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TWO-POINT TAYLOR SERIES EXPANSIONS

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INTRODUCTION

The Taylor series has been generalized in many directions. It is our purpose here to extend the series to include two points about which a function may be expanded. The virtue of this procedure lies in the fact that the accuracy of using $2n$ derivatives in a one-point Taylor series is attained using only n derivatives with the two-point expansion.

The classical interpolation problem is concerned with approximating a function, f, with a polynomial of degree n such that the polynomial is in agreement with the values of the function at each of the points x_i, $i=0, 1, 2, \ldots, n$. If in addition to the function values at each point there are available a given number of derivatives of the function at each point, we have the well-known general Hermite interpolation formula. Thus the n+1 term Taylor series is simply a special case of the Hermite formula with only one point x_1 and n derivatives of f given at x_1. The special case of the Hermite formula for two points x_1 and x_2 can be written in the form[1]

$$y(x) = (x-x_1)^n \sum_{i=0}^{n-1} \frac{B_i(x-x_2)^i}{i!} + (x-x_2)^n \sum_{i=0}^{n-1} \frac{A_i(x-x_1)^i}{i!}$$

where $$A_i = \frac{d^i}{dx^i}\left[\frac{f(x)}{(x-x_2)^n}\right]\Bigg|_{x=x_1}$$

and $$B_i = \frac{d^i}{dx^i}\left[\frac{f(x)}{(x-x_1)^n}\right]\Bigg|_{x=x_2}$$

This form suffers from the fact that for each n the coefficients must be recalculated, i.e., the coefficients are non-final.

It is the purpose of this report to develop a series expansion of a function f about two points, x_1 and x_2, such that the coefficients are final. Hence increasing the number of derivatives used involves only the addition of more terms to the original series.

CALCULATION OF SERIES COEFFICIENTS:

Assume that $\frac{d^i f}{dx^i}$, i = 0, 1, ..., k, are known at the two points x_1 and x_2. Then

$$y(x) = \sum_{i=0}^{k-1} [a_i(x-x_2) + b_i(x-x_1)] [(x-x_1)(x-x_2)]^i \qquad (1)$$

is a (2k-1)th degree polynomial approximation of f(x) if a_i and b_i are determined so that $\frac{d^i y}{dx^i}$ and $\frac{d^i f}{dx^i}$ agree at x_1 and x_2 for i = 0, 1, ..., k-1.

For ease in the calculations which follow, we introduce the transformation

$$t = \frac{x-x_1}{x_2-x_1} = \frac{x-x_1}{h}, \text{ letting } x_2-x_1 = h.$$

Then

$$x-x_1 = ht$$

$$x-x_2 = h(t-1),$$

and

$$y(t) = \sum_{i=0}^{k-1} [A_i(t-1) + B_i t] [t(t-1)]^i \qquad (2)$$

The coefficients of the transformed series (2) are related to those of the original series (1) by

$$A_i = h^{2i+1} a_i$$
$$B_i = h^{2i+1} b_i . \qquad (3)$$

This transformation effectively transforms the interval $[x_1, x_2]$ onto $[0, 1]$. Now $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = h\frac{dy}{dx}$ and in general $\frac{d^jy}{dt^j} = h^j \frac{d^jy}{dx^j}$.

Rearranging (2) in the form

$$y(t) = \sum_{i=0}^{k-1} A_i \, t^i(t-1)^{i+1} + \sum_{i=0}^{k-1} B_i t^{i+1}(t-1)^i$$

and differentiating we obtain

$$\frac{d^jy(t)}{d\,t^j} = \sum_{i=0}^{k-1} A_i \frac{d^j}{dt^j}[t^i(t-1)^{i+1}] + \sum_{i=0}^{k-1} B_i \frac{d^j}{dt^j}[t^{i+1}(t-1)^i] .$$

It can be shown from the binomial theorem that

$$\left.\frac{d^j}{dt^j}[t^{i+1}(t-1)^i]\right|_{t=0} = (-1)^{j+1} j! \binom{i}{j-i-1}$$

when $2i+1 \geq j \geq i+1$

$$\left.\frac{d^j}{dt^j}[t^i(t-1)^{i+1}]\right|_{t=0} = (-1)^{j+1}\binom{i+1}{j-i} j!$$

when $2i+1 \geq j \geq i$

$$\frac{d^j}{dt^j} [t^{i+1}(t-1)^i] \Big|_{t=1} = \binom{i+1}{j-i} j!$$

when $2i+1 \geq j \geq i$

$$\frac{d^j}{dt^j} [t^i(t-1)^{i+1}] \Big|_{t=1} = \binom{i}{j-i-1} j!$$

when $2i+1 \geq j \geq i+1$.

These expressions vanish when j falls outside of the given range.

Hence we have

$$\frac{d^jy(t)}{dt^j}\Big|_{t=0} = \sum_{i=[\frac{j}{2}]}^{j} (-1)^{j+1} j! \binom{i+1}{j-i} A_i + \sum_{i=[\frac{j}{2}]}^{j-1} (-1)^{j+1} j! \binom{i}{j-i-1} B_i$$

and $$\frac{d^jy(t)}{dt^j}\Big|_{t=1} = \sum_{i=[\frac{j}{2}]}^{j-1} j! \binom{i}{j-i-1} A_i + \sum_{i=[\frac{j}{2}]}^{j} j! \binom{i+1}{j-i} B_i$$

from which

$$A_j = \frac{(-1)^{j+1}\frac{d^jy}{dt^j}\Big|_0}{j!} - \sum_{i=[\frac{j}{2}]}^{j-1} \left[\binom{i+1}{j-i} A_i + \binom{i}{j-i-1} B_i \right]$$

$$B_j = \frac{\left.\frac{d^j y}{dt^j}\right|_1}{j!} - \sum_{i=[\frac{j}{2}]}^{j-1} \left[\binom{i}{j-i-1} A_i + \binom{i+1}{j-i} B_i \right] \tag{4}$$

for $j = 0, 1, 2, \ldots, k$

where $\sum$ is deleted for $j=0$ and $[\frac{j}{2}]$ denotes the integer part of $\frac{j}{2}$.

The remainder for the classical interpolation problem involving m distinct data points with α_j-1, $j=1, 2, \ldots, m$, derivatives available at each point is well-known[2]. For our case where $m = 2$ and $\alpha_1 = \alpha_2 = k$ the remainder is

$$R(x) = \frac{f^{(2k)}(\xi)}{(2k)!} [(x-x_1)(x-x_2)]^k$$

where ξ lies in the interval defined by x, x_1, and x_2, and $f^{(2k)}$ is continuous on that interval. Thus the finite form of the two-point Taylor series is

$$f(x) = \sum_{i=0}^{k-1} [a_i(x-x_2) + b_i(x-x_1)] [(x-x_1)(x-x_2)]^i + \frac{f^{(2k)}(\xi)}{(2k)!} [(x-x_1)(x-x_2)]^k, \tag{5}$$

where the a_i and b_i are obtained from formulas (3) and (4).

EXAMPLE OF APPLICATION:

The solution of the two-body problem, given initial coordinates and velocity components at a time $t=t_0$ is

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

and for short intervals of time these solutions may be expanded in a Taylor series about t_o:

$$x(t) = x_o + \dot{x}_o\,(t-t_o) + \frac{\ddot{x}_o}{2}\,(t-t_o)^2 + \ldots$$

$$y(t) = y_o + \dot{y}_o\,(t-t_o) + \frac{\ddot{y}_o}{2}\,(t-t_o)^2 + \ldots$$

$$z(t) = z_o + \dot{z}_o\,(t-t_o) + \frac{\ddot{z}_o}{2}\,(t-t_o)^2 + \ldots$$

From the differential equations representing the motion of the two body problem we have

$$\ddot{x} = -\frac{\mu x}{r^3}, \quad \ddot{y} = -\frac{\mu y}{r^3}, \quad \ddot{z} = -\frac{\mu z}{r^3}$$

or

$$\ddot{x} = -\mu v x, \quad \ddot{y} = -\mu v y, \quad \ddot{z} = -\mu v z$$

where $v = \frac{1}{r^3}$, and μ is the gravitational constant times the mass of the attracting body.

To obtain an algorithm for the $n^{\underline{th}}$ time derivative of x, y, and z we consider a power series solution of the equations of motion.

Let

$$x = \sum_{n=0}^{\infty} X_n\,(t-t_o)^n$$

$$y = \sum_{n=0}^{\infty} Y_n\,(t-t_o)^n$$

$$z = \sum_{n=0}^{\infty} Z_n\,(t-t_o)^n$$

$$v = \sum_{n=0}^{\infty} V_n\,(t-t_o)^n$$

$$r = \sum_{n=0}^{\infty} R_n (t-t_o)^n$$

Then

$$\dot{v} = \sum_{k=0}^{\infty} (k+1) \quad V_{k+1} \; (t-t_o)^k$$

$$\dot{r} = \sum_{k=0}^{\infty} (k+1) \quad R_{k+1} \; (t-t_o)^k$$

$$\ddot{x} = \sum_{k=0}^{\infty} (k+2) \quad (k+1) \quad X_{k+2} \; (t-t_o)^k$$

$$\ddot{y} = \sum_{k=0}^{\infty} (k+2) \quad (k+1) \quad Y_{k+2} \; (t-t_o)^k$$

$$\ddot{z} = \sum_{k=0}^{\infty} (k+2) \quad (k+1) \quad Z_{k+2} \; (t-t_o)^k$$

Substituting into the equations of motion and equating coefficients for the powers of $(t-t_o)$ we obtain

$$R_o = r_o$$

$$V_o = v_o = \frac{1}{r_o^{\,3}}$$

$$X_o = x_o$$

$$Y_o = y_o$$

$$Z_o = z_o$$

$$X_1 = \dot{x}_o$$

$$Y_1 = \dot{y}_o$$

$$Z_1 = \dot{z}_o$$

and

$$R_j = \frac{1}{jr_0} \{j(X_j X_0 + Y_j Y_0 + Z_j Z_0)$$

$$+ \sum_{n=1}^{j-1} n \, [X_{j-n} X_n + Y_{j-n} Y_n + Z_{j-n} Z_n - R_{j-n} R_n]\}$$

$$V_j = - \frac{1}{jr_0} \left\{ \frac{3jR_j}{r_0^{\,5}} + \sum_{n=1}^{j-1} [3nR_n V_{j-n} + nV_n R_{j-n}] \right\}$$

$$X_{j+2} = - \frac{\mu}{(j+1)(j+2)} \sum_{n=0}^{j} X_{j-n} V_n$$

$$Y_{j+2} = - \frac{\mu}{(j+1)(j+2)} \sum_{n=0}^{j} Y_{j-n} V_n$$

$$Z_{j+2} = - \frac{\mu}{(j+1)(j+2)} \sum_{n=0}^{j} Z_{j-n} V_n$$

$j = 0, 1, 2, 3, \ldots$

and hence

$$\left.\frac{d^n x}{dt^n}\right|_{t=t_0} = n! \; X_n$$

$$\left.\frac{d^n y}{dt^n}\right|_{t=t_0} = n! \; Y_n$$

$$\left.\frac{d^n z}{dt^n}\right|_{t=t_0} = n! \; Z_n$$

As a special case of the preceding two-body problem consider the motion to be confined to the x-y plane and suppose that we are given

$$x_1,\ y_1,\ \dot{x}_1,\ \text{and}\ \dot{y}_1\ \ \text{at time}\ t_1$$

and

$$x_2,\ y_2,\ \dot{x}_2,\ \text{and}\ \dot{y}_2\ \ \text{at time}\ t_2.$$

For convenience we set the period of the orbit $T = 2\pi$ and the semi-major axis $a = 1$ so that the constant $\mu = 1$. Obtaining the quantities

$$\left.\frac{d^n x}{dt^n}\right|_{t=t_1},\ \left.\frac{d^n y}{dt^n}\right|_{t=t_1},\ \left.\frac{d^n x}{dt^n}\right|_{t=t_2},\ \text{and}\ \left.\frac{d^n y}{dt^n}\right|_{t=t_2}$$

by the general procedure outlined above, we may in turn calculate the coefficients a_i and b_i for the two-point Taylor expansion from equations (3) and (4).

A comparison of the resulting solutions for the two-body problem from the two-point Taylor series and one-point Taylor series expanded about t_1 is presented in Figure I. Here the orbit is of eccentricity $e = \frac{1}{10}$ and t_1, t_2 were chosen to be time at perigee and time at apogee (0 and π) respectively. The figure is a plot of the error in the calculation of the magnitude of the radius vector vs. the time of the orbit. All calculations were performed in double precision on an IBM 7094.

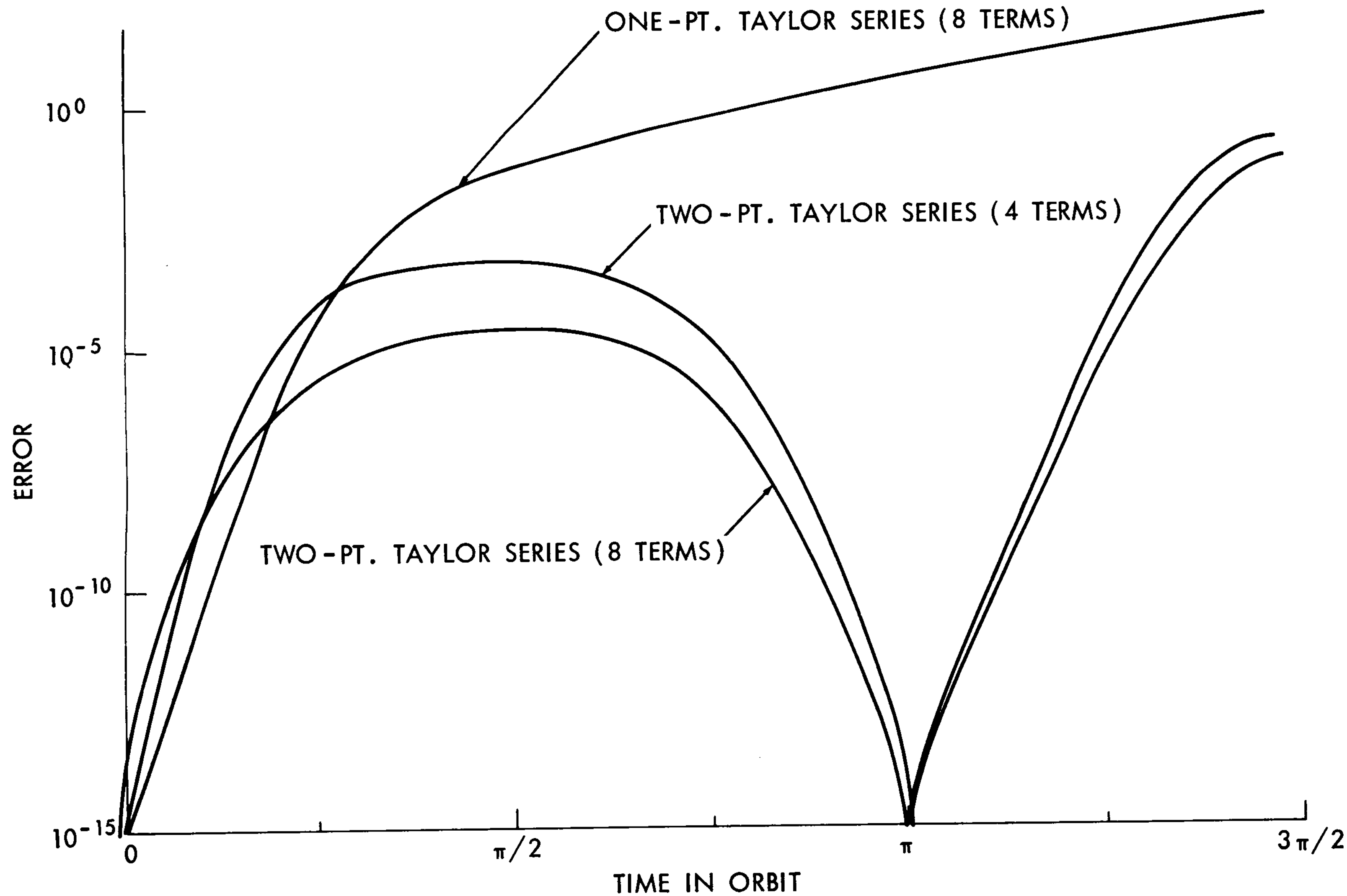
ONE-PT. TAYLOR SERIES (8 TERMS)
TWO-PT. TAYLOR SERIES (4 TERMS)
TWO-PT. TAYLOR SERIES (8 TERMS)
ERROR
10^{0}
10^{-5}
10^{-10}
10^{-15}
0
$\pi/2$
π
$3\pi/2$
TIME IN ORBIT

FIG. 1

REFERENCES

1. Davis, Philip J., "Interpolation and Approximation", Blaisdell Publishing Co., New York, 1963, page 37.

2. Fort, Tomlinson, "Finite Differences and Difference Equations in the Real Domain", Oxford Press, 1948, page 86.