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THE EQUATIONS OF MOTION AND KINETIC ENERGY FOR THIRD ORDER FLUIDS

by

Bruce Caswell

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1. Introduction

Constitutive equations for the stress in viscoelastic fluids have recently received much attention. However, except for the uncomplicated simple shearing flows, few motions have been investigated with the proposed stress equations. One obstacle to the study of more complex motions is the lack of equations of motion corresponding to the Navier-Stokes equations of the linearly viscous case. In this paper we derive such equations for the third order approximation [1] to the constitutive equation of Rivlin and Ericksen [2]. This equation can also be derived from Noll's more general theory of simple fluids via the principle of fading memory [3].

To obtain the dynamical equations we require the divergence of the stress tensor. For non-linear constitutive relations the divergence of the stress can be expressed in many forms; however, only a few of these are useful in the analysis of flows. The particular set of divergence expressions derived below permit a physical interpretation of each term in the constitutive equation, and appear to be analytically useful.

2. The Stress Approximations

Under the assumption of slow motion the stress of an incompressible Rivlin-Ericksen fluid can be successively approximated as follows [1]:

$$\bar{\bar{T}}^{i} = -\bar{\bar{1}}p + \mu_o \bar{\bar{A}}^{(1)} \qquad \text{1st Order (Newtonian),} \qquad (2.1)$$

$$\bar{\bar{T}}^{ii} = \bar{\bar{T}}^{i} + \alpha_1 \bar{\bar{A}}^{(2)} + \alpha_2 \left(\bar{\bar{A}}^{(1)}\right)^2 \qquad \text{2nd Order,} \qquad (2.2)$$

$$\bar{\bar{T}}^{iii} = \bar{\bar{T}}^{ii} + \beta_1 \bar{\bar{A}}^{(3)} + \beta_2\left(\bar{\bar{A}}^{(1)} \cdot \bar{\bar{A}}^{(2)} + \bar{\bar{A}}^{(2)} \cdot \bar{\bar{A}}^{(1)}\right) + \beta_3 II\, \bar{\bar{A}}^{(1)} \qquad \text{3rd Order,} \qquad (2.3)$$

where the $\bar{\bar{A}}^{(N)}$ are the Rivlin-Ericksen tensors defined below, $II = \bar{\bar{A}}^{(1)}:\bar{\bar{A}}^{(1)}$, and μ_o, the α's and β's are constants under isothermal conditions.

Generally we denote the stress by $\bar{\bar{T}}$; the superscripts i, ii, and iii in (2.1)-(2.3) are used to indicate the order of the approximation. Equations (2.2) and (2.3) are usually said to describe the second and third order fluids respectively; they may be regarded as exact constitutive equations defining these classes of fluids or as approximations to more general constitutive relations valid in the regime of slow flow.

The following recursive formulae for the Rivlin-Ericksen tensors will be useful in the subsequent sections of the paper [4].

$$A_{ij}^{(N)} = v_{i,j}^{(N)} + v_{j,i}^{(N)} + \sum_{Q=1}^{N-1} \binom{N}{Q} v_{m,i}^{(N-Q)} v^{(Q)m}{}_{,j} \quad , \tag{2.4}$$

where the $v_i^{(N)}$ are the covariant components of the $(N-1)^{th}$ acceleration. Setting $N = 1$ we immediately recognize $\bar{\bar{A}}^{(1)}$ as twice the usual rate of deformation tensor. The $\bar{\bar{A}}^{(N)}$ can also be expressed as

$$\bar{\bar{A}}^{(N)} = \frac{\mathcal{D}\bar{\bar{A}}^{(N-1)}}{\mathcal{D}t} + \frac{1}{2}\left(\bar{\bar{A}}^{(1)} \cdot \bar{\bar{A}}^{(N-1)} + \bar{\bar{A}}^{(N-1)} \cdot \bar{\bar{A}}^{(1)}\right) \quad . \tag{2.5}$$

Here t denotes time, and $\mathcal{D}/\mathcal{D}t$ is the corotational derivative which can be written as the operator

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{D}{Dt} + \frac{1}{2}\left(x\bar{\omega} - \bar{\omega}x\right) \quad , \tag{2.6}$$

where the usual material derivative is denoted by D/Dt and $\bar{\omega}$ is the vorticity vector.

From (2.5) and the Cayley-Hamilton theorem [4] we find the identity

$$\bar{\bar{A}}^{(1)} \cdot \bar{\bar{A}}^{(2)} + \bar{\bar{A}}^{(2)} \cdot \bar{\bar{A}}^{(1)} = \frac{\mathcal{D}\left(\bar{\bar{A}}^{(1)}\right)^2}{\mathcal{D}t} + II\,\bar{\bar{A}}^{(1)} + \frac{2}{3}\,III\,\bar{\bar{I}} \quad , \tag{2.7}$$

where $III = \bar{\bar{I}} : \left(\bar{\bar{A}}^{(1)}\right)^3$. (2.8)

Substituting (2.7) into (2.3) and absorbing the last term into the scalar pressure we have an alternative form for the third order fluid

$$\bar{\bar{T}}^{iii} = \bar{\bar{T}}^{ii} + \beta_1 \bar{\bar{A}}^{(3)} + \beta_2 \frac{\mathcal{D}\left(\bar{\bar{A}}^{(1)}\right)^2}{\mathcal{D}t} + \beta_3' \, II \, \bar{\bar{A}}^{(1)} \quad , \qquad (2.9)$$

where $\beta_3' = \beta_2 + \beta_3$.

Some of the coefficients of the third order fluid can now be given physical interpretation from their role in steady simple shearing motions. From (2.1), (2.2) and (2.9) we have respectively for the material functions of simple shear

$$\tau^{i}(\kappa) = \mu_o \kappa = \tau^{ii}(\kappa) \quad , \quad \tau^{iii}(\kappa) = \tau^{ii}(\kappa) + 2\beta_3' \kappa^2 \qquad \text{shear stress function,} \qquad (2.10)$$

$$\left.\begin{aligned} \sigma_1^{i}(\kappa) &= 0 \quad , \quad \sigma_1^{ii}(\kappa) = \sigma_1^{iii}(\kappa) = (2\alpha_1 + \alpha_2)\kappa^2 \\ \sigma_2^{i}(\kappa) &= 0 \quad , \quad \sigma_2^{i}(\kappa) = \sigma_2^{iii}(\kappa) = \alpha_2 \kappa^2 \end{aligned}\right\} \text{normal stress functions,} \qquad (2.11)$$

where κ is the rate of strain.

Comparison of (2.10) and (2.11) with data on polymer solutions [7] clearly shows that the domain of the third order fluid is rather limited. In spite of this limitation the theory can be used to extrapolate data on the material functions τ, σ_1 and σ_2 in the low shear rate regime. The zero-shear viscosity, μ_o, has frequently been measured, and in at least one case [6] α_1 and α_2 have been determined from normal stress measurements. The shear function coefficient β_3', while seldom reported, can be readily evaluated from shear function measurements at sufficiently small values of κ.

Liquids whose molecules are small have long been known to obey (2.1) over the entire range of measurable shear rates. It is reasonable to expect (2.3) to be valid for very dilute polymer solutions and suspensions. However, at very low dilutions the non-Newtonian effects tend to dissappear sufficiently fast to make difficult the experimental verification of this conjecture. Hence even though the mathematical theory of second and third order fluids can be regarded as exact constitutive equations, its relation to real materials is that of an approximate theory at best valid over a limited range of deformation rates. Although these restrictions appear to be severe, it will be demonstrated in a future publication [5] that the theory can be successfully used to interpret data on flows which are more complicated than the simple shearing motions.

3. The Divergence of the Stress Tensor

Before taking the divergence of (2.2) and (2.9) we record two useful identities. If b_i are the covariant components of any vector field, then

$$b_{i,j} = b_{j,i} + \varepsilon_{jik}(\nabla \times \bar{b})^k \quad . \tag{3.1}$$

For any symmetric tensor field $\bar{\bar{B}}$ the following identity holds for incompressible motion.

$$\nabla \cdot \frac{\mathcal{D}\bar{\bar{B}}}{\mathcal{D}t} = \frac{\partial \nabla\cdot\bar{\bar{B}}}{\partial t} - \bar{V} \times [\nabla\times\nabla\cdot\bar{\bar{B}}]+\nabla[\bar{V}\cdot(\nabla\cdot\bar{\bar{B}})] + \frac{1}{2}\left(\nabla\bar{\bar{A}}^{(1)}:\bar{\bar{B}} + \nabla \times [\bar{\bar{A}}^{(1)} \overset{\times}{\cdot} \bar{\bar{B}}]\right) \cdot \tag{3.2}$$

This identity is derived in the Appendix where it is given in index notation. In all formulae derived from (2.4), (2.5) and (2.6) the velocity and vorticity appearing in them are, because of invariance requirements, to be taken as relative to the coordinate system to which the various field quantities have been referred. Hence, in problems such as the flow exterior to a body where coordinates imbedded in the moving body are frequently employed, it is necessary to use the velocity relative to the body in the Rivlin-Ericksen tensors, and formulae derived from them.

To find $\nabla\cdot\bar{\bar{T}}^{ii}$ we begin with $\nabla\cdot\bar{\bar{A}}^{(2)}$. From (2.4) with (N=2) we have

$$A^{(2)k}_{i,k} = \nabla^2 v^{(2)}_i + (\nabla\cdot\bar{V}^{(2)} + V^2)_{,i} - 2(v^j v_{j,ik})^{,k} \quad . \qquad (3.3)$$

Expanding the last term and employing (3.1), we finally obtain

$$\nabla\cdot\bar{\bar{A}}^{(2)} = \nabla^2\bar{V}^{(2)} + 2\left(\frac{\partial\nabla^2\bar{V}}{\partial t} - \bar{V}\times\nabla^2\bar{\omega} - \frac{D\nabla^2\bar{V}}{Dt}\right) + 2\nabla\left(\nabla\cdot(\bar{V}\cdot\bar{\bar{A}}^{(1)}) - \frac{1}{4}\,II\right) , \qquad (3.4)$$

where $\bar{V}^{(2)}$ is the first acceleration.

To find $\nabla\cdot(\bar{\bar{A}}^{(1)})^2$ we proceed indirectly by first finding $\nabla\cdot\mathcal{D}\bar{\bar{A}}^{(1)}/\mathcal{D}t$, and then use (2.5) and (3.4) to obtain the desired result by difference. Putting $\bar{\bar{B}} = \bar{\bar{A}}^{(1)}$ in (3.2), the last term vanishes leaving us with

$$\nabla\cdot\frac{\mathcal{D}\bar{\bar{A}}^{(1)}}{\mathcal{D}t} = \frac{\partial\nabla^2\bar{V}}{\partial t} - \bar{V}\times\nabla^2\bar{\omega} + \nabla\left(\nabla\cdot(\bar{V}\cdot\bar{\bar{A}}^{(1)}) - \frac{1}{4}\,II\right) \quad . \qquad (3.5)$$

Hence from (3.4) and (3.5) it follows that

$$\nabla\cdot\left(\bar{\bar{A}}^{(1)}\right)^2 = \nabla\cdot\left(\bar{\bar{A}}^{(2)} - \frac{\mathcal{D}\bar{\bar{A}}^{(1)}}{\mathcal{D}t}\right) = \nabla^2\bar{V}^{(2)} - 2\,\frac{D\nabla^2\bar{V}}{Dt} + \nabla\cdot\frac{\mathcal{D}\bar{\bar{A}}^{(1)}}{\mathcal{D}t} \quad . \qquad (3.6)$$

From (2.3), (3.4) and (3.6) we have for the second order fluid

$$\nabla\cdot\bar{\bar{T}}^{ii} = -\nabla[p - (2\alpha_1+\alpha_2)\left(\nabla\cdot(\bar{V}\cdot\bar{\bar{A}}^{(1)}) - \frac{II}{4}\right)] + \mu_o\nabla^2\bar{V} +$$

$$+(2\alpha_1+\alpha_2)\left(\frac{\partial\nabla^2\bar{V}}{\partial t} - \bar{V}\times\nabla^2\bar{\omega}\right) + (\alpha_1+\alpha_2)\left(\nabla^2\bar{V}^{(2)} - 2\,\frac{D\nabla^2\bar{V}}{Dt}\right) \quad . \qquad (3.7)$$

It is worth noting that the restrictions on the velocity vector given above for equation (3.2) apply here also.

An alternative form of (3.7) can be obtained with the use of Lagrange's formula for the acceleration

$$\bar{V}^{(2)} = \frac{\partial\bar{V}}{\partial t} + \bar{\omega}\times\bar{V} + \frac{1}{2}\,\nabla(V^2) \qquad (3.8)$$

together with the identity

$$\nabla^2(V^2) = 2\overline{V} \cdot \nabla^2\overline{V} + \omega^2 + \frac{II}{2} \quad . \tag{3.9}$$

On substituting (3.8) and (3.9) into (3.7), we find

$$\nabla \cdot \overline{\overline{T}}^{ii} = -\nabla[p-(3\alpha_1+2\alpha_2)\left(\overline{V}\cdot\nabla^2\overline{V} + \frac{II}{4}\right)]+\mu_o\nabla^2\overline{V} + (2\alpha_1+\alpha_2)\left(\frac{\partial\nabla^2\overline{V}}{\partial t} - \overline{V} \times \nabla^2\bar{\omega}\right)$$

$$+(\alpha_1+\alpha_2)\left(\nabla^2\left(\frac{\partial\overline{V}}{\partial t} + \bar{\omega} \times \overline{V}\right) + \frac{1}{2}\nabla(\omega^2) - 2\frac{D\nabla^2\overline{V}}{Dt}\right) \quad . \tag{3.10}$$

When (3.10) is put into the equation of motion we can, by recalling the identity $\nabla^2\overline{V} = -\nabla \times \bar{\omega}$, read off the following theorem:

Any isochoric irrotational motion satisfies the dynamical equations of the second order fluid with Bernoulli's equation given by

$$\frac{p}{\rho} - (3\alpha_1 + 2\alpha_2)\frac{II}{4} + \Omega + \frac{1}{2}V^2 = \phi(t) \quad , \tag{3.11}$$

where Ω is the body force potential, and $\phi(t)$ is an arbitrary function of time. For polymer solutions $3\alpha_1 + 2\alpha_2$ is positive [7] so that viscoelasticity tends to reduce the pressure.

In planar flows the Cayley-Hamilton theorem permits (2.2) to be written in the form

$$\overline{\overline{T}}^{ii} = -\left(p - \frac{II}{2}(\alpha_1+\alpha_2)\right)\overline{\overline{I}} + \mu_o\overline{\overline{A}}^{(1)} + \alpha_1\frac{\mathcal{D}\overline{\overline{A}}^{(1)}}{\mathcal{D}t} \quad . \tag{3.12}$$

Hence for such motions $\nabla \cdot \overline{\overline{T}}^{ii}$ becomes

$$\nabla \cdot \overline{\overline{T}}^{ii} = -\nabla[p-(3\alpha_1+2\alpha_2)\frac{II}{4}] + \mu_o\nabla^2\overline{V} + \alpha_1\left(\frac{\partial\nabla^2\overline{V}}{\partial t} - \overline{V}\times\nabla^2\bar{\omega} + \nabla(\overline{V}\cdot\nabla^2\overline{V})\right) \quad . \tag{3.13}$$

The curl of (3.13) can easily be shown to reduce to the planar flow vorticity equation of P. L. and R. K. Bhatnagar [8]. With this equation the Bhatnagars proved that the velocity field in planar flow is independent of α_2, a result which follows more directly from (3.12), and is a special case of the results for planar flows of general Rivlin-Ericksen fluids given by Rivlin and Ericksen in their original paper [2]. For two-dimensional flow the Bhatnagars extended the above theorem on irrotational motion to the case where the vorticity is a function of the stream function. However, their pressure equation is not in agreement with (3.11), and they give no result for the pressure in axisymmetric flow which was included in their investigation.

Another result which follows directly from (3.13) is Tanner's theorem [9] on inertialess plane flow of second order fluids. According to this theorem any two-dimensional solution to the linear Stokes equations satisfies the inertialess dynamical equation obtained by setting (2.12) equal to zero.

Proceeding to the third order terms in (2.9) we find that the divergence of $\bar{\bar{A}}^{(3)}$ cannot be obtained in a form which leads to any new physical interpretation. In perturbation calculations it is most helpful to have the equations in their simplest form, and with this in mind we have from (2.4)

$$\nabla \cdot \bar{\bar{A}}^{(3)} = \nabla^2 \bar{V}^{(3)} + 3\left(\nabla \bar{V}^{(2)} \cdot \nabla^2 \bar{V} + \nabla \bar{V} \cdot \nabla^2 \bar{V}^{(2)}\right.$$
$$\left. + \nabla(\nabla \cdot \bar{V}^{(3)} + 3\nabla \bar{V}^{(2)} : \bar{V}\nabla)\right) \quad . \qquad (3.14)$$

Many equivalent rearrangements of (3.14) are possible. A somewhat different result can be obtained by using (2.5) and (3.2).

Writing (3.2) with $\bar{\bar{B}} = \bar{\bar{A}}^{(1)} \cdot \bar{\bar{A}}^{(1)}$ gives us

$$\nabla \cdot \frac{\mathcal{D}(\bar{\bar{A}}^{(1)})^2}{\mathcal{D}t} = \frac{\partial}{\partial t}\left(\nabla \cdot (\bar{\bar{A}}^{(1)})^2\right) - \bar{V} \times [\nabla \times \left(\nabla \cdot (\bar{\bar{A}}^{(1)})^2\right)] + \nabla[\nabla \cdot \left(\bar{V} \cdot (\bar{\bar{A}}^{(1)})^2\right) - \frac{III}{3}] \quad .$$

(3.15)

Clearly any terms in the stress equation of the form $\mathcal{D}(\bar{\bar{A}}^{(1)})^n/\mathcal{D}t$ will give a result similar to (3.15). That is, in steady flow, apart from a pressure term, the only forces generated are normal to the streamlines. The significance of this property will be investigated further in the next section which deals with the kinetic energy equation.

Adding the divergence of the third order terms to (3.7), we write

$$\nabla \cdot \bar{\bar{T}}^{iii} = -\nabla p^{*} + \nabla\cdot[(\mu_o+\beta_3^1 II)\bar{\bar{A}}^{(1)}]+(\alpha_1+\alpha_2)\left(\nabla^2\bar{V}^{(2)} - 2\,\frac{D\nabla^2\bar{V}}{Dt}\right)$$
$$+\beta_1\nabla\cdot\bar{\bar{A}}^{(3)} + \bar{E}\cdot[(2\alpha_1+\alpha_2)\bar{\bar{A}}^{(1)} + \beta_2(\bar{\bar{A}}^{(1)})^2] \quad , \tag{3.16}$$

where the operator $\bar{E}\cdot$ is given by

$$\bar{E}\cdot = \frac{\partial\nabla\cdot}{\partial t} - \bar{V}\times\left(\nabla\times(\nabla\cdot\quad)\right) + \nabla[\nabla\cdot(\bar{V}\cdot\quad)] \quad , \tag{3.17}$$

and

$$p^{*} = p + (2\alpha_1+\alpha_2)\,\frac{II}{4} + \beta_2\,\frac{III}{3} \quad . \tag{3.18}$$

Equating the sum of (3.16) and appropriate body force terms to $\rho\bar{V}^{(2)}$, where ρ is the fluid density, we obtain the dynamical equation governing the motion of the incompressible third order fluid.

4. The Equation of Kinetic Energy

In this section we seek further interpretation of the constitutive equation (2.9) by investigating the equation of kinetic energy. In the absence of body forces the kinetic energy equation takes the form (Serrin [10] p. 138)

$$\frac{d}{dt}\int_V \frac{1}{2}\,\rho V^2 dV = \int_S \bar{\bar{T}}\cdot\bar{V}dA - \frac{1}{2}\int_V \bar{\bar{T}} : \bar{\bar{A}}^{(1)}dV \quad , \tag{4.1}$$

where V is an arbitrary material volume enclosed by the surface S. The constitutive equation enters (4.1) through the terms involving $\bar{\bar{T}}$ which we now evaluate with (2.9).

For the stress power we find

$$\bar{\bar{T}}^{iii} : \bar{\bar{A}}^{(1)} = (\mu_o+\beta_3 II)II + (\alpha_1+\alpha_2)III + \beta_1\left(\frac{\mathcal{D}A^{(1)}}{\mathcal{D}t^2} : \bar{\bar{A}}^{(1)} + \frac{DIII}{Dt} + \frac{II^2}{2}\right)$$

$$+ \frac{D}{Dt}\left(\alpha_1 \frac{II}{2} + \beta_2 \frac{2}{3} III\right) \quad . \qquad (4.2)$$

The last term suggests that we rewrite (2.9) as

$$\bar{\bar{T}}^{iii} = \bar{\bar{T}}_D + \bar{\bar{T}}_E \quad , \qquad (4.3)$$

where $$\bar{\bar{T}}_D = -\bar{\bar{1}}p + (\mu_o+\beta_3' II)\bar{\bar{A}}^{(1)} + (\alpha_1+\alpha_2)\left(\bar{\bar{A}}^{(1)}\right)^2 + \beta_1\bar{\bar{A}}^{(3)} \quad , \qquad (4.4)$$

and $$\bar{\bar{T}}_E = \frac{\mathcal{D}[\alpha_1\bar{\bar{A}}^{(1)} + \beta_2\left(\bar{\bar{A}}^{(1)}\right)^2}{\mathcal{D}t} \quad . \qquad (4.5)$$

Writing the kinetic energy equation (4.1) using (4.2) and (4.3) we have

$$\frac{d}{dt}\int_V \rho\left(\frac{1}{2}V^2 + \frac{1}{4}\frac{\alpha_1 II}{\rho} + \frac{1}{3}\frac{\beta_2 III}{\rho}\right) dV = \int_S (\bar{\bar{T}}_D + \bar{\bar{T}}_E)\cdot\bar{V}dA - \frac{1}{2}\int_V \bar{\bar{T}}_D : \bar{\bar{A}}^{(1)}dV \quad . \qquad (4.6)$$

We can interpret (4.6) as indicating the existence of a "rate-of-strain energy function" $\frac{1}{4}\frac{\alpha_1}{\rho} II + \frac{1}{3}\frac{\beta_2}{\rho} III$ which is associated with the stress field (4.5). In steady flow this field contributes, apart from an isotropic pressure term, only a force normal to the streamlines. Without introducing formal thermodynamic arguments it is not possible to label the last term in (4.6) the dissipation. Since thermodynamics is beyond the scope of this paper, we will remain content with our purely mechanical interpretation of fluid elasticity.

5. The Perturbation Analysis of Drag

As an example of the use of some of our previous results, we consider the flow exterior to a rigid body undergoing steady translation and rotation. Assuming the condition of adherence we have on the surface of the body S_o, and on the exterior surface Σ

$$\left.\begin{aligned} \bar{V} &= \bar{U} + \bar{\Omega} \times \bar{r} \quad \text{on } S_o \ , \\ \bar{V} &= 0 \qquad\qquad\quad \text{on } \Sigma \ , \end{aligned}\right\} \tag{5.1}$$

where $\bar{r}$ is the radius vector, and $\bar{U}$ and $\bar{\Omega}$ are respectively the translational and angular velocities. The surface Σ can have any shape consistent with steady motion. If Σ is at a finite distance from S_o steady motion is possible only if Σ and S_o have a common axis of symmetry. From (4.8), we have

$$-(\bar{F} \cdot \bar{U} + \bar{L} \cdot \bar{\Omega}) = \frac{1}{2} \int_V \bar{\bar{T}}_D : \bar{\bar{A}}^{(1)} dV \geq 0^* \ , \tag{5.2}$$

where the force $\bar{F}$ and torque $\bar{L}$ on S_o are given by

$$\bar{F} = \int_{S_o} (\bar{\bar{T}}_D + \bar{\bar{T}}_E) \cdot \bar{n} dA \quad (5.3a); \qquad \bar{L} = \int_{S_o} \bar{r} \times (\bar{\bar{T}}_D + \bar{\bar{T}}_E) \cdot n dA \ . \quad (5.3b)$$

Perturbation analysis of the problem begins with the assumption of a velocity expansion of the form

$$\bar{V} = \bar{V}_o + \lambda(\bar{V}_{11} + \varepsilon_2 \bar{V}_{12}) + \lambda^2(\bar{V}_{21} + \varepsilon_2 \bar{V}_{22} + \varepsilon_3 \bar{V}_{23} + \varepsilon_4 \bar{V}_{24} + \varepsilon_5 \bar{V}_{25}) + O(\lambda^3) \ , \tag{5.4}$$

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*It is not necessary to invoke formal thermodynamics to see that the volume integral of $\bar{\bar{T}}_D : \bar{\bar{A}}^{(1)}$ is the global dissipation in this case.

where λ and the ε_i are dimensionless perturbation parameters given by

$$\lambda = \frac{\alpha_1+\alpha_2}{\mu_o}\,\frac{M}{a} \;, \quad \varepsilon_2 = \frac{\alpha_1}{\alpha_1+\alpha_2} \;, \quad \varepsilon_3 = \frac{\beta_1\mu_o}{(\alpha_1+\alpha_2)^2} \;,$$

$$\varepsilon_4 = \frac{\beta_2\mu_o}{(\alpha_1+\alpha_2)^2} \;, \qquad \varepsilon_5 = \frac{\beta_3'\mu_o}{(\alpha_1+\alpha_2)^2} \;, \qquad (5.5)$$

Here M is a suitable characteristic speed and a the characteristic length. The boundary conditions (5.1) are satisfied as follows

$$M\overline{V}_o = \overline{U} + \overline{\Omega} \times \bar{r} \quad \text{on} \quad S_o \;, \quad V_o = 0 \quad \text{on} \quad \Sigma \;, \qquad (5.6a)$$

$$\overline{V}_{ij} = 0 \quad \text{on} \quad S_o \quad \text{and on} \quad \Sigma \quad \text{for all} \quad i,j \quad . \qquad (5.6b)$$

The stress field can also be expanded in a series similar to (5.4)

$$\bar{\bar{T}} = \bar{\bar{T}}_o + \lambda(\bar{\bar{T}}_{11}+\varepsilon_2\bar{\bar{T}}_{12}) + \lambda^2(\bar{\bar{T}}_{21}+\varepsilon_2\bar{\bar{T}}_{22}+ \ldots) + O(\lambda^3) \quad . \qquad (5.7)$$

The $\bar{\bar{T}}_{ij}$ can be identified by substituting (5.4) into (2.9) and equating coefficients of λ, $\lambda\varepsilon_2$ and so on. Putting (5.7) into (5.3) we write for the force and torque respectively

$$\overline{F} = \overline{F}_o + \lambda(\overline{F}_{11}+\varepsilon_2\overline{F}_{12}) + \lambda^2(\overline{F}_{21}+\varepsilon_2\overline{F}_{22}+ \ldots) + O(\lambda^3) \;, \qquad (5.8a)$$

$$\overline{L} = \overline{L}_o + \lambda(\overline{L}_{11}+\varepsilon_2\overline{L}_{12}) + \lambda^2(\overline{L}_{21}+\bar{\varepsilon}_2\overline{L}_{22}+ \ldots) + O(\lambda^3) \;, \qquad (5.8b)$$

where $\overline{F}_{ij}$ and $\overline{L}_{ij}$ are the force and torque due to the stress $\bar{\bar{T}}_{ij}$.

Most solutions for the fields $\overline{V}_{ij}$ which have been published so far have involved the neglect of inertia in the dynamical equations. In fact, this assumption is implicit in the choice of perturbation parameters in (5.4) since none of them contain the fluid density. It is possible to include the effect of small Reynolds numbers [11,12], but we will neglect inertia completely. It is shown elsewhere [11] that (2.9), (5.4) and (5.7) lead to the following governing equations for the $\overline{V}_{ij}$ and the corresponding pressure fields p_{ij}.

$$0 = \nabla \cdot \overline{\overline{T}}_o = -\nabla p_o + \nabla^2 \overline{V}_o \quad , \qquad (5.9a)$$

$$0 = \nabla \cdot \overline{\overline{T}}_{11} = -\nabla p_{11} + \nabla^2 \overline{V}_1 + \nabla \cdot \overline{\overline{\tau}}_{11} \quad , \qquad (5.9b)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$0 = \nabla \cdot \overline{\overline{T}}_{ij} = -\nabla p_{ij} + \nabla^2 \overline{V}_{ij} + \nabla \cdot \overline{\overline{\tau}}_{ij} \quad . \qquad (5.9c)$$

These equations are linear in $\overline{V}_{ij}$ since the inhomogeneous stress tensors $\overline{\overline{\tau}}_{ij}$ are evaluated with the previously determined velocity fields $\overline{V}_{i-1j}$, $\overline{V}_{i-2j}$, ... , $\overline{V}_o$. Clearly, $\overline{V}_o$ is the solution to the corresponding Newtonian problem.

After the $\overline{\overline{T}}_{ij}$ have been expressed in terms of the $\overline{V}_{ij}$ with the help of (2.9), we substitute (5.7) into (5.2) and, by equating coefficients of λ, λ^2, $\lambda^2 \varepsilon_i$, we obtain

$$\overline{F}_{22} \cdot \overline{U} + \overline{L}_{22} \cdot \overline{\Omega} = \int_V \overline{\overline{A}}^{(1)}_{22} : \overline{\overline{A}}^{(1)}_o \, dV \quad , \qquad (5.10a)$$

$$\overline{F}_{24} \cdot \overline{U} + \overline{L}_{24} \cdot \overline{\Omega} = \int_V \overline{\overline{A}}^{(1)}_{24} : \overline{\overline{A}}^{(1)}_o \, dV \quad . \qquad (5.10b)$$

Here the notation $\overline{\overline{A}}^{(1)}_{ij}$ indicates that $\overline{\overline{A}}^{(1)}$ is to be evaluated with $\overline{V}_{ij}$. Using (5.9a), we can easily find the following identity

$$\overline{\overline{A}}^{(1)}_{nm} : \overline{\overline{A}}^{(1)}_o = 2\nabla \cdot (\overline{V}_{nm} \cdot \overline{\overline{T}}_o) \quad . \qquad (5.11)$$

Employing (5.11) into (5.10) together with the divergence theorem and the boundary conditions (5.6b), we have

$$\overline{F}_{22} \cdot \overline{U} + \overline{L}_{22} \cdot \overline{\Omega} = \overline{F}_{24} \cdot \overline{U} + \overline{L}_{24} \cdot \overline{\Omega} = 0 \quad . \qquad (5.12)$$

The vorticity equation obtained from (5.9a) is simply

$$\nabla^2 \bar{\omega}_o = 0 \quad . \tag{5.13}$$

When (5.4) is substituted into the dynamical equations, it follows from (3.7), (5.13) and the boundary conditions (5.6b) that $\overline{V}_{12}$ will always vanish. Hence we also have

$$\overline{F}_{12} \cdot \overline{U} + \overline{L}_{12} \cdot \overline{\Omega} = 0 \quad . \tag{5.14}$$

The series (5.8) and (5.9) have been written so that the material constants appear only in λ and the ε_i. The vectors $\overline{F}_{ij}$ and $\overline{L}_{ij}$ depend only on $\overline{U}$ and $\overline{\Omega}$, thus (5.12) and (5.14) mean that to within terms $O(\lambda^3)$ the dissipation is independent of α_1 and β_2, the material constants which appear in the rate of strain energy (see (4.8)).

The case of a sphere in an unbounded fluid has been solved by Giesekus [13]. For the special cases of pure translation ($\overline{\Omega}$=0) and pure rotation ($\overline{U}$=0) his results are in accord with (5.12) and (5.14). However, for combined rotation and translation (5.12) is not satisfied which indicates an error in this part of the calculation. The details of these perturbation calculations are extremely complicated, and errors are difficult to avoid. Hence checks for internal consistency such as (5.12) can be very helpful.

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Appendix

To obtain (3.2), we start with

$$\left(\frac{DB_{ik}}{Dt}\right)^{,k} = \frac{DB_{ik}^{,k}}{Dt} + (v^{k,m}B_{im})_{,k} - v^{k,m}_{,k}B_{im} \quad . \tag{A.1}$$

Using (3.1) in the middle term of (A.1) gives us

$$\left(\frac{DB_{ik}}{Dt}\right)^{,k} = \frac{DB_{ik}^{,k}}{Dt} + \frac{1}{2}\left(A^{(1)}_{km}B^m_i + \varepsilon_{\ell mk}\omega^{\ell}B^m_i\right)^{,k} - v^{k,m}_{,k}B_{im} \quad . \tag{A.2}$$

Combining (2.6) and (A.2) we have

$$\left(\frac{\mathcal{D}B_{ik}}{\mathcal{D}t}\right)^{,k} = \frac{DB_{ik}^{,k}}{Dt} + \left(\frac{1}{2}A^{(1)}_{km}B^m_i + \varepsilon_{\ell im}\omega^{\ell}B^m_k\right)^{,k} - v^{k,m}_{,k}B_{im} \quad . \tag{A.3}$$

Again using (3.1) on the last term we obtain

$$\left(\frac{\mathcal{D}B_{ik}}{\mathcal{D}t}\right)^{,k} = \frac{DB_{ik}^{,k}}{Dt} + \frac{1}{2}\left(A^{(1)}_{km}B^m_i - A^{(1)}_{im}B^m_k\right)^{,k} + \left(v_{m,i}B^m_k\right)^{,k} - v^k_{,km}B^m_i \, . \tag{A.4}$$

Expanding the next to last term gives us

$$\left(v_{m,i}B^m_k\right)^{,k} = v^{m,k}_{,i}B_{mk} + \left(v^m B_{mk}^{,k}\right)_{,i} - v^m B_{mk,i}^{,k} \quad . \tag{A.5}$$

Putting (A.5) into (A.4) and assuming B_{mk} is symmetric, we find for the incompressible case

$$\left(\frac{\mathcal{D}B_{ik}}{\mathcal{D}t}\right)^{,k} = \frac{\partial B_{ik}^{,k}}{\partial t} + \delta^{rs}_{i\ell}\, v^{\ell}B^k_{r,ks} + \left(v^m B_{mk}^{,k}\right)_{,i} + \frac{1}{2}\left(A^{(1)}_{mk,i}B^{mk} + \delta^{rs}_{ki}\left(A^{(1)}_{rm}B^m_s\right)^{,k}\right). \tag{A.6}$$

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