1 and replacing it within (11) we realize that we have

$$\ddot{q}_0 + Q_0 q_0 = 0$$

so that

$$q_0(t; Q_0) \equiv \cos\left(\sqrt{Q_0}\, t\right)$$

and

$$q_0(0) = 1 \quad ; \quad \dot{q}_0(0) = 0 .$$

The higher-order approximations satisfy the system

$$\ddot{q}_k + Q_0 q_k = 2 q_{k-1} \cos(2t)$$

for $k = 1, 2, \ldots,$ with initial conditions

$$q_k(0) = \dot{q}_k(0) = 0 .$$

It is easily seen that the first-order approximation is

$$q_1(t) \equiv -\frac{1}{2} \frac{1}{Q_0 - 1} \cos\left(\sqrt{Q_0}\, t\right)$$

$$-\frac{1}{4} \frac{1}{\sqrt{Q_0} + 1} \cos\left[\left(\sqrt{Q_0} + 2\right)t\right] + \frac{1}{4} \frac{1}{\sqrt{Q_0} - 1} \cos\left[\left(\sqrt{Q_0} - 2\right)t\right]$$

and this shows that

$$q_1(\pi) = 0 .$$

The second-order approximation is of the form

$$q_2(t) \equiv \sum_{-2}^{2} j \; B_{2j} \cos\left[\left(\sqrt{Q_0} + 2j\right)t\right]$$

$$+ \frac{t}{4\sqrt{Q_0}\,(Q_0 - 1)} \sin\left(\sqrt{Q_0}\, t\right)$$

with $\sum_{-2}^{2} j \; B_{2j} = 0,$ consequently one gets

$$q_2(\pi) = \frac{\pi \sin\left(\pi\sqrt{Q_0}\right)}{4\sqrt{Q_0}\,(Q_0 - 1)} \, .$$

Also, by straightforward calculations, one can find

$$q_3(\pi) = 0$$

and

$$q_4(\pi) = \frac{15\,Q_0^2 - 32\,Q_0 + 8}{64\,(Q_0-1)^3\,(Q_0-4)\,Q_0\sqrt{Q_0}} \,\pi \sin\left(\pi\sqrt{Q_0}\right)$$

$$- \frac{\pi^2 \cos\left(\pi\sqrt{Q_0}\right)}{32\,Q_0\,(Q_0-1)^2}$$

The general trend of these approximations, however, has not been investigated. These results give an explicit representation for $\cos(\pi c)$.

PERIODIC SOLUTIONS ABOUT A LAGRANGIAN POINT

The purpose of this task has been to ascertain certain properties of the periodic solutions about the Lagrangian points in the circular, restricted three-body problem. This task is meant to generalize certain results obtained by this author in an earlier investigation and which have appeared in a recent publication, see Lanzano (1965). The results of this generalization will be summarized in what follows.

It is interesting to announce that the procedure in question has lent itself to an application to a more complicated case of the restricted problem, i.e. the elliptic case. This is the case when the two massive bodies describe elliptic orbits about their common centroid. The results pertaining to the elliptic case constitute the material for the paper "Contributions to the Elliptic Restricted Three-Body Problem" which has been accepted for publication in Icarus (International Journal of the Solar System).

We shall describe, therefore, only the results concerning the circular problem which are not covered in the aforementioned paper. For obvious reasons, we have been compelled, in describing this task, to use a notation different from the one employed in the other task.

Canonical transformation

Consider the coordinate system $(L_4; y_1, y_2)$ having the Lagrangian triangular point as its origin and its axes y_1, y_2 parallel to the synodic axes. If we designate by y_3, y_4 the two momenta, the Hamiltonian form of the equations of motion for the third body can be written, using matrix notation, as follows:

$$\dot{y} = J\, H_y \tag{17}$$

Here

$$y \equiv col\,(y_j)$$
$$H_y \equiv col\,(\partial H / \partial y_j)$$

are column vectors. The Hamiltonian is an infinite double series in the y_j's

$$\begin{aligned} H(y_j) \equiv \tfrac{1}{2}(\mu_1\mu_2 - 3) + \tfrac{1}{2}(y_3^2 + y_4^2) + y_2 y_3 \\ - y_1 y_4 + \tfrac{1}{8} y_1^2 + \gamma\, y_1 y_2 - \tfrac{5}{8} y_2^2 - H^*(y_1, y_2) \end{aligned} \tag{18}$$

where

$$\gamma = \frac{3\sqrt{3}}{4}(\mu_1 - \mu_2)$$

and the infinite series H^*, representing the contribution due to third and higher order terms, has been obtained by expressing the spherical harmonics as homogeneous polynomials of the cartesian coordinates. J is a (4×4) skew symmetric matrix, characteristic of Hamiltonian systems, which can be represented as follows

$$J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where 0 and I stand for the zero and unit matrices of size (2×2).

The first fundamental problem which had to be solved consisted in determining a nonsingular (4×4) matrix C, with constant elements, such that the linear transformation

$$y = C z$$

would achieve the objectives of: (a) preserving the Hamiltonian form of the differential system (17), and (b) reducing its linear terms to diagonal form.

The first condition requires that C be a symplectic matrix, i.e. that

$$C^T J C = J$$

where the upper T is used to denote the transpose of a matrix.

The second requirement is equivalent to

$$C^{-1}\, B\, C \equiv \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

where B stands for the matrix of the linear terms, in the original system, obtainable from (18) by differentiation. The previous matrix equation gives rise to the linear systems

$$(B - \lambda_k I)\, C_k = 0 \quad , \quad (k = 1 \text{ to } 4) \tag{19}$$

where the C_k's are the column vectors of C. The eigen-values λ_j, which are the roots of the characteristic equation

$$\mathrm{Det}\,(B - \lambda I) = 0,$$

can be written as

$$2\lambda^2 = -1 \pm \sqrt{\delta}$$

with

$$\delta = 1 - 27\,\mu_1 \mu_2 .$$

We consider only the case when the mass ratio of the primaries is such that the eigenvalues are pure imaginary numbers. We shall designate them as follows

$$\lambda_1 = i\,\delta_1 \qquad \lambda_3 = -\lambda_1$$

$$\lambda_2 = i\,\delta_2 \qquad \lambda_4 = -\lambda_2$$

where i is the imaginary unit and

$$\delta_1 = \left(\frac{1-\sqrt{\delta}}{2}\right)^{1/2} \quad ; \quad \delta_2 = \left(\frac{1+\sqrt{\delta}}{2}\right)^{1/2}$$

are real quantities.

The solution of (19) can be written as

$$c_{1k} = \gamma + 2\lambda_k \quad ; \quad c_{2k} = -\tfrac{3}{4} + \lambda_k^2$$

$$c_{3k} = \tfrac{3}{4} + \gamma\lambda_k + \lambda_k^2 \quad ; \quad c_{4k} = \gamma + \tfrac{5}{4}\lambda_k + \lambda_k^3$$

for $k = 1$ to 4. However, these elements do not constitute a symplectic matrix. Various matrix operations had to be performed on C to ultimately achieve a symplectic matrix and still satisfy the diagonalization condition. It was found that such symplectic matrix D can be written in terms of the column vectors of C as follows

$$D \equiv \left(\frac{1}{P_1}C_1 \,;\, \frac{1}{P_2}C_2 \,;\, \bar{C}_1 \,;\, \bar{C}_2\right)$$

where

$$P_1 = i\,\delta_1\sqrt{\delta}\left(\frac{5}{2} - \sqrt{\delta}\right)$$

$$P_2 = -i\,\delta_2\sqrt{\delta}\left(\frac{5}{2} + \sqrt{\delta}\right)$$

are two scalars and the bar atop a quantity denotes its complex conjugate.

To epitomize: The linear transformation

$$y = D\,z$$

takes the original differential system (17) into

$$\dot{z} = J\,H_z \tag{20}$$

and the quadratic terms of the Hamiltonian are

$$H_2 \equiv \lambda_1 z_1 z_3 + \lambda_2 z_2 z_4 \ . \tag{21}$$

Series solutions

We have obtained solutions to the differential system (20) in the form of infinite power series in two variables

$$z_j(u,v) \equiv \sum_{p,q}^{\infty} A_{j;p,q}\, u^p v^q, \quad (j = 1 \text{ to } 4). \tag{22}$$

The summation for the indices p, q extends to all positive integers, including zero, with $p+q \geqslant 1$. u and v are two complex variables which satisfy the auxiliary differential system

$$\dot{u} = u\,\alpha(uv) \quad ; \quad \dot{v} = v\,\beta(uv) \tag{23}$$

where α and β are infinite power series in the product uv alone:

$$\alpha(uv) \equiv \sum_{0}^{\infty}{}_r\ \alpha_r\,(uv)^r$$

$$\beta(uv) \equiv \sum_{0}^{\infty}{}_r\ \beta_r\,(uv)^r.$$

In order to generate a consistent recurrent procedure, appropriate conditions had to be imposed upon the proposed series solutions (22). They are as follows:

(a) the series $z_1 - u$; $z_3 - v$; z_2 ; z_4 begin with quadratic terms in u, v

(b) the series $z_1 - u$ and $z_3 - v$ do not contain terms in $u^p v^q$ with $p - q = 1$ and $p - q = -1$ respectively.

In symbols

$$A_{1;1,0} = A_{3;0,1} = 1 \quad ; \quad A_{1;0,1} = A_{3;1,0} = 0$$

$$A_{2;1,0} = A_{2;0,1} = A_{4;1,0} = A_{4;0,1} = 0 \tag{24}$$

$$A_{1;k+1,k} = A_{3;k,k+1} = 0 , \quad k > 0 .$$

Because of the auxiliary system (23) and of the normalization of the quadratic terms expressed by (21), we are led to consider the system of partial differential equations

$$u \, \alpha(uv) \frac{\partial z_j}{\partial u} + v \, \beta(uv) \frac{\partial z_j}{\partial v} \tag{25}$$

$$- \lambda_j z_j = G_j (u,v) \quad ; \quad (j = 1 \text{ to } 4).$$

The G_j's are infinite power series in y_1, y_2 starting with quadratic terms and depending on the derivatives of H^*. They can ultimately be expressed as infinite power series in u, v by means of the canonical transformation $y = Dz$ and the trial series (22). We shall denote by $G_{j;p,q}$ the numerical coefficient of $u^p v^q$ in the series $G_j(u,v)$.

By equating the coefficients of similar powers of u, v appearing in the left and right-hand side of (25), we have been able to ascertain in a unique fashion the coefficients $A_{j;p,q}$ of the trial series and also the α_r, β_r of the auxiliary system. When $j=1,\ p=1,\ q=0$ and $j=3,\ p=0,\ q=1$ we have

$$\alpha_0 = \lambda_1 \quad ; \quad \beta_0 = \lambda_3 = -\lambda_1 .$$

For $p+q>1$, we get the system of linear equations

$$\begin{aligned} &\left[(p-q)\lambda_1 - \lambda_j \right] A_{j;p,q} \\ &+ \sum_1^N{}_r \left[(p-r)\alpha_r + (q-r)\beta_r \right] A_{j;p-r,q-r} = G_{j;p,q} \end{aligned} \tag{26}$$

N, the upper limit of the summation, is the smaller of the two integers p, q. If either p or q is zero, the summation will not appear.

Under the assumptions expressed by the (24) and by imposing as additional condition that the ratio λ_2/λ_1 of the two eigenvalues is not an integer, we have shown that formula (26) affords a recurrent determination of the various coefficients in question. The crux of the matter is that the series $G_j(u,v)$ initiate with quadratic terms and consequently it follows that $G_{j;p,q}$ with $p+q=s$, depends only upon the $A_{j;p,q}$ with $p+q<s$. In particular we get

$$\alpha_k = G_{1;k+1,k} \quad ; \quad \beta_k = G_{3;k,k+1} .$$

For a rigorous proof of this statement, one should see Lanzano (1965).

Using the fact that the differential system is a Hamiltonian one, we have also proved that

$$\beta(uv) \equiv -\alpha(uv).$$

This means that, for the auxiliary system, the following property holds

$$u(t)\, v(t) \equiv u_0 v_0 = \text{constant}$$

hence we can choose solutions of the form

$$u = u_0 \exp(i\alpha_1 t)$$

$$v = i\,\bar{u}_0 \exp(-i\alpha_1 t)$$

where α_1 is a real number. This ultimately guarantees the existence of periodic solutions (in the synodic plane) with period $2\pi/\alpha_1$ and having real coefficients.

The first-order approximation of this series development has been shown to be an ellipse having the Lagrangian point as center of symmetry. All the relevant elements of the ellipse have been determined in terms of initial conditions, eigenvalues and mass ratio of the primaries. It is interesting to notice that the eccentricity depends only on the eigenvalues. The inclination of the major axis of this ellipse (on the y_1-axis) depends on the inequality of the masses: if $\mu_1 > \mu_2$ the major axis is located in the second and fourth quadrants, if $\mu_2 > \mu_1$ it is located in the first and third quadrants. (Notice that the positive direction of the synodic x_1-axis goes from μ_1 to μ_2).

The second-order approximation is constituted by families of oval shaped quartics.

Let us finally emphasize here that by interchanging the relative role played by the two eigenvalues λ_1 and λ_2 in the established procedure, we can reach the analytic representation for a second family of periodic solutions.

Conclusions

Under the assumption that the smaller of the two masses is less than 0.03852 of the total mass of the system, the eigenvalues $\lambda_1 = i\,\delta_1$, $\lambda_2 = i\,\delta_2$ are pure imaginary numbers. Furthermore, supposing $|\delta_1| < |\delta_2|$, if the ratio δ_2/δ_1 is not an integer, the previous formulation has proved that there are two families of periodic orbits about a triangular point, which can be expressed analytically within the radius of convergence of their series representation.

We have shown that the cartesian coordinates $\eta_k(t)$, $k = 1, 2$, of the solutions can be expressed as trigonometric series of time, the coefficients of the various harmonics depending on a real parameter $k^2 = u_0 \bar{u}_0$. The synodic period of the orbits is given by $\tau = 2\pi/\alpha_1$, where α_1 is an infinite series in k^2, whose leading terms are δ_1 and δ_2, respectively, for the two families in question. The formulation of this procedure is general and provides an algorithm for any order of approximation.

Finally, we wish to point out that among the topics that should be considered for a logical future investigation, we shall mention:

(a) a detailed study of the algebraic curves representing the second and third-order approximations of these periodic orbits,

(b) use of this analytic representation of periodic orbits to study the asymptotic approach of orbits to a Lagrangian point,

(c) a stability study of orbits about a Lagrangian point.

REFERENCES

Flugge, S. (1956). "Handbuch der Physik," Vol. I, Springer Verlag, Berlin.

Kopal, Z. (1959). "Close Binary Systems," Chapman and Hall, Ltd., London.

Lanzano, P. (1960). Application of the Jacobi Integral of Celestial Mechanics to the Terminal Guidance of Space Probes, Proceedings of the XI-th International Astronautical Congress, Stockholm, Vol. I, pp. 114-124, Springer Verlag, Vienna.

Lanzano, P. (1961). Application of Hill's Lunar Theory to the Motion of Satellites, Journal of Astronautical Sciences, Vol. 8, pp. 40-47.

Lanzano, P. (1965). Periodic Motion about a Lagrangian Triangular Point, Icarus, Vol. 4, No. 3, pp. 223-241.

Lanzano, P. Contributions to the Elliptic Restricted Three-Body Problem. Accepted for publication in Icarus.

Lanzano, P. Stability of a Class of Periodic Orbits in the Restricted Three-Body Problem. Accepted for publication in Icarus.

Meixner, J. and Schafke, F. W. (1954). "Mathieusche Funktionen und Spharoidfunktionen," Springer Verlag, Berlin.

Smart, W. M. (1953). "Celestial Mechanics," Longmans, Green and Co., London.

Struik, D. J. (1950). "Classical Differential Geometry," Addison-Wesley, Cambridge, Massachusetts.

NASA-Langley, 1967 —— 19 CR-765

10-5-67

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546