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by

K. Schiffner

Deutsche Luft und Raumfahrt,
Forschungsbericht 66-24, April 1966

Translated from the German

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ABSTRACT

The prebuckling state and the stability behaviour of thin-walled isotropic and orthotropic shells under axisymmetric loads (axial load and internal pressure) is investigated, applying a nonlinear shell theory (finite deformations).

At first the unbuckled equilibrium state which is axisymmetric under the given load is described. The derived system of ordinary nonlinear differential equations is solved by a difference method with approximation of higher order. For isotropic conical shells, an exact solution for the linearized system can be obtained (linear bending theory).

In the second part of this paper the buckling loads are calculated. The buckling (bifurcation) state is governed by an eigen-value problem, the eigen-values of which are the buckling loads and the eigen-functions of which are the additional stresses and deformations. The eigen-value problem is also solved iteratively by the difference method.

The influence of shell parameters and of an axisymmetric predeformation on the buckling load is discussed. The results are in good agreement with the experiments, particularly for a conical shell under axial load and high internal pressure.

CONTENTS

1.1 Introduction

The stressed structures in aviation and rocket construction consist to a large part of thin-wall isotropic, orthotropic, and reinforced shells. Detailed knowledge of the carrying capacity of thin-wall shells is required if the construction elements are to be designed as light as possible in favor of a high payload.

The elastic stability behavior of cylinder shells has been intensively investigated during the last years according to the linear as well as to the nonlinear theory of stability. For conical shells only buckling loads have been determined so far according to the linear theory, because the basic equations describing the stress and deformation state of the conical shell are more complicated compared with those for the cylinder shell. These loads have been determined among others by Mushtari and Sachenkov[9], Schnell[5], and Seide[15,18,21] for the isotropic conical shell and by Leyko[12], Serpico[19], and Singer[6,20,22] for the orthotropic shell.

By means of the linear theory the qualitative effect of the geometrical characteristics and of the rigidity parameter on the buckling load can be determined. However, the resulting buckling loads deviate greatly from the buckling loads obtained experimentally, especially in the case, herein considered, of the conical shell under axial load and inner pressure. This discrepancy between experiment and theory is greatly due to the fact that according to the linear theory of stability the basic state of the stresses of the shell--that is the state of the shell before reaching the critical load--is replaced with the state of diaphragm stress, which neither satisfies the compatibility condition nor all boundary conditions (no circumferential expansion).

In the following discussion, first the basic state of the stresses of the orthotropic conical shell (with the special case "isotropic") shall be described, taking into account nonlinear members. The derived nonlinear differential equations are approximately solved with the difference method. With the given boundary conditions and for the load investigated (axial load and inner pressure), the solution is axially symmetric.

For the linearized basic equations of the isotropic cone describing the state of the stresses and of the deformation under small loads at the edge, a general solution in closed form can be stated.

The second part of the discussion deals with the determination of the buckling loads. After exceeding the critical load, the axially symmetric basic state of the stresses becomes unstable and the shell buckles into a nonaxially symmetric state of equilibrium. The determination of the additional stresses and deformations occurring in the instance of buckling leads to an eigen-value problem, the eigen-value of which is the critical load. The iterative solution of this eigen-value problem is obtained by means of the difference method.

If the calculation of the buckling loads is based on the basic state of the stresses which results from the basic nonlinear equations when the boundary conditions are completely satisfied, the buckling loads determined are in good agreement with the experimental values for high inner pressure. The discrepancy between the buckling loads calculated for an ideal cone and the experimental buckling values for low inner pressure (especially for $p = 0$) is essentially traced to the existence of prebuckling in the test shells.

In order to be able to investigate the effect of such a predeformation on the stability, the differential equations of the prebuckled conical shell are derived and iteratively solved. The applied method of solving permits us to take into account any arbitrary axially symmetric prebuckling.

The investigations described here principally apply also to the conical shell under outer pressure.

1.2 Glossary of Symbols

b_A	Width of prebuckling
$B(y, \varphi_0)$	Differential operator
C_{ik}, $\bar{C}_{ik}$	Stretching and shearing rigidity parameter
D_{ik}	Flexural stiffness and torsional stiffness
E	Modulus of elasticity
f, F_1, F_{1n}	Stress functions
G	Modulus of rigidity
h	Distance between the equidistant intermediate points
h_k	Height of the cone
L(y)	Differential operator
m	Number of the equidistant intermediate points

$\overline{m}$	Number of bucklings along the generatrix
M_s, M_θ, $M_{s\theta}$	Section momentum
$\overline{M}_s$, $\overline{M}_\theta$, $\overline{M}_{s\theta}$	Edge momentum
n	Number of bucklings in the direction of circumference
N_s, N_θ, $N_{s\theta}$	Section forces
$\overline{N}_s$, $\overline{N}_\theta$, $\overline{N}_{s\theta}$	Edge forces
N_0	Section force parameter $N_0 = \frac{s_1^2 e^2}{D_{11}} N_{s\theta}$
N_v	Dimensionless loadparameter $N_v = \frac{-N_0}{2\lambda}$
q_0, Q_0	Deformation parameter of the basic state of stresses $q_0 = \frac{\partial w_0}{\partial s}$, $Q_0 = \frac{s_1}{\sqrt{D_{11} C_{22}}} q_0$
$\bar{Q}_s$, $\bar{Q}_\theta$	Transversal edge forces
p	Inner pressure
$\bar{p}$	Dimensionless inner pressure parameter $\bar{p} = \frac{1}{2} p \frac{s_1^2}{\cot^2 \alpha} \sqrt{\frac{C_{22}}{D_{11}}}$
P	Axial load
r_1, r_2	End radii of the cone
s	Coordinate along the generatrix
s_1, s_2	Length of the generatrix to the ends of the cone
t	Wall thickness of the shell
u	Local displacement of a point along the generatrix
v	Local displacement of a point in the direction of the circumference
w	Local displacement of a point perpendicular with respect to the central surface of the shell
w_A	Predeformation perpendicular with respect to the central surface of the shell
z	Dimensionless coordinate along the generatrix $z = \ln \frac{s}{s_1}$

z_0	$= \ln \frac{s_2}{s_1}$
z_A	Position of the prebuckling maximum along the generatrix
α	Half-angle of the cone
$\gamma_{s\theta}$	Shearing of a shell element $s\, d\theta\, ds$
ϵ_s, ϵ_θ	Stretchings of the central surface of the shell
η	$= \frac{n}{\sin \alpha}$
θ, $\bar{\theta}$	Circumference coordinates $\theta = \bar{\theta} \sin \alpha$
k	$= \frac{\bar{m}\bar{k}}{z_0}$
k_s, k_θ, $k_{s\theta}$	Curvatures of the central surface of the shell
λ	Shell parameter $\lambda = \frac{s_1 \cot \alpha}{\sqrt{D_{11} C_{22}}}$
ν	Poisson's ratio
Π	Potential energy of the deformed shell

Index $(\;)_0$ for stresses and deformations designates the state of the basic stresses

Index $(\;)_1$ designates the additional stresses and deformations occurring in the instance of buckling.

2. Basic Equations of the Orthotropic Conical Shell

2.1 The Law of Elasticity of the Orthotropic Shell

The geometrical designations used in the investigations are given in Figure 1. Each point of the central surface of the shell can be fixed by the distance s and the circumferential angle $\bar{\theta}$; s and the angle θ--where $\theta = \bar{\theta} \sin \alpha$--represent the polar coordinate system of the development of the shell. The arrows of the coordinate axes point in the direction of positive values. Because of the singularities occurring at the tip of an axially loaded cone only truncated cones are investigated. The geometrical configuration of a truncated cone is uniquely determined by α, s_1, s_2.

The law of elasticity of the orthotropic shell, the main axes of rigidity of which coincide with the coordinate axes s and θ, can be stated in decoupled form for the section forces N_s, N_θ, $N_{s\theta}$ and for

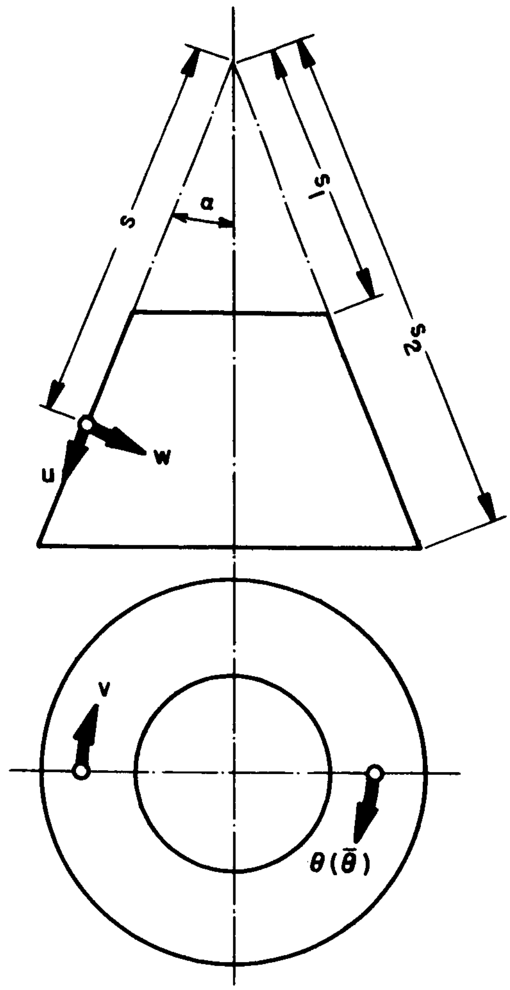


Figure 1. Dimensions and Designations of the Truncated Cone

the section momentums M_s, M_θ, $M_{s\theta}$ utilizing the usual assumptions[23,7] for the investigation of plates and shells. (In Figure 2 the forces and momentums with positive sign are indicated for an element of the shell.)

$$\begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \end{bmatrix} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{12} & \bar{C}_{22} & 0 \\ 0 & 0 & \bar{C}_{33} \end{bmatrix} \begin{bmatrix} \epsilon_s \\ \epsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M_s \\ M_\theta \\ M_{s\theta} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & 2D_{33} \end{bmatrix} \begin{bmatrix} k_s \\ k_\theta \\ k_{s\theta} \end{bmatrix} \tag{2.1}$$

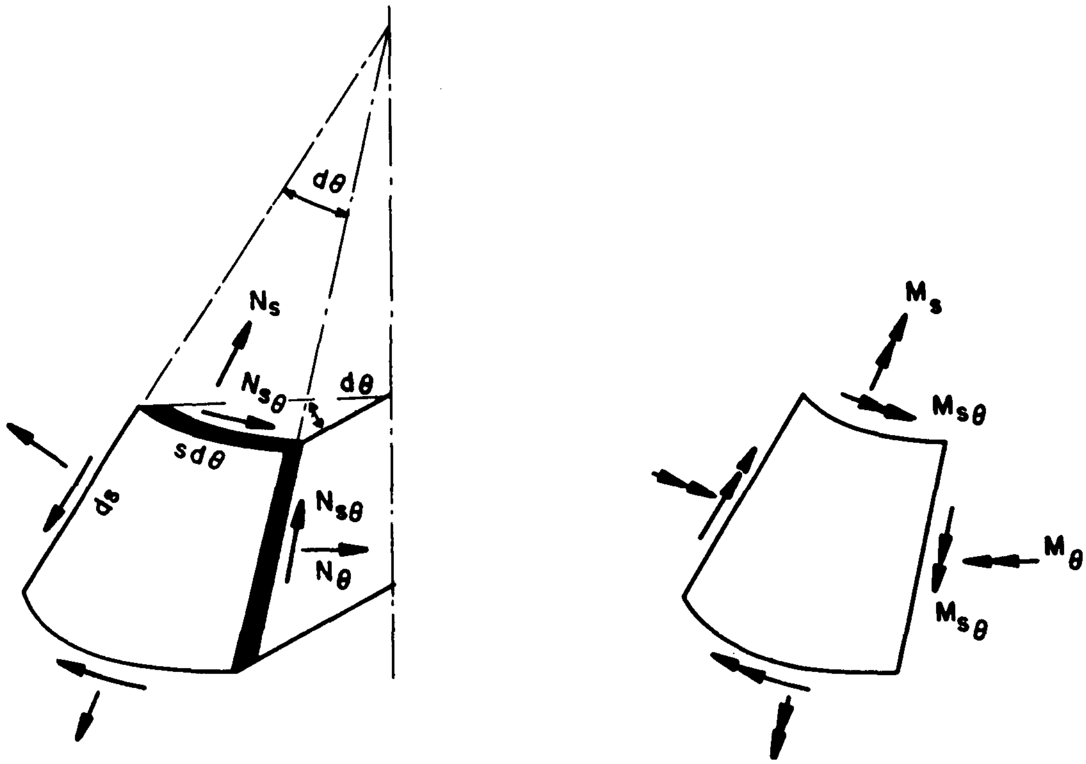


Figure 2. Section Forces and Section Momentums on an Element of a Conical Shell

Here ϵ_s and ϵ_θ represent the expansions of the central surface of the shell in the direction of the generatrix and in the direction of the circumference. $\gamma_{s\theta}$ is the shear of one element of the shell s dθ ds (Figure 2). k_s and k_θ are the curvatures; $k_{s\theta}$ is the torsion of the central surface of the shell. The C_{ik} designates the expansion and shear rigidities, the D_{ik} the flexural and torsional rigidities.

In the further investigations, the first elasticity equations of (2. 1) are used in inverse form

$$\begin{bmatrix} \epsilon_s \\ \epsilon_\theta \\ \gamma_{s\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} N_s \\ N_\theta \\ N_{s\theta} \end{bmatrix} \qquad (2.2)$$

where the relation between the $\overline{C}_{ik}$ and the C_{ik} is given by the following equations:

$$\bar{C}_{11} = \frac{C_{22}}{C_{11} C_{22} - C_{12}^2}, \qquad \bar{C}_{22} = \frac{C_{11}}{C_{11} C_{22} - C_{12}^2},$$

$$\bar{C}_{12} = \frac{-C_{12}}{C_{11} C_{22} - C_{12}^2}, \qquad \bar{C}_{33} = \frac{1}{C_{33}}. \tag{2.3}$$

For the isotropic conical shell, the rigidity parameters assume the values

$$C_{11} = C_{22} = \frac{1}{Et}, \quad C_{12} = -\frac{\nu}{Et}, \quad C_{33} = \frac{1}{Gt}$$

$$D_{11} = D_{22} = \frac{Et^3}{12\left(1-\nu^2\right)}, \quad D_{12} = \frac{\nu Et^3}{12\left(1-\nu^2\right)}, \quad 2D_{33} = \frac{Et^3}{12\left(1+\nu\right)} \tag{2.4}$$

where t designates the wall thickness, E the modulus of elasticity and G the modulus of rigidity, and ν is Poisson's ratio.

2.2 The Kinematic Relations

The expansions of the central surface of the conical shell utilizing the essential members of second order in the displacement derivatives[5] and taking into account the predeformation can be expressed by the following relations:

$$\left.\begin{aligned}
\epsilon_s &= \frac{\partial u}{\partial s} + \frac{1}{2}\left(\frac{\partial w}{\partial s}\right)^2 + \frac{\partial w_A}{\partial s}\frac{\partial w}{\partial s}, \\
\epsilon_\theta &= \frac{u - w\cot\alpha}{s} + \frac{1}{s\sin\alpha}\frac{\partial v}{\partial \bar{\theta}} + \frac{1}{2}\left(\frac{1}{s\sin\alpha}\frac{\partial w}{\partial \bar{\theta}}\right)^2 \\
&\quad + \left(\frac{1}{s\sin\alpha}\right)^2 \frac{\partial w}{\partial \bar{\theta}}\frac{\partial w_A}{\partial \bar{\theta}}, \\
\gamma_{s\theta} &= \frac{\partial v}{\partial s} - \frac{v}{s} + \frac{1}{s\sin\alpha}\frac{\partial u}{\partial \bar{\theta}} + \frac{\partial w}{\partial s}\left(\frac{1}{s\sin\alpha}\frac{\partial w}{\partial \bar{\theta}}\right) \\
&\quad + \frac{1}{s\sin\alpha}\left(\frac{\partial w_A}{\partial \bar{\theta}}\frac{\partial w}{\partial s} + \frac{\partial w_A}{\partial s}\frac{\partial w}{\partial \bar{\theta}}\right).
\end{aligned}\right\} \tag{2.5}$$

Here u, v, w designate the local displacements of a point along the generatrix in direction of the circumference and perpendicularly with respect to the central surface of the shell (Figure 1). w_A (A - initial deformation) represents a predeformation of the shell perpendicular with respect to the central surface of the shell in the same sense of direction as the displacement w.

The displacement-expansion relations of the ideal conical shell ($w_A = 0$) reduce to the familiar equations of the theory of thin plates[24] for the case $\alpha = 90°$ ($s_1 \rightarrow r_1$, $s_2 \rightarrow r_2$, $d_s \rightarrow d_r$).

Making the transition to the cylinder ($\alpha \rightarrow 0$, $s \rightarrow \infty$, $s \sin \alpha \rightarrow R$, $ds \rightarrow dx$) from (2. 5), the relations follow which Marguerre[1] and Donnell[7] have derived for the predeformed cylinder shell.

With Donnell's simplifications[7] extended for the conical shell[5], for the curvatures of the central surface of the shell follows

$$\left.\begin{aligned} k_s &= \frac{\partial^2 w}{\partial s^2} \\ k_\theta &= \frac{1}{s}\frac{\partial w}{\partial s} + \frac{1}{s^2 \sin \alpha}\frac{\partial^2 w}{\partial \bar{\theta}^2} \\ k_{s\theta} &= \frac{\partial}{\partial s}\left(\frac{1}{s \sin \alpha}\frac{\partial w}{\partial \bar{\theta}}\right) \end{aligned}\right\} \qquad (2.6)$$

By means of the equations of elasticity (2. 1) and the relations (2. 5) and (2. 6), the section forces N_s, N_θ, $N_{s\theta}$ and the section momentums M_s, M_θ, $M_{s\theta}$ can be expressed with the displacements u, v, w, and their derivatives. The differential equations from which the displacements u, v, w can be determined can be derived from equilibrium considerations on the deformed shell element or by means of the calculus of variations from the energy. In the present discussion the second way is used.

2. 3 The Potential Energy of the Deformed Conical Shell

The total potential energy of a conical shell deformed by edge loads and inner pressure

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 \qquad (2.7)$$

is composed of the following contributions: The expansion energy of the central surface of the shell

$$\Pi_1 = \frac{1}{2}\int_{\bar{\theta}_1}^{\bar{\theta}_2}\int_{s_1}^{s_2} \left\{ N_s \epsilon_s + N_\theta \epsilon_\theta + N_{s\theta}\, \gamma_{s\theta} \right\} s \sin\alpha \, ds \, d\bar{\theta} ,$$

the bending energy

$$\Pi_2 = \frac{1}{2}\int_{\bar{\theta}_1}^{\bar{\theta}_2}\int_{s_1}^{s_2} \left\{ M_s k_s + M_\theta k_\theta + 2M_{s\theta} k_{s\theta} \right\} s \sin\alpha \, ds \, d\bar{\theta} ,$$

the potential of the edge load at the edges s = constant

$$\Pi_3 = -\int_{\theta_1}^{\bar{\theta}_2} \left\{ s \left(\bar{N}_s u + \bar{N}_{s\theta} v + \bar{M}_s \frac{\partial w}{\partial s} + \bar{M}_{s\theta} \frac{1}{s \sin\alpha} \frac{\partial w}{\partial \bar{\theta}} \right.\right.$$

$$\left.\left. + \; \bar{Q}_s w \right) \right\} \Bigg|_{s_1}^{s_2} \sin\alpha \, d\bar{\theta} ,$$

the potential of the edge load at the edges $\bar{\theta}$ = constant

$$\Pi_4 = -\int_{s_1}^{s_2} \left\{ \bar{N}_\theta v + \bar{N}_{s\theta} u + \bar{M}_\theta \frac{1}{s \sin\alpha} \frac{\partial w}{\partial \bar{\theta}} \quad \bar{M}_{s\theta} \frac{\partial w}{\partial s} \right.$$

$$\left. + \; \bar{Q}_\theta w \right\} \Bigg|_{\bar{\theta}_1}^{\bar{\theta}_2} ds$$

and of the potential energy of the inner pressure

$$\Pi_5 = \int_{\theta_1}^{\bar{\theta}_2}\int_{s_1}^{s_2} p w \; s \sin\alpha \, ds \, d\bar{\theta} ,$$

where $\bar{N}_s \ldots, \bar{M}_s \ldots, \bar{Q}_s \ldots, \bar{Q}_\theta$ are the edge loads.

2.4 The Differential Equations of the Stress and Deformation State

2.4.1 Derivation of the Differential Equations

The system of differential equations describing the state of the stresses and deformation of the orthotropic conical shell is derived via the potential energy, the first variation $\delta \Pi$ of which must go to zero for all states of equilibrium of the deformed shell. With (2.5), (2.6), and (2.7) and the substitution (2.8),

$$\theta = \bar{\theta} \sin \alpha \quad , \tag{2.8}$$

the first variation of the total energy is formed and is arranged according to the displacement differentials δu, δv, δw.

In order to satisfy $\delta \Pi = 0$, the double integrals over the surface of the shell have to vanish individually since δu, δv, and δw are independent virtual displacements. This leads to the following equilibrium conditions:

$$\left.\begin{aligned}
&-\frac{\partial}{\partial s}\left(s N_s\right) + N_\theta - \frac{\partial}{\partial \theta} N_{s\theta} = 0 \\
&+\frac{\partial}{\partial \theta} N_\theta + \frac{\partial}{\partial s}\left(s N_{s\theta}\right) + N_{s\theta} = 0 \\
&-\frac{\partial}{\partial s}\left[s N_s\left(\frac{\partial w}{\partial s} + \frac{\partial w_A}{\partial s}\right)\right] - N_\theta \cot \alpha - \frac{\partial}{\partial \theta}\left[\frac{1}{s} N_\theta \left(\frac{\partial w}{\partial \theta} + \frac{\partial w_A}{\partial \theta}\right)\right] \\
&-\frac{\partial}{\partial s}\left[N_{s\theta}\left(\frac{\partial w}{\partial \theta} + \frac{\partial w_A}{\partial \theta}\right)\right] - \frac{\partial}{\partial \theta}\left[N_{s\theta}\left(\frac{\partial w}{\partial s} + \frac{\partial w_A}{\partial s}\right)\right] + \frac{\partial^2}{\partial s^2}\left(s M_s\right) \\
&-\frac{\partial}{\partial s} M_\theta + \frac{\partial^2}{\partial \theta^2}\left(\frac{M_\theta}{s}\right) + 2\frac{\partial}{\partial \theta}\left(\frac{M_{s\theta}}{s}\right) + 2\frac{\partial^2}{\partial s \partial \theta} M_{s\theta} + p \cdot s = 0.
\end{aligned}\right\} \tag{2.9}$$

The boundary conditions at the edges s = constant and θ = constant are obtained from the single integrals in $\delta \Pi$. For the boundary conditions at the upper ($s = s_1$) and lower ($s = s_2$) edge of the shell follows:

$$\left.\begin{array}{ll}
M_\theta + s N_s \left(\frac{\partial w}{\partial s} + \frac{\partial w_A}{\partial s}\right) + N_{s\theta}\left(\frac{\partial w}{\partial \theta} + \frac{\partial w_A}{\partial \theta}\right) & \\
- \frac{\partial s}{\partial s}\left(s M_s\right) - 2 \frac{\partial}{\partial \theta} M_{s\theta} + \frac{\partial}{\partial \theta} \bar{M}_{s\theta} - \bar{Q}_s s = 0 & \text{or} \;\; \delta w = 0 \\
M_s = \bar{M}_s & \text{or} \;\; \delta\left(\frac{\partial w}{\partial s}\right) = 0 \\
N_s = \bar{N}_s & \text{or} \;\; \delta u = 0 \\
N_{s\theta} = \bar{N}_{s\theta} & \text{or} \;\; \delta v = 0
\end{array}\right\} \quad (2.10)$$

For the closed conical shell ($0 \leqslant \theta \leqslant 2\pi \sin \alpha$), which is treated here, the boundary conditions at the edges θ = constant are replaced with the periodicity requirement with the period $2\pi \sin \alpha$.

In (2.10) the general static and geometrical boundary conditions of a conical shell are given for arbitrary edge loads. The further investigations are restricted to shells under axially symmetric load (axial load and inner pressure, Figure 3).

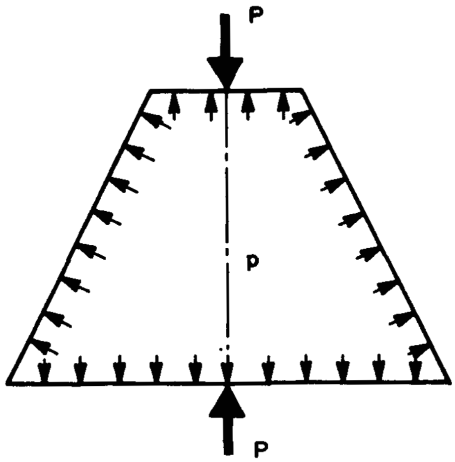


Figure 3. The Load of the Conical Shell Investigated

Corresponding to the test conditions[13], the following necessary geometrical conditions shall apply to the edges of the shell caused by the stiff end-disks bordering the shells at the edges $s = s_1$ and $s = s_2$ (Figure 4):

$$u - w \cot \alpha = 0. \tag{2.11}$$

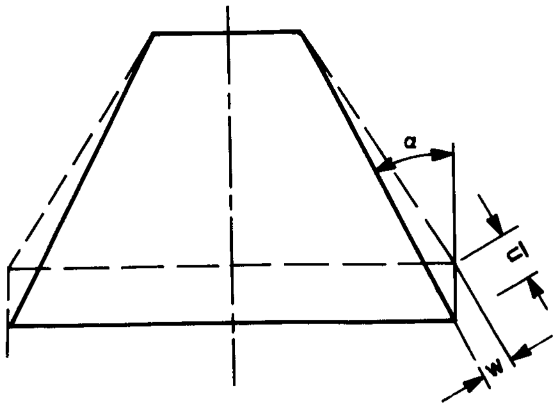


Figure 4. Displacement of the Lower Edge of the Cone ($s = s_2$) for Constant Radius of the End

For such a conical shell under axially symmetric load, the following axially symmetric boundary conditions are obtained at the edges $s = s_1$ and $s = s_2$.

(2.12):

$$w = \text{const} = \frac{u}{\cot \alpha} \quad (1)$$

$$M_s = 0 \quad (2a)$$

or

$$\frac{\partial w}{\partial s} = 0 \quad (2b)$$

$$N_s = \bar{N}_s \quad (3a)$$

or

$$u = \text{const} \quad (3b)$$

$$v = 0 \quad (4)$$

In order to be able to investigate the effect of the boundary conditions on the buckling load, the condition (2a)--pin-jointed edge--as well as the condition (2b)--fully restrained edge--is taken into account in the further considerations. The boundary condition (2b) corresponds to the test conditions[13] since the shell has been lap-glued to the end disks. The system of differential equations (2.9) with the boundary condition (2.12) generally describes the stress and deformation states of the axially loaded orthotropic conical shell. These are:

1) The state of the shell before buckling--here called basic state of the stresses.

2) The states of equilibrium at the branching point starting from which the shell buckles to assume another configuration--the edge load $\bar{N}_s$ of the branching point is called critical load or buckling load in the further investigations.

3) The postbuckling range.

The following considerations are restricted to the determination of the basic state of the stresses and to the determination of the branching point. For these states (1 and 2) special forms of the system of differential equations (2.9) can be given.

First the state of the shell prior to buckling shall be investigated.

2.4.2 The Differential Equations for the Stress and Deformation State Prior to Buckling

In the further investigations a predeformation $w_A(s)$ is taken into account which is independent of θ and includes the special case $w_A \equiv 0$ (ideal cone).

Considering first the state of the shell for small load (axial load and inner pressure) it can be assumed that the stresses and deformations are small of first order. Thus the system of equations (2.9) is linearized. For the system of linear differential equations with the boundary conditions (2.12) only one solution exists. Under the given boundary conditions (2.12) the solution is independent of the circumferential angle θ, it is axially symmetric. For the system of differential equations of the isotropic conical shell obtained from (2.9) by linearization, a solution in closed form is given in Section 3.1 of this paper.

In the following part of the paper, the axially symmetric basic state of the stresses shall be investigated for large loads as well (e.g., in the region of the buckling load). Thus the nonlinear members in the basic equations--corresponding to equation (2.9)--are taken into account. Because of the axial symmetry, all derivatives of the section forces and section momentums and deformations with respect to θ go to zero. Thus the system of differential equations (2.9) simplifies

$$\left.\begin{aligned} & -\frac{d}{ds}\left(s N_{s_0}\right) + N_{\theta_0} = 0 \,, \\ & \frac{d}{ds}\left(s\ N_{s\theta_0}\right) + N_{s\theta_0} = 0, \\ & -\frac{d}{ds}\left[s N_{s_0}\left(\frac{dw_0}{ds} + \frac{dw_A}{ds}\right)\right] - N_{\theta_0} \cot a + \frac{d^2}{ds^2}\left(s M_{s_0}\right) \\ & -\frac{d}{ds} M_{\theta_0} + p\cdot s = 0 \end{aligned}\right\} \quad (2.13)$$

with the boundary conditions (2.12). The index $(\ )_0$ of the section forces, section momentums, and displacements designates the basic state of the stresses.

$$N_{s\theta_0} \equiv 0 \qquad \text{and} \qquad v_0 \equiv 0 \qquad (2.14)$$

respectively.

Due to the axial symmetry of the stresses and deformations, v_0 and $N_{s\theta_0}$ become identically zero. Thus the differential equation (2.13.2) and the boundary condition (2.12.4) are satisfied. The two differential equations (2.13.1) and (2.13.3) represent the basic equations of the axially symmetric state of the stresses which in the further investigations shall be described by the section forces N_{s_0} and the displacement w_0.

From equation (2.13.3), the section force N_{θ_0} can be eliminated by means of (2.13.1). After integrating once from (2.13.3) follows

$$-\ s N_{s_0}\left(\cot\ a + \frac{dw_0}{ds} + \frac{dw_A}{ds}\right) + \frac{d}{ds}\left(s M_{s_0}\right) - M_{\theta_0} + \frac{1}{2}\ ps^2 + K_0 = 0\,, \qquad (2.15)$$

where K_0 designates an integration constant still to be determined. As additional differential equation, the compatibility condition from the law of elasticity (2.2) is derived by replacing the expansions ϵ_{s0} and ϵ_{θ_0} with the displacements u_0 and w_0 corresponding to (2.5) and by then eliminating u_0. Taking into account the relationship (2.13.1) it follows that

$$-C_{22}\, s^2 \frac{d^2 N_{s_0}}{ds^2} - 3C_{22}\, s \frac{dN_{s_0}}{ds} + \left(C_{11} - C_{22}\right) N_{s_0}$$

$$= \frac{dw_0}{ds}\left(\cot\alpha + \frac{1}{2}\frac{dw_0}{ds} + \frac{dw_A}{ds}\right). \tag{2.16}$$

With (2. 1) and (2. 6), M_{s_0} and M_{θ_0} in (2. 15) are replaced with the displacement w_0 and its derivatives. Then two coupled nonlinear differential equations for N_{s_0} and w_0 are obtained in which w_0 appears only in its derivatives with respect to s $\left(\frac{d^\varrho w}{d s^\varrho}\ ; \varrho = 1,\ 2,\ 3\right)$ and the axially symmetric prebuckling w_A only in the form dw_A/ds.

To reduce the order of the differential equation (2. 15) it is thus substituted:

$$\frac{dw_0}{ds} = q_0\ ; \qquad \frac{dw_A}{ds} = q_A. \tag{2.17}$$

The resulting system of differential equations is further simplified by the coordinate transformation

$$s = s_1\, e^z \qquad 0 \leqslant z \leqslant z_0$$
$$z_0 = \ell n\left(\frac{s_2}{s_1}\right). \tag{2.18}$$

A system of two nonlinear coupled differential equations of second order for q_0 and N_{s_0} is obtained,

$$-C_{22}N''_{s_0} - 3C_{22}N'_{s_0} + \left(C_{11} - C_{22}\right) N_{s_0} = q_0\left(\cot\alpha + \frac{1}{2} q_0 + q_A\right)$$

$$D_{11}\, q''_0 - D_{22} q_0 = s_1 e^z\left[s_1 e^z N_{s_0}\left(\cot\alpha + q_0 + q_A\right) - \frac{1}{2} p s_1^2 e^{2z} - K_0\right], \tag{2.19}$$

where $(\ )' = \frac{d(\ )}{dz}$.

Introducing the dimensionless functions N_0 and Q_0, the new parameters $\left(C_{22},\ C_{11},\ C_{12};\ D_{11},\ D_{12},\ D_{22};\ s_1\ z_0,\ \cot\alpha\right)$ essential for the behavior of the solution of (2. 19) with the boundary conditions (2. 12) can be combined to six characteristic quantities $\left(C_{11}/C_{22},\ C_{12}/C_{22},\ D_{22}/D_{11},\ D_{12}/D_{11},\ z_0,\ \lambda\right)$. With the substitution

$$
\left.
\begin{aligned}
N_{s_0} &= \frac{1}{s_1 e^2} \frac{D_{11}}{s_1} N_0 \\
q_0 &= \frac{\sqrt{D_{11}\, C_{22}}}{s_1} Q_0 \\
q_A &= \frac{\sqrt{D_{11}\, C_{22}}}{s_1} Q_A ,
\end{aligned}
\right\} \quad (2.20)
$$

system (2. 19) becomes

$$
\begin{aligned}
N''_0 - \frac{C_{11}}{C_{22}} N_0 &= - e^z Q_0 \left(\lambda + \frac{1}{2} Q_0 + Q_A\right) \\
Q''_0 - \frac{D_{22}}{D_{11}} Q_0 &= e^z \left\{ N_0 \left(\lambda + Q_0 + Q_A\right) - \lambda^2 \left(\bar{p} e^{2z} + \bar{K}_0\right) \right\} ,
\end{aligned}
\quad (2.21)
$$

where

$$
\left.
\begin{aligned}
\lambda &= \frac{s_1 \cot \alpha}{\sqrt{D_{11}\, C_{22}}} \\
\bar{p} &= \frac{1}{2} p \frac{s_1{}^2}{\cot^2 \alpha} \sqrt{\frac{C_{22}}{D_{11}}} \\
\bar{K}_0 &= \frac{K_0}{\cot^2 \alpha} \sqrt{\frac{C_{22}}{D_{11}}} .
\end{aligned}
\right\} \quad (2.22)
$$

The solutions N_0 and Q_0 of the system of differential equations (2. 21) must satisfy the boundary conditions (2. 12. 1) and (2. 12. 2). Due to the axial symmetry--$v_0 \equiv 0$--condition (2. 12. 1)-- $w_0 = u_0/\cot \alpha$-- means that the circumferential expansion ϵ_{θ_0} at the edges $z = 0$ and $z = z_0$ becomes zero. With the aid of the substitution (2. 17) and (2. 20) and the coordinate transformation (2. 18), the boundary conditions (2. 12. 1) and (2. 12. 2) are expressed by the variables N_0 and Q_0:

$$\left.\begin{array}{lr}
C_{22}\, N'_0 + C_{12}\, N_0 = 0 & (1) \\
\quad \text{(this means } \epsilon_{\theta_0} = 0) & \\
D_{11}\, Q'_0 + D_{12}\, Q_0 = 0 & (2a) \\
\quad \text{(pint jointed edge)} & \\
\text{or} & \\
Q_0 = 0 & (2b) \\
\quad \text{(fully restrained edge)} &
\end{array}\right\} \quad (2.23)$$

From the condition (2.12.3a) and (2.12.3b) respectively, the integration constant $\bar{K}_0$ in (2.21) is determined. For the range of the axially symmetric state of the stresses up to buckling which will be investigated in this portion of the paper for each edge load $\bar{N}_0$ there exists a one-to-one correlation of an edge displacement $\bar{u}_0$. For the numerical evaluation $\bar{K}_0$ is given, and from the solution of the system of differential equations (2.21) the edge displacement $\bar{u}_0$ and the edge load $\bar{N}_0$ are determined.

The determining equation for the displacement w_0 perpendicularly with respect to the central surface of the shell is derived from the relation (2.17),

$$w_0 = \int_0^z \sqrt{D_{11}\, C_{22}}\; e^{\xi}\; Q_0\,(\xi)\; d\xi + K_1 \quad . \tag{2.24}$$

Without affecting the generality, the integration constant K_1 can be set equal to zero. This means that the edge $z = 0$ is retained while the shell is deformed, while the edge $z = z_0$ can move (Figure 4):

$$\begin{array}{llll}
z = 0: & w_0 = 0 & \text{and} & u_0 = 0 \\
z = z_0: & w_0 = \bar{w}_0 = \dfrac{\bar{u}_0}{\cot \alpha} & \text{and} & u_0 = \bar{u}_0
\end{array} \tag{2.25}$$

2.4.3 The System of Differential Equations for the Additional Stresses and Deformations at the Branching Point

The axially symmetric basic state of the stresses which has been described in the last section is maintained as long as the edge load $\bar{N}_0$ is smaller than the critical load $\bar{N}_{cr}$. When reaching the critical load (branching point) at least two positions of the equilibrium exist, one axially symmetric (identical with the basic state of the stresses) and one more or less nonaxially symmetric (starting of buckling). At the branching point the smallest external perturbations render the basic state of the stresses unstable; a nonaxially symmetric state of equilibrium is assumed.

In this portion of the paper the system of differential equations shall be derived from which the nonaxially symmetric stresses and deformations occuring for buckling can be determined.

With the equation of elasticity (2. 1), the relations concerning the state of deformation (2. 5) and (2. 6), and the equilibrium relations (2. 9), the section forces N_s, N_θ, $N_{s\theta}$, the section momentums M_s, M_θ, $M_{s\theta}$, the deformations ϵ_s, ϵ_θ, $\gamma_{s\theta}$, k_s, k_θ, $k_{s\theta}$, and the displacements u, v, w can be determined for every state of equilibrium of the shell. By introducing a stress function,

$$\left.\begin{aligned} N_s &= \frac{1}{s}\frac{\partial f}{\partial s} + \frac{1}{s^2}\frac{\partial^2 f}{\partial \theta^2} \\ N_\theta &= \frac{\partial^2 f}{\partial s^2} \quad . \\ N_{s\theta} &= -\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial f}{\partial \theta}\right), \end{aligned}\right\} \qquad (2.26)$$

which, corresponding to equation (2. 26), is defined in such a way that the two first equations of (2. 9) are identically satisfied; the system of nonlinear partial differential equations can be reduced to two equations for the stress function f and the displacement w. The first relation for f and w follows from equation (2. 9. 3) by expressing the section momentums by means of the law of elasticity (2. 1) and by means of the equations (2. 6) by w and its derivatives, and the section forces are eliminated by means of (2. 6).

Due to the condition of compatibility, the second differential equation for f and w is given. From the relations concerning the state of deformation (2. 5), u and v are eliminated. The expansions ϵ_s and ϵ_θ and the shear $\gamma_{s\theta}$ are replaced by means of the law of elasticity (2. 2) and the substitution (2. 26) with the stress function f. Equation (2. 9. 3) and the condition of compatibility form a system of two nonlinear, partial differential equations of fourth order in s and θ for the stress function f and the displacement w:

$$L_0(w) = \left(\frac{1}{s}\frac{\partial f}{\partial s} + \frac{1}{s^2}\frac{\partial^2 f}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w_A}{\partial s^2}\right)$$

$$+ \frac{\partial^2 f}{\partial s^2}\left[\frac{1}{s}\left(\cot\alpha + \frac{\partial w}{\partial s} + \frac{\partial w_A}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 w}{\partial \theta^2}\right]$$

$$- 2\left(\frac{1}{s^2}\frac{\partial f}{\partial \theta} - \frac{1}{s}\frac{\partial^2 f}{\partial s \partial \theta}\right)\left(\frac{1}{s^2}\frac{\partial w}{\partial \theta} - \frac{1}{s}\frac{\partial^2 w}{\partial s \partial \theta}\right) - p \tag{2.27}$$

$$L_0(f)'' = -\left(\frac{1}{s}\frac{\partial w}{\partial s} + \frac{1}{s^2}\frac{\partial^2 w}{\partial \theta^2}\right)\left(\frac{1}{2}\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w_A}{\partial s^2}\right)$$

$$- \frac{\partial^2 w}{\partial s^2}\left[\frac{1}{s}\left(\cot\alpha + \frac{1}{2}\frac{\partial w}{\partial s} + \frac{\partial w_A}{\partial s}\right) + \frac{1}{2}\frac{1}{s^2}\frac{\partial^2 w}{\partial \theta^2}\right]$$

$$+ \left(\frac{1}{s^2}\frac{\partial w}{\partial \theta} - \frac{1}{s}\frac{\partial^2 w}{\partial s \partial \theta}\right)^2 .$$

$L_0(y)$ represents a linear differential operator:

$$L_0(y) = \alpha_y \frac{\partial^4 y}{\partial s^4} + 2\alpha_y \frac{1}{s}\frac{\partial^3 y}{\partial s^3} - \beta_y \frac{1}{s^2}\frac{\partial^2 y}{\partial s^2} + \beta_y \frac{1}{s^3}\frac{\partial y}{\partial s}$$

$$+ \gamma_y\left(\frac{1}{s^2}\frac{\partial^4 y}{\partial s^2 \partial \theta^2} - \frac{1}{s^3}\frac{\partial^3 y}{\partial s \partial \theta^2}\right)$$

$$+ \left(2\beta_y + \gamma_y\right)\frac{1}{s^4}\frac{\partial^2 y}{\partial \theta^2} + \beta_y \frac{1}{s^4}\frac{\partial^4 y}{\partial \theta^4}$$

where

$\alpha_w = D_{11}$ $\qquad\qquad$ $\alpha_f = C_{22}$

$\beta_w = D_{22}$ $\qquad\qquad$ $\beta_f = C_{11}$

$\gamma_w = 2\left(D_{12} + 2D_{33}\right)$ $\qquad\qquad$ $\gamma_f = 2C_{12} + C_{33}$.

The boundary conditions for the system (2.27) are given in (2.12).

The system of differential equations (2.27) generally describing the states of stresses and deformation of the orthotropic conical shell agrees with the system of differential equations given by Schnell[5] for the ideal isotropic conical shell ($w_A \equiv 0$).

As described in Section 2.4.2, the conical shell first deforms axially symmetrically for the given geometrical boundary conditions and under the load of axial load and inner pressure. When reaching the critical load nonaxially, symmetric contributions are added to the axially symmetric stresses and deformations so that for the branching point it can be set:

$$\begin{aligned} w &= w_0\,(s) + w_1\,(s,\theta) \\ f &= f_0\,(s) + f_1\,(s,\theta)\ , \end{aligned} \qquad (2.28)$$

where f_0 and w_0 designate the stresses and deformations of the axially symmetric basic state of the stresses. f_1 and w_1 are the nonaxially symmetric stresses and deformations which are added when making the transition to the buckled state. If these expressions (2.28) are inserted in the system of differential equations (2.27) it is obtained:

$$L_0\left(w_0 + w_1\right) = \left(\frac{1}{s}\frac{\partial f_1}{\partial s} + \frac{1}{s}\frac{\partial f_0}{\partial s} + \frac{1}{s^2}\frac{\partial^2 f_1}{\partial \theta^2}\right)\left(\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_0}{\partial s^2} + \frac{\partial^2 wA}{\partial s^2}\right)$$

$$-2\left(\frac{1}{s^2}\frac{\partial f_1}{\partial \theta} - \frac{1}{s}\frac{\partial^2 f_1}{\partial s \partial \theta}\right)\left(\frac{1}{s^2}\frac{\partial w_1}{\partial \theta} - \frac{1}{s}\frac{\partial^2 w_1}{\partial s \partial \theta}\right)$$

$$+\left(\frac{\partial^2 f_1}{\partial s^2} + \frac{\partial^2 f_0}{\partial s^2}\right)\left[\frac{1}{s}\left(\cot \alpha + \frac{\partial w_1}{\partial s} + \frac{\partial w_0}{\partial s} + \frac{\partial w A}{\partial s}\right)\right.$$

$$\left. + \frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2}\right] - p$$

and (2.29)

$$L_0\left(f_0 + f_1\right) = -\left(\frac{1}{s}\frac{\partial w_1}{\partial s} + \frac{1}{s}\frac{\partial w_0}{\partial s} + \frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2}\right)\left(\frac{1}{2}\frac{\partial^2 w_1}{\partial s^2}\right.$$

$$\left. + \frac{1}{2}\frac{\partial^2 w_0}{\partial s^2} + \frac{\partial^2 w_A}{\partial s^2}\right) + \left(\frac{1}{s^2}\frac{\partial w_1}{\partial \theta} - \frac{1}{s}\frac{\partial^2 w_1}{\partial s \partial \theta}\right)^2$$

$$-\left(\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_0}{\partial s^2}\right)\left[\frac{1}{s}\left(\cot \alpha + \frac{1}{2}\frac{\partial w_1}{\partial s} + \frac{1}{2}\frac{\partial w_0}{\partial s}\right.\right.$$

$$\left.\left. + \frac{\partial w_A}{\partial s}\right) + \frac{1}{2}\frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2}\right] .$$

The axially symmetric contributions f_0, w_0 may be expressed with the relations (2.26) and (2.17) by N_{s0} and q_0.

For the branching point, system (2.19) can be split into two determining equations for N_{s0} and q_0 (and f_0 and w_0, respectively)--identical with equations (2.19) of the axially symmetric state of stresses and deformations--and into two determining equations for the nonaxially symmetric contributions f_1 and w_1 which may be written in the following form:

$$L_0(w_1) = \left(\frac{1}{s}\frac{\partial f_1}{\partial s} + \frac{1}{s^2}\frac{\partial^2 f_1}{\partial \theta^2} + N_{s0}\right)\frac{\partial^2 w_1}{\partial s^2} + \left(\frac{\partial q_0}{\partial s} + \right.$$

$$\left. + \frac{\partial q_A}{\partial s}\right)\left(\frac{1}{s}\frac{\partial f_1}{\partial s} + \frac{1}{s^2}\frac{\partial^2 f_1}{\partial \theta^2}\right) + \frac{\partial^2 f_1}{\partial s^2}\left[\frac{1}{s}\left(\cot\alpha + q_0\right.\right.$$

$$\left.\left. + q_A + \frac{\partial w_1}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2}\right] + \frac{\partial}{\partial s}\left(s N_{s0}\right)\left(\frac{1}{s}\frac{\partial w_1}{\partial s} + \frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2}\right)$$

$$+ 2\left(\frac{1}{s^2}\frac{\partial f_1}{\partial \theta} - \frac{1}{s}\frac{\partial^2 f_1}{\partial s \partial \theta}\right)\left(\frac{1}{s^2}\frac{\partial w_1}{\partial \theta} - \frac{1}{s}\frac{\partial^2 w_1}{\partial s \partial \theta}\right)$$

and (2. 30)

$$L_0(f_1) = -\left(\frac{1}{s}\frac{\partial w_1}{\partial s} + \frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2} + \frac{1}{s} q_0\right)\left(\frac{1}{2}\frac{\partial^2 w_1}{\partial s^2} + \frac{1}{2}\frac{\partial q_0}{\partial s}\right.$$

$$\left. + \frac{\partial q_A}{\partial s}\right) + \left(\frac{1}{s^2}\frac{\partial w_1}{\partial \theta} - \frac{1}{s}\frac{\partial^2 w_1}{\partial s \partial \theta}\right)^2 - \left(\frac{\partial^2 w_1}{\partial s^2} + \frac{\partial q_0}{\partial s}\right)\left[\frac{1}{s}\left(\cot\alpha\right.\right.$$

$$\left.\left. + \frac{1}{2} q_0 + q_A + \frac{1}{2}\frac{\partial w_1}{\partial s}\right) + \frac{1}{2}\frac{1}{s^2}\frac{\partial^2 w_1}{\partial \theta^2}\right] .$$

If we restrict ourselves to the determination of the buckling loads, it can be assumed that in the region of the branching point the functions $f_1(s, \theta)$ and $w_1(s, \theta)$ as well as their derivatives are small of first order. Neglecting members of second order, the following system of linear differential equations is obtained for the nonaxially symmetric contributions f_1 and w_1.

$$L_0(w_1) = B_0\left(w_1,\ s N_{s0}\right) + B_0\left(f_1,\ \cot\alpha + q_0 + q_A\right)$$

$$L_0(f_1) = -\, B_0\left(w_1,\ \cot\alpha + q_0 + q_A\right) \qquad (2.\ 31)$$

where $B_0(y_1, \varphi_0)$ represents the following differential operator:

$$B_0\left(y_1, \varphi_0\right) = s\left\{\frac{\partial^2 y_1}{\partial s^2}\ \frac{1}{s^2}\ \varphi_0 + \left(\frac{1}{s}\ \frac{\partial y_1}{\partial s} + \frac{1}{s^2}\ \frac{\partial^2 y_1}{\partial \theta^2}\right)\frac{1}{s}\ \frac{\partial \dot{\varphi_0}}{\partial s}\right\} .$$

In order to obtain an overall picture of the essential parameters of the system, dimensionless functions F_1 and W_1 are introduced.

With equations (2.20) with the substitution

$$w_1 = e^z \sqrt{D_{11}\,C_{22}}\ W_1$$

$$f_1 = e^z\, D_{11}\, F_1 \tag{2.32}$$

and with the coordinate transformation (2.18), the system of differential equations (2.32) becomes

$$L_1\left(W_1\right) = B_1\left(W_1,\ N_0\right) + B_1\left(F_1, \lambda + Q_0 + Q_A\right)$$

$$L_1\left(F_1\right) = -B_1\left(W_1, \lambda + Q_0 + Q_A\right) , \tag{2.33}$$

where

$$L_1\left(y_1\right) = y_1^{(4)} - \frac{\alpha y + \beta y}{\alpha y}\ y_1'' + \frac{\gamma y}{\alpha y}\ y_1''^{\cdot\cdot} + \frac{\beta y}{\alpha y}\left(y_1 + 2y_1^{\cdot\cdot} + y_1^{::}\right)$$

is a differential operator with the constant coefficients

$$\alpha_F = C_{22} , \qquad \beta_F = C_{11} , \qquad \gamma_F = 2\,C_{12} + C_{33}$$

$$\alpha_W = D_{11} , \qquad \beta_W = D_{22} , \qquad \gamma_W = 2\,(D_{12} + 2\,D_{33})$$

and

$$B_1\left(y_1,\ \varphi_0\right) = e^z\left\{\varphi_0\, y_1'' + \left(\varphi_0 + \varphi_0'\right) y_1' + \varphi_0'\left(y + y_1^{\cdot\cdot}\right)\right\} .$$

In (2.33) the differential quotients $\partial(\ )/\partial z$ and $\partial(\ )/\partial\theta$ have been replaced with the following symbols:

$$\frac{\partial(\ )}{\partial z} = (\ )' \quad \text{and} \quad \frac{\partial(\ )}{\partial \theta} = (\ )^{\cdot} .$$

The determination of the nonaxially symmetric stress and deformation and contributions F_1 and W_1 is performed under the condition that in the instance of buckling--during the transition from the axially symmetric basic state of the stresses to an infinitesimally adjacent nonaxially symmetric state of the equilibrium--the axial edge load $\bar{N}_s$ (2.12.3a) and the radial edge displacement $\bar{w}$ (2.12.1) remain constant. The axially symmetric stress and deformation contributions N_0 and Q_0 satisfy the boundary conditions corresponding to (2.12). Thus the additionally occurring nonaxially symmetric stresses and deformations appearing in the instance of buckling must satisfy the following homogeneous boundary conditions at the edges $z = 0$ and $z = z_0$:

$$\left.\begin{array}{ll} W_1 = 0 & (1) \\ (M_s)_1 = 0 & (2a) \\ \text{or} & \\ W_1' = 0 & (2b) \\ (N_s)_1 = 0 & (3) \\ v_1 = 0 & (4) \end{array}\right\} \quad (2.34)$$

Index $(\ )_1$ designates the additional stresses and deformations occurring in the instance of buckling.

From the boundary condition (2.34.1)--$W_1 = 0$, i.e., also $\partial W_1/\partial\theta = 0$--and condition (2.34.4)--$v_1 = 0$, likewise $\partial v_1/\partial\theta = 0$--it follows, due to the necessary geometrical condition (2.17), that for the edges $z = 0$ and $z = z_0$

$$(\epsilon_\theta)_1 = 0 \quad \text{and because} \quad (N_s)_1 = 0 \quad \text{it follows that } (N_\theta)_1 = 0. \quad (2.35)$$

With the aid of the law of elasticity (2.1), the definition of the stress function (2.26), and the substitution (2.32) from (2.34) and (2.35), the following boundary conditions for functions F_1 and W_1 can be formulated:

$$\left.\begin{array}{lll} W_1 = 0 & (1) & \\ \dfrac{D_{11}}{D_{11}+D_{12}} W_1'' + W_1' + \dfrac{D_{12}}{D_{11}+D_{12}} \ddot{W}_1 = 0 & (2a) & \\ & \text{means } (M_s)_1 = 0 & \\ \text{or} & & \\ W_1' = 0 & (2b) & \\ F_1' + F_1 + \ddot{F}_1 = 0 & (3) & \\ & \text{means } (N_s)_1 = 0 & \\ F_1'' + F_1' = 0 & (4) & \\ & \text{means } (N_\theta)_1 = 0 & \end{array}\right\} \quad (2.36)$$

Since the system of linear partial differential equations (2.33) as well as the boundary conditions (2.36) are homogeneous, the determination of the nonaxially symmetric stress and deformation contributions F_1 and W_1 is an eigen-value problem. F_1 and W_1 are the associated eigen functions; the axial edge load N_0 (and the edge displacement u_0 or the constant $\bar{K}_0$, respectively) of the axially symmetric basic state of the stresses represents the eigen-value of the system.

To solve the system of differential equations (2.33), the functions F_1 and W_1 which are periodic with $2\pi \sin\alpha$ in the direction of the circumference are expanded with $\eta = n/\sin\alpha$ into series of the form

$$\begin{aligned} F_1 &= \sum_{n=1}^{\infty} F_{1n}(z) \cos \eta\theta \\ W_1 &= \sum_{n=1}^{\infty} W_{1n}(z) \cos \eta\theta \end{aligned} \qquad (2.37)$$

and are inserted into the system of differential equations (2.33). By comparing the coefficients, systems of linear, homogeneous ordinary differential equations are obtained,

$$L_2\left(W_{1n}\right) = B_2\left(W_{1n}, N_0\right) + B_2\left(F_{1n}, \lambda + Q_0 + Q_A\right)$$

$$L_2\left(F_{1n}\right) = - B_2\left(W_{1n}, \lambda + Q_0 + Q_A\right) \quad , \qquad (2.38)$$

where $L_2\left(y_{1n}\right)$ and $B_2\left(y_{1n}, \varphi_0\right)$ represent the ordinary differential operators,

$$L_2\left(y_{1n}\right) = y_{1n}^{(6)} - \frac{\alpha y + \beta y + \delta y}{\alpha y} y_{1n}'' + \frac{\beta y}{\alpha y}\left(1 - \eta^2\right)^2 y_{1n}$$

with $\qquad \alpha_F = C_{22}, \ \beta_F = C_{11}, \ \delta_F = \eta^2 \left(2 C_{12} + C_{33}\right)$

$$\alpha_W = D_{11}, \ \beta_W = D_{22}, \ \delta_W = 2\eta^2 \left(D_{12} + 2D_{33}\right)$$

$$B_2\left(y_{1n}, \varphi_0\right) = e^z \left\{\varphi_0 \, y_{1n}'' + \left(\varphi_0 + \varphi_0'\right) y_{1n}' + \varphi_0' \left(1 - \eta^2\right) y_{1n}\right\} \quad .$$

The functions $F_{1n}(z)$ and $W_{1n}(z)$ have to satisfy the following boundary conditions according to (2.36):

$$\left.\begin{array}{lr} W_{1n} = 0 & (1) \\ \frac{D_{11}}{D_{11} + D_{12}} W_{1n}'' + W_{1n}' = 0 & (2a) \\ \text{or} & \\ W_{1n}' = 0 & (2b) \\ F_{1n}' + \left(1 - \eta^2\right) F_{1n} = 0 & (3) \\ F_{1n}'' + F_{1n}' = 0 \ . & (4) \end{array}\right\} \quad (2.39)$$

For each n (number of bucklings in the direction of the circumference in the instance of buckling) a system of differential equations (2.38) with the boundary conditions (2.39) exists, the solutions of which F_{1n} and W_{1n} represent the eigen functions with the associated eigen-value N_0 (n).

3. Solutions and Solution Methods

3.1 Exact Solutions of the Simplified System of Differential Equations for the Axially Symmetric State of Stresses and Deformation

The determination of the buckling stress, i.e., a solution of the system of differential equations (2.38), assumes the knowledge of the axially symmetric stresses and deformations (N_0, Q_0) of the unbuckled region--here called the basic state of stresses--corresponding to equation (2.21). For the cylinder shell[3], the differential equations of the axially symmetric state of stresses and deformations can be solved linearly and in closed form. The corresponding system of differential equations of the conical shell (2.21) remains nonlinear even in the axially symmetric case. A general closed solution of this nonlinear system of differential equations for the basic state of the stresses is not possible with the known methods. Therefore, first, two methods shall be stated (a and b) by means of which solutions for N_0 and Q_0 can be found when the system of differential equations is simplified.

a) Diaphragm stress state

Under the assumption that the deformations of the shell perpendicular with respect to the central surface of the shell are constant over the entire shell of the cone--this means that Q_0 is identically zero--and that no prebuckling occurs ($Q_A \equiv 0$), N_0 is determined from equation (2.21.2):

$$N_0 = \lambda\left(\overline{p}\, e^{2z} + \overline{K}_0\right) . \tag{3.1}$$

With the abbreviations (2.20) and (2.22), this becomes

$$N_{s0} = \frac{1}{2}\, p \,\frac{s}{\cot \alpha} + \frac{K_0}{s \cot \alpha} \ , \tag{3.2}$$

the solution for the flexural-free state of diaphragm stress corresponding to Schnell's[5] Equation (2) and Schiffner's[8] Equation (7) if the axial load (Figure 3) is set

$$p = -2\pi \sin^2 \alpha \, K_0 \ . \tag{3.3}$$

For N_{θ_0} follows then from equation (2.13.1):

$$N_{\theta_0} = \frac{p.s}{\cot \alpha} \quad . \tag{3.4}$$

The diaphragm stresses N_{s_0} and N_{θ_0} corresponding to equations (3.2) and (3.4) satisfy the equilibrium conditions (2.13) and the boundary conditions

$$N_{s_0} = \overline{N}_s \text{ at the edges } z = 0 \text{ and } z = z_0 \ ,$$

not, however, the compatibility condition and the boundary condition $\epsilon_{\theta_0} = 0$ and thus are no valid solution of the boundary value problem investigated here.

b) Linearization of the System of Differential Equations

For small edge loads and small inner pressure, the deformations and stresses of the ideal cone $\left(Q_A \equiv 0\right)$ are small of first order. If, therefore, the system of differential equations (2.21) is linearized--this linearization means a neglect of Q_0 compared with λ--the following system of differential equations is obtained:

$$\begin{aligned} N_0'' - \frac{C_{11}}{C_{22}} N_0 &= -e^z Q_0 \lambda \\ Q_0'' - \frac{D_{22}}{D_{11}} Q_0 &= e^z \lambda \left[N_0 - \lambda\left(\overline{p}\, e^{2z} + \overline{K}_0\right)\right] \end{aligned} \tag{3.5}$$

and after eliminating Q_0, respectively

$$N_0^{IV} - 2 N_0''' + \left(1 - \frac{C_{11}}{C_{22}} - \frac{D_{22}}{D_{11}}\right) N_0'' + 2 \frac{C_{11}}{C_{22}} N_0' + \left(\frac{D_{22}}{D_{11}} - \frac{C_{11}}{C_{22}}\right) N_0$$

$$= - e^{2z} \lambda^2 \left[N_0 - \lambda\left(\overline{p}\, e^{2z} + \overline{K}_0\right)\right] \quad . \tag{3.6}$$

Equations (3.5) and (3.6), respectively, correspond to the differential equations of the linear bending theory[25] for axially symmetric stresses and deformations (container theory).

For the isotropic conical shell the rigidity parameters of which may assume the value $C_{11}/C_{12} = D_{22}/D_{11} = 1$ according to (2.4) the differential equation (3.6) can be solved in closed form. As for the cylinder shell, the particular solution of (3.6) is identical with the diaphragm stress (3.1). Together with the solution (modified cylinder function--Kelvin function[16]) of the homogeneous differential equation it is obtained

$$N_0 = \lambda\left(\frac{\bar{p}}{s_1^2} s^2 + \bar{K}_0\right) + K_1 \,\mathrm{ber}_2\left(2\sqrt{\lambda s}\right) + K_2 \,\mathrm{bei}_2\left(2\sqrt{\lambda s}\right)$$

$$+ K_3 \,\mathrm{her}_2\left(2\sqrt{\lambda s}\right) + K_4 \,\mathrm{hei}_2\left(2\sqrt{\lambda s}\right) \quad . \tag{3.7}$$

For the range of parameters (Section 4.1) investigated here--argument $x = 2\sqrt{\lambda s} > 10^3$, order $\bar{\nu} = 2$--these functions are not tabulated. However, Jahnke[16] gives a first approximation of the Kelvin functions for the parameters $x \gg 1$, $x \gg \bar{\nu}^2$. Inserting this approximation into solution (3.7) for N_0 is obtained

$$N_0 = \lambda\left(\frac{\bar{p}}{s_1^2} s^2 + \bar{K}_0\right) + \frac{1}{\sqrt[4]{2\lambda s}}\left\{\sinh\sqrt{2\lambda s}\left[\bar{K}_1 \sin\sqrt{2\lambda s}\right.\right.$$

$$\left.\left. + \bar{K}_2 \cos\sqrt{2\lambda s}\right] + \cosh\sqrt{2\lambda s}\left[\bar{K}_3 \sin\sqrt{2\lambda s} + \bar{K}_4 \cos\sqrt{2\lambda s}\right]\right\} . \tag{3.8}$$

Here the constants $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$, $\bar{K}_4$ are to be determined from the boundary conditions (2.23) and $\bar{K}_0$ from the condition (2.12.3a) or (2.12.3b). For the numerical evaluation, however, $\bar{K}_0$ is given so that then the edge load $\bar{N}_0$ and the edge displacement $\bar{u}_0$ is calculated, respectively.

The approximate solution (3.7) and (3.8), respectively, of the linearized system of differential equations from (2.21) represents a good approximation for the basic state of the stresses for small edge loads and small inner pressure. However, it cannot be used for determining the buckling load since the system of differential equations (2.31) for the additional stresses and deformations f_1 and w_1 appearing in the instance of buckling has been derived under the condition that N_0 and Q_0 (and N_{s_0} and w_0, respectively) are solutions of the nonlinear system of differential equations (2.21).

3.2 Approximate Solution of the Linear Buckling Equations by Means of the Energy Method

In the following portion of the discussion, the state of the diaphragm stress (3.1) described in Section 3.1 shall be applied to the determination of the buckling load. Inserting the diaphragm stresses (3.1) into the system of differential equations (2.38), the buckling equations of the linear theory of shells is obtained:

$$\left.\begin{aligned} & L_2\left(W_{1n}\right) = B_2\left(W_{1n},\ N_0\right) + B_2\left(F_{1n},\lambda\right) \\ & L_2\left(F_{1n}\right) = -\ B_2\left(W_{1n},\lambda\right) \\ \text{where} \quad & B_2\left(Y_{1n},\ \lambda\right) = \lambda e^{z}\left(Y''_{1n} + Y'_{1n}\right) \\ \text{and} \quad & N_0 = \lambda\left(\bar{p}\ e^{2z} + \bar{K}_0\right). \end{aligned}\right\} \quad (3.9)$$

The system of differential equations agrees with the buckling equations given in several sources[5,6,8,12,15] provided the appropriate substitutions are observed.

Except for the special case $\eta = 0$ (ring buckling) which has been treated by Seide[15] for the isotropic conical shell with the aid of further simplifications,no solution in closed form is known for the system of differential equations (3.9).

It is stated once more that the state of the diaphragm stress does not represent an axially symmetric solution of the shell problem investigated here since the compatibility condition as well as the boundary condition $\epsilon_{\theta_0} = 0$ are not satisfied. The further investigation according to the linear theory has only the aim to derive a reference parameter for the buckling loads calculated according to the nonlinear shell theory.

Since a general solution of the linear buckling differential equations (3.9) is not known, an approximate solution according to the energy method is determined in the following. An appropriate formulation for the deformations perpendicular with respect to the central surface of the shell is chosen, and from the condition of compatibility a stress function is determined with the formulation for W_{1n}. The free values A_k of the formulation are determined from the condition for the energy minimum $\partial\pi/\partial A_k = 0$ corresponding to the variation requirement $\delta\pi = 0$.

For a basic state of the stresses according to equations (3.2) and (3.4)--diaphragm stresses--the following expression is obtained for the total energy of the deformed shell in the branching point.

$$\Pi = \frac{1}{2}\iint_{(o)} \left\{ N_{s_1}\, \epsilon_{s_1}^{(L)} + N_{\theta_1}\epsilon_{\theta_1}^{(L)} + N_{s\theta_1}\, \gamma_{s\theta_1}^{(L)} \right\} s\, ds\, d\theta$$

$$+ \frac{1}{2}\iint_{(o)} \left\{ M_{s_1}\, k_{s_1} + M_{\theta_1}\, k_{\theta_1} + 2\, M_{s\theta_1}\, k_{s\theta_1} \right\} s\, ds\, d\theta$$

$$+ \iint_{(o)} \left\{ N_{s_0}\, \epsilon_{s_1} + N_{\theta_0}\, \epsilon_{s_1} \right\} s\, ds\, d\theta$$

$$- \int_0^{2\bar{a}\sin\alpha} \left\{ s \left[\bar{\bar{N}}_s\, u_1 + \overline{N}_{s\theta}\, v_1 + \overline{M}_s \frac{\partial w_1}{\partial s} + \overline{M}_{s\theta} \frac{1}{s} \frac{\partial w_1}{\partial \theta} \right] \right\} \Bigg|_{s_1}^{s_2} d\theta$$

$$+ \iint_{(o)} p w_1\, s\, ds\, d\theta \qquad , \qquad (3.10)$$

where the first two double integrals represent the deformation energy of the forces and deformations occurring additionally during buckling. Since these additional forces and deformations are small of first order, the nonlinear contributions in ϵ_{s_1}, ϵ_{θ_1}, $\gamma_{s\theta_1}$ are neglected $\left(\epsilon_{s_1}^{(L)}, \epsilon_{\theta_1}^{(L)}, \gamma_{s\theta_1}^{(L)}\right)$.

For the displacement W_{1n} appearing in the instance of buckling perpendicular with respect to the central surface of the shell, the formulation

$$W_{1n} = A e^{-z} \sin k z \quad \text{with} \quad k = \frac{\bar{m}\,\bar{a}}{z_0} \qquad (3.11)$$

is chosen. This formulation satisfies the boundary condition (2.39.1) at the edges $z = 0$ and $z = z_0$; $W_{1n} = 0$ and corresponds to the so-called "checkerboard buckling configuration" with $\bar{m}$-half-waves in the longitudinal direction and n-waves in the circumferential direction, where because of $s = s_1\, e^z$ the half-wavelength increases along the coordinate s. For a given W_{1n} the stress function can be found from the compatibility condition (3.9.2) by integration. It is composed of a particular solution,

$$\left.\begin{aligned} &\left(F_{1n}\right)_p = \bar{A}\,k\,\Omega\,(\cos k z + k \sin k z) \\ &\text{where} \\ &\Omega = \frac{\lambda}{k^6 + \dfrac{(C_{22} + C_{11}) + \eta^2 (2C_{12} + C_{33})}{C_{22}}\, k^2 + \dfrac{C_{11}}{C_{22}}\left(1 - \eta^2\right)^2} \end{aligned}\right\} \quad (3.12)$$

and of the solutions

$$\left(F_{1n}\right)_n = K e^{\mu z}$$

of the homogeneous differential equation.

With these solutions the boundary conditions for the stress function can be satisfied. Investigations on isotropic conical shells, however, indicated that the effect of the boundary conditions on the buckling loads is small according to the linear theory. For this reason only the particular solution is taken into account.

Inserting (3.11) and (3.12) into (3.10) and after integration, the total energy of the orthotropic conical shell is calculated from (3.10).

From the minimum condition

$$\frac{\partial \Pi}{\partial \bar{A}^2} = 0 \qquad (3.13)$$

in addition to the trivial solution $k^2 = 0$ (no buckling),

$$N_{s_0} = - \frac{P - p k s_1^2 \sin^2 \alpha}{2 k s_1 \sin \alpha \cos \alpha}$$

$$N_{s_0} = - \frac{1 + 4k^2}{\left(1 - e^{-z_0}\right)\left(1 + 2k^2\right)} \left\{ \frac{z_0}{2} \cot^2 \alpha \right.$$

$$\times \frac{1 + k^2}{C_{11} \left(1 - \eta^2\right)^2 + \left(C_{11} + C_{22}\right) k^2 + C_{22} k^4 + \left(2C_{12} + C_{33}\right) k^2 \eta^2} \qquad (3.14)$$

$$+ \frac{e^{z_0} - 1}{1 + 4k^2} \; \frac{p s_1}{2 \cot \alpha} \left[\left(1 + 2k^2\right)\left(1 - e^{-z_0}\right) + 4 \eta^2 \right]$$

$$+ \frac{1}{4 s_1^2} \left(1 - e^{-2z_0}\right) D_{11} \left(2 + k^2\right) - 4D_{12} + D_{22} \left(1 + \frac{\left(1 - \eta^2\right)^2}{4 + k^2}\right)$$

$$\left. + 2 \eta^2 \left(D_{12} + 2 D_{33}\right) \Big] \right\}$$

is obtained for the force at the edge $z = 0$ (tension positive according to Figure 2).

A general evaluation of (3. 14) is involved while by restricting to certain shell parameters equation (3. 14) can be essentially simplified. In the following only shells shall be considered the geometry of which satisfies the equation

$$z_0 = \ln \frac{s_2}{s_1} \leqslant 1 \qquad\qquad \frac{s_2}{s_1} \leqslant e \qquad (3.15)$$

and the opening angle α of which is smaller than 45° (see Figure 5).

For this range of the shell parameters it is

$$k^2 \gg 1, \qquad \eta^2 \gg 1$$

so that it may be set

$$k^2 \pm 1 \approx k^2$$

$$\eta^2 \pm 1 \approx \eta^2 \ .$$

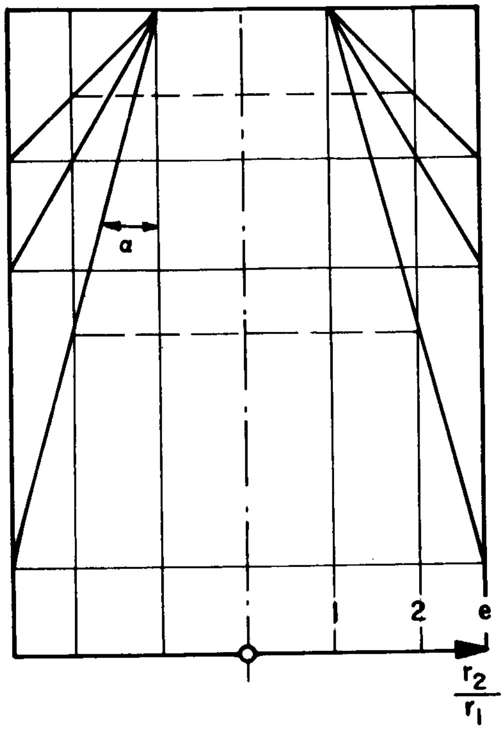


Figure 5. Possible Cone Configurations With $r_2/r_1 = s_2/s_1 = 2$ and e, Respectively, for Various Opening Angles of the Cone $\alpha \leqslant 45°$

From (3.14) follows then

$$N_{s_0} = -\frac{2}{1 - e^{-z_0}} \left\{ \frac{z_0}{2} \cot^2 \alpha \frac{k^2}{\eta^4 C_{11} + C_{11} k^2 + C_{22} k^4 + \left(2C_{12} + C_{33}\right) k^2 \eta^2} \right.$$

$$+ \frac{1}{4} \left(e^{z_0} - 1\right) \frac{p s_1}{\cot \alpha} \left[\left(1 - e^{-z_0}\right) + \frac{2\eta^2}{k^2}\right]$$

$$\left. + \frac{1}{4 s_1^2} \left(1 - e^{-2z_0}\right) \left[D_{11} k^2 + 2\left(D_{12} + 2D_{33}\right)\eta^2 + D_{22}\left(1 + \frac{\eta^4}{k^2}\right)\right] \right\}. \quad (3.16)$$

If first the special case of the axially symmetric buckling configuration (ring buckling) is investigated, i.e., $\eta = 0$, from

$$\frac{\partial N_{s_0}}{\partial k^2} = 0$$

a number of the half-waves in longitudinal direction can be determined

$$k^2 = \sqrt{\frac{2z_0}{C_{22}\, D_{11} \left(1 - e^{-2z_0}\right)}}\; s_1 \cot \alpha - \frac{C_{11}}{C_{22}} \tag{3.17}$$

with the associated ring buckling stress

$$N_{s_{0cr}} = - \frac{\cot \alpha}{s_1} \sqrt{2z_0 \frac{1+e^{-z_0}}{1-e^{-z_0}} \frac{D_{11}}{C_{22}}}$$

$$+ \frac{1}{2s_1}\left(1+e^{-z_0}\right) D_{22}\left(1 - \frac{C_{11}\, D_{11}}{C_{22}\, D_{22}}\right) - \frac{1}{2}\left(e^{z_0} - 1\right) \frac{p s_1}{\cot \alpha} \, . \tag{3.18}$$

From this formula it follows that the buckling load increases linearly with the inner pressure. This solution agrees with the ring buckling stress of the linear theory of stability stated by Schnell[5] for the special case of the isotropic shell.

The general investigation of the buckling equation (3.16) is restricted to isotropic and longitudinally reinforced orthotropic shells. For these shells it is

$$C_{11} \leqslant C_{22} \;; \qquad D_{11} \geqslant D_{22} \tag{3.19}$$

and because of (3.15)

$$C_{11} + C_{22}\, k^2 \approx C_{22}\, k^2$$

$$D_{11}\, k^2 + D_{22} \approx D_{11}\, k^2 \, .$$

For a further calculation, it turns out to be practical to combine the orthotropic rigidities to three rigidity characteristics[4,10,11]:

$$\vartheta_s = \frac{C_{12} + \frac{1}{2} C_{33}}{\sqrt{C_{11}\, C_{22}}}, \qquad \vartheta_p = \frac{D_{12} + 2D_{23}}{\sqrt{D_{11}\, D_{22}}}, \qquad \gamma_N = \frac{D_{11}\, C_{11}}{D_{22}\, C_{22}}; \tag{3.20}$$

ϑ_s = disc characteristic, ϑ_p = plate characteristic γ_N = main rigidity characteristic. For the isotropic shell it is $\vartheta_s = \vartheta_p = \gamma_N = 1$. Equation (3.16) can be further simplified with (3.19) and (3.20):

$$N_{s_0} = - \frac{2}{1 - e^{-2 z_0}} \left\{ \frac{z_0 \cot^2 a}{2 \sqrt{C_{11} C_{22}}} \frac{1}{\lambda_0 \bar{\phi}_1} + \frac{\left(1 - e^{-z_0}\right) \sqrt{D_{11} D_{22}}}{4 s_1^2 \sqrt{\gamma_N}} \lambda_0 \bar{\phi}_2 \right.$$

$$\left. + \frac{1}{4} \left(e^{z_0} - 1\right) \frac{p s_1}{\cot a} \left[\left(1 - e^{-z_0}\right) + \frac{2}{\mu^2} \sqrt{\frac{C_{22}}{C_{11}}}\right] \right\} \qquad (3.21)$$

where

$$\mu^2 = \left(\frac{k}{\eta}\right)^2 \sqrt{\frac{C_{22}}{C_{11}}}, \qquad \bar{\phi}_1 = 1 + 2 \vartheta_s \mu^2 + \mu^4$$

$$\lambda_0 = \frac{\eta^2}{\mu^2} = \frac{\eta^4}{k^2} \sqrt{\frac{C_{11}}{C_{22}}}, \qquad \bar{\phi}_2 = 1 + 2 \vartheta_p \sqrt{\gamma_N} + \gamma_N \mu^{\phi} .$$

Now the minimum buckling load is determined in dependence of the wave number ratios λ_0 and μ. From

$$\frac{\partial N_{s_0}}{\partial \lambda_0} = 0$$

follows

$$\lambda_0 = \overset{+}{(-)} \cot a \; s_1 \sqrt{\frac{2 z_0}{1 - e^{-z_0}} \frac{1}{C_{22} D_{22}} \frac{1}{\sqrt{\bar{\phi}_1 \bar{\phi}_2}}} \; . \qquad (3.22)$$

In (3.23) only the positive root is of significance since $\lambda_0 < 0$ results in complex $\bar{m}$ and n which is geometrically meaningless. With (3.22) it follows then that for the critical buckling load σ_{cr} (dimensionless load parameter)

$$\sigma_{cr} = \sqrt{\frac{\bar{\phi}_2}{\bar{\phi}_1}} + \hat{p} \frac{1}{\mu^2} \qquad (3.23)$$

where

$$\hat{p} = \frac{\frac{ps_1}{\cot \alpha} e^{z_0}}{2 \frac{\cot \alpha}{s_1} \sqrt{\frac{D_{22}}{C_{22}}} \vartheta_T}$$

$$\sigma_{cr} = \frac{- N_{s_0} - \frac{1}{2} \frac{ps_1}{\cot \alpha}\left(e^{z_0} - 1\right)}{2 \frac{\cot \alpha}{s_1} \sqrt{\frac{D_{22}}{C_{11}}} \vartheta_T}$$

$$\vartheta_T = \sqrt{\frac{z_0}{2} \frac{1 + e^{-z_0}}{1 - e^{-z_0}}} \; .$$

For the conical shells ($z_0 \leqslant 1$) considered here, the value of the root ϑ_T is about 1,

$$\vartheta_T = \sqrt{\frac{z_0}{2} \frac{1 + e^{-z_0}}{1 - e^{-z_0}}} \approx 1,$$

and when making the transition to the cylinder ($z_0 \rightarrow 0$) it is exactly 1.

The ratio of the wave numbers μ is determined from the minimum condition $\partial\sigma_{cr}/\partial\mu^2 = 0$:

$$\hat{p} = \frac{\mu^4 \left[\left(\vartheta_p \gamma_N - \vartheta_p \sqrt{\gamma_N}\right) \mu^2 + \left(\gamma_N - 1\right)\mu^2 + \left(\vartheta_p \sqrt{\gamma_N} - \vartheta_s\right)\right]}{\bar{\phi}_0 \sqrt{\bar{\phi}_1 \bar{\phi}_2}} \; . \qquad (3.24)$$

For a given orthotropic conical shell satisfying the geometrical requirements according to (3. 15) and the rigidities of which are within the limits given by (3. 19), from (3. 23) the critical buckling stress σ_{cr} can be determined for any inner pressure where μ^2 is to be determined from equation (3. 24).

Observing the parameters matched to the geometry of the cone equations, (3. 23) and (3. 24) are identical with the buckling conditions of the longitudinally compressed orthotropic cylinder shell discussed by Schnell[11]. Therefore equations (3. 23) and (3. 24) are not further investigated in this paper.

3.3 Iterative Solution by Means of the Difference Method

Closed solutions of the system of nonlinear differential equations (2.21) for the basic state of stresses and of the eigen-value problem (2.38) for the stress and deformation contributions in the instance of buckling are not known. Thus an approximation method for solving both problems (2.21) and (2.38) is described in this section of the paper. The derivatives with respect to z and the differential operators $L(y_{1n})$ and $B(y_{1n}, \varphi_0)$ --generally described by

$$G\left(\frac{h^4}{4!} y_n^{(4)}, \frac{h^2}{2!} y_n'', \frac{h}{1!} y_n', y_n\right)$$

--appearing in the systems of differential equations (2.21) and (2.38) are expanded into difference expressions of the form

$$G\left(\frac{h^4}{4!} U_r^{(4)}, \frac{h^2}{2!} U_k'', \frac{h}{1!} U_k', U_k\right) + \sum_k \alpha_k U_k + 0(h^8) = 0. \qquad (3.25)$$

The U_k are the function values at m equidistant intermediate points in the interval $0 \leqslant z \leqslant z_0$. The distance h between the points is

$$h = \frac{z_0}{m+1} .$$

The residual member $0(h^8)$ gives the order of the first nonvanishing member of G if the Taylor expansions are balanced, i.e., balancing has been performed up to the members with h^7 inclusively. We first consider the axially symmetric state of the stresses and deformation.

3.3.1 Approximate Solution for the Basic State of Stresses

Finite expressions of the form

$$G\left(\frac{h^2}{2!} U_i'' + \delta_u U_i\right) + \sum_{k=1}^{7} a_{ik} U_k + r_{10}\left(\frac{h}{1!} k_u U_0' + U_0\right) = 0$$

for i = 1, 2, 3

$$G\left(\frac{n^2}{2!} U_i'' + \delta_u U_i\right) + \sum_{k=i-3}^{i+3} a_{ik} U_k = 0$$

for i = 4, 5. . . (m-3)

$$G\left(\frac{h^2}{2!} U_i'' + \delta_u U_i\right) + \sum_{k=m-6}^{m} a_{ik} U_k + r_{im+1}\left(\frac{h}{1!} k_u U_{m+1}' + U_{m+1}\right) = 0$$

for i = m-2, m-1, m (3.26)

are derived for the system of differential equations. U_i are the function values N_0 and Q_0 at the m intermediate points; δ_u follows from the coefficients of the differential equation (2.21).

$$\delta_N = \frac{2}{h^2} \frac{C_{11}}{C_{22}}$$

$$\delta_Q = \frac{2}{h^2} \frac{D_{22}}{D_{11}}$$

The boundary conditions (2.23) are taken into account in the finite expressions for the edge-near points (i = 1, 2, 3; i = m - 2, m - 1, m) in the form

$$k_u \frac{h}{1!} U_i' + U_i = 0 \qquad (i = 0,\ i = m + 1).$$

The factor means

$$k_N = \frac{1}{h} \frac{C_{22}}{C_{12}}$$ - corresponding to (2.23.1)

$$k_Q = \frac{1}{h} \frac{D_{11}}{D_{12}}$$ - corresponding to (2.23.2a)

or

$k_Q = 0$ - corresponding to (2.23.2b).

If the differential equations at any intermediate point are replaced with the finite expressions formed according to equations (3.26) a system of differential equations is obtained which can be combined to matrix equations[17]:

$$\left.\begin{aligned} &\left(a_{ik}^{(1)}\right)\left(Q_{0\,i}\right) - \left(b_{kk}\right)\left(N_{0\,i}\right) - \left(Q_{0\,i}\,N_{0\,i}\right) + \left(c_i\right) = 0 \\ &\left(a_{ik}^{(2)}\right)\left(N_{0\,i}\right) + \left(b_{kk}\right)\left(Q_{0\,i}\right) + \frac{1}{2}\left(Q_{0\,i}^{\,2}\right) = 0 \\ &\qquad\qquad 1 \leqslant i,\ k \leqslant m \end{aligned}\right\} \quad (3.28)$$

$\left(a_{ik}^{(j)}\right)$ represent quadratic matrices

$$\left(a_{ik}^{(j)}\right) = \begin{bmatrix} a_{11}^{(j)} & a_{12}^{(j)} & a_{13}^{(j)} & \cdots \\ a_{21}^{(j)} & a_{22}^{(j)} & a_{23}^{(j)} & \cdots \\ a_{31}^{(j)} & a_{32}^{(j)} & a_{33}^{(j)} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

with the elements $a_{ik}^{(j)}$ which are composed of the coefficients of the finite expressions (3.25).

$\left(Q_{0\,i}\right)$, $\left(N_{0\,i}\right)$, $\left(Q_{0\,i}\,N_{0\,i}\right)$, $\left(Q_{0\,i}^{\,2}\right)$, $\left(c_i\right)$ are vectors the elements $Q_{0\,i}$ and $N_{0\,i}$ of which are the function values Q_0 and N_0 at the m intermediate points and the elements c_i of which are determined from the equation

$$c_i = \left(\bar{p}\,e^{2hi} + \bar{K}_0\right)\lambda^2 .$$

$\left(b_{kk}\right)$ is a diagonal matrix

$$\left(b_{kk}\right) = \begin{bmatrix} b_{11} & & & 0 \\ & b_{22} & & \\ & & b_{33} & \\ 0 & & & \ddots \end{bmatrix}$$

with the elements

$$b_{kk} = \lambda + Q_{Ak} \; .$$

Furthermore, the vector $\left(Q_{0\,i}\; N_{0\,i}\right)$ can be replaced with

$$\left(Q_{0i}\; N_{0i}\right) = \begin{bmatrix} Q_{0_1} & N_{0_1} \\ Q_{0_2} & N_{0\,2} \\ Q_{0_3} & N_{0_3} \\ \vdots & \end{bmatrix} = \left(Q_{0\,kk}\right)\left(N_{0\,i}\right) = \begin{bmatrix} Q_{0_{11}} & & & 0 \\ & Q_{0_{22}} & & \\ & & Q_{0_{33}} & \\ 0 & & & \ddots \end{bmatrix} \begin{bmatrix} N_{0_1} \\ N_{0_2} \\ N_{0_3} \\ \vdots \end{bmatrix}$$

whereby the following relation exists between the elements Q_{0j} of the vector and the elements $Q_{0\,jj}$ of the diagonal matrix

$$Q_{0\,j} \equiv Q_{0\,jj} \; .$$

From equation (3.28.1) vector $(N_{0\,i})$ can be eliminated with (3.28.2) so that a matrix equation $\bar{\phi}\;(Q_{0\,i})$ is obtained for the vector $(Q_{0\,i})$.

This nonlinear matrix equation is solved by means of Newton's iteration: For a νth approximation $\left(Q_{0\,i}{}^{(\nu)}\right)$, the matrix equation $\bar{\phi}$ is satisfied except for the defects $\left(\Delta_i{}^{(\nu)}\right)$.

$$\bar{\phi}\left(Q_{0\,i}{}^{(\nu)}\right) = \left(\Delta_i{}^{(\nu)}\right) \; . \qquad (3.29)$$

By means of corrections $\left(\Delta\, Q_{0\,i}{}^{(\nu)}\right)$ which are determined from the system of linear equations

$$\frac{\partial \bar{\phi}\left(Q_{0i}{}^{(\nu)}\right)}{\partial Q_{0j}} \Delta Q_{0j} + \Delta_j{}^{(\nu)} = 0 \tag{3.30}$$

and are added to the νth approximation $\left(Q_{0i}{}^{(\nu)}\right)$, the nonlinear matrix equation (3.28) is iteratively solved step-by-step. The iteration is carried out until two successive approximate solutions differ by more than a previously specified amount. In matrix form the following iteration scheme can be given:

$$\left.\begin{aligned}
&\left(a_{ik}^{(1)}\right)\left(Q_{0i}{}^{(\nu)}\right) + \left(d_{kk}{}^{(\nu)}\right)\left(e_i{}^{(\nu)}\right) + \left(c_i\right) = \left(\Delta_i{}^{(\nu)}\right) \\
&\left\{\left(a_{ik}^{(1)}\right)\left(d_{kk}{}^{(\nu)}\right)\left(a_{ik}^{(2)}\right)^{-1}\left(d_{kk}{}^{(\nu)}\right) + \left(e_{kk}{}^{(\nu)}\right)\right\}\left(\Delta Q_{0i}{}^{(\nu)}\right) \\
&\qquad\qquad = -\left(\Delta_i{}^{(\nu)}\right) \\
&\left(Q_{0i}{}^{(\nu)}\right) + \left(\Delta Q_{0i}{}^{(\nu)}\right) = \left(Q_{0i}{}^{(\nu+1)}\right)
\end{aligned}\right\} \tag{3.31}$$

where

$$\left(d_{kk}{}^{(\nu)}\right) = \left[\left(b_{kk}\right) + \left(Q_{0kk}{}^{(\nu)}\right)\right]$$

$$\left(e_i{}^{(\nu)}\right) = \left(a_{ik}{}^{(2)}\right)^{-1}\left[\left(b_{kk}\right)\left(Q_{0i}{}^{(\nu)}\right) + \frac{1}{2}\left(Q_{0i}{}^{(\nu)^2}\right)\right] .$$

If the initial approximation of this iteration is set $Q_{0i}^{(0)} = 0$, the approximate solution $Q_{0i}{}^{(1)}$ represents an approximate solution of the linearized differential equations for the axially symmetric state of stress and deformation according to equation (3.5).

3.3.2 Approximate Solution for the Eigen-Value Problem

The differential equations (2.38) for the determination of the stresses and deformations in the instance of buckling are replaced with difference expressions of the form given in (3.25):

$$
\left.
\begin{aligned}
& G\left[L\left(U_i\right),\; B\left(U_i,\varphi_{0\,i}\right)\right] + \sum_{k=1}^{6} a_{ik}\, U_k + r_{i_0}\left[\varrho_{1u}\frac{h}{1!}U_0' + U_0\right] \\
& \qquad + s_{i_0}\left[\varrho_{2u}\frac{h^2}{2!}U_0'' + \frac{h}{1!}U_0'\right] = 0 \\
& \qquad\qquad \text{for } i = 1,2,3,4 \\
& G\left[L\left(U_i\right),\; B\left(U_i,\varphi_{0\,i}\right)\right] + \sum_{k=i-4}^{i+4} a_{ik}\, U_k = 0 \\
& \qquad\qquad \text{for } i = 5,6\ldots m=4 \\
& G\left[L\left(U_i\right),\; B\left(U_i,\varphi_{0\,i}\right)\right] + \sum_{k=m-5}^{m} a_{ik}\, U_k + a_{m+1}\left[\varrho_{1u}\frac{h}{1!}U_{m+1}'\right. \\
& \qquad \left. + U_{m+1}\right] + s_{im+1}\left[\varrho_{2u}\frac{h^2}{2!}U_{m+1}''\right. \\
& \qquad \left. + \frac{h}{1!}U_{m+1}'\right] = 0 \\
& \qquad\qquad \text{for } i = m-3,\ m-2,\ m-1,\ m
\end{aligned}
\right\}
\tag{3.32}
$$

The U_i are the function values F_{1n} and W_{1n} at m equidistant intermediate points. For $\varphi_{0\,i}$ the stresses and deformations of the basic state of stresses have to be inserted.

In (3.32) the homogeneous boundary conditions (2.39) were combined to the form

$$
\begin{aligned}
& \varrho_{2u}\frac{h}{1!}U_k' + U_k = 0 \\
& \varrho_{2u}\frac{h^2}{2!}U_k'' + \frac{h}{1!}U_k' = 0.
\end{aligned}
\tag{3.33}
$$

For the coefficients ϑ_{1u}, ϑ_{2u} it is then

$$\left.\begin{array}{lll}
\vartheta_{1w} = 0 & & (1) \\
\text{meaning} \quad W_{1n} = 0 & (2.39.1) & \\
\vartheta_{2w} = \dfrac{2}{h} \dfrac{D_{11}}{D_{11} + D_{22}} & & (2a) \\
\text{meaning} \quad \dfrac{D_{11}}{D_{11} + D_{22}} W''_{1n} + W'_{1n} = 0 & (2.39.2a) & \\
\text{or} & & \\
\vartheta_{2F} = 0 & & (2b) \\
\text{meaning} \quad W'_{1n} = 0 & (2.39.2b) & \\
\vartheta_{1F} = \dfrac{1}{h} \dfrac{1}{1 - \eta^2} & & (3) \\
\text{meaning} \quad F'_{1n} + \left(1 - \eta^2\right) F''_{1n} = 0 & (2.39.3) & \\
\vartheta_{2F} = \dfrac{2}{h} & & (4) \\
\text{meaning } F'_{1n} + F''_{1n} = 0. & (2.39.4) &
\end{array}\right\} \quad (3.34)$$

From the difference equation at each intermediate point, a system of linear equations for the F_{1n_i} and W_{1n_i} is obtained which is combined in matrix equations.

$$\begin{array}{l}
\left(b_{ik}^{(1)}\right) \left(W_{1n_i}\right) = \left(c_{ik}^{(1)}\right) \left(F_{1n_i}\right) - \left(c_{ik}^{(2)}\right) \left(W_{1n_i}\right) \\
\left(b_{ik}^{(2)}\right) \left(F_{1n_i}\right) = - \left(c_{ik}^{(3)}\right) \left(W_{1n_i}\right) \qquad 1 \leqslant i \, , \; k \leqslant m
\end{array} \qquad (3.35)$$

where

$$c_{ik}^{(1)} = c_{ik}^{(1)} \left(\lambda + Q_{0k} + QA_k\right), \quad c_{ik}^{(2)} = c_{ik}^{(2)} \left(N_{0k}\right), \quad c_{ik}^{(3)} = c_{ik}^{(3)} \left(\lambda + Q_{0k} + QA_k\right).$$

The elements $b_{ik}^{(j)}$ and $c_{ik}^{(j)}$ of the matrices are composed of the coefficients of the finite expressions. Nontrivial solutions of this system of linear homogeneous equations (3. 35) exist only if the coefficient determinant vanishes.

$$\det \left\{ \left(b_{ik}^{(1)}\right) + \left(c_{ik}^{(2)}\right) + \left(c_{ik}^{(1)}\right) \left(b_{ik}^{(2)}\right)^{-1} \left(c_{ik}^{(3)}\right) \right\} = 0 \qquad (3.36)$$

The critical eigen-value is the smallest edge load N_0 with the associated displacement of the edge u_0 of the axially symmetric state of stresses and deformation for which the determinant (3. 36) become zero. Of the eigen-values determined for various numbers n of bucklings in the direction of the circumference the absolute smallest yields the buckling values (buckling load N_{0cr}).

With the approximation methods given here, the boundary conditions can be completely taken into account without difficulty.

For the numerical evaluation, the number m of the equidistant intermediate points--i. e., the number of the unknowns--is increased as long as the difference of the critical values of two successive approximations N_{0cr} (n, m) - N_{0cr} (n, m + 1) stays above a given error limit.

4. Numerical Evaluation and Discussion of the Results

4.1 Dimensions of the Shell and Parameters

The systems of differential equations (2. 21)--for the determination of the basic state of stresses--and (2. 38)--for the determination of the critical load--with the associated boundary conditions (2. 23) and (2. 39) contain the following eight parameters

$$z_0 \ , \ \lambda = \frac{s_1 \cot \alpha}{\sqrt{D_{11} C_{22}}} , \ \frac{C_{11}}{C_{22}} , \ \frac{D_{22}}{D_{11}} , \ \frac{2 C_{12} + C_{33}}{C_{22}} , \ \frac{2(D_{12} + 2D_{33})}{D_{11}} ,$$

$$\frac{C_{12}}{C_{22}} \text{ and } \frac{D_{12}}{D_{11}} \ ,$$

which are composed of the three geometrical characteristics s_1, z_0, and $\cot \alpha$ and the eight rigidity parameters C_{22}, C_{11}, C_{12}, C_{33}, D_{11}, D_{22}, D_{12}, and D_{23}.

For the isotropic conical shell, the rigidity parameters of which are given in equations (2.4), the eight parameters reduce to

$$z_0, \quad \lambda = \frac{s_1 \cot \alpha}{t} \sqrt{12(1-\nu^2)}, \quad \nu .$$

The mentioned shell parameter λ is identical with the cone characteristic stated by Weingarten et al[14]. As can be seen from the systems of differential equations (2.21) and (2.38) with the boundary conditions (2.23) and (2.39) λ is not sufficient, however, for the determination of the buckling load of the isotropic cone (as stated by Weingarten et al[14]) but--aside from Poisson's ratio ν--also the ratio of the end radii $\left(r_2/r_1 = s_2/s_1 = e^{z_0}\right)$ has an effect.

The numerical evaluation is performed for conical shells for the stability behavior of which experimental results are available[13,8]. Because of the extent of the computation--compared with the computer IBM 1620/23 available--the investigation was limited to conical shells (Figure 6) with constant height h_k and constant radius r_2 of the base while the half angle α was varied ($\alpha = 10°, 20°, 30°$). The investigated conical shells have a wall thickness of $t = 0.255$ mm. The elasticity modulus has been regarded as constant, $E = 525$ kg/mm^2. Poisson's number was $\nu = 0.3$.

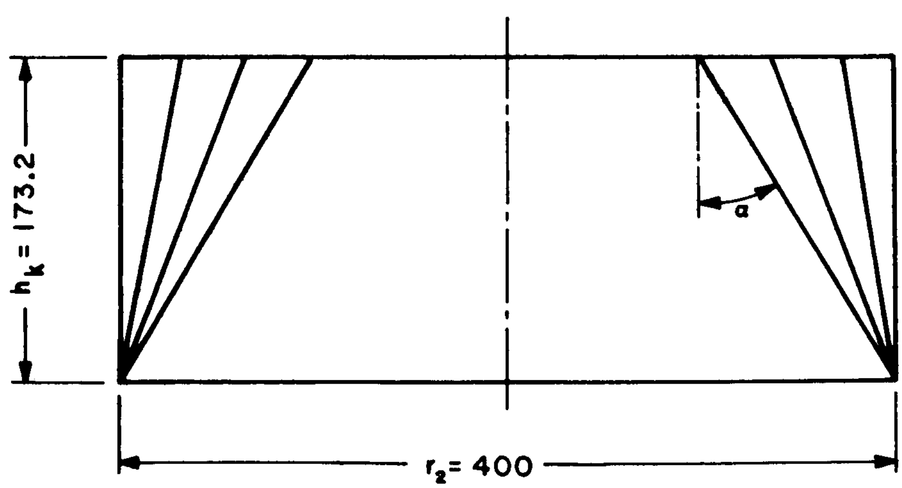


Figure 6. Dimensions of the Conical Shells Investigated

In Table I the dimensions of the investigated shells are listed. For the isotropic shells the shell parameters z_0 and λ in the last two columns suffice for the determination of the stress and stability behavior. For comparison, the cylinder ($\alpha = 0$) with radius r_2 and height h_k for which the buckling load has been determined by Fischer[3] is used.

Table I. List of the Dimensions and Parameters of the Cone

Designation	Cone Geometry			Cone Parameter	
	s_1 [mm]	$\frac{s_2}{s_1}$	cot α	z_0	λ
J00	∞	1	∞	0	∞
J10	$9.75877 \cdot 10^2$	1.18022	5.67128	0.165705	$7.17212 \cdot 10^4$
J20	$4.00440 \cdot 10^2$	1.46030	2.74748	0.378639	$1.42575 \cdot 10^4$
J30	$2.00000 \cdot 10^2$	2.00000	1.732051	0.693147	$4.48913 \cdot 10^3$

In order to be able to compare the results of the calculation for various shells, dimensionless load and inner pressure parameters (N_V and $\bar{P}$) are used in the graphic presentations. The computed edge loads at the edge of the cone z = 0 are related to the critical ring buckling load of an isotropic cone under pure axial load which had been obtained according to the linear theory according to equation (3.18).

$$N_{s_0}(p = 0) = -\frac{2 \cot \alpha}{s_1} \sqrt{\frac{z_0}{2} \frac{e^{z_0} + 1}{e^{z_0} - 1}} \sqrt{\left(\frac{D_{11}}{C_{22}}\right)_{\text{isotrope}}} \qquad (4.1)$$

For the orthotropic shells, the ring buckling load of the comparison shell (J 30, shell of the same weight and of the same geometrical dimensions) is used as reference load.

For the investigated conical shells ($0 \leqq z_0 \leqq \ln 2$) the root can be set

$$\sqrt{\frac{z_0}{2} \frac{e^{z_0} + 1}{e^{z_0} - 1}} \approx 1 \qquad (4.2)$$

so that

$$N_{s_0}(p = 0) = -\frac{2 \cot \alpha}{s_1} \sqrt{\left(\frac{D_{11}}{C_{22}}\right)_{\text{isotrope}}} = -\frac{\cot \alpha}{s_1} \frac{Et^2}{\sqrt{3(1-\nu^2)}} \qquad (4.3)$$

applies for the ring buckling load (linear theory) of an isotropic conical shell[18].

In Table II the exact values of root (4.2) are listed for the conical shells of Table I.

Table II

Designation	z_0	$\sqrt{\frac{z_0}{2} \frac{e^{z_0}+1}{e^{z_0}-1}}$
J00	0	1
J10	0.165705	1.00114
J20	0.578639	1.00594
J30	0.693147	1.01967

Going in the limit to the cylinder ($\alpha \rightarrow 0$, $\cot \alpha / s_1 \rightarrow 1/R$, $z_0 \rightarrow 0$) from the equation (4.3) follows the so-called classic buckling load of an infinitely long cylinder[3]:

$$N_{cl} = \frac{Et^2}{R} \frac{1}{\sqrt{3(1-\nu^2)}} \approx 0.605 \frac{Et^2}{R} . \qquad (4.4)$$

For the dimensionless load parameter N_V, the reference value

$$N_V = \frac{N_s}{N_{scl}(p = 0)} = -\frac{N_0}{2\lambda} \qquad (4.5)$$

is obtained with the classic ring buckling load according to formula (4,3) and for the orthotropic shells

$$N_V = -\frac{N_0}{2\lambda} \sqrt{\left(\frac{D_{11}}{C_{22}}\right) \Big/ \left(\frac{D_{11}}{C_{22}}\right)_{\text{isotrope}}} . \qquad (4.6)$$

As inner pressure parameter $\bar{p}$ is used according to formula (2.22),

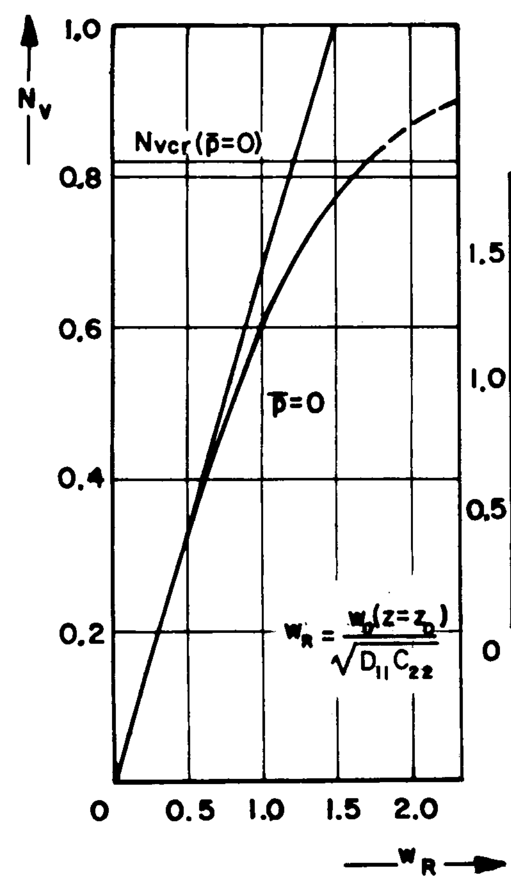
$$\bar{p} = \frac{1}{2} \left(\frac{s_1}{\cot \alpha}\right)^2 \sqrt{\frac{C_{22}}{D_{11}}}\, p \ .$$

In the case of the orthotropic shell for D_{11}, C_{22} the rigidity parameter of the isotropic comparison shell are to be inserted.

4.2 Axially Symmetric Basic State of Stresses and Buckling Loads of Ideal Isotropic Conical Shells

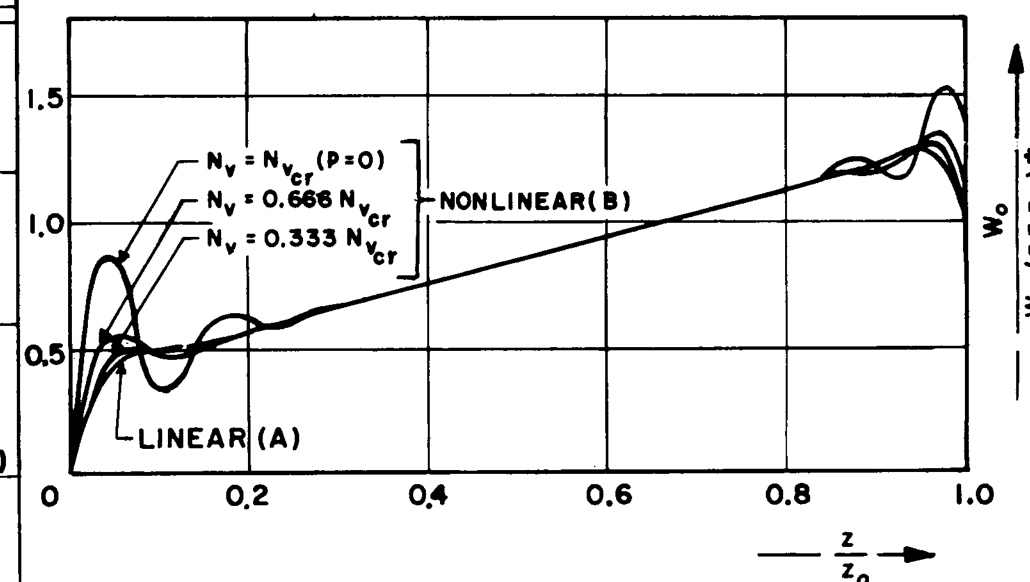
The axially symmetric basic state of stresses of the isotropic conical shell can be described with a closed solution of the linearized system of differential equations (3.5) for small edge load and small inner pressure. From Figure 7 where the edge load N_V of an isotropic cone J30 is depicted for pure axial load ($\bar{p} = 0$) as a function of the radial displacement of the edge w_R (proportional to the compression of the cone) and the axially symmetric configurations of buckling along the generatrix for various edge loads, it is seen that for small edge loads N_V the curves A (linearized, system of differential equations (3.5)) and B (nonlinear, system of differential equations (2.21)) are in good agreement. With increasing edge load, ring buckling appears in the edge zones of the cone--analogously to the cylinder shell--which can be comprehended only by the solution of the nonlinear differential equations (2.21).

In Figure 8 curves N_V vs. $|Q_{0max}|$ are plotted for various inner pressures for the cone J30 (pin jointed edge) (for comparison the curve calculated from the system of linearized differential equations for $\bar{p} = 0$ is included). First $|Q_{0max}|$ increases linearly with the axial edge load, for higher edge load it increases nonlinearly. When reaching the critical load, the axially symmetric basic state of stress becomes unstable (for this reason the curve above the critical load is dashed). The shell starts to assume a new state of equilibrium. In the framework of the calculation performed here (linearization at the branching point) the branching point with the number, n, of bucklings over the circumference appearing in the beginning of buckling is determined. Concerning the further course of the curves $N_V(|Q_{0max}|)$, nothing can be stated. In the experiment[13] as well as in the calculation the decrease of the number of bucklings, n, with increasing inner pressure could be observed (however, in the experiment the number of bucklings was determined only for the stable postbuckling region).



(A) Basic state of stress calculated from the linearized system of differential equations (3.5)

(B) Basic state of stress calculated from the nonlinear system of differential equations (2.21)



* Subscript characters (illegible in translator's copy of original) are omitted.

Figure 7. Axially Symmetric Basic State of Stress of an Isotropic Cone J30 Under Pure Axial Load and Pin-Jointed Edge

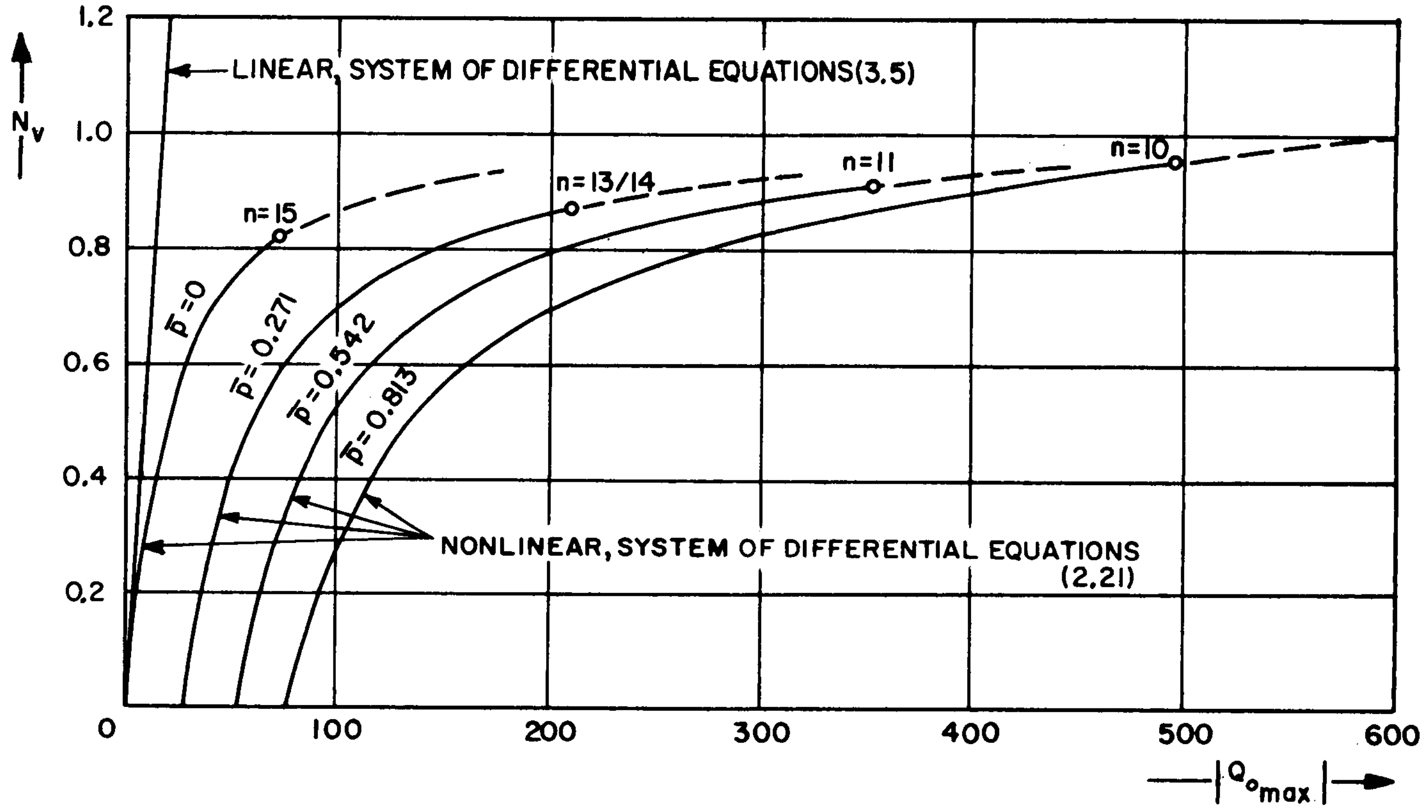


Figure 8. The Axially Symmetric Basic State of Stress, $\left[N_V = f\left(|Q_{0max}|\right)\right]$, of an Isotropic Pin-Jointed Conical Shell for Various Inner Pressures (n - Number of Bucklings in Circumferential Direction)

The axially symmetric configurations of the bucklings for various inner pressures are plotted in Figure 9 for a constant edge load which equals the buckling load of the conical shell for $\overline{p} = 0$, i.e., $N_V = N_{Vcr}$ at ($\overline{p} = 0$). For a pure axial load ($\overline{p} = 0$), axially symmetric buckling configurations begin to form beginning at the edges, decaying towards the center of the cone. Due to the inner pressure, the radial stress increases with increasing radius, i.e., with increasing z for which reason the radial deformation in the edge zone $z = z_0$ is suppressed. The tendency that with increasing inner pressure the axially symmetric buckling configurations are formed only at the edge $z = 0$ and there with increasing amplitude, typically for cones, is seen. For high inner pressure, ring buckling could be observed at the edge $z = 0$ of the test shells before reaching the critical load.

A comparison of the measured buckling loads (four test shells J30 of the same dimensions[13]) shown in Figure 10 with the critical loads calculated according to the nonlinear theory for a fully restrained ideal cone (curve III) shows good agreement for large inner pressure. However, for small inner pressure--especially for $\overline{p} = 0$--the experimental values deviate from the theoretical buckling loads for an ideal conical shell. The discrepancy between theory and experiment can be essentially traced to the existence of prebuckling in the test shells. An investigation of the effect of the predeformation on the buckling loads is performed in Section 4.4 of this paper.

The theoretical buckling loads of the pin-jointed cone (curve II) are--like for other stability problems (buckling of a rod, cylinder buckling)--below those calculated buckling values for the fully restrained shell (curve III). Curve (I) contains the critical loads which were calculated according to the linear theory for one-member approximation formulation (ring buckling)--corresponding to formula (3.18). Compared with curves (II) and (III), the buckling loads determined according to the linear theory represent only a course approximation to the stability problem investigated here.

In Figure 11 the buckling loads according to the linear (curve I) and according to the nonlinear theory (curve II) for pin-jointed conical shells of the same height and the same base radius r_2--according to Table I--are plotted versus the opening angle α. It is seen, as shown by the corresponding experiments[13], that the thus normalized buckling load N_V--equation (4.4)--is practically independent of the cone angle which on the other side means that the linear buckling load (3.18) used as reference quantity properly represents the effect of the cone angle on the buckling load. The values for the cylinder shell (curve II, $\alpha = 0$) entered into the diagram are taken from Fischer[3] and

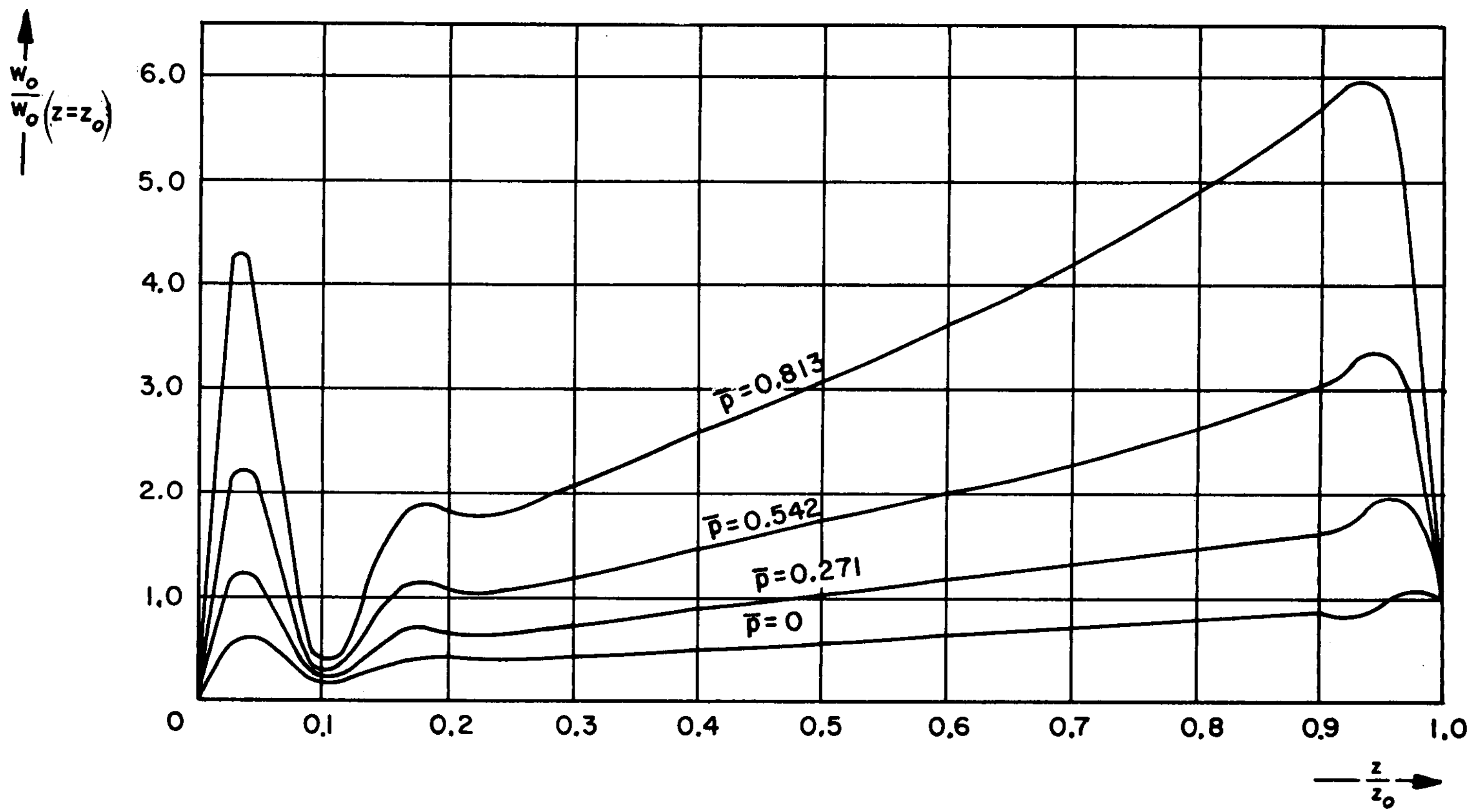
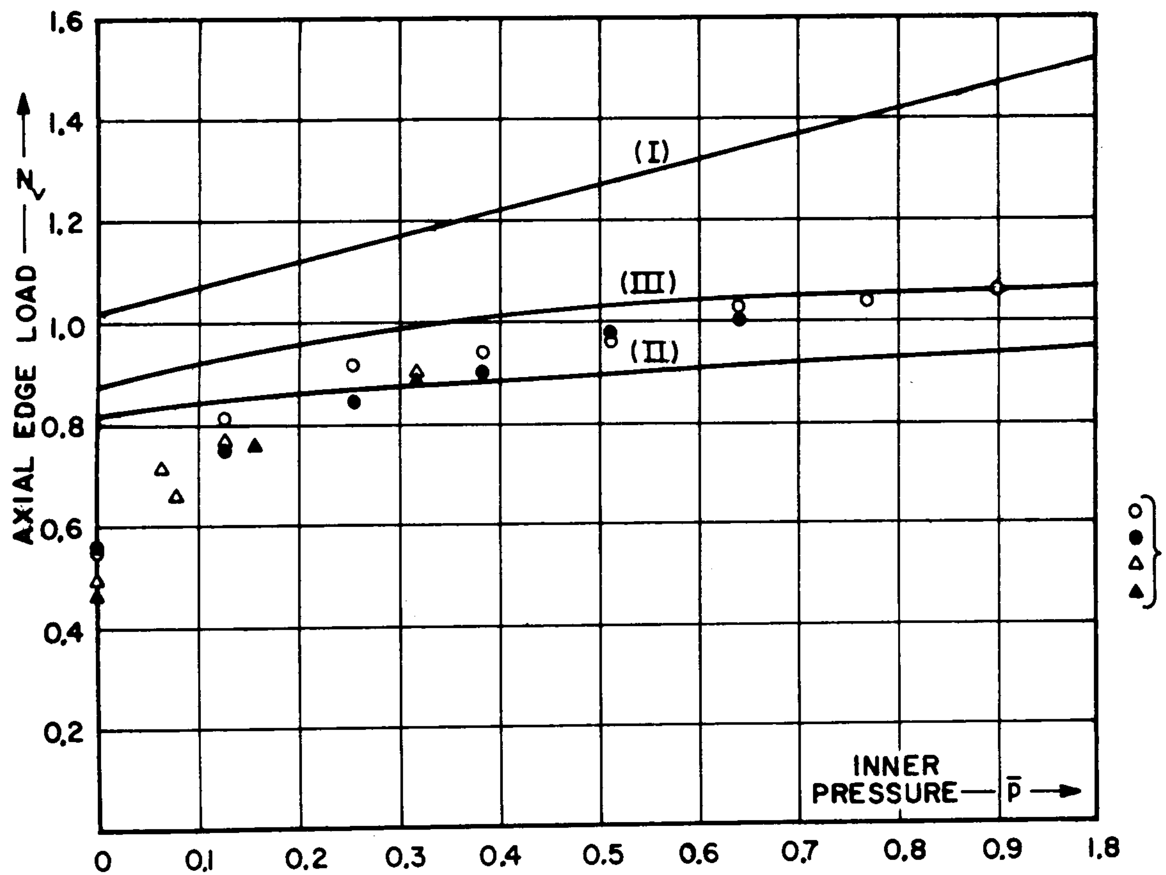


Figure 9. Axially Symmetric Buckling Configurations of an Isotropic Conical Shell (J30) $N_v = N_{vcr}$ ($\bar{p}$ = 0) for Various Inner Pressures

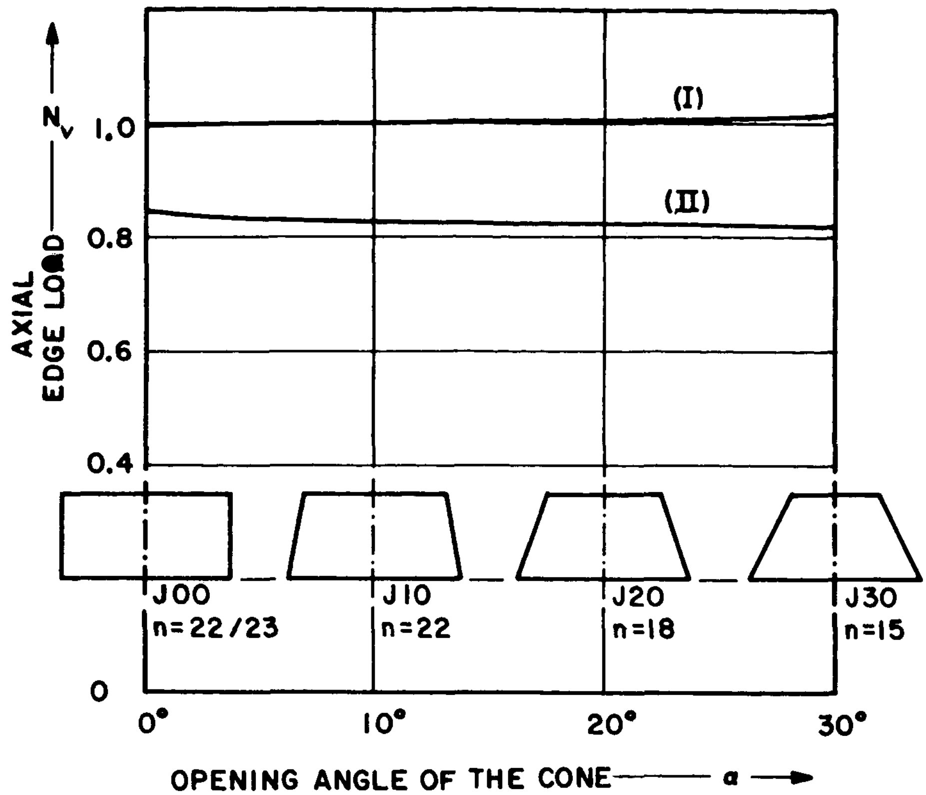


(I) Buckling loads according to the linear theory [formula (3.18)[5]].

Buckling loads according to the nonlinear theory [formulas (2.21) and (2.38)].
(II) Ideal cone, pin jointed.
(III) Ideal cone, fully restrained.

Experiments DVL[13].

Figure 10. Comparison of the Experimental Critical Loads With the Theoretical Values for an Ideal Isotropic Conical Shell (J30)



(I) Buckling loads according to the linear theory [formula (3. 18)[5]].

(II) Buckling loads according to the nonlinear theory [formulas (2.21) and (2. 38)] for pin-jointed edge buckling load of cylinder J00[3].

n: Number of bucklings in circumferential direction.

Figure 11. The Effect of the Opening Angle of the Cone on the Buckling Load of Pin-Jointed Isotropic Conical Shells of the Same Height and the Same Base Radius

satisfactorily supplement the here performed cone investigations (α = 10°, 20°, 30°) concerning the buckling load as well as with respect to the number of bucklings n.

4.3 Investigation of an Orthotropic Conical Shell

With the approximation methods given here (Section 3.4) orthotropic conical shells can be investigated as well without additional difficulties, provided the neutral plane coincides with the central surface of the shell (classic orthotropic[23]) or is so close that the eccentricity can be neglected.

The polar orthotropic test shells, which are shells with constant rigidity along the main rigidity axes forming a polar coordinate system, have been fabricated by cementing 72 conical strips uniformly distributed over the circumference along the generatrix to an isotropic skin (Mylar). These orthotropic shells had the same geometrical dimensions and the same cross-sectional area (same weight) as the isotropic conical shell J30.

The calculation of the rigidities of these test shells has been done under the following assumptions:

1) The reinforcements are free from the effect of transverse contraction.

2) Shearing is exclusively transmitted by the isotropic skin. The contribution of the reinforcement to the torsional rigidity of the shell is calculated according to Flügge[25].

3) The flexural rigidity D_{11} is determined for one beam element--consisting of skin and reinforcement--around the common neutral plane (Figure 13a).

4) The effect of the eccentricity of the neutral plane from the central surface of the shell on the law of elasticity is not taken into account so that the elasticity equations may be written in the decoupled form (2.1).

5) It is assumed that when reaching the critical load skin and reinforcement buckle simultaneously.

For the orthotropic test shell--designation 0 30-1--the following shell parameters have been determined.

Geometry parameters:

$z_0 = \ln 2 = 0.693147 \qquad \lambda = 2.24471 \cdot 10^3$.

Rigidity characteristics:

$$\frac{C_{11}}{C_{22}} = 0.519\,, \quad \frac{C_{12}}{C_{22}} = -\,0.154\,, \quad \frac{2C_{12} + C_{33}}{C_{22}} = 2.41$$

$$\frac{D_{22}}{D_{11}} = 0.0573\,, \quad \frac{D_{12}}{D_{11}} = 0.0172\,, \quad \frac{2\left(D_{12} + 2D_{33}\right)}{D_{11}} = 0.435\,.$$

The results of the stability calculation together with the measured values are depicted in Figure 12. The straight line (I) represents the buckling values according to the linear theory, according to formula (3.18), and illustrates properly the qualitative relationship between buckling load and inner pressure. The buckling loads themselves, however, are above the experimental values.

The critical loads for a pin-jointed (III) and a fully restrained cone (IV) determined according to the nonlinear shell theory--formulas (2.21) and (2.38)--agree with the measured buckling values for small inner pressure. On the one side this can be explained in such a way that in the reinforced test shells local prebuckling can occur only between the reinforcements which have respectively less effect on the buckling load. On the other side, the strong decrease of the theoretical buckling values for $\overline{p} = 0$ is to be traced to the orthotropic rigidity parameters which have been calculated only approximately (especially the assumption that the reinforcements are free from transverse contraction and that shearing is transmitted only by the isotropic skin represent a course approximation to the here investigated reinforced shells).

For high inner pressure, the measured values increase more than the theoretically determined buckling loads (curve IV for the boundary condition: fully restrained). This can be explained in such a way that due to the inner pressure the isotropic skin is stretched between the reinforcements (Figure 13b), whereby the axial flexural rigidity is additionally increased (corrugated sheet metal). The increase of the flexural rigidity D_{11} with increasing inner pressure has been attempted to comprehend theoretically by calculating the rigidity parameter D_{11} around the common neutral plane of an infinitely long segment of a cylinder shell with sectionally constant rigidity precurved under inner pressure (Figure 13b: Section I - isotropic skin, Section II - skin and reinforcement). The buckling loads (curve V) determined according to the nonlinear theory of shells taking into account the flexural rigidity dependence on the inner pressure agree well with the experimental values even for high inner pressure.

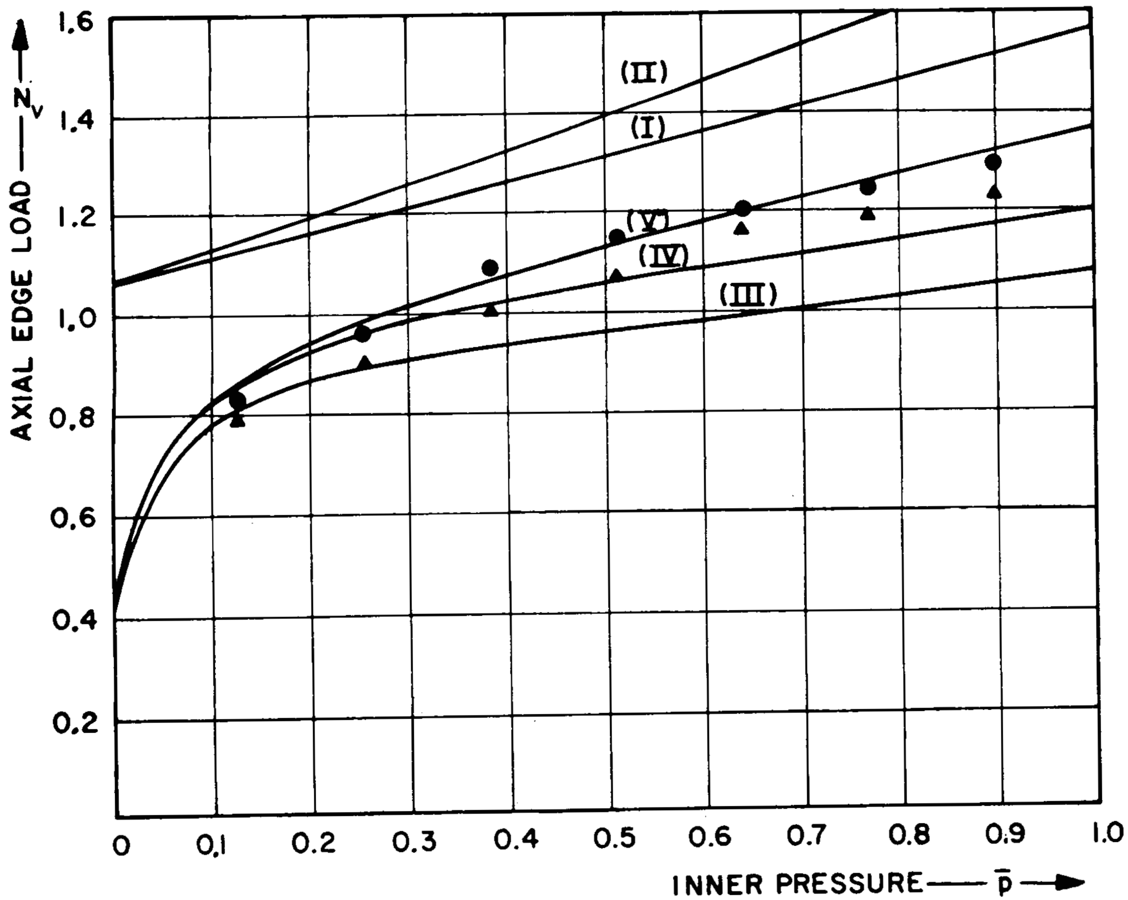


Figure 12. Theoretical and Experimental Buckling Loads of an Orthotropic Conical Shell (030-1)

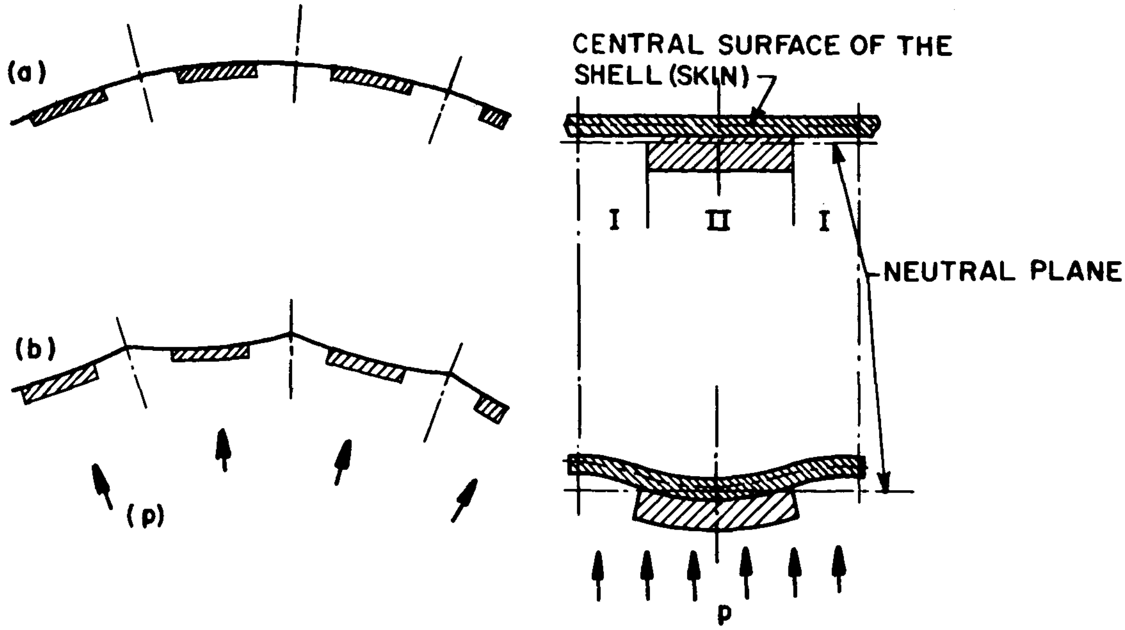


Figure 13. The Deformation of the Reinforced Conical Shell Due to Inner Pressure

A comparison between the buckling loads (Figures 10 and 12) found for the orthotropic (longitudinally reinforced) conical shell 030-1 and that for the isotropic comparison shell I30 (same weight) shows that under inner pressure the orthotropic shell buckles at higher loads than the isotropic shell.

4.4 The Effect of Axially Symmetric Prebuckling on the Stability Behavior

In the system of differential equations (2.21)--for the determination of the basic state of stresses--and (2.38)--to determine the buckling load--axially symmetric prebuckling $Q_A(z)$ has been taken into account. For the approximation method (difference method) described here it does not present any difficulties to investigate arbitrarily axially symmetrically precurved conical shells.

Since the predeformation of the test shells is not known, a fictitious axially symmetric prebuckling of the following form is assumed for the theoretical investigations:

$$\frac{w_A}{\xi_0 t} = \begin{cases} 0 & \text{for } \left|z - z_A\right| \geqslant \frac{b_A}{2} \\ \cos^3 \frac{(z - z_A)}{b_A} \pi & \text{for } \left|z - z_A\right| \leqslant \frac{b_A}{2} \end{cases} \tag{4.6}$$

Here ξ_0 designates the prebuckling amplitude related to the wall thickness, b_A represents the width of the prebuckling, and z_A gives the position of the prebuckling maximum along the generatrix.

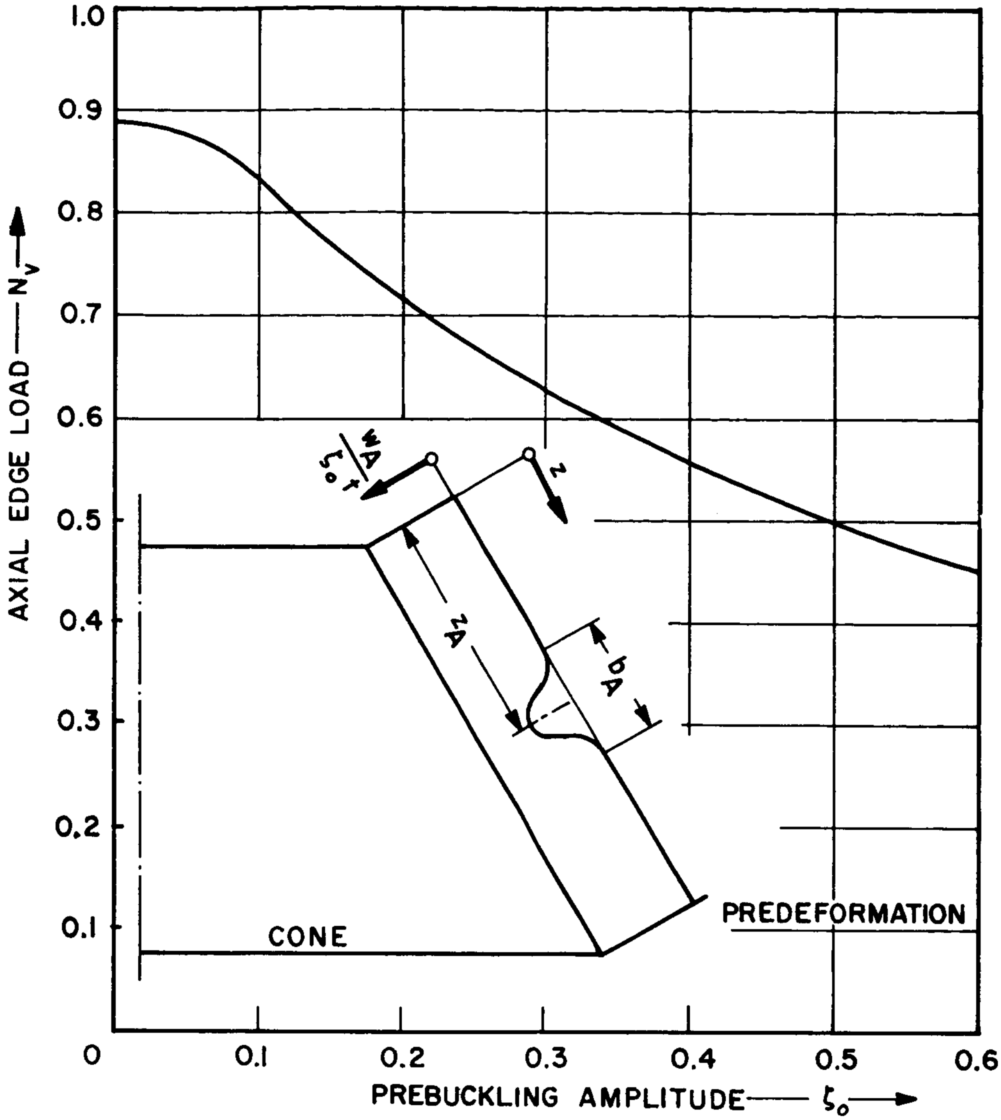
The numerical evaluation of the systems of differential equations (2.21) and (2.38) results in the critical loads plotted versus the prebuckling amplitude in Figure 14 for a prebuckling according to Equation (4.6) with $b_A/z_0 = 0.25$, $z_A/z_0 = 0.5$. As was to be expected, the buckling load decreases with increasing amplitude ξ_0.

Figure 15 illustrates the effect of the position of prebuckling on the critical load. It is seen that prebuckling at the edge $z = 0$ results in a greater decrease of the load than the same prebuckling at the edge $z = z_0$. This tendency is even increased with inner pressure as is seen from a comparison of the curves II (prebuckling at the edge $z = 0$) and III (prebuckling at the edge $z = z_0$) in Figure 16.

An additional phenomenon is to be noted in Figure 16.

With increasing inner pressure in agreement with experiment and theory, the effect of prebuckling on the critical load becomes smaller and smaller. The buckling values measured for four different test shells scatter greatly for low inner pressure--due to the predeformation which is different for each test shell--while for high inner pressure the experimental buckling loads are about the same and agree well with the theoretical buckling loads of the ideal as well as of the precurved fully restrained conical shell.

Taking into account prebuckling, the decrease of the buckling loads can be explained for low inner pressure. An absolute comparison between measured and calculated buckling loads is, however, not permitted since no measured results are available concerning the predeformation of test shells and prebuckling on which the calculation has been based has been assumed arbitrarily.

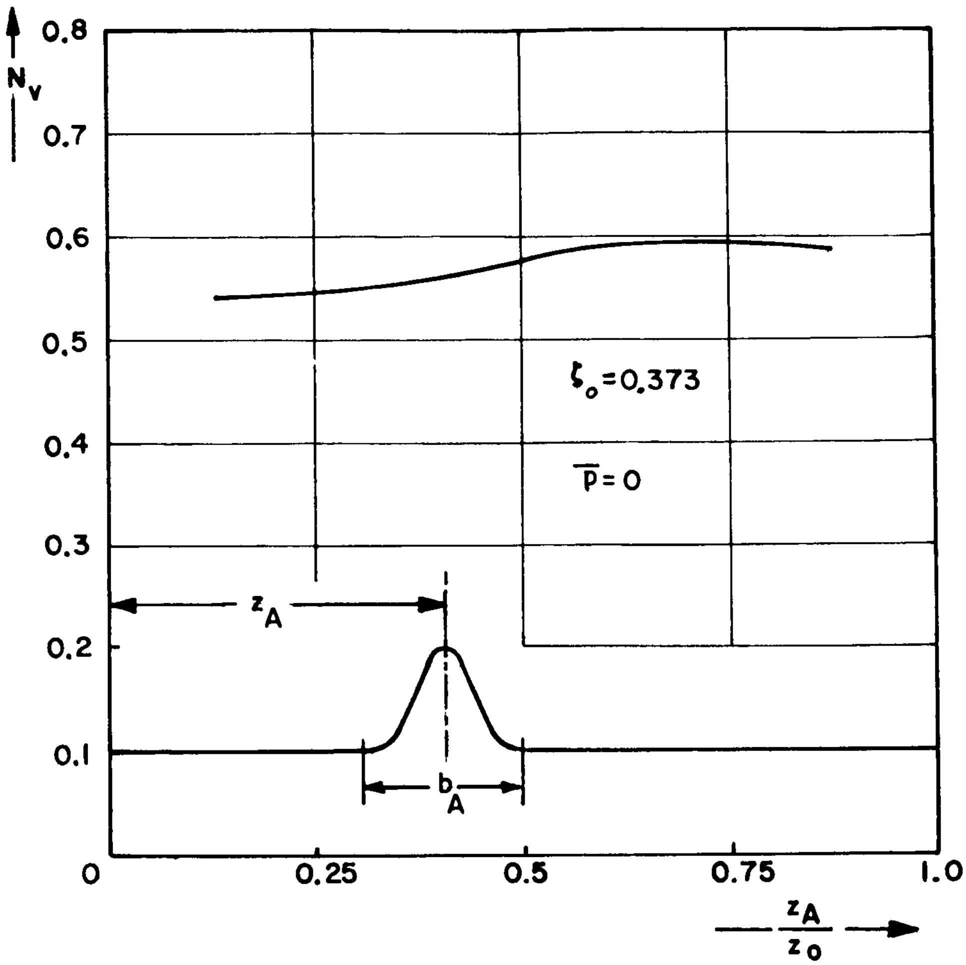


Configuration of the predeformation:

$$\frac{w_A}{\xi_0 \cdot t} = \begin{cases} 0 & |z - z_A| \geqslant \frac{b_A}{2} \\ \cos^3 \frac{\pi}{b_A}(z - z_A) & |z - z_A| \leqslant \frac{b_A}{2} \end{cases}$$

$$b_A = \frac{z_0}{4}; \qquad z_A = \frac{z_0}{2}.$$

Figure 14. Effect of the Prebuckling Amplitude on the Buckling Load of an Isotropic Cone (J30, $\overline{p}$ = 0)

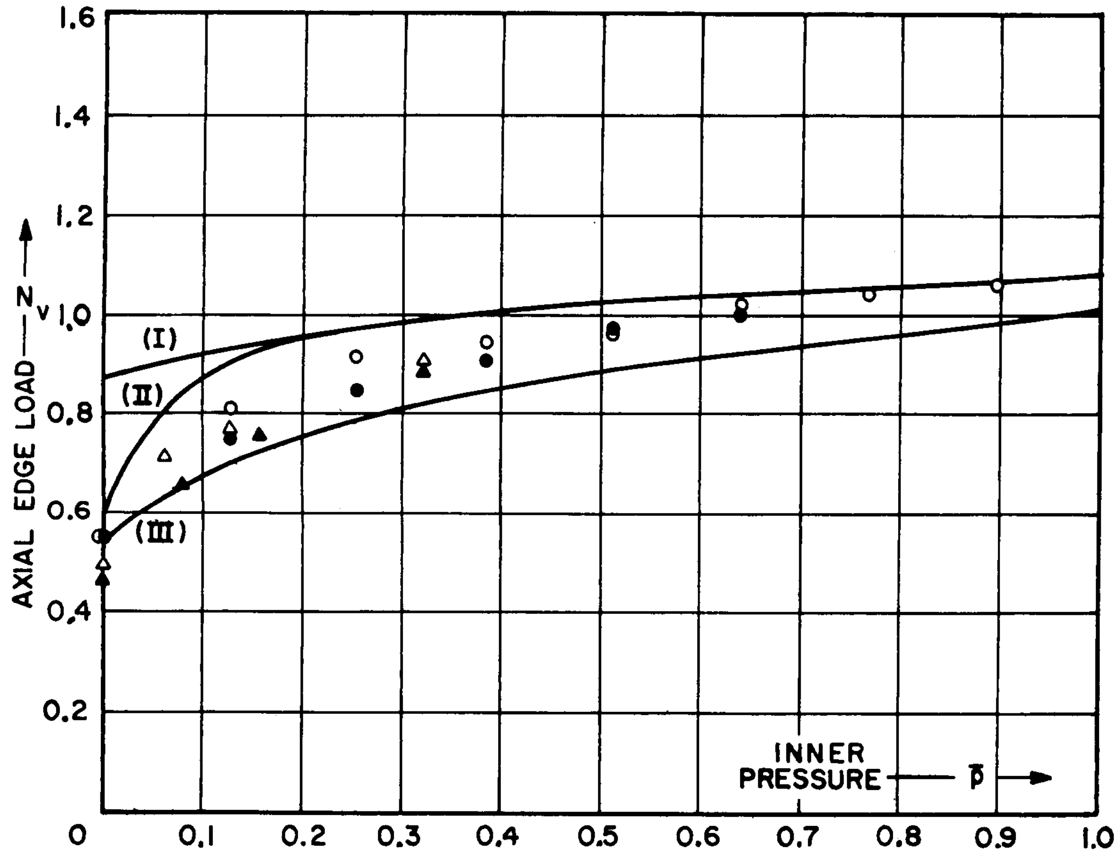


Configuration of the Predeformation:

$$\frac{w_A}{\xi_0 \cdot t} = \begin{cases} 0 & |z - z_A| \geqslant \frac{b_A}{2} \\ \cos^3 \frac{\pi}{b_A}(z - z_A) & |z - z_A| \leqslant \frac{b_A}{2} \end{cases}$$

$$b_A = \frac{z_0}{4}$$

Figure 15. Effect of the Position of Prebuckling on the Buckling Load of an Isotropic Cone (J30, $\overline{p}$ = 0)



Theoretical buckling load according to the nonlinear theory (fully restraint).

(I) Ideal cone (J30).

(II) Predeformed cone.

$$\frac{z_A}{z_0} = 0.875, \quad \frac{b_A}{z_0} = 0.25$$

$$\xi_0 = 0.373$$

(III) Predeformed cone.

$$\frac{z_A}{z_0} = 0.125, \quad \frac{b_A}{z_0} = 0.25$$

$$\xi_0 = 0.373$$

Experimental values[13]: ○ ● △ ▲

Figure 16. Comparison of the Theoretical Critical Loads of Ideal and Predeformed Isotropic Conical Shells (J30) Dependent on the Inner Pressure With the Experimental Values

5. Summary

The previously known investigations of the stability behavior of thin-wall conical shells under axial load and inner pressure have shown that by means of the linear theory the qualitative effect of the geometrical characteristics and the rigidity parameters on the buckling load can be determined. However, the calculated critical loads deviate greatly from the buckling values found experimentally. To gain a better agreement between experiment and theory, the derivation of the differential equations of the state of stresses and deformation in this paper takes into account nonlinear members. The derived system of nonlinear differential equations generally describes the stress and deformation states of a conical shell-deformed, prebuckled orthotropic (special case "isotropic") conical shell under inner pressure and arbitrary edge loads.

By restricting ourselves to axially symmetric edge loads and to an axially symmetric predeformation, special forms of this system of differential equations for the basic state of stress and for the additional stresses and deformations at the branching point can be stated. The investigations in this paper are limited to these special forms of system of differential equations.

The system describing the axially symmetric basic state of stresses consists of two coupled nonlinear ordinary differential equations of second order in z for the section force N_0 and for the deformation parameter Q_0 (1st derivative of the displacement w perpendicular with respect to the central surface of the shell with respect to z). In the linear theory of stability it has been assumed that the conical shell assumes a state of flexure-free diaphragm stress until buckling occurs. It can be shown that by means of the diaphragm stress the equilibrium conditions at the shell element, but not the compatibility condition and all boundary conditions, can be satisfied. The state of diaphragm stress is thus no solution of the shell problem considered here.

A general solution in closed form of the system of nonlinear differential equations for the basic state of stress is not possible with the familiar methods. For the linearized system describing the stresses and deformations for small loads, a solution in closed form has been found for the isotropic ideal conical shell. (The particular solution of the linearized system is identical with the state of diaphragm stress.) The iterative solution of the nonlinear system is performed with a difference method enabling the satisfaction of all boundary conditions. The finite equations at each intermediate point can be combined to a nonlinear matrix equation which was iteratively

solved by means of Newton's method. By calculating suitable initial values (extrapolation with the approximate solution of next lower load steps) the approximate solutions according to Newton's method converge rapidly (after 3 to 4 iteration steps the agreement includes the fifth decimal). The convergence of the difference method is assured if the approximate solutions with increasing number of intermediate points tend toward a common limiting value. The number of the equidistant intermediate points was increased until the difference between two successive approximations remained below a specified error limit. The required number of points to maintain this limit increased with increasing load. Because of the size of the available computer (a maximum of 45 intermediate points could be used in the calculation) the axially symmetric basic state of the stresses could not be determined for arbitrarily large loads. For the range of inner pressures ($0 \leqslant \overline{p} \leqslant 1.0$) investigated here, this load limit is above the critical load, i. e., in the unstable range of the basic state of stress.

Because of the investigation of the basic state of stress according to the nonlinear theory the appearance of ring buckling below the critical load--as has been observed in the experiment for high inner pressure--can be theoretically explained as well.

A second special form of the general system of differential equations can be stated for the additional stresses and deformations in the branching point. When reaching the critical load (branching point) two states of equilibrium exist which are infinitesimally close: one axially symmetric--identical with the basic state of stress--and one nonaxially symmetric characterized by the existence of nonaxially symmetric stress and deformation contributions in addition to the basic state of stresses. For the nonaxially symmetric additional stresses and deformations, a system of two linear homogeneous partial differential equations (of fourth order for the stress function F and the deformation W) with homogeneous boundary conditions can be derived. The solution of this linear system of equations (eigen-value problem, the critical load represents the eigen-value) requires the knowledge of the stresses and deformations of the basic state of stresses, i. e., the solution of the nonlinear system for N_0 and Q_0. If, instead, the diaphragm stresses are inserted for the basic state of stresses, the familiar equations of the linear theory of stability are obtained.

Since a general solution of this eigen-value problem in closed form is not known, an approximate solution has been determined by means of the difference method. Solving iteratively the eigen-value problem a smaller number of equidistant intermediate points resulted in the same degree of approximations as for the determination of the axially symmetric basic stresses and deformations.

The numerical evaluation of the buckling equations of the nonlinear theory--nonlinear because the axially symmetric stresses and deformations determined from the nonlinear basic equations have been inserted in the eigen-value problem--shows that for high inner pressure the determined buckling loads agree well with the experimental values. The deviation of the critical loads calculated for ideal conical shells and the experimental buckling values for low inner pressure (especially for $p = 0$) is to be traced to prebuckling of the test shells. Investigations where the effect of such predeformations on the buckling behavior has been taken into account are so far known only for the cylinder shell where mostly an affinitive prebuckling with respect to the buckling configuration occurring has been assumed in the calculation. In the present paper the differential equation of the predeformed conical shell has been derived, and it has been solved approximately for the case of an arbitrary axially symmetric fixed (nonaffinitive) prebuckling.

Taking into account a predeformation, the decrease of the buckling loads for low inner pressure could be proved. An absolute comparison between the measured results and the buckling loads calculated for a predeformed shell is not permitted since the arbitrarily assumed predeformation is certainly not in agreement with the predeformation of the test shell.

Because of the large extent of the calculation (operations with large matrices) the numerical evaluation was limited to a few conical shells. The approximation method stated can be combined to a general computer program with which--assuming a suitable computer--an arbitrary degree of approximation can be achieved.

LITERATURE CITED

1. K. Marguerre, CONCERNING THE THEORY OF CURVED PLATES WITH GREAT DEFORMATION, Jahrbuch 1939 der deutschen Luftfahrtforschung (1939 Yearbook of German Aviation Research), pp. 413-418.

2. W. Thielemnan and H. J. Dreyer, CONTRIBUTION TO THE QUESTION OF BUCKLING OF THIN-WALL AXIALLY COMPRESSED CIRCULAR CYLINDERS, DVL-Bericht Nr. 17 (German Test Installation for Aviation and Space Flight, Report No. 17) Westdeutscher Verlag, Cologne and Opladen, 1956.

3. G. Fischer, CONCERNING THE EFFECT OF PIN JOINTS ON THE STABILITY OF THIN-WALL CIRCULAR CYLINDER SHELLS UNDER AXIAL LOAD AND INNER PRESSURE, ZfW, Vol. 11, No. 3, 1963, pp. 111-119.

4. W. Thielemann, W. Schnell, and G. Fischer, BUCKLING AND POSTBUCKLING BEHAVIOR OF ORTHOTROPIC CIRCULAR CYLINDERS UNDER AXIAL AND INNER PRESSURE, ZfW, Vol. 8, Nos. 10/11, pp. 284-293.

5. W. Schnell, THE THIN-WALL CONICAL SHELL UNDER AXIAL AND INNER PRESSURE, ZfW, Vol. 10/11, No. 4/5, 8, 1962, pp. 154-160 and 314-321.

6. J. Singer, A DONNELL TYPE THEORY FOR BENDING AND BUCKLING OF ORTHOTROPIC CONICAL SHELLS, TAE-Report 18, Technion - Israel Institute of Technology, Haifa 1961.

7. L. H. Donnell, A NEW THEORY FOR THE BUCKLING OF THIN CYLINDERS UNDER AXIAL COMPRESSION AND BENDING, Amer. Soc. Mech. Eng., Vol. 56, 1943, pp. 795-806.

8. K. Schiffner, CONCERNING THE ELASTIC STABILITY BEHAVIOR OF THIN-WALL ORTHOTROPIC CONICAL SHELLS UNDER AXIAL LOAD AND INNER PRESSURE, WGL-Jahrbuch, 1962.

9. Kh. M. Mushtari and A. V. Sachenkov, STABILITY OF CYLINDRICAL AND CONICAL SHELLS OF CIRCULAR CROSS SECTION, WITH SIMULTANEOUS ACTION OF AXIAL COMPRESSION AND EXTERNAL NORMAL PRESSURE (In Russian), Priklad. Math. Mech., Vol. 18, No. 6, 1945; Am. Ubersetzung: NACA TM 1433, 1958.

10. W. Thielemann CONCERNING THE BUCKLING OF ANISOTROPIC STRIPS OF PLATES, Report 16 of DLV (German Test Installation for Aviation and Space Flight), Muhlheim/Ruhr, Westdeutscher Verlag, Cologne and Opladen, 1956.

11. W. Schnell and Ch. Bruhl, THE LONGITUDINALLY COMPRESSED ORTHOTROPIC CIRCULAR CYLINDER SHELL UNDER INNER PRESSURE, ZfW, Vol. 7, No. 7, 1959, pp. 201-207.

12. Jerzy Leyko, STABILITY OF AN ORTHOTROPOUS CONESECTOR SHAPED SHELL COMPRESSED ALONG THE GENERATRICES, (In Polnisch), Archiwun Budowy Maszyn, TOM VIII, 1961, Heft 4, pp. 447-460.

13. W. Schnell and K. Schiffner, EXPERIMENTAL INVESTIGATIONS OF THE STABILITY BEHAVIOR OF THIN-WALL CONICAL SHELLS UNDER AXIAL LOAD AND INNER PRESSURE, DVL-Bericht No. 243, (German Test Installation for Aviation and Space Flight, Report No. 243), 1962.

14. V. J. Weingarten, E. J. Morgan, and P. Seide, FINAL REPORT ON DEVELOPMENT OF DESIGN CRITERIA FOR ELASTIC STABILITY OF THIN SHELL STRUCTURES, Space Techn. Lab., TH-60-0000-19425, December 1960.

15. P. Seide, A DONNELL TYPE THEORY FOR ASYMMETRICAL BENDING AND BUCKLING OF THIN CONICAL SHELLS, J. Appl. Mech., Vol. 24, 1957, pp. 547-552.

16. Jahnke, Emde, and F. Lösch, TABLES OF HIGHER FUNCTIONS, 6th Edition, Stuttgart, B. G. Teubner Verlagsgesellschaft, 1960.

17. R. Zurmühl, MATRICES, 3rd Edition, Berlin, Gottingen, Heidelberg, Springer Verlag, 1961.

18. P. Seide, NOTES ON ANALYSIS AND DESIGN OF RIGHT CIRCULAR CONICAL SHELLS, The Ramo - Wooldridge Corp., 1955, Report No. AM 5 - 10.

19. I. C. Serpico, ELASTIC STABILITY OF ORTHOTROPIC CONICAL AND CYLINDRICAL SHELLS SUBJECTED TO AXISYMMETRIC LOADING CONDITIONS, Journal of the American Institute of Aeronautics and Astronautics, Vol. 1, No. 1, January 1963, p. 128.

20. J. Singer, BUCKLING OF CIRCULAR CONICAL SHELLS UNDER AXISYMMETRICAL EXTERNAL PRESSURE, Journal of Mechanical Engineering Sciences, Vol. 3, No. 4, December 1962, p. 330.

21. P. Seide, ON THE BUCKLING OF TRUNCATED CONICAL SHELLS UNDER UNIFORM HYDROSTATIC PRESSURE, Proceedings of IUTAM Symposium of the Theory of Thin Elastic Shells, Delft, Holland, August 1959, North-Holland Publishing Company, Amsterdam,1960, p. 363.

22. J. Singer, BUCKLING OF ORTHOTROPIC AND STIFFENED CONICAL SHELLS, Collected Papers on Instability of Shell Structures, NASA TN-D-1510, December 1962, p. 463.

23. I. Wiedemann, A CONTRIBUTION TO THE PROBLEM OF ORTHOTROPIC PLATES WITHOUT GENERAL NEUTRAL PLANE, Luftfahrttechnik (Aeronautical Engineering), Vol. 8, No. 10, 1962, pp. 283-289; Vol. 9, No. 3, 1963, pp. 73-82; Vol. 9, No. 4, 1963, pp. 118-130.

24. T. von Karman, STRENGTH PROBLEMS IN MACHINE BUILDING, Enc. d. Math. Wiss. (Encyclopedia of the Mathematical Sciences), Vol. 4, 1910, Teubner, Leipzig, pp. 348-352.

25. W. Flügge, STRESSES IN SHELLS, Berlin, Springer-Verlag, 1960.

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The prebuckling state and the stability behavior of thin-walled isotropic and orthotropic shells under axisymmetric loads (axial load and internal pressure) is investigated, applying a nonlinear shell theory (finite deformations).

At first the unbuckled equilibrium state which is axisymmetric under the given load is described. The derived system of ordinary nonlinear differential equations is solved by a difference method with approximation of higher order. For isotropic conical shells, an exact solution for the linearized system can be obtained (linear bending theory).

In the second part of this paper, the buckling loads are calculated. The buckling (bifurcation) state is governed by an eigen-value problem, the eigen-values of which are the buckling loads and the eigen-functions of which are the additional stresses and deformations. The eigen-value problem is also solved iteratively by the difference method.

The influence of shell parameters and of an axisymmetric predeformation on the buckling load is discussed. The results are in good agreement with the experiments, particularly for a conical shell under axial load and high internal pressure.

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