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ON THE EQUATIONS DESCRIBING THE ORBITAL PLANE'S ROTATION

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SUMMARY

The general properties are investigated of equations utilized in the work [1] for describing the rotation of the orbital plane. Analogy is established between the equation describing the variation of the lateral angular deflection of the satellite from the fixed plane, and the equation for quasi inear oscillations of the mechanical system with one degree of freedom.

Certain characteristic peculiarities of spatial motion in the central field are illustrated on examples of isolated solutions.

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1. In the work [1] we find the equations describing the rotation of satellite orbit plane:

$$\begin{aligned}\frac{d\gamma_1}{dv} &= \gamma_2,\\ \frac{d\gamma_2}{dv} &= -\gamma_1 + \frac{j_z}{\mu p\, y^3}\gamma_3,\\ \frac{d\gamma_3}{dv} &= -\frac{j_z}{\mu p\, y^3}\gamma_2.\end{aligned} \tag{1.1}$$

Here j_z is the vector component of the perturbing acceleration,normal to the osculating plane of the orbit, $y = 1/r$, $\underline{r}$ is the geocentrical radius-vector of the satellite, $\underline{p}$ is the focal parameter of the orbit, μ is the product of the gravitational constant by the mass of the Earth. The variable $\underline{v}$ is determined by the equation

$$\frac{dv}{dt} = \sqrt{\mu p}\, y^2, \tag{1.2}$$

* [OB URAVNENIYAKH OPISYVAYUSHCHIKH POVOROT ORBITAL'NOY PLOSKOSTI].

the variables $\gamma_1, \gamma_2, \gamma_3$ are expressed through orbit inclination i and the latitude argument u (Fig.1) with the aid of the formulas

$$\gamma_1 = \sin i \sin u,$$
$$\gamma_2 = \sin i \cos u, \qquad (1.3)$$
$$\gamma_3 = \cos i.$$

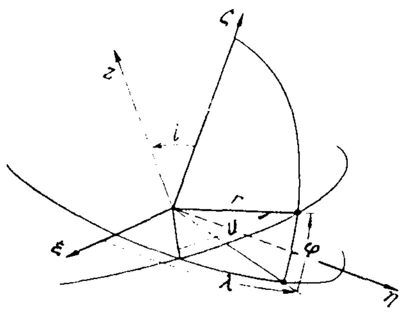

Fig.1

As follows from Fig.1, variable γ_1 determines the value of the sine of satellite's deflection angle from the fundamental coordinate plane $\xi\eta$

$$\gamma_1 = \sin \varphi. \qquad (1.4)$$

Let us assume the terrestrial equatorial plane being the coordinate plane $\xi\eta$. Then φ will represent the geographical latitude of the satellite.

Let us ascertain the geographical significance of variable γ_2. From the spherical triangle in Fig.1 follows the dependence

$$\operatorname{tg} i \cos u = \operatorname{ctg} \delta, \qquad (1.5)$$

where δ is the angle between the orbit and the meridional planes. Taking into account (1.3), formula (1.5) may be rewritten in the form

$$\gamma_2 = \gamma_3 \operatorname{ctg} \delta. \qquad (1.6)$$

For the angle λ, constituted by the meridional plane and the plane (Fig.1), we may obtain the following equation:

$$\frac{d\lambda}{dv} = \frac{\gamma_3}{1-\gamma_1^2}. \qquad (1.7)$$

Let Ω be the angular velocity of Earth's rotation; then, equation

$$\frac{d\lambda_\Gamma}{dv} = \frac{\gamma_3}{1-\gamma_1^2} - \Omega \frac{dt}{dv}$$

will determine the variation of satellite's geographical longitude λ_Γ.

For the sake of illustration we shall consider two examples.

1) Assume that the orbit plane is not perturbed.

Postulating $j_z = 0$, we shall obtain from Eqs.(1)

$$\gamma_1 = A \sin (v+B), \quad \gamma_2 = A \cos (v+B), \quad \gamma_3 = \gamma_{30} = \cos i_0, \qquad (1.8)$$

where $A = \sqrt{1-\gamma_{30}^2} = \sin i_0$, $B = \operatorname{arctg}\frac{\gamma_{10}}{\gamma_{20}} = u_0$ (the index 0 corresponding to the initial data at v = 0).

According to (1.6), (1.8), the variation of the angle δ will be determined by the formula

$$\operatorname{ctg}\delta = \operatorname{tg} i_0 \cos(v+u_0). \tag{1.9}$$

From Eq.(1.7) we shall obtain the expression for the angle λ

$$\lambda = \operatorname{arctg}[\cos i_0 \operatorname{tg}(v+u_0)] + C. \tag{1.10}$$

2) Let us consider the motion of the satellite in the assumption that $k = j_z/\mu p \gamma^3$ maintains a constant value. In this case the satellite's radius-vector describes in the inertial space a cone with an aperture angle equal to $\pi - 2\operatorname{arc\,tg} k$ (see [1]). Therefore, independently from the variation of orbit shape in its instantaneous plane, the projection of satellite's trajectory on the Earth's surface will have the shape of a small circle (without taking into account the Earth's rotation). Let us consider the solution corresponding to the initial conditions $\gamma_1(0) = \gamma_2(0) = 0, \gamma_3(0) = 1$ $(i(0) = 0)$,

$$\begin{aligned}
\gamma_1 &= \frac{k}{1+k^2}(1-\cos\sqrt{1+k^2}\,v),\\
\gamma_2 &= \frac{k}{\sqrt{1+k^2}}\sin\sqrt{1+k^2}\,v,\\
\gamma_3 &= \frac{1}{1+k^2}(1+k^2\cos\sqrt{1+k^2}\,v).
\end{aligned} \tag{1.11}$$

Hence we find the expressions for the angles ϕ, λ, δ (see (1.4), (1.6), (1.7) at the additional condition $\lambda(0) = 0$).

$$\varphi = \arcsin\left[\frac{k}{1+k^2}(1-\cos\sqrt{1+k^2}\,v)\right], \tag{1.12}$$

$$\lambda = \operatorname{arctg}\left[\sqrt{1+k^2}\,\frac{\sin\sqrt{1+k^2}\,v}{k^2+\cos\sqrt{1+k^2}\,v}\right], \tag{1.13}$$

$$\delta = \operatorname{arcctg}\left[\frac{\sqrt{1+k^2}}{k}\,\frac{\sin\sqrt{1+k^2}\,v}{1+k^2\cos\sqrt{1+k^2}\,v}\right], \tag{1.14}$$

Formulas (1.12), (1.13) generalize the results obtained in the work [2] for the case of circular orbit.

2. Eqs.(1) admit the first integral

$$\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1, \tag{2.1}$$

with the aid of which the system of Eqs.(1) may be reduced to a single differential equation of second order

$$\frac{d^2\gamma_1}{dv^2}+\gamma_1 = U(v)\sqrt{1-\gamma_1^2-\left(\frac{d\gamma_1}{dv}\right)^2}. \tag{2.2}$$

where $U(v) = j_z / \mu p \gamma^3$.

Eq.(2.2) corresponds to the case $\gamma_3 > 0$ $(0 \leqslant i < 90°)$; for $\gamma_3 < 0$ $(90° \leqslant i < 180°)$ the square root in the right-hand part must be taken with the sign minus.

According to the solution found for $\gamma_1(v)$, the angles i and λ are determined by the formulas

$$i = \arccos \sqrt{1 - \gamma_1^2 - (d\gamma_1/dv)^2},$$

$$\lambda = \int_0^v \frac{\sqrt{1 - \gamma_1^2 - (d\gamma_1/dv)^2}}{1 - \gamma_1^2} dv + C.$$

Eq.(2.2) may be considered as the equation of oscillations of a certain mechanical system of unit mass with natural frequency $\omega = 1$, being under the action of an external nonlinear perturbation.

The analogy given is quite remarkable. Hence it follows that the solution of Eq,(2.2) will have properties inherent to the so-called quasilinear oscillating systems. At the same time the appearance of such effects as various-types of resonance oscillations, periodical or nearly periodical oscillations, autooscillations, is possible [3, 4].

Let us consider as an example the problem of rotation of satellite's elliptical orbit plane with the help of time-constant in magnitude small perturbing acceleration j_z, normal to the osculating orbital plane. In this case the variable coefficient U(v) in the right-hand part of Eq,(2.2) has the form

$$U(v) = \frac{j_z p^2}{\mu} \frac{1}{[1 + e\cos(v - v_\pi)]^3}, \tag{2.3}$$

where j_z, μ, p, e, v_π are constants [1]. Inasmuch as the frequency of the perturbing force coincides with system's (2.2) natural frequency, it is obvious that a resonance case takes place. Indeed, let us consider the solution corresponding to zero initial conditions

$$(\gamma_{10} = 0, \left.\frac{d\gamma_1}{dv}\right|_{v=0} = 0,$$

i. e., the initial orbit plane coincides with the plane $\xi\eta$). This solution should be represented in the first approximation in the form [5]

$$\gamma_1 = A \sin (v + B), \tag{2.4}$$

where

$$A = \sin l\varepsilon v, \quad B = v_\pi = \text{const},$$

$$l = -\frac{3}{2} \frac{e}{(1 - e^2)^{3/2}}, \quad \varepsilon = \frac{j_z}{g_0}, \quad g_0 = \frac{\mu}{p^2}.$$

Therefore, a typical manifestation of resonance takes place: the small perturbing force leads to a significant variation (from 0 to 1) of coordinate's γ_1 oscillation amplitude.

The physical significance of the solution (2.4) consists in that the influence of the considered perturbing acceleration leads to monotonic rotation of the orbital plane around the line of apsides, remaining fixed in the inertial space.

The analogy of Eqs.(1.1) with the equation of forced oscillations of the quasilinear mechanical system allows us to apply the methods of the theory of nonlinear oscillations when investigating the problems linked with orbit plane rotation [3, 4].

3. The case presented in section 2 shows that the oscillating character of quantity γ_1 or angle δ variation during the action of a small perturbing force is maintained over a long time interval even in the resonance case

$$\phi = \arcsin [A \sin (v + B)].$$

It is clear, a priori, that in order to obtain, for example, the monotonic dependence $\phi(v)$, it is necessary to assign in the right-hand part of Eq.(2.2) a function $j_z(v)$, either great in magnitude or rapidly rising. Such a qualitative variation in the character of dependence $\phi(v)$ will obviously require significant expenditures of characteristic velocity

$$\Delta V = \int |j_z| dt.$$

Let us consider the case when the dependence $\phi(v)$ has the form

$$\phi = \alpha v + \phi_0, \tag{3.1}$$

where α and ϕ are constants ($|\alpha| < 1$).

According to (3.1), we shall have

$$\begin{aligned} \gamma_1 &= \sin (\alpha v + \varphi_0), \\ \gamma_2 &= \alpha \cos (\alpha v + \varphi_0), \\ \gamma_3 &= \sqrt{1 - \alpha^2} \cos (\alpha v + \varphi_0). \end{aligned} \tag{3.2}$$

As may be seen from (1.6). at an assigned law of latitude ϕ variation the angle δ maintains the constant value $\operatorname{ctg} \delta = \alpha / \sqrt{1 - \alpha^2}$, i. e., the projection of satellite's orbit on the ground surface has the form of loxodrome (Fig.2). The constant α is linked with the angle δ by the simple dependence

$$\alpha = \cos \delta \tag{3.3}$$

Taking into account (3.2), (3.3), we shall find the angle λ from Eq.(1.7)

$$\lambda = \frac{1}{2}\operatorname{tg}\delta \ln\left[\frac{1+\sin(\alpha v+\varphi_0)}{1-\sin(\alpha v+\varphi_0)}\right] + C. \tag{3.4}$$

It is evident that the preassignment of either the dependence (3.1) or (3.2) leads to the inverse problem of the determination of the law of component j_z variation of the perturbing acceleration (or, to be more precise, of the quantity $j_z/\mu p y^3$). Indeed from Eq.(2.2) we obtain outright

$$\frac{j_z}{\mu p y^3} = \sqrt{1-\alpha^2}\operatorname{tg}(\alpha v+\varphi_0). \tag{3.5}$$

Therefore, in the case considered the component j_z must increase boundlessly over the interval $v - v_0 = (\pi/2 - \phi_0)/\alpha$, corresponding to the attainment of latitude's maximum value $\phi = \pi/2$.

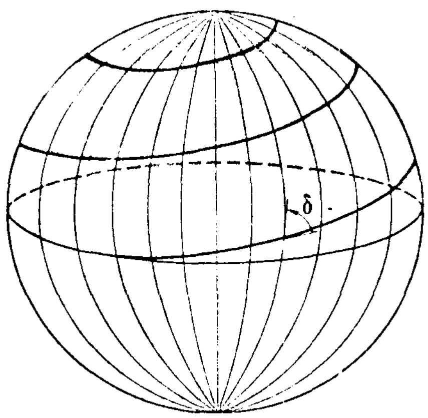

Fig.2

Note that the orbit may, in general case, change arbitrarily its shape under the influence of the comp nent of the perturbing acceleration acting in the osculating plane of the orbit, whereupon the quantities p and y will be certain functions of the variable v. In the simplest case, when the orbit maintains in its instantaneous plane a circular form, we shall have p = const.; y = const. At the same time formula (3.5) will determine the law of lateral overload variation j_z/g_0 ($g_0 = \mu/p^2$ being the gravitation acceleration at circular orbit height).

Shown in Fig.3 are the dependences of the quantity $j_z/\mu p y^3$ on time (the number of satellite revolution $T = v/2\pi$ being plotted in abscissa), corresponding to various values of the angle δ. The initial value of the latitude λ_0 was taken equal to zero (for $\phi_0 > 0$ the origin of the coordinates shifts along the abscissa axis by the quantity $\phi_0/2\pi\cos\delta$).

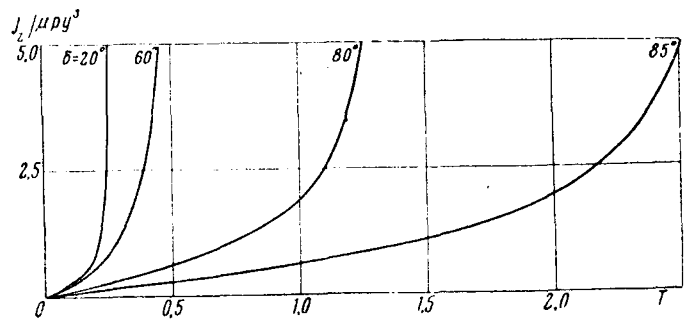

Fig.3

It is obvious that with the help of Eq.(2.2) we may analogously find the law of variation of the component j_z of the perturbing acceleration and in the general case, when the projection of the satellite trajectory $\phi(v)$ is preassigned in an arbitrary form.

***** THE END *****

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