RADIATED POWER AND OHMIC LOSS OF THE INFINITELY LONG CYLINDRICAL ANTENNA

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By

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SUMMARY

The input conductance of an infinite antenna formed by a cylindrical tubular conductor of internal impedance z^i is obtained, together with the ohmic loss and the radiated power which contribute to the input conductance. It is found that the ohmic loss on the antenna behaves like $1/[\log|1/z^i|]$ for small z^i, rather than a small perturbation of order z^i. Several conclusions may be drawn: (1) The infinite antenna with zero internal impedance is of very different character from that with non-vanishing z^i, even when z^i is quite small; (2) The present theory may be verified experimentally with an antenna of finite length provided the internal impedance is not too low; (3) In the theory of the very long antenna the internal impedance is not negligible for all practically available metals. A brief qualitative discussion of the very long antenna is given to indicate the need for further study on that subject.

INTRODUCTION

In the theory of the antenna as a boundary-value problem, the antenna has usually been assumed to be made of a perfect conductor since the conductivities of practically available metals are very high. The current distribution and the input admittance of an antenna made of metal are often obtained under the assumption of infinite conductivity and the effect of the finite conductivity is often regarded as a small perturbation. The perturbation is indeed small for a short antenna. As the antenna becomes longer, or in the extreme when it is infinitely long, the similarity between an antenna with infinite condictivity and that with high but finite conductivity disappears. Since the current on the infinitely long antenna with infinite conductivity is not square integrable, perturbation theory cannot be applied in a straightforward manner.

In order to see the effect of finite conductivity on long antennas, it seems logical to try to study the infinite antenna first. In this report the input conductance of an infinitely long antenna formed by a cylindrical tubular conductor of internal impedance z^i is obtained, together with the ohmic loss and the radiated power which contribute to the input conductance. It is found that the ohmic loss on the antenna behaves like $1/[\log|1/z^i|]$ for small z^i, rather than like a small perturbation of order z^i.

THE SOLUTION

Let the radius of the infinitely long tubular antenna be a, the internal impedance per unit length be z^i. The following integral equation for the current I(z) on the antenna holds [1]:

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_{-\infty}^{\infty} K(z-z')I(z')dz' = \frac{4\pi ik}{\zeta_o} [\delta(z) - z^i\, I(z)] \tag{1}$$

where

$$K(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \frac{e^{ikR}}{R}, \qquad R = \sqrt{(z-z')^2 + (2a \sin\theta/2)^2}$$

and z is the axial coordinate; ζ_o is the intrinsic impedance of free space, and k is the free-space wave number. The time dependence is taken to be $e^{-i\omega t}$ where ω is the angular frequency of the driving source which is assumed to be a delta-function generator of unit voltage.

Define

$$\overline{I}(\zeta) = \int_{-\infty}^{\infty} I(z)\, e^{i\zeta z}\, dz$$

then from (1)

$$\overline{I}(\zeta) = \frac{4\pi ik}{\zeta_o} \frac{1}{(k^2-\zeta^2)\overline{K}(\zeta) + \frac{4\pi ik}{\zeta_o} z^i} \tag{2}$$

In (2), $\overline{K}(\zeta) = \pi i J_o(a\sqrt{k^2-\zeta^2})\, H_o^{(1)}(a\sqrt{k^2-\zeta^2})$ [2], with branch cuts chosen as shown in Fig. 1. The input conductance G which is defined as $G = \lim_{z\to 0} \mathrm{Re}\, I(z)$ is readily obtained from (2) with the Fourier integral theorem:

$$G = \frac{1}{2\pi}\left\{ \int_{-k}^{k} \mathrm{Re} \frac{4\pi ik/\zeta_o}{(k^2-\zeta^2)\pi i(J_o+iY_o)J_o + \frac{4\pi ikz^i}{\zeta_o}}\, d\zeta \right.$$

$$\left. + 2\int_{k}^{\infty} \mathrm{Re}\, \frac{4\pi ik/\zeta_o}{(\zeta^2-k^2)2I_oK_o + \frac{4\pi ikz^i}{\zeta_o}}\, d\zeta \right\} \tag{3}$$

In (3), the Bessel functions J_o and Y_o have arguments $a\sqrt{k^2-\zeta^2}$ and the modified Bessel functions I_o and K_o have arguments $a\sqrt{\zeta^2-k^2}$.

The input conductance G can be separated into three parts, namely,

$$G = G_R + G_{H2} + G_{H3}$$

where

$$G_R = \frac{4\pi}{\zeta_o}\int_0^1 \frac{(1-y^2)J_o^2\, dy}{[(1-y^2)\pi J_o^2 + Z_R]^2 + [(1-y^2)\pi J_oY_o - Z_I]^2} = \frac{4\pi}{\zeta_o} I_1 \tag{4a}$$

$$G_{H2} = \frac{4Z_R}{\zeta_o}\int_0^1 \frac{dy}{[(1-y^2)\pi J_o^2 + Z_R]^2 + [(1-y^2)\pi J_oY_o - Z_I]^2} = \frac{4Z_R}{\zeta_o} I_2 \tag{4b}$$

$$G_{H3} = \frac{4Z_R}{\zeta_o}\int_1^\infty \frac{dy}{[(y^2-1)2I_oK_o + Z_I]^2 + Z_R^2} = \frac{4Z_R}{\zeta_o} I_3 \tag{4c}$$

$$Z_R = \frac{2\lambda}{\zeta_o}\,\mathrm{Re}\, z^i, \qquad Z_I = -\frac{2\lambda}{\zeta_o}\,\mathrm{Im}\, z^i$$

The arguments of J_o and Y_o are $ak\sqrt{1-y^2}$, the arguments of I_o and K_o are $ak\sqrt{y^2-1}$. Z_I is positive if z^i is inductive.

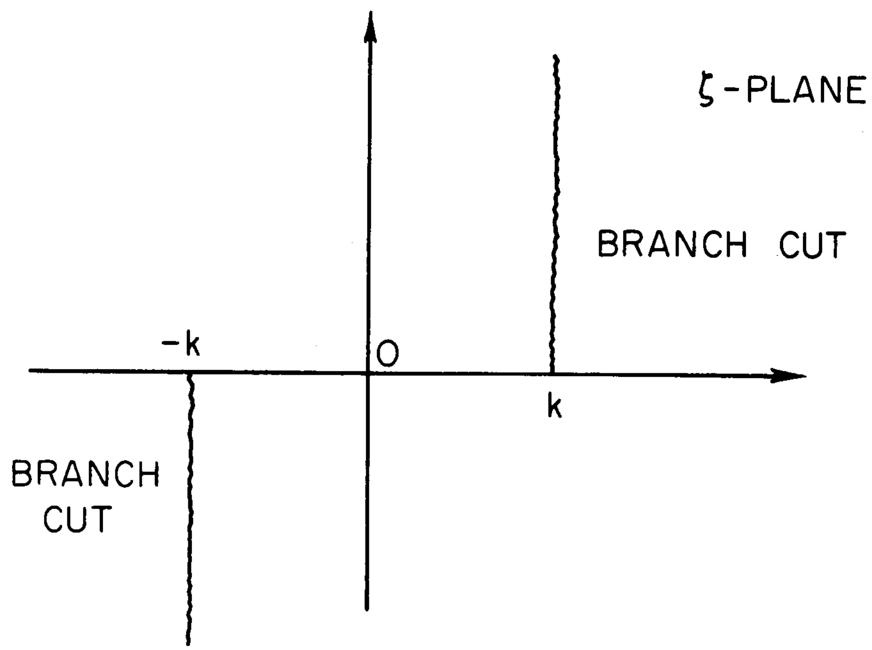

FIG. 1 THE BRANCH CUTS OF $\bar{k}(\zeta)$

The ohmic loss on the antenna can be expressed as the integral of the square of the absolute value of $I(z)$ along z:

$$\text{Ohmic loss} = 2\int_0^\infty \frac{1}{2} \operatorname{Re} z^i \, |I(z)|^2 \, dz \tag{5}$$

According to Parseval's formula, (5) can be written as the integral of the square of the absolute value of $\overline{I}(\zeta)$ along real ζ:

$$\text{Ohmic loss} = \frac{1}{2\pi} \operatorname{Re} z^i \int_0^\infty |\overline{I}(\zeta)|^2 \, d\zeta = \frac{1}{2} (G_{H2} + G_{H3})$$

Thus $G_H = G_{H2} + G_{H3}$ can be identified as one component of the input conductance representing the ohmic loss on the antenna. The other component, the radiated power, is thus represented by G_R. That is,

$$\text{Radiated power} = \frac{1}{2} G_R.$$

The above relation can also be obtained by integrating the Poynting vector over a large sphere.

APPROXIMATIONS

The integrals I_1 through I_3 defined in (4) can be evaluated approximately in the limit of small z^i provided the antenna is "thin" in the sense that

$$\Omega = 2 \log \frac{2}{ka} >> 1 \tag{6a}$$

z^i is said to be small in the sense that

$$|z|^2 = z_R^2 + z_I^2 << \Omega^2. \tag{6b}$$

Under condition (6a), the Bessel functions in the integrands of I_1 and I_2 can be replaced by their limiting forms for small arguments. Similarly, the modified Bessel functions in the integrand of I_3 can also be replaced by their small argument counterparts since the major contribution to I_3 comes from near $y = 1$.

Therefore, under (6a), I_1 through I_3 can be expressed as follows:

$$I_1 = \int_0^1 \frac{(1-y^2)\,dy}{[(1-y^2)\pi + Z_R]^2 + [(1-y^2)Q_1(y) + Z_I]^2} \tag{7a}$$

$$I_2 = \int_0^1 \frac{dy}{[(1-y^2)\pi + Z_R]^2 + [(1-y^2)Q_1(y) + Z_I]^2} \tag{7b}$$

$$I_3 = \int_1^\infty \frac{dy}{[(y^2-1)Q_3(y) + Z_I]^2 + Z_R^2} \tag{7c}$$

where

$$Q_1(y) = \Omega - 2\gamma - \log(1-y^2)$$

$$Q_3(y) = \Omega - \log(y^2-1)$$

$$\gamma = 0.57721566$$

The logarithmic functions in (7b) and (7c) are further approximated by constants such that $\log(1-y^2)$ is replaced by $\log \xi_2$ in (7b) where ξ_2 satisfies

$$(\xi_2\pi + Z_R)^2 + [\xi_2(\Omega - 2\gamma - \log \xi_2) + Z_I]^2 = (K^2 + 1)|Z|^2 \tag{8a}$$

and $\log(y^2-1)$ is replaced by $\log \xi_3$ in (7c) where ξ_3 satisfies

$$[\xi_3(\Omega - \log \xi_3) + Z_I]^2 + Z_R^2 = (K^2 + 1)|Z|^2. \tag{8b}$$

If K is treated as a constant, the above two equations can be solved for ξ_2 and ξ_3 by iteration. Generally, one or two iterations are good enough for the purpose of caluclating Q_2 and Q_3 which are defined by

$$\left.\begin{aligned} Q_2(y) &\simeq Q_2 = \Omega - 2\gamma - \log \xi_2 \\ Q_3(y) &\simeq Q_3 = \Omega - \log \xi_3 \end{aligned}\right\} \tag{8c}$$

In the limit as both Z_R and Z_I approach zero,

$$\left.\begin{aligned} \log \xi_2 &\rightarrow \log \left[\frac{\sqrt{(K^2+1)(Z_R^2+Z_I^2)} - Z_I}{\Omega - 2\gamma} \right] \\ \log \xi_3 &\rightarrow \log \left[\frac{\sqrt{(K^2+1)Z_I^2 + K^2 Z_R^2}}{\Omega} \right] \end{aligned}\right\} \tag{8d}$$

Since K appears only in the argument of the logarithmic function, no critical accuracy is required for its solution. The determination of K is carried out in the Appendix, where it is found to be approximately e, or 2.71828... . Up to this point the Q(y) functions in (7b) and (7c) are replaced by the constants Q_2 and Q_3 respectively [(8c)], and no further approximation is needed to carry out the integrations. Note that as the logarithmic functions replace the Bessel functions [from (4) to (7)], equivalently an extra zero is introduced in the denominator of (2). This extra zero is located near the real axis though far away from $y = 1$. Therefore, strictly speaking, the limits of the integral I_3 in the form of (7c) should have been written from 1 to A, where A is a constant not large enough to bring about the contribution from the extra zero but large enough to take into account the contribution near $y = 1$. But after the approximation (8b) has been introduced, the extra zero is eliminated and the precaution of writing A instead of ∞ in the upper limit of the integral (7c) becomes unnecessary.

The denominators of the integrands of I_2 and I_3 are polynomials. These two integrals can both be transformed into the following form:

$$J(A,p) = 2A \int_p^\infty \frac{dy}{(y^2-1)^2 + A^2} = \int_{\frac{p^2-1}{A}}^\infty \frac{dt}{\sqrt{1+At}\,(t^2+1)} \tag{9}$$

where A and p are both real. This is integrable and the result is

$$J(A,p) = \frac{1}{2\sqrt{1+A^2}} \left\{ r(\theta_{41} + \theta_{32}) - b \log\left[\frac{(p+r)^2 + b^2}{(p-r)^2 + b^2}\right] \right\} \tag{10}$$

where

$$r = (1+A^2)^{1/4} \cos\left(\frac{\tan^{-1}A}{2}\right)$$

$$b = (1+A^2)^{1/4} \sin\left(\frac{\tan^{-1}A}{2}\right)$$

$$0 \le \tan^{-1}A \le \frac{\pi}{2}$$

$$\theta_{41} = 2 \tan^{-1}\left(\frac{b}{p-r}\right), \qquad 0 \le \theta_{41} \le \pi$$

$$\theta_{32} = 2 \tan^{-1}\left(\frac{b}{p+r}\right), \qquad 0 \le \theta_{32} \le \pi$$

I_2 and I_3 can then be expressed in terms of $J(A,p)$

$$I_2 = \frac{J(a_2,0) - J(a_2,\frac{1}{\beta_2})}{2\beta_2[(Z_R^2 + Z_I^2)(Q_2^2 + \pi^2) - (Z_R\pi + Q_2Z_I)^2]^{1/2}} \tag{11a}$$

$$I_3 = \frac{J\left(\frac{Z_R}{Q_3\beta_3^2}, \frac{1}{\beta_3}\right)}{2Q_3Z_R\,\beta_3} \tag{11b}$$

where

$$Q_2 = \Omega - 2\gamma - \log \xi_2$$

$$Q_3 = \Omega - \log \xi_3$$

$$\beta_2 = \left(1 + \frac{Z_R\pi + Z_IQ_2}{Q_2^2 + \pi^2}\right)^{1/2}$$

$$a_2 = \frac{[(Z_R^2 + Z_I^2)(Q_2^2 + \pi^2) - (Z_R\pi + Z_IQ_2)^2]^{1/2}}{\beta_2^2(Q_2^2 + \pi^2)}$$

$$\beta_3 = \left(1 - \frac{Z_I}{Q_3}\right)^{1/2}.$$

In the limit as both Z_R and Z_I approach zero, I_2 and I_3 can be expressed as follows:

$$I_2 \approx \frac{(1 - \frac{Z_I}{2Q_2})}{2Q_2 Z_R (1 - \frac{Z_I}{Q_2 Z_R})} \left[\theta + \frac{Z_R}{4Q_2} \log \frac{16Q_2^2}{|z|^2}\right] \tag{11c}$$

$$I_3 \approx \frac{(1 + \frac{Z_I}{2Q_3})}{2Q_3 Z_R} \left[\theta - \frac{Z_R}{4Q_3} \log \frac{16Q_3^2}{|z|^2}\right] \tag{11d}$$

where

$$0 \leq \theta = \tan^{-1}\left(\frac{Z_R}{Z_I}\right) \leq \frac{\pi}{2} .$$

Thus,

$$G_{H2} \approx \frac{4}{\zeta_o} \frac{1}{2Q_2} \left(\frac{1 - \frac{Z_I}{2Q_2}}{1 - \frac{Z_I}{Q_2 Z_R}} \right) \left[\theta + \frac{Z_R}{2Q_2} \log \frac{4Q_2}{|Z|}\right]$$

$$G_{H3} \approx \frac{4}{\zeta_o} \frac{1}{2Q_3} \left(1 + \frac{Z_I}{2Q_3}\right) \left[\theta - \frac{Z_R}{2Q_3} \log \frac{4Q_3}{|Z|}\right]$$

where the limiting forms of the Q's are given in (8c). It can then be concluded that the ohmic loss on the antenna behaves in the same manner as $1/(\log|\frac{1}{z^i}|)$ for small z^i.

Although the foregoing approximations are intended for small $|Z|$, numerical checks show that (11a) and (11b) are good approximations for (4b) and (4c), respectively, up to $Z_R \sim 10$.

The I_1 integral in the form (7a) is of different character from I_2 and I_3; that is, unlike the latter two, I_1 is integrable in the limit as $Z_R = Z_I$ approaches zero. This limit is exactly the input conductance of a perfectly conducting, infinitely long antenna if it is multiplied by the constant $4\pi/\zeta_o$. Let

$$I_o = \int_0^1 \frac{(1-y^2)\,dy}{[(1-y^2)\pi]^2 + [(1-y^2)Q_1(y)]^2}$$

then

$$I_o = \int_0^1 \frac{y\,dy}{(1-y^2)[\pi^2 + \left(C-\log(1-y^2)\right)^2]} + \int_0^1 \frac{dy}{(1+y)[\pi^2 + \left(C-\log(1-y^2)\right)^2]}$$

The first integral can be integrated exactly and is equal to 1/2C where $C = \Omega - 2\gamma$. The second integral is obtained approximately by first neglecting the term π^2 and then expanding in powers of 1/C. It is approximately equal to

$$\frac{1}{C^2}\log 2 + \frac{1}{C^3}\left[(\log 2)^2 + 2\int_1^2 \frac{\log(2-x)}{x}dx\right] .$$

Thus

$$I_o \approx \frac{1}{2C} + \frac{\log 2}{C^2} .$$

The following two facts should be noted: (1) I_1 approaches I_o in the limit of vanishing $|Z|$ and (2) the main difference between the integrand of I_1 and that of I_o for small $|Z|$ is in the small region from $y = \sqrt{1-\xi_1}$ to $y = 1$, where ξ_1 is a small quantity of the order of $|Z|/C$. Thus, I_1 can be expressed as follows for small $|Z|$:

$$I_1 = \int_0^{\sqrt{1-\xi_1}} \frac{(1-y^2)\,dy}{[(1-y^2)\pi]^2 + [(1-y^2)\left(C-\log(1-y^2)\right)]^2}$$

This can be integrated by following the same steps of integration as for I_o. The result is

$$I_1 = \frac{1}{2\pi}\left[\tan^{-1}\frac{\pi}{C} - \tan^{-1}\frac{\pi}{C-\log\xi_1}\right] + \frac{1}{C^2}\log(1+\sqrt{1-\xi_1}) + \frac{1}{C^3}\left[\log(1+\sqrt{1-\xi_1})\right]^2$$
$$+ \frac{2}{C^3}\int_1^{1+\sqrt{1-\xi_1}} \frac{\log(2-x)}{x}\,dx \qquad (12a)$$

Here ξ_1 is treated as a constant. Since ξ_1 appears only in the arguments of the logarithmic functions, its value cannot critically affect the value of I_1 given in (12a). As mentioned, ξ_1 is of the order of $|Z|/C$ for small $|Z|$, which is a property similar to ξ_2 and ξ_3 defined in (8). Numerical checks show that if the value of ξ_2 is used for ξ_1, I_1 in the exact form of (4a) is represented satisfactorily by the approximate form in (12a) up to $Z_R = 1$. For $Z_R < 1$, (12a) can be written as

$$I_1 = \frac{1}{2} \frac{\log(\frac{1}{\xi_1})}{C(C + \log\frac{1}{\xi_1})} + \frac{1}{C^2} \log(1 + \sqrt{1-\xi_1}) \tag{12b}$$

For Z_R in the range of 1 to 10, I_1, I_2, and I_3 are of the same character. Since the latter two are found to be well approximated by the forms of (11a) and (11b) up to $Z_R = 10$, the same thing can be done for I_1 in the range $1 < Z_R < 10$. Thus,

$$I_1 = \int_0^1 \frac{dy}{[(1-y^2)\pi+Z_R]^2+[(1-y^2)Q_1+Z_I]^2} - \int_0^1 \frac{y^2\,dy}{[(1-y^2)\pi+Z_R]^2+[(1-y^2)Q_1+Z_I]^2}$$

The first integral is exactly I_2, the second integral can be transformed into the form of (9) if the variable y is changed to $1/z$. The result is

$$I_1 = I_2 - \frac{p_1}{2} \frac{1}{Q_2Z_R - Z_I\pi} \; J\left(\frac{Q_2Z_R - Z_I\pi}{a_1^2}, p_1\right) \qquad \text{for } 1 < Z_R < 10 \tag{12c}$$

where

$$p_1 = \left\{ \frac{(Z_R+\pi)^2 + (Q_2+Z_I)^2}{\pi(Z_R+\pi) + Q_2(Q_2+Z_I)} \right\}^{1/2}$$

$$a_1^2 = \pi(Z_R+\pi) + Q_2(Q_2+Z_I).$$

Equation (12c) can be checked numerically against the exact form of (4a). It is found to be in good agreement with the latter. Furthermore, (12b) and (12c) overlap near $Z_R = 1$.

Fig. 2 shows some numerical results. The input conductance G together with G_R (radiated power) and G_H (ohmic loss) are plotted against the normalized internal impedance Z_R which ranges from 10 to 10^{-5} with $Z_I = 0$. This corresponds to an antenna formed by a thin layer of conductive coating. If a good conductor is used and skin effect exists, $Z_I = Z_R$ and they are of the order 10^{-2} to 10^{-3}. This case is also shown graphically in Fig. 2. The results show that in the case $Z_R >> Z_I$, the ohmic loss is of the same order of magnitude as the radiated power unless Z_R is extremely small (say, less than 10^{-6}). When skin effect is significant $Z_I = Z_R$, and G_H is only about one-half as great as when $Z_I = 0$, so that the input conductance is quite different from that of a perfectly conducting antenna.

CONCLUSIONS

The following conclusions can be drawn from the foregoing calculation:

(1) The infinite antenna has quite different properties when its internal impedance is zero than when z^i is non-vanishing, even when z^i is quite small. (The input impedance of the former has been obtained, for example by Papas [3]; the current distribution has been worked out, for example by Kunz [4].)

(2) The present theory may be verified experimentally with an antenna of finite length provided the internal impedance is not too low.

(3) In the theory of the very long antenna the internal impedance is not negligible for all practically available metals. The current distribution is greatly modified when z^i differs from zero. This effect should be more apparent as the antenna becomes longer or as the impedance per unit length becomes larger. The above statement is illustrated by the qualitative sketch of Fig. 3. The initial slope of the loss curve of a long antenna is determined by the antenna length; a perturbation method may be used to calculate it. For larger z^i, the loss curve

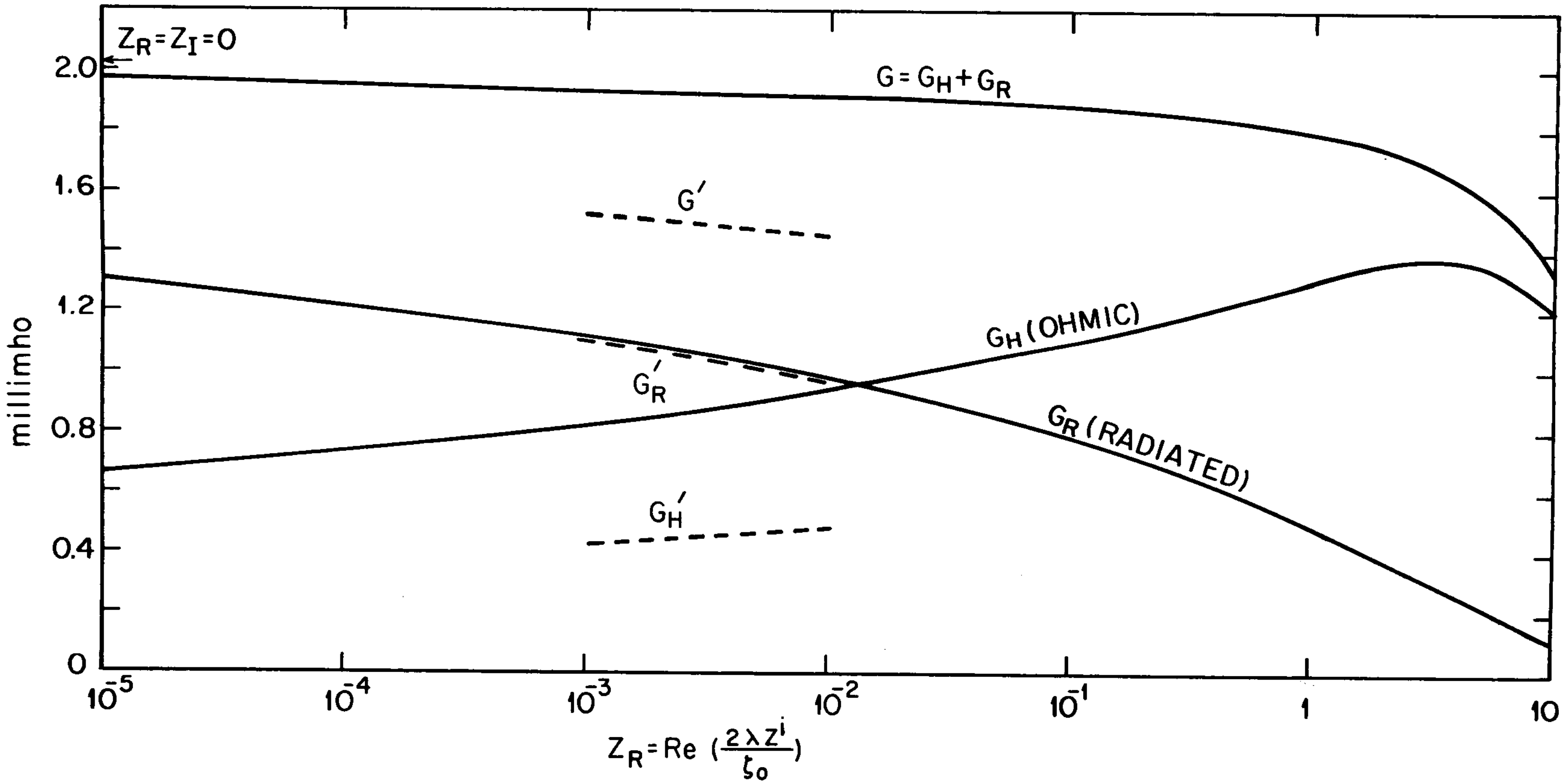

FIG. 2 INPUT CONDUCTANCE OF AN INFINITE ANTENNA, $ka = 0.01$. SOLID CURVE FOR $Z_I = 0$, DASHED CURVE FOR $Z_I = Z_R$

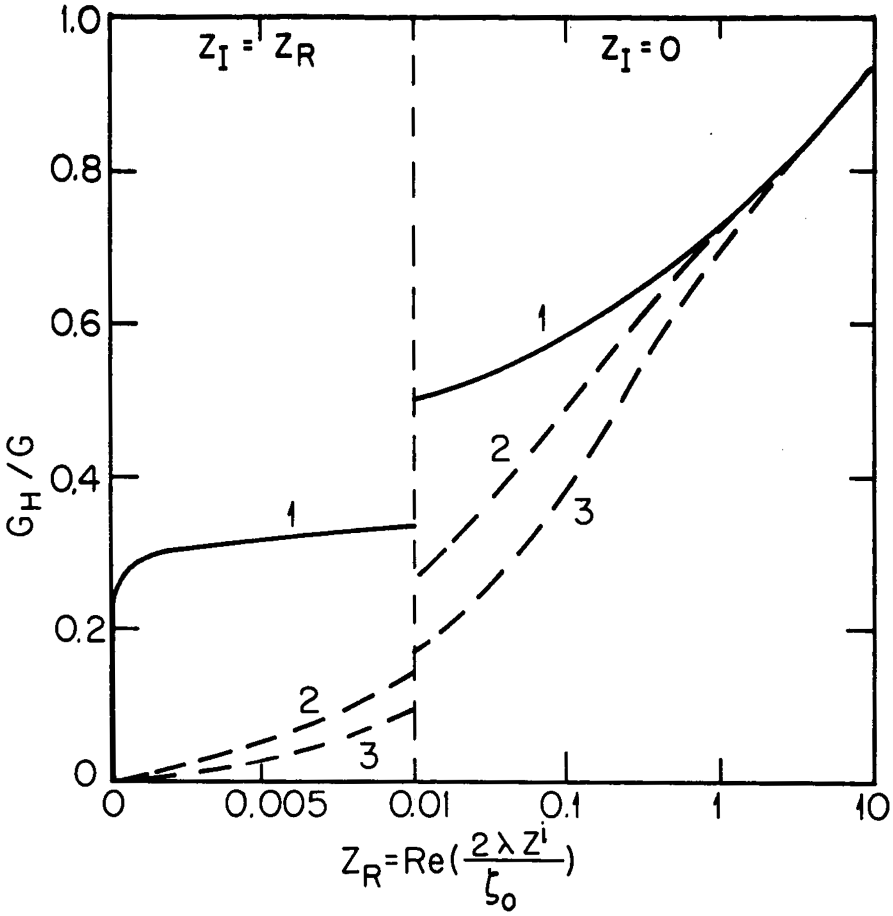

FIG. 3 LOSS CURVE OF ANTENNAS

1: INFINITE ANTENNA; 2 AND 3: LONG ANTENNAS (SKETCH) $kh_2 > kh_3$

joins the curve of the infinitely long antenna with the same z^1 since the current on the long antenna should be attenuated to such a small quantity at the ends that any increase of the antenna length would not affect the current distribution. But, for moderate z^1, or for very long antennas, the behavior requires further study.

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APPENDIX

This appendix is devoted to the estimation of the constant K in the approximations associated with (8). It is assumed that $Z_I = 0$. The problem is to determine K so that the logarithmic function can be replaced by a constant determined by (8) without excessive error. The question is actually equivalent to approximating

$$\int_0^\infty \frac{dx}{\sqrt{1+z'x}\,[x^2(C-\log x)^2+1]} \qquad \text{(A-1)}$$

by

$$\int_0^\infty \frac{dx}{\sqrt{1+z'x}\,(x^2Q^2+1)}$$

with $Q = C - \log x_o$, $x_o(C - \log x_o) = K$.

The integral (A-1) is essentially I_3 in (7c); z' is proportional to Z_R, C corresponds to Ω, and x is proportional to y^2-1.

Take the difference of the above two integrals

$$\Delta = \int_0^\infty \frac{dx}{\{x^2[C-\log x]^2+1\}\sqrt{1+z'x}} - \int_0^\infty \frac{dx}{\{x^2Q^2+1\}\sqrt{1+z'x}}$$

$$= \int_0^\infty \frac{dx}{\{x^2[Q-\log \frac{x}{x_o}]^2+1\}\sqrt{1+z'x}} - \int_0^\infty \frac{dx}{\{x^2Q^2+1\}\sqrt{1+z'x}} \qquad \text{(A-2)}$$

$$\Delta \approx \int_0^\infty \frac{dx}{\{(x^2Q^2+1)-2x^2Q\log\frac{x}{x_o}\}\sqrt{1+z'x}} - \int_0^\infty \frac{dx}{(x^2Q^2+1)\sqrt{1+z'x}}$$

Expand the first integral so that the first term cancels the second integral, and the result is

$$\Delta \approx 2Q\int_0^\infty \frac{x^2\log\frac{x}{x_o}}{\sqrt{1+z'x}\,(x^2Q^2+1)^2}\,dx \qquad \text{(A-3)}$$

x_o is determined by K as defined in (A-1). K is to be chosen so that the right hand side of (A-3) is zero. Let $t = x^2Q^2$,

$$\int_0^\infty \frac{\sqrt{t}\,dt}{(1+t)^2\sqrt{1+z'\sqrt{t}/Q}} \log\frac{t}{K^2} = 0. \qquad \text{(A-4)}$$

It is difficult to integrate (A-4) in general, but it is known for two special cases:

<u>Case I</u>. $z' = 0$

$$\int_0^\infty \frac{\sqrt{t}\,dt}{(1+t)^2} \log\frac{t}{K^2} = 0 \qquad \text{(A-5)}$$

Let

$$F_1(\tau) = \int_0^\infty \frac{\sqrt{t}\,dt\; t^\tau}{(1+t)^2} \qquad \text{(A-6)}$$

$$F_1'(\tau) = \int_0^\infty \frac{\sqrt{t}\,dt}{(1+t)^2} \log t \; e^{\tau \log t} \qquad \text{(A-7)}$$

Therefore, (A-5) is equivalent to

$$F_1'(0) - F_1(0) \log K^2 = 0.$$

The integral (A-6) is known [Ref. 5, p. 9]

$$F_1(\tau) = (\tau + \tfrac{1}{2})\pi \sec \pi\tau$$

and

$$\log K^2 = \frac{\pi}{\pi/2} = 2, \quad K = e = 2.71828$$

<u>Case II</u>. $z' \to \infty$

$$\int_0^\infty \frac{t^{1/4}\,dt}{(1+t)^2} \log\frac{t}{K^2} = 0 \qquad \text{(A-8)}$$

$$F_2(\tau) = \int_0^\infty \frac{t^{\frac{1}{4}+\tau}\,dt}{(1+t)^2}$$

$$\log K^2 = \frac{F_2'(0)}{F_2(0)}$$

$F_2(\tau)$ is also known [Ref. 5, p. 9]

$$F_2(\tau) = (\tau + \tfrac{1}{4})\pi \sec \pi(\tau - \tfrac{1}{4}), \qquad F_2(0) = \frac{\pi}{4}\sqrt{2}$$

$$F_2'(0) = \pi\sqrt{2}\,(1 - \tfrac{\pi}{4})$$

$$\log K^2 = 4 - \pi, \qquad K = 1.536.$$

Thus, it seems plausible to choose K equal to e = 2.71828 when $Z_R = 10^{-5}$ to 10 in the numerical calculations.

Correction to

RADIATED POWER AND OHMIC LOSS OF THE INFINITELY LONG CYLINDRICAL ANTENNA

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Correction to

RADIATED POWER AND OHMIC LOSS OF THE INFINITELY LONG CYLINDRICAL ANTENNA

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In the above report the authors would like to make the following corrections: In (3), (ζ^2-k^2) in the second integral should be changed to $(k^2-\zeta^2)$. Consequently Z_I in (4c) and in (7c) should be preceded by minus signs. Subsequent calculations are valid except that for I_3 when $Z_I \neq 0$. Thus, (8b) is valid for $Z_I = 0$ only; and in (8d), the limit of log ξ_3 should be changed to

$$\log \xi_3 \rightarrow \log(KZ_R/\Omega) \quad \text{for} \quad Z_I = 0.$$

The limit of log ξ_2 is correct. The expression (11b) and (11d) for I_3 and the expression for G_{H3} below (11d) are correct for $Z_I = 0$ only. The last sentence above the Conclusion should be deleted.

What remains to be done is to evaluate I_3, or equivalently, G_{H3}, for $Z_I \neq 0$. Only the case when $0 \leq Z_I \leq Z_R << 1$ is studied since $Z_I = Z_R << 1$ for metal as a result of skin effect.

It is learned from previous calculations that I_3 as defined by (4c) can be approximated by (11b) for $Z_I = 0$ if the conditions (6) are satisfied. Let

$$\begin{aligned}\Delta(Z_I) &= I_3(Z_I) - I_3(0) \\ &= \int_1^\infty \left\{ \frac{1}{[(y^2-1)Q_3(y)-Z_I]^2 + Z_R^2} - \frac{1}{[(y^2-1)Q_3(y)]^2 + Z_R^2} \right\} dy \qquad \text{(C-1)}\end{aligned}$$

Note that the contribution to $\Delta(Z_I)$ is mainly from a small region near $y = 1$. The logarithmic function $Q_3(y)$ may then be replaced by a proper constant:

$$Q_3(y) \simeq Q_3' = \Omega - \log \xi_3' \qquad \text{(C-2)}$$

Under the condition that $Z_I \leq Z_R \ll 1$, ξ_3' can be approximately chosen to be Z_I/Ω since the main contribution to $\Delta(Z_I)$ is from the region $0 < y^2-1 \lesssim \xi_3'$.

$\Delta(Z_I)$ is readily obtained in terms of the J-functions defined in (9) when $Q_3(y)$ is replaced by Q_3', and the result is

$$\Delta(Z_I) = \frac{J\left(\frac{Z_R}{Q_3'\beta_3'^2}, \frac{1}{\beta_3'}\right)}{2Z_R Q_3'\beta_3'} - \frac{J\left(\frac{Z_R}{Q_3'}, 1\right)}{2Z_R Q_3'} \qquad \text{(C-3)}$$

where

$$\beta_3' = \left(1 + \frac{Z_I}{Q_3'}\right)^{1/2} .$$

Therefore

$$I_3 = \Delta(Z_I) + \frac{J\left(\frac{Z_R}{Q_3}, 1\right)}{2Z_R Q_3} . \qquad \text{(C-4)}$$

And for $0 \leq Z_I \leq Z_R \ll 1$, and also under the conditions (6),

$$I_3 \simeq \frac{1}{2Z_R Q_3'} \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{Z_R}{Z_I}\right)\right] + \frac{1}{2Z_R Q_3}\frac{\pi}{2}$$

$$G_{H3} \simeq \frac{4}{\zeta_o}\left\{ \frac{1}{2Q_3'}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{Z_R}{Z_I}\right)\right] + \frac{\pi}{4Q_3} \right\} \qquad \text{(C-5)}$$

$$G_H = \frac{2}{\zeta_o}\left\{ \frac{\pi}{2}\left(\frac{1}{Q_3'} + \frac{1}{Q_3}\right) + \left(\tan^{-1}\frac{Z_R}{Z_I}\right)\left(\frac{1}{Q_2} - \frac{1}{Q_3'}\right)\right\}. \qquad \text{(C-6)}$$

Thus, G_H does not change significantly as Z_I is changed from 0 to Z_R, as long as $Z_R \ll 1$. All above approximations have been checked by numerical integrations.

Fig. 2 and Fig. 3 have been corrected and are shown in the following pages. Fig. 4 shows the comparison between the approximate formula (C-5) and the numerical integration of the exact formula (4c) for G_{H3}. The agreement is found to be very good.

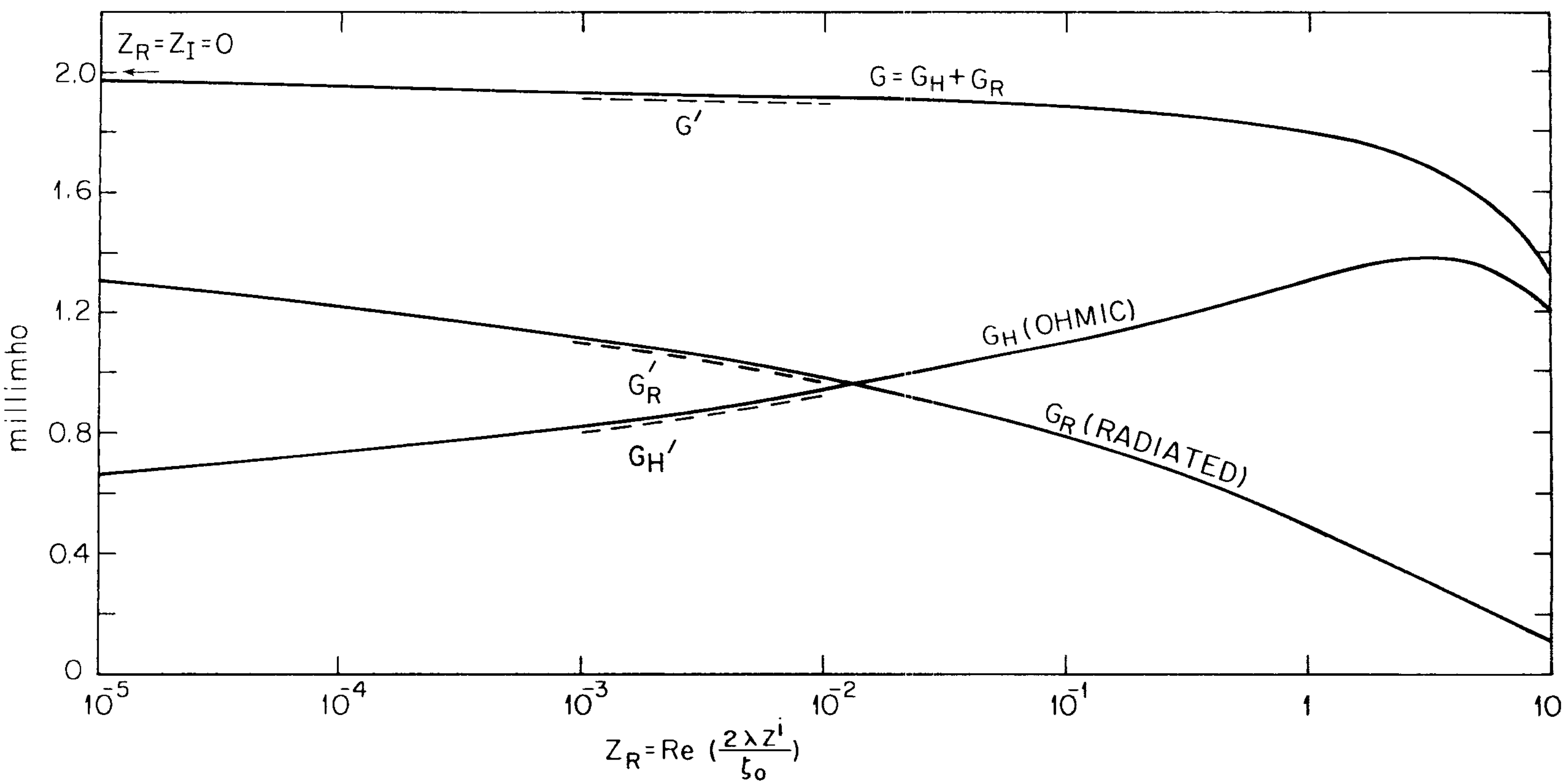

FIG. 2 INPUT CONDUCTANCE OF AN INFINITE ANTENNA, $ka = 0.01$. SOLID CURVE FOR $Z_I = 0$, DASHED CURVE FOR $Z_I = Z_R$

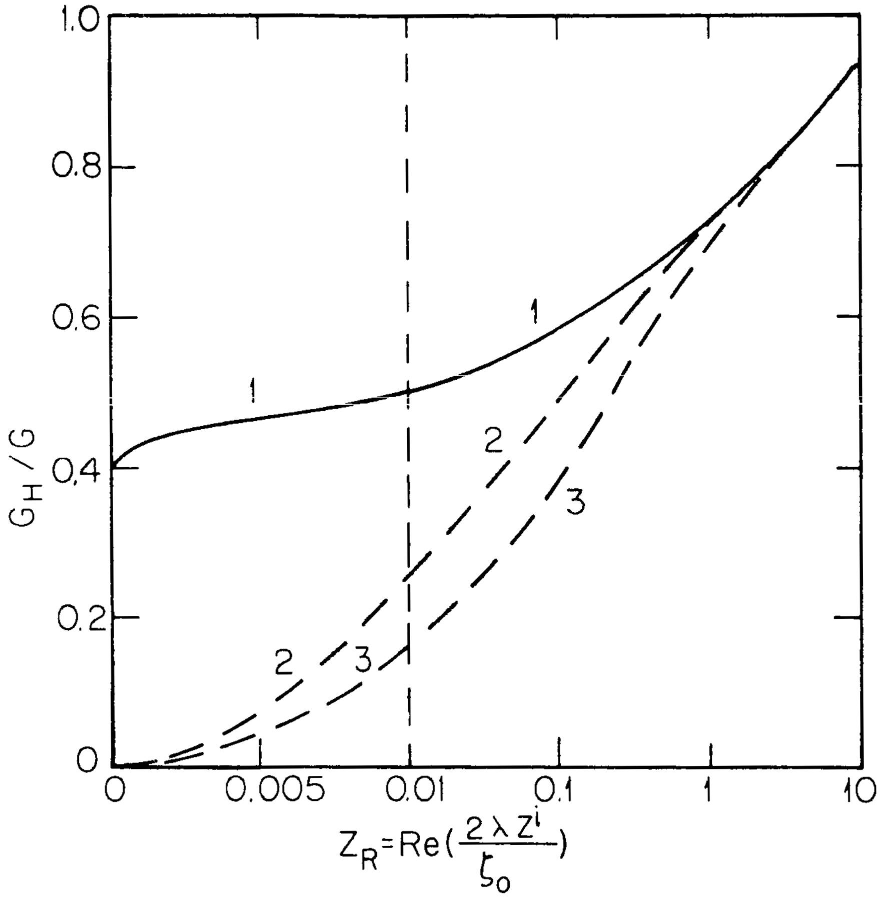

FIG. 3 LOSS CURVE OF ANTENNAS

1 : INFINITE ANTENNA; 2 AND 3 : LONG ANTENNAS (SKETCH) $kh_2 > kh_3$

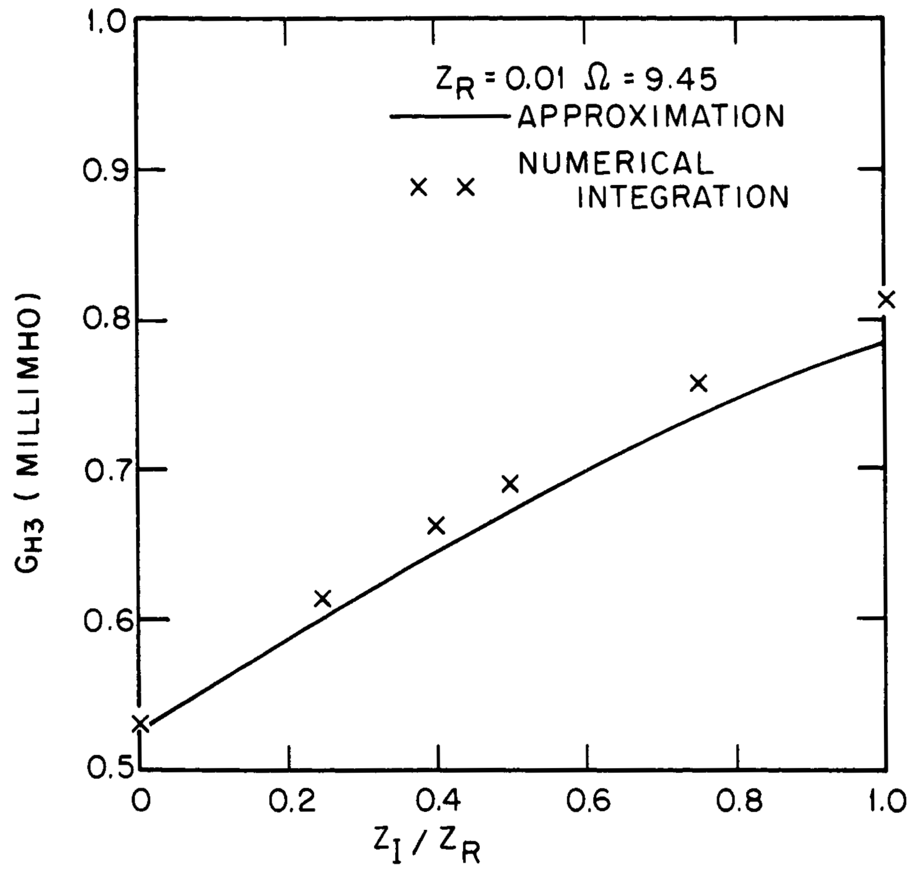

FIG. 4 COMPARISON OF THE APPROXIMATE FORMULA (C-5) AND THE NUMERICAL INTEGRATION OF THE EXACT FORMULA (4C) FOR G_{H3}