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VOLUME III

A STUDY OF HEREDITARY SPRINGS IN RELATION TO HYSTERETIC DAMPING

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FOREWORD

This Technical Memorandum was prepared by LMSC/HREC for NASA/MSFC under Contract NAS8-20387, entitled "Non-Linear Dynamic Analysis of Structures."

This report is Volume III of three volumes which comprise the Final Report under Contract NAS8-20387 as follows:

Volume I - "Synthesis of Structural Damping," by C. S. Chang and R. E. Bieber (LMSC/HREC A783975)

Volume II - "Non-linear Dynamic Analysis," by R. O. Hultgren (LMSC/HREC A783963)

Volume III - "A Study of Hereditary Springs in Relation to Hysteretic Damping," by G. A. Ramerez (LMSC/HREC A783201)

INTRODUCTION

There are two mathematical models which are commonly used to represent the restoring force in a spring-mass system showing hysteretic damping. The first is the complex stiffness model where the restoring force is taken as

$$F(x) = (a + ib)x$$

where a and b are constants. The second is the frequency dependent damping model where the force is taken as

$$F(x) = \frac{h}{\omega} \dot{x} + rx$$

where h and r are constants. Both models yield an energy loss per cycle proportional to amplitude square and independent of frequency under forced sinusoidal motion.

By considering a combination of springs and dashpots, Caughey (Reference 4) introduced a third model giving the restoring force as

$$F(t) = rx(t) - A \int_{t_o}^{t} Ei\left[- t\ (t - \tau)\right] \frac{d}{d\tau} x(\tau)\, d\tau$$

where r and A are constants and Ei(u) is the exponential integral. Caughey also discusses the deficiency of the above first two models.

In this report we investigate the hysteretic spring and find that, under a suitable choice of the memory function, energy loss per cycle proportional to amplitude square and independent of frequency for a spring-mass system can be predicted. We find four such functions which are equivalent to four models for the restoring force. One of these of course, is the one proposed by Caughey in Reference 4. However, we must mention that our mathematical considerations are independent of Caughey's representation of springs and dashpots. In addition, we find that we can predict energy loss per cycle that is given by a polynomial of even powers of the amplitude and independent of frequency.

Section 1
HEREDITARY SPRING

Let us recall that a relation written as

$$Z(t) = \underset{-\infty}{\overset{t}{M}}\left[f(\tau);\ b\right] \tag{1-1}$$

means that Z depends on the values taken by one or more functions of the time, $f(\tau)$, in the interval between $-\infty$ and the present time, t. The quantities Z are said to be functionals of $f(\tau)$ and are also functions of t and the parameter b in the ordinary sense (cf. Volterra in Reference 1).

It is known that the force displacement relation of a linear spring is given by

$$F = rx$$

where F is the force, r the spring constant and x the displacement (or elongation) measured from some reference position where the force is equal to zero. We wish to explore some of the more obvious consequences when the force displacement relation of the spring has the general form

$$F(t) = rx(t) + \mathscr{F}\left[x(\tau)\right]_{-\infty}^{t} \tag{1-2}$$

i.e., the force depends not only on the actual displacement but also on the values taken by the displacement in the interval between $-\infty$ and the present time, t. Equation (1-2) is the force displacement relation of a hereditary spring, i.e., a spring with memory. It is then said that the function $x(\tau)$, $\tau \epsilon (-\infty, t)$, determines the history of the displacement x (the primary history), and the function F(t) determines the history of the force F (the hereditary history). We often find it more convenient to introduce the difference history $x_d(\tau)$ of the displacement x defined by

$$x_d(\tau) = x(\tau) - x(t) \qquad (1\text{-}3)$$

Let us use the convention that whenever the argument variable t is not exhibited, it means that the value of the parameter is that at the present time t.

We may write Equation (1-2) in the form

$$\begin{aligned} F &= rx + \mathscr{F}\Big[x(\tau)\Big]_{-\infty}^{t} \\ &= rx + \mathscr{F}\Big[x_d(\tau);\ x\Big]_{-\infty}^{t} \end{aligned} \qquad (1\text{-}4)$$

We remark that for a spring which has always been held at a constant value x, the difference history $x_d(\tau)$ becomes the zero history $x_d(\tau) \equiv 0$.

Under the proper smoothness conditions, the functional $\mathscr{F}$ can be expanded in a series analogous to a Taylor's series (Reference 1) and if all the terms in it of higher order than the first can be neglected, we obtain a linear functional, i.e.,

$$\mathscr{F}\Big[x_d(\tau);\ x\Big]_{-\infty}^{t} = \int_{-\infty}^{t} \varphi(t, \tau, x)\, x_d(\tau)\, d\tau \qquad (1\text{-}5)$$

where the kernal, φ, is a continuous function of t, τ, and x. Thus, for this special case Equation (1-4) reduces to

$$F = rx + \int_{-\infty}^{t} \varphi(t, \tau, x)\, x_d(\tau)\, d\tau \qquad (1\text{-}6)$$

We shall further assume that the kernel φ satisfies an order condition of the form

$$\varphi(t, \tau, x) < \frac{M}{(t - \tau)^{1+\delta}} \quad , \qquad \delta > 0 \qquad (1\text{-}7)$$

Thus, in Volterra's terminology, (Reference 1) $\mathcal{F}$ is a linear functional with order of continuity δ. Now in Reference 2, Volterra has proven that with the order-condition, Equation (1-7), F will be a periodic function of t whenever x(t) is a periodic function (principe du cycle ferme) if and only if the kernel function φ depends on the two variables, t and τ, through their difference, $t - \tau$. We shall delimit the the class of spring under consideration by assuming that F is periodic when x is periodic. Volterra calls the hereditary of the nature we assume here "invariable hereditary."

It follows that under the above assumptions Equation (1-6) has the reduced form

$$F = rx + \int_{-\infty}^{t} \varphi(t - \tau, x)\, x_d(\tau)\, d\tau \qquad (1\text{-}8)$$

It is convenient to introduce the change of variable

$$s = t - \tau \quad , \qquad 0 \leq s < \infty \qquad (1\text{-}9)$$

and write Equation (1-8) in the form

$$F = rx + \int_0^{\infty} \varphi(s,x)\, x_d^{(t)}(s)\, ds \tag{1-10}$$

where

$$x_d^{(t)}(s) = x(t - s) - x(t) \tag{1-11}$$

Let us further introduce the function $\Psi(s,x)$ through

$$\Psi(s,x) = -\int_s^{\infty} \varphi(\lambda, x) d\lambda \tag{1-12}$$

whence

$$\frac{d}{ds}\Psi(s,x) = \varphi(s,x) \tag{1-13}$$

We note that in view of Equation (1-12)

$$\lim_{s \to \infty} \Psi(s,x) = 0 \tag{1-14}$$

Introduction of Equation (1-13) into Equation (1-9) and integration by parts yields

$$F = rx - \int_0^{\infty} \Psi(s,x) \frac{d}{ds}\, x(t - s)\, ds \tag{1-15}$$

where we have used Equation (1-14) and that in view of Equation (1-11)

$$x_d^{(t)}(0) = x(t) - x(t) = 0$$

$$\frac{d}{ds} x_d^{(t)}(s) = \frac{d}{ds} x(t - s)$$

Let us further assume that the kernel function Ψ can be approximated by a Taylor series expansion about $x = 0$ and retain only terms up to x^2, i.e., we assume

$$\Psi(s, x) = \Psi_0(s) + \Psi_1(s)x + \Psi_2(s)x^2 \tag{1-16}$$

It follows that with Ψ given by Equation (1-16), the force displacement relation becomes

$$F = rx - \int_0^\infty \Psi_0(s) \frac{d}{ds} x(t - s)\, ds - x \int_0^\infty \Psi_1(s) \frac{d}{ds} x(t - s)\, ds - x^2 \int_0^\infty \Psi_2(s) \frac{d}{ds} x(t - s)\, ds \tag{1-17}$$

If we take the memory functions Ψ_1 and Ψ_2 to be zero, then we have

$$F = rx - \int_0^\infty \Psi_0(s) \frac{d}{ds} x(t - s)\, ds \tag{1-18}$$

Section 2
ENERGY LOSS PER CYCLE

Let us assume as a displacement history

$$x(\tau) = A \sin(\omega\tau + \phi) \tag{2-1}$$

where

$$x(t - s) = A \sin(\omega t + \phi) \cos\omega s - A \cos(\omega t + \phi) \sin\omega s \tag{2-2}$$

Substituting Equation (2-2) into the force displacement relation, Equation (1-17), yields

$$\begin{aligned} F(t) = {} & A \sin(\omega t + \phi) \Big[r + \omega \overset{1}{\gamma}_0(\omega) + A\omega \sin(\omega t + \phi) \overset{1}{\gamma}_1(\omega) \\ & + A^2\omega \sin(\omega t + \phi) \overset{1}{\gamma}_2(\omega) \Big] + A \cos(\omega t + \phi) \Big[\omega \overset{2}{\gamma}_0(\omega) \\ & + A\omega \sin(\omega t + \phi) \overset{2}{\gamma}_1(\omega) + A^2\omega \sin^2(\omega t + \phi) \overset{2}{\gamma}_2(\omega) \Big] \end{aligned} \tag{2-3}$$

where

$$\begin{aligned} \overset{1}{\gamma}_n(\omega) &= \int_0^\infty \Psi_n(s) \sin\omega s \, ds \quad , \quad n = 0, 1, 2 \\ \overset{2}{\gamma}_n(\omega) &= \int_0^\infty \Psi_n(s) \cos\omega s \, ds \quad , \quad n = 0, 1, 2 \end{aligned} \tag{2-4}$$

The energy loss per cycle corresponding to the history, Equation (2-1), is given by

$$\Delta W = \int_{T}^{T + 2\pi/\omega} F(\tau) \frac{d}{d\tau} x(\tau)\, d\tau \tag{2-6}$$

Performing the integration of Equation (2-6), we obtain

$$\Delta W = \pi A^2 \omega \left[\overset{2}{\gamma}_0 (\omega) + \frac{A^2}{4} \overset{2}{\gamma}_2 (\omega) \right] \tag{2-7}$$

We note that the absence of $\overset{2}{\gamma}_1 (\omega)$ in Equation (2-7) implies that for the history, Equation (2-1), the term

$$x(t) \int_0^\infty \Psi_1 (s) \frac{d}{dt} x (t - s)\, ds$$

does not contribute to the energy dissipated per cycle.

Suppose that we were to require that under the history, the hereditary spring give an energy loss per cycle that was proportional to A^2 and independent of ω for all $\omega > 0$. Inspection of Equation (2-7) shows that for this spring the memory function, $\Psi_2(s)$ must be equal to zero and that

$$\overset{2}{\gamma}_0 (\omega) = \frac{1}{\omega} \tag{2-8}$$

Now we know that

$$\overset{0}{\gamma}_2(\omega) = \sqrt{\frac{\pi}{2}}\ \overset{0}{\Gamma}_2(\omega) \tag{2-9}$$

where $\overset{0}{\Gamma}_2(\omega)$ is the Fourier cosine transform of $\Psi_0(s)$. In view of Equations (2-8) and (2-9), it follows that

$$\overset{0}{\Gamma}_2(\omega) \approx \frac{1}{\omega} \tag{2-10}$$

or equivalently

$$\overset{0}{\Gamma}_2(\omega) = \frac{B}{\omega} \tag{2-11}$$

where B is a positive constant. Formally then, we would have

$$\Psi_0(s) = \sqrt{\frac{2}{\pi}}\ B \int_0^\infty \frac{\cos \omega s}{\omega}\, d\omega \tag{2-12}$$

However, the integral on the right-hand side of Equation (2-12) has the value

$$\int_0^\infty \frac{\cos \omega s}{\omega}\, d\omega = \infty \tag{2-13}$$

independent of s. This means that our considerations so far cannot yield such a spring.

Consider the memory function given by

$$\Psi_0(s) = - B_0 \, Y_0 \, (\alpha s) \quad , \qquad \alpha > 0 \qquad (2\text{-}14)$$

where B_0 is a positive constant and $Y_0(\alpha s)$ is Weber's Bessel function of the second kind of order zero. It can be shown (Reference 3) that

$$- B_0 \int_0^{\infty} Y_0(\alpha s) \cos \omega s \, ds = \frac{B_0}{\omega} \frac{1}{\left[1 - \left(\frac{\alpha}{\omega}\right)^2\right]^{1/2}} \quad , \qquad \alpha<\omega<\infty \qquad (2\text{-}15)$$

We have for $0<\alpha/\omega < 0.1$,

$$1.0 < \frac{1}{\left[1 - \left(\frac{\alpha}{\omega}\right)^2\right]^{1/2}} < 1.005 \qquad (2\text{-}16)$$

Thus by choosing $\Psi_0(s)$ to be given by Equation (2-14) for $0<\alpha/\omega<0.1$, the energy lost per cycle is essentially proportional to A^2 and independent of ω. Moreover, if we further choose

$$\Psi_2(s) = - B_2 \, Y_0(\alpha s) \quad , \qquad \alpha>0 \qquad (2\text{-}17)$$

where B_2 is a positive constant, the energy loss per cycle, Equation (2-7), is given by

$$\Delta W = K_0 \, A^2 + K_2 \, A^4 \qquad (2\text{-}18)$$

where

$$K_0 = \frac{\pi B_0}{\left[1 - \left(\frac{\alpha}{\omega}\right)^2\right]^{1/2}} , \quad K_2 = \frac{\pi B_2}{4\left[1 - \left(\frac{\alpha}{\omega}\right)^2\right]^{1/2}} . \tag{2-19}$$

Thus, for $0<\alpha/\omega<0.1$, the choice of Equations (2-14) and (2-17) for $\Psi_0(s)$ respectively, yields an energy loss per cycle which is independent of ω and "non-linear" in A^2.

Considering that in the force-displacement relation of the hereditary spring, Equation (1-17), the term associated with the memory function $\Psi_1(s)$ does not contribute to the energy dissipated per cycle, we might choose $\Psi_1(s) = 0$. It follows that the force-displacement relation

$$F(t) = rx(t) + B_0 \int_0^{\infty} Y_0(\alpha s) \frac{d}{ds} x(t-s)\, ds \tag{2-20}$$

will, under the history, Equation (2-1), and $0<\alpha/\omega<0.1$, yield an energy dissipated per cycle that is linear in A^2 and independent of ω, while the force-displacement relation

$$F(t) = rx(t) + \left[B_0 + B_2\, x^2(t)\right] \int_0^{\infty} Y_0(\alpha s) \frac{d}{ds} x(t-s)\, ds \tag{2-21}$$

will yield an energy dissipated per cycle which is non-linear in A^2 and independent of ω.

Let

$$\Psi_i(s) = - \int_s^{\infty} \varphi_i(\lambda)\, d\lambda \quad , \qquad i = 0, 2 \tag{2-22}$$

Then

$$\overset{2}{\gamma}_n(\omega) = - \int_0^\infty \int_s^\infty \varphi_n(\xi)\, d\xi \, \cos\omega s \, ds$$

$$= - \int_0^\infty \int_0^\xi \cos\omega s \, ds \;\; \varphi_n(\xi)\, d\xi \tag{2-23}$$

$$= - \frac{1}{\omega} \int_0^\infty \varphi_n(\xi) \sin\omega\xi \, d\xi \quad , \quad n = 0, 2$$

Equation (2-7) now reads

$$\Delta W = - \pi A^2 \left[\overset{1}{\lambda}_0(\omega) + \frac{A^2}{4} \overset{1}{\lambda}_2(\omega) \right] \tag{2-24}$$

where

$$\overset{1}{\lambda}_n(\omega) = \int_0^\infty \varphi_n(s) \sin\omega s \, ds \quad , \quad n = 0, 2 \tag{2-25}$$

For the choice

$$\varphi_0(s) = B_0 \, e^{-\frac{1}{2}\alpha s} \left(1 - e^{-\alpha s}\right)^{-1} = 2\, B_0 \left(\sinh \frac{\alpha s}{2}\right)^{-1} , \quad \alpha > 0 \tag{2-26}$$

where B_0 is a constant, we have

$$\overset{1}{\lambda}_0(\omega) = -\frac{B_0}{2\alpha} \tanh\left(\pi\frac{\omega}{\alpha}\right) \tag{2-27}$$

For $\omega/\alpha \geq 6.5/\pi$, we have

$$\overset{1}{\lambda}_0(\omega) = -\frac{B_0}{2\alpha} \tag{2-28}$$

For the choice

$$\varphi_0(s) = -B_0 \frac{\cos\alpha s}{s}, \qquad \alpha > 0 \tag{2-29}$$

we have

$$\overset{1}{\lambda}_0(\omega) = \begin{cases} 0 & 0<\omega<\alpha \\ -\dfrac{B_0\pi}{4} & \omega=\alpha \\ -\dfrac{B_0\pi}{2} & \alpha<\omega<\infty \end{cases} \tag{2-30}$$

For the choice

$$\varphi_0(s) = -B_0 \frac{e^{-\alpha s}}{s}, \qquad \alpha>0 \tag{2-31}$$

we have

$$\overset{1}{\lambda}_0(\omega) = -B_0 \tan^{-1}\left(\frac{\omega}{\alpha}\right) \tag{2-32}$$

For $\omega/\alpha > 10$ and the choice of Equation (2-31), $\overset{1}{\lambda}_0(\omega)$ becomes

$$\overset{1}{\lambda}_0(\omega) \approx - B_0 \quad , \qquad \omega/\alpha > 10 \tag{2-33}$$

For each one of the choices of $\varphi_0(s)$, Equations (2-26), (2-29) and (2-31), the energy loss per cycle will, for some range of ω, be proportional to A^2 and independent of frequency.

For $\varphi_0(s)$ given by Equation (2-26), we have

$$\Psi_0(s) = - \frac{4B_0}{\alpha} \log \tanh \frac{\alpha s}{4} \tag{2-34}$$

For $\varphi_0(s)$ given by Equation (2-30), we have

$$\Psi_0(s) = B_0 \int_s^\infty \frac{\cos \alpha\xi}{\xi} \, d\xi = - B_0 \, \mathrm{Ci}\,(\alpha s) \tag{2-35}$$

where $\mathrm{Ci}\,(\alpha s)$ is the cosine integral.

For $\varphi_0(s)$ given by Equation (2-31), we have

$$\Psi_0(s) = B_0 \int_s^\infty \frac{e^{-\alpha\xi}}{\xi} \, d\xi = - B_0 \, \mathrm{Ei}(-\alpha s) \tag{2-36}$$

where $\mathrm{Ei}(-\alpha s)$ is the exponential integral.

Section 3
EQUATION OF MOTION FOR A BEAM

Under the change in variable $\tau = t - s$ we can write the spring force-displacement relation, Equation (1-18), in the form

$$F(t) = rx(t) + \int_{-\infty}^{t} \Psi_0(t - \tau) \frac{d}{d\tau} x(\tau)\, d\tau \tag{3-1}$$

where $\Psi_0(t)$ is any one of the memory functions discussed in the preceeding section.

Let us consider a beam made of material that has a stress-strain relation of the form

$$\sigma(x, z, t) = E \left[\epsilon(x, z, t) + \int_{-\infty}^{t} \Psi_0(t - \tau) \frac{d}{d\tau} \epsilon(x, z, \tau)\, d\tau \right] \tag{3-2}$$

where $\sigma(x, z, t)$ and $\epsilon(x, z, \tau)$ are the stress at the present time and engineering strain at the generic time τ at a point $P(x, z)$ located at a distance z from the neutral axis. Let $\rho(x,\tau)$ be the radius of curvature of the neutral axis of the beam at the generic time τ. With the assumption that plane sections before deformation remain plane after deformation, it can be established that

$$\epsilon(x, z, \tau) = \frac{z}{\rho(x,\tau)} \tag{3-3}$$

From this we find

$$\frac{d}{d\tau}\epsilon(x, z, \tau) = z \frac{\partial}{\partial \tau}\left[\frac{1}{\rho(x,\tau)}\right] \tag{3-4}$$

We may therefore write Equation (3-2) in the form

$$\sigma(x, z, t) = Ez\left\{\frac{1}{\rho(x, t)} + \int_{-\infty}^{t} \Psi_0(t - \tau) \frac{\partial}{\partial \tau}\left[\frac{1}{\rho(x,\tau)}\right] d\tau\right\} \tag{3-5}$$

The bending moment, $M(x, t)$, at the point x may then be obtained from

$$M(x, t) = \int_{A(x)} z\sigma(x, z, t)\, dA$$

as

$$M(x, t) = EI(x)\left\{\frac{1}{\rho(x, t)} + \int_{-\infty}^{t} \Psi_0(t - \tau) \frac{\partial}{\partial \tau}\left[\frac{1}{\rho(x,\tau)}\right] d\tau\right\} \tag{3-6}$$

where $I(x)$ is the moment of inertia of the beam.

The relation between the displacement $y(x,\tau)$ of the neutral axis of the beam and $\rho(x,\tau)$ is given by

$$\frac{1}{\rho(x,\tau)} = \frac{y''(x,\tau)}{\left\{1 + [y'(x,\tau)]^2\right\}^{3/2}} \tag{3-7}$$

where a prime indicates partial derivative with respect to x. It follows that

$$\frac{\partial}{\partial \tau}\left[\frac{1}{\rho(x,\tau)}\right] = \frac{\dot{y}''(x,\tau)}{\left\{1+\left[y'(x,\tau)\right]^2\right\}^{3/2}} - \frac{3y''(x,\tau)\, y'(x,\tau)\, \dot{y}'(x,\tau)}{\left\{1+\left[y'(x,\tau)\right]^2\right\}^{5/2}} \tag{3-8}$$

where a dot over the quantity indicates the partial derivative with respect to τ.

It is customary to assume that the displacement history $y(x,\tau)$ of the neutral axis of the beam is such that

$$\left[y'(x,\tau)\right]^2 << 1 \tag{3-9}$$

for all $\tau \leq t$. Using Equation (3-9) we may approximate Equation (3-7) by

$$\frac{1}{\rho(x,\tau)} \approx y''(x,\tau) \tag{3-10}$$

and Equation (3-8) by

$$\frac{\partial}{\partial \tau}\left[\frac{1}{\rho(x,\tau)}\right] \approx \dot{y}''(x,\tau) - \frac{3}{2}\frac{\partial}{\partial x}\left[y'(x,\tau)\right]^2 \dot{y}'(x,\tau) \tag{3-11}$$

Substitution of Equations (3-10) and (3-11) into Equation (3-6) yields

$$\begin{aligned} M(x,t) = EI(x)\, y''(x,t) &+ \int_{-\infty}^{t} \Psi_0(t-\tau)\, \dot{y}''(x,\tau)\, d\tau \\ &- \frac{3}{2}\int_{-\infty}^{t} \Psi_0(t-\tau)\, \frac{\partial}{\partial x}\left[y'(x,\tau)\right]^2 \dot{y}'(x,\tau)\, d\tau \end{aligned} \tag{3-12}$$

Let

$$\hat{M}(x,t) = -\frac{3}{2}\ EI(x) \int_{-\infty}^{t} \Psi_0(t-\tau)\ \frac{\partial}{\partial x}\left[y'(x,\tau)\right]^2 \dot{y}'(x,\tau)\ d\tau \tag{3-13}$$

and write Equation (3-12) as

$$M(x,t) = EI(x)\left[y''(x,t) + \int_{-\infty}^{t} \Psi_0(t-\tau)\ \dot{y}''(x,\tau)\ d\tau\right] + \hat{M}(x,t) \tag{3-14}$$

Note that had we taken the partial derivative of $1/\rho(x,\tau)$ with respect to τ "after" we had made the approximation by using Equation (3-9), we would have lost $\hat{M}(x,t)$. Moreover, at this stage of the formulation the assumption of Equation (3-9) does not yield sufficient grounds for the neglect of this term. In addition, a little thought will show that the assumption of Equation (3-9) and the assumption used to write Equation (3-3), i. e., plane sections remain plane, are indeed independent of each other. Therefore, the grounds (if they exist) for neglect of $\hat{M}(x,t)$ will have to be sought elsewhere.

The equation of motion of the beam may now be obtained from

$$\frac{d^2}{dx^2}\ M(x,t) + \mu(x)\ \frac{\partial^2}{\partial t^2}\ y(x,t) = f(x,t) \tag{3-15}$$

where $\mu(x)$ is the mass density per unit length. Introducing Equation (3-14) into (3-15), we obtain

$$\frac{d^2}{dx^2}\left\{EI(x)\left[y''(x,t) + \int_{-\infty}^{t} \Psi_0(t-\tau)\ \dot{y}''(x,\tau)\ d\tau\right]\right\} + \frac{d^2}{dx^2}\ \hat{M}(x,t)$$

$$+\ \mu(x)\ \frac{\partial^2}{\partial t^2}\ y\ (x,t) = f\ (x,t) \tag{3-16}$$

as the equation of motion of the beam.

Suppose we had found grounds for the neglect of $\hat{M}(x,t)$. The equation of motion would then be

$$\frac{d^2}{dx^2}\left\{EI(x)\left[y''(x,t) + \int_{-\infty}^{t} \Psi_0(t-\tau)\,\dot{y}''(x,\tau)\,d\tau\right]\right\} + \mu(x)\,\frac{\partial^2}{\partial t^2}y(x,t) = f(x,t) \tag{3-17}$$

Let us consider the case of free vibration, i.e., $f(x,t) = 0$, and assume the solution of Equation (3-17) in the form

$$y(x,t) = X(x)\;T(t) \tag{3-18}$$

Substitution of Equation (3-18) into (3-17), with $f(x,t) = 0$, yields

$$\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2X}{dx^2}\right]\left[T + \int_{-\infty}^{t} \Psi_0(t-\tau)\,\frac{dT}{d\tau}\,d\tau\right] + \mu(x)\,X\,\frac{d^2T}{dt^2} = 0 \tag{3-19}$$

This equation suggests that if X is the solution of

$$\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2X}{dx^2}\right] - \mu(x)\,\Omega^2\,X = 0 \tag{3-20}$$

then Equation (3-18) will be the solution to the homogeneous equation of (3-17) if $T(t)$ is the solution of

$$\frac{d^2T}{dt^2} + \Omega^2\left[T + \int_{-\infty}^{t} \Psi_0(t-\tau)\,\frac{dT}{d\tau}\,d\tau\right] = 0 \tag{3-21}$$

When the beam is subjected to homogeneous boundary conditions, Equation (3-20) has an infinite number of solutions X_i $(i = 1, \ldots, \infty)$, i.e., X_i are the eigenfunctions of

$$\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2X_i}{dx^2}\right] - \mu(x)\,\Omega_i^2\,X_i = 0 \tag{3-22}$$

We then have, instead of Equation (3-18),

$$y(x,t) = \sum_{i=1}^{\infty} X_i(x)\,T_i(t) \tag{3-23}$$

where $T_i(t)$ are now the solutions of

$$\frac{d^2T_i}{dt^2} + \Omega_i^2\left[T_i + \int_{-\infty}^{t} \psi_0(t-\tau)\,\frac{d}{d\tau}\,T_i(\tau)\,d\tau\right] = 0 \tag{3-24}$$

subject to prescribed initial conditions. Thus, when the vibration of the beam is governed by Equation (3-17), normal uncoupled modes exist but they are damped.

If instead of Equation (3-17) we have Equation (3-16), then separation of variable technique is not applicable since we have a non-linear integro-differential equation. The solution must then be sought by other means and it is senseless to talk about normal modes of vibration for the beam.

By assuming that the hereditary effects in Equation (3-24) prior to an instant $t_0 < t$ are negligible and taking $t_0 = 0$ we can use Laplace tranform techniques to solve Equation (3-24). The problem then reduces to finding the inverse transform of

$$\bar{T}_i = \frac{T_i(0)\ (p + \bar{\Psi}_0) + \dot{T}_i(0)}{p^2 + \Omega_i^2\ (1 + p\ \bar{\Psi}_0)} \tag{3-25}$$

where p is the Laplace transform parameter and a bar over a quantity denotes the Laplace transform. Thus, we have

$$T_i = \frac{1}{2\pi i} \int_{Br} \bar{T}_i(p)\ e^{pt}\ dp \tag{3-26}$$

where $\int_{Br}$ is the Bromwich contour integral. The evaluation of the integral, Equation (3-26) with

$$\Psi_0(t - \tau) = -\ B_0\ \mathrm{Ei}\left[-\alpha(t - \tau)\right] \tag{3-27}$$

has been obtained by Caughey (Reference 4).

The longitudinal vibrations of a rod are governed by the equation

$$\frac{\partial}{\partial x}\sigma(x, t) = \rho(x, t)\ \frac{\partial^2}{\partial t^2}\ u\ (x, t) \tag{3-28}$$

where $\rho(x, t)$ is the mass density per unit volume and $u(x, t)$ is the longitudinal displacement. The strain is now given by

$$\epsilon(x, t) = \frac{\partial}{\partial x}\ u\ (x, t) \tag{3-29}$$

The stress-displacement relation is now given by

$$\sigma(x,t) = E\left[\frac{\partial}{\partial x} u\,(x,t) + \int_{-\infty}^{t} \Psi_0(t-\tau)\,\frac{\partial}{\partial x}\,\frac{\partial u}{\partial \tau}\,d\tau\right] \tag{3-30}$$

To within the first order in $\partial u/\partial t$, the mass density per unit volume is approximately constant in its time variable. Thus, upon substitution of Equation (3-25) into (3-28) we obtain

$$\frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{t} \Psi_0(t-\tau)\,\frac{\partial^2}{\partial x^2}\,\frac{\partial u}{\partial t}\,d\tau = \frac{1}{C^2}\,\frac{\partial^2 u}{\partial t^2} \tag{3-31}$$

where $C^2 = E/\rho$. Concentrating on steady state we seek solutions of Equation (3-31) in the form

$$u(x,t) = X(x)\;T(t) \tag{3-32}$$

Thus Equation (3-31) separates into the two equations, namely

$$\frac{d^2X}{dx^2} + \frac{\Omega^2}{C^2}\,X = 0 \tag{3-33}$$

and

$$\frac{d^2T}{dt^2} + \Omega^2\left[T + \int_{-\infty}^{t} \Psi_0(t-\tau)\,\frac{dT}{d\tau}\right] = 0 \tag{3-34}$$

Thus, normal uncoupled damped modes exist for longitudinal vibrations.

Section 4
CONCLUDING REMARKS

In a real vibratory system described by a spring mass system, the restoring force of the "spring" is not entirely conservative. Under cyclic deformation of the spring, mechanical energy is consumed. The presence of such dissipative forces is generally described by the general term, structural damping. Forces due to structural damping are, as a rule, small, but often their presence affects the dynamic behavior of a vibratory system.

It is generally accepted that when structural damping is caused by the material of the spring in a spring-mass system, the hysteresis of the spring material under cyclic deformation is responsible for the energy dissipation. Precise measurements on the stress-strain relationship of most materials show that even at stress levels much below accepted elastic limits cyclic straining will produce a hysteresis loop.

Many experimentalists have confirmed through their experiments that numerous engineering materials will, for a wide range of frequency, show that the energy loss per cycle is proportional to the square of the strain and independent of the frequency at which the strain is applied.

We have shown that by considering a hereditary spring, energy loss per cycle (proportional to amplitude square and independent of frequency) can be predicted with sufficient accuracy provided we choose an appropriate memory function. Four of these memory functions were found. It is imperative to point out that an equivalent representation of our hereditary spring by an appropriate combination of springs and dashpots need not exist.

The equation of motion for a beam under the Euler-Bernoulli hypothesis was briefly discussed. The stress-strain relation for the beam material was without justification other than by analogy, taken to be in the same form as the hereditary spring.

In a continuous system the energy dissipated must, according to the principle of conservation of energy, be transformed into other forms of energy. In this case the dissipated mechanical energy is transformed into heat. This strongly suggests that the stress-strain relation for a continuous system that shows hysteretic damping be investigated such that it be in accordance with all the conservation principles which unify continuum physics. In particular it must be obtained through proper thermodynamic considerations.

It is strongly recommended that the constitutive equation for this material be obtained through a proper linearization of the constitutive equation of an ideal material under finite strain. In this way all the methodological principles which unify constitutive theory of non-linear continuum mechanics are available in order to arrive at the constitutive equation for the ideal material.

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