FINAL REPORT

PROJECT A-918

APPLICATION OF DIMENSIONAL ANALYSIS AND GROUP THEORY TO THE SOLUTION OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

L. J. GALLAHER AND M. J. RUSSELL

Contract NAS8-20286

17 January 1966 to 16 January 1967

Performed for
George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama

1967

Engineering Experiment Station
GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Georgia

GEORGIA INSTITUTE OF TECHNOLOGY
Engineering Experiment Station
Atlanta, Georgia

FINAL REPORT

PROJECT A-918

APPLICATION OF DIMENSIONAL ANALYSIS AND GROUP THEORY TO THE SOLUTION OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

By

L. J. GALLAHER and M. J. RUSSELL

CONTRACT NAS8-20286

17 January 1966 to 16 January 1967

Performed for
GEORGE C. MARSHALL SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
HUNTSVILLE, ALABAMA

ABSTRACT

This report gives a survey and analysis of the application of one parameter transformation groups to the solution of ordinary and partial differential equations.

The first part considers ordinary differential equations. Lie's method for finding an integrating factor for a single ordinary differential equation is discussed and examples given. It is then shown how Lie's method can be extended to total differential equations, and systems of total differential equations an extension thought to be new. Examples are given and the connection with dimensional analysis is pointed out.

The second part of this report deals with partial differential equations. Here Morgan's theorems for reducing the number of independent variables are discussed and applications given. It is shown that Morgan's theorems can also be applied to ordinary differential equations but are much less useful in this case.

A brief discussion is given of the connection between Hamiltonian, or Euler-Lagrange equations and Lie algebras and Lie groups, but no examples are given.

Finally, there are some recommendations for further study in this field.

TABLE OF CONTENTS

LIST OF APPENDICES

I. INTRODUCTION

Group theory has come to the forefront of applied mathematics in recent times. Best known or most publicized of these applications has been the study of Lie groups in connection with particle physics and symmetry groups in crystallography and chemistry. The applications of groups discussed in this report are generally less well known than those just mentioned and possibly less well developed. But in a historic perspective they are much older, the basic ideas going back to Sophus Lie (1872).

This report is concerned with the application of transformation groups to the solution of ordinary and partial differential equations. These groups are in fact "Lie groups" in the sense the term is generally used today but the application is such that no particular use is made of the usual properties associated with Lie groups. It is the transformation properties that are exploited and not the detailed structure of the group or associated algebra.

The first part of this report is concerned with ordinary differential equations and the application of one-parameter transformation groups to their solution. The principal theorem here, referred to as Lie's theorem, gives a method for finding an integration factor when an invariance group for the differential equation is known. This technique is then extended to total differential equations and to systems of total differential equations. The connection with dimensional analysis, in particular Brand's work, is pointed out, dimensional analysis being the study of the nonuniform magnification groups.

Partial differential equations are taken up in the second part of the report. Here it is Morgan's theorems that are most significant. Morgan showed that if a system of partial differential equations was invariant with respect to a one-parameter transformation group, the number of independent variables can be reduced by one. Morgan's results are the most significant achieved so far in applying group theory to partial differential equations. The disadvantage of Morgan's method is that the transformed equations are not as general as the original set. Thus there is no assurance that the reduced equations have solutions obeying the original boundary conditions. Each problem must be considered individually, the boundary conditions together with the partial differential equations.

It is also shown here that Morgan's theorems can be applied to ordinary differential equations. The results are not however so interesting, giving particular solutions to the differential equations which are seldom those sought.

In a third section a brief outline is given of how Lie algebras and Lie groups are used in Hamiltonian theory. This is the area in which most of the present activity in particles physics takes place. Here it is the detailed structure of the individual groups that is important and the goals are not so much to solve the equations as to discover their structure from the symmetry considerations. This section is quite brief and no examples are given.

Appendices on the proof of some of the theorems are given. Also included as appendices are the definitions and some examples of groups, Lie groups, and Lie algebras.

II. ORDINARY DIFFERENTIAL EQUATIONS

A. Introduction

Lie introduced the theory of continuous groups into the study of differential equations* and thereby unified and illuminated in a striking way the earlier techniques for handling them. This section will give a short description of the application of the one-parameter transformation group to the solution of a single first order ordinary differential equation. Extension to systems of equations and higher order equations is given later in this chapter. Most of the material in this section is contained in Ince [15].

B. One-parameter transformation groups

Consider the aggregate of transformations included in the family

$$\bar{x} = \phi(x,y;a), \ \bar{y} = \psi(x,y;a) \ .$$

Here x and y are an initial set of coordinates and $\bar{x}$ and $\bar{y}$ are the transformed set, a is a parameter that characterizes the particular transformation. Now whenever two successive transformations of the family are equivalent to a single transformation of the family, then the transformations form a group.** That is, if

$$\phi(x,y;a_3) = \phi\big(\phi(x,y;a_1), \ \psi(x,y;a_1); \ a_2\big)$$

$$\psi(x,y;a_3) = \psi\big(\phi(x,y;a_1), \ \psi(x,y;a_1); \ a_2\big)$$

such that the set of a's are closed (every a_1, a_2 pair has an a_3 in the set)

*[16a], [16b]

**See Appendix I for definition of a group and some examples.

then the transformations form a one parameter group. Note that this means that the inverse of every transformation is present.

Examples of one-parameter transformation groups are the following.

1) The group of rotations about the origin:

$$\bar{x} = x \cos a - y \sin a, \; \bar{y} = x \sin a + y \cos a$$

Two successive rotations characterized by a_1 and a_2 are equivalent to a rotation characterized by a_3 where $a_3 = a_1 + a_2$, and the inverse of the rotation a is - a.

2) The magnification group:

$$\bar{x} = a^j x, \qquad \bar{y} = a^k y$$

Here j and k are constants and if k = j this is called the uniform magnification group. The transformation determined by a_3 that is equivalent to the successive transformations determined by a_1 and a_2 is such that $a_3 = a_1 a_2$. The inverse of the transformation characterized by a is characterized by 1/a.

1. Infinitesimal transformations

Let a_0 be the value of the parameter which characterizes the identity transformation of family so that

$$x = \phi(x,y;a_0), \qquad y = \psi(x,y;a_0) \; .$$

Then if ϵ is small (an infinitesimal), the transformation

$$\bar{x} = \phi(x,y;a_0+\epsilon), \qquad \bar{y} = \psi(x,y;a_0+\epsilon)$$

will be such that $\bar{x}$ and $\bar{y}$ differ only infinitesimally from $\bar{x}$ and $\bar{y}$ or

$$\bar{x} \approx x + \frac{\partial\phi\ (x,y;a_0)\epsilon}{\partial a_0} = x + \alpha(x,y)\epsilon$$

$$\bar{y} \approx y + \frac{\partial\psi\ (x,y;a_0)\epsilon}{\partial a_0} = x + \beta(x,y)\epsilon \quad .$$

This transformation is then said to be an infinitesimal transformation. Now it can be proved* that every one-parameter transformation group contains one and only one infinitesimal transformation. Thus a group of transformations can be characterized either by the pair of functions ϕ and ψ or by the pair of functions α and β where

$$\alpha(x,y) = \frac{\partial\phi\ (x,y;a)}{\partial a}\Bigg|_{a\,=\,a_0}$$

$$\beta(x,y) = \frac{\partial\psi\ (x,y;a)}{\partial a}\Bigg|_{a\,=\,a_0} ,$$

and a_0 characterizes the identity transformation.

Some examples of infinitesimal transformations are the following:

1) The rotation group mentioned above is defined by $\bar{x} = x \cos a - y \sin a$, $\bar{y} = x \sin a + y \cos a$ and the infinitesimal rotation by

$$\bar{x} = x - y\epsilon, \ \bar{y} = y + x\epsilon$$

since

$$\frac{\partial}{\partial a}\,(x \cos a - y \sin a)\Bigg|_{a=0} = -\,y$$

and

$$\frac{\partial}{\partial a}\,(x \sin a - y \cos a)\Bigg|_{a=0} = x\ .$$

*Ince [15] page 95.

2) The magnification group mentioned above is defined by

$$\bar{x} = a^j x, \quad \bar{y} = a^k y$$

then

$$\frac{\partial}{\partial a}(a^j x)\Big|_{a=1} = jx \text{ and } \frac{\partial}{\partial a}(a^k y)\Big|_{a=1} = ky$$

so the infinitesimal transformation is

$$\bar{x} = x(1 + j\epsilon), \ \bar{y} = y(1 + k\epsilon).$$

Consider now the infinitesimal change in the function $f(x,y)$ due to an infinitesimal transformation of x and y

$$\bar{x} = x + \alpha(x,y)\epsilon, \ \bar{y} = y + \beta(x,y)\epsilon,$$

$$f(\bar{x},\bar{y}) \approx f(x,y) + \left(\alpha(x,y) \frac{\partial f}{\partial x} + \beta(x,y) \frac{\partial f}{\partial y} \right) \epsilon$$

to first order in ϵ. Thus the infinitesimal transformation (and hence the entire group) can be represented by the operator U where

$$U \equiv \alpha(x,y) \frac{\partial}{\partial x} + \beta(x,y) \frac{\partial}{\partial y} .$$

$Uf\epsilon$ is the infinitesimal change in the function $f(x,y)$ produced by the infinitesimal transformation of x and y.

Now let the finite equations of a one parameter transformation group be

$$\bar{x} = \phi(x,y;a_0 + t), \ \bar{y} = \psi(x,y;a_0 + t)$$

where a_0 characterizes the identity transformation. Then

$$f(\bar{x},\bar{y}) = f_0 + f_0' t + \frac{1}{2!} f_0'' t^2 + \ . \ . \ .$$

where

$$f_0 = f(\bar{x},\bar{y}) \Big|_{t=0} = f(x,y)$$

$$f_0' = \frac{d}{dt} f(\bar{x},\bar{y}) \Big|_{t=0} = \left(\frac{\partial f}{\partial \bar{x}} \frac{d\bar{x}}{dt} + \frac{\partial f}{\partial \bar{y}} \frac{d\bar{y}}{dt}\right) \Big|_{t=0}$$

$$= \left(\frac{\partial f}{\partial \bar{x}} \alpha(\bar{x},\bar{y}) + \frac{\partial f}{\partial \bar{y}} \beta(\bar{x},\bar{y})\right) \Big|_{t=0} = Uf$$

$$f_0'' = \frac{d^2}{dt^2} f(\bar{x},\bar{y}) \Big|_{t=0} = U^2 f \quad \text{etc.}$$

Thus

$$f(\bar{x},\bar{y}) = f(x,y) + tUf + \frac{t^2}{2!} U^2 f + \ldots$$

$$= e^{tU} f(x,y)$$

where $U^n f$ symbolizes the result of operating n times on f(x,y) and e^{tU} symbolizes the operator

$$e^{tU} \equiv 1 + tU + \frac{t^2}{2!} U^2 + \ldots .$$

Thus the operator e^{tU} represents the finite transformation corresponding to the infinitesimal transformation

$$U \equiv \alpha(x,y) \frac{\partial}{\partial x} + \beta(x,y) \frac{\partial}{\partial y}$$

Some examples of obtaining the finite transformation from the infinitesimal are as follows:

1) Given the infinitesimal transformation

$$U \equiv - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} .$$

Then

$$\bar{x} = e^{tU}x$$

$$= x + tUx + \frac{t^2}{2!} U^2x + \dots$$

$$= x - yt - \frac{t^2}{2!} x + \frac{t^3}{3!} y + \frac{t^4}{4!} x - \dots$$

$$= x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots) - y(t - \frac{t^3}{3!} + \dots)$$

$$= x \cos t - y \sin t,$$

$$\bar{y} = e^{tU}y$$

$$= x \sin t + y \cos t.$$

This corresponds to the rotation group.

2) Let $U = cx \frac{\partial}{\partial x} + by \frac{\partial}{\partial y}$

then

$$\bar{x} = e^{tU}x = x + ctx + \frac{(ct)^2}{2!} x \dots = xe^{ct}$$

$$\bar{y} = e^{tU}y = y + bty + \frac{(by)^2}{2!} y \dots = ye^{bt}$$

Letting $a = e^t$ it is seen that this is the magnification group

$$\bar{x} = a^c x, \; \bar{y} = a^b y.$$

If $b = c$, it is the uniform magnification group.

2. Invariants

$F(x,y)$ is said to be invariant if, when $\bar{x}$ and $\bar{y}$ are derived from x and y by a one-parameter group of transformations, one has

$$F(\bar{x},\bar{y}) = F(x,y) .$$

A necessary and sufficient condition for $F(x,y)$ to be invariant is that

$$UF = 0$$

where $U = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$ characterizes the group. Then $F(x,y)$ is a solution to the partial differential equation

$$\alpha\frac{\partial Z}{\partial x} + \beta\frac{\partial Z}{\partial y} = 0$$

and

$$F(x,y) = \text{constant}$$

is a solution of the equivalent ordinary differential equation

$$\frac{dx}{\alpha} = \frac{dy}{\beta} \quad .$$

This differential equation has only one solution depending on an arbitrary constant; thus every other invariant of the group can be expressed in terms of F.

A family of curves is said to be invariant under a transformation group if

$$F(x,y) = c \text{ (a constant)}$$

and

$$F(\bar{x},\bar{y}) = \bar{c} \text{ (another constant)}$$

where $\bar{x},\bar{y}$ are derived from x,y by that transformation. A necessary and sufficient condition that $F(x,y) = \text{const}$ represents a family invariant under the transformation group U is that UF be a function of F, i.e., $UF = g(F)$.

C. Integration of a differential equation using group properties

The principal theorem for use in the solution of ordinary differential equations and which will be referred to as Lie's theorem, is the following:

Let the differential equation be given by

$$P(x,y)dx + Q(x,y)dy = 0 .$$

Then if the family of solutions $\phi(x,y)$ = const is invariant under the transformation $U = \alpha(x,y) \frac{\partial}{\partial x} + \beta(x,y) \frac{\partial}{\partial y}$, the quantity $(P\alpha + Q\beta)^{-1}$ is an integration factor of the differential equation, provided $P\alpha + Q\beta$ is not identically zero. That is, the solution can be reduced to a quadrature, and is

$$\int \frac{Pdx + Qdy}{P\alpha + Q\beta} = K$$

where K is a constant. A proof will not be given here*, but a proof of a more general theorem that includes this as a special case is given in Appendix II.

Furthermore if the family of solutions is invariant under two distinct transformations U_1 and U_2

$$U_1 = \alpha_1 \frac{\partial}{\partial x} + \beta_1 \frac{\partial}{\partial y}$$

and

$$U_2 = \alpha_2 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial y}$$

then the solution is

$$\frac{\alpha_1 P + \beta_1 Q}{\alpha_2 P + \beta_2 Q} = K$$

*See Ince [15] pages 106-107.

where K is a constant. This is just the application of a well-known result in differential equations* that if two distinct integration factors for a differential equation are known, say λ and μ, then provided that their ratio is not a constant, λ/μ = const is a general solution. But practically speaking, it is not always easy to find two distinct transformation groups for a differential equation.

Example 1: $2xydy + (x - y^2)\, dx = 0$.

This is invariant under the transformation

$$\bar{x} = a^j x, \quad \bar{y} = a^k y$$

if

$$j + 2k = 2j \quad \text{or} \quad j = 2k \ .$$

Thus

$$U = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}$$

represents the invariance group and λ, the integration factor is

$$\lambda = \frac{1}{xy^2 + (x - y^2)x} \ .$$

The quadrature problem becomes

$$\int \frac{2xydy + (x - y^2)\, dx}{xy^2 + (x - y^2)x} = K \ .$$

Since this is a perfect differential the limits of integration can be chosen as those most convenient. Here we choose y = 0, x = 1 to x,y along the path y = 0, x = 1 to x and then along x = x, y = 0 to y: that is,

$$\int_0^y \frac{2xy\, dy}{xy^2 + (x - y^2)x} + \int_1^x \frac{(x - y^2)}{xy^2 + (x - y^2)x} \Bigg|_{y=0} dx = K$$

*See Ford [9] page 58.

or

$$\int_0^y \frac{2y\,dy}{x} + \int_1^x \frac{dx}{x} = K$$

$$\frac{y^2}{x} + \ln x = K$$

$$y^2 = -x\ln cx$$

where c is an arbitrary constant.

Example 2: $x\,dy-(y + x^m)dx = 0$.

This is invariant under the transformation

$$\bar{x} = a^k x,\ \bar{y} = a^j y$$

if $j = mk$. Thus $U = x\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}$ and $\lambda = \dfrac{1}{-(y + x^m)x + xmy}$. The quadrature problem is

$$\int \frac{x dy - (y + x^m)dx}{-(y + x^m)x + xmy} = K$$

Choose the path of integration as $y = 0$, x from 1 to x and then $y = 0$ to y or

$$\int_{x=1}^x \frac{dx}{x} + \int_0^y \frac{dy}{(m-1)y - x^m} = K$$

giving

$$\ln\left(\frac{y(m-1) + x^m}{x}\right) = K$$

or

$$y = (x^m - cx)/(m-1)$$

where c is an arbitrary constant. Both of these examples are of equations invariant under a nonuniform magnification and this treatment is equivalent to Brand's dimensional analysis [4] approach.

Example 3: $dy - y^2\, dt = 0$

This equation is invariant under a translation along the t axis, that is,

$$\bar{t} = t + a \quad \text{or} \quad U_1 = \frac{\partial}{\partial t}$$

and the nonuniform magnification

$$\bar{y} = ay$$

$$\bar{t} = a^{-1}t$$

for which $U_2 = y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t}$. Thus the two integration factors are $\lambda = -y^{-2}$ and $\mu = (y + ty^2)^{-1}$. Their ratio is

$$\lambda/\mu = -(y + ty^2)/y^2 = \text{const},$$

and the solution is $\frac{1}{y} + t = c$. In this case the answer could have been easily obtained by direct integration.

D. Total differential equations and transformation groups

A total differential equation in n variables is a relation of the form

$$\sum_{k=1}^{n} P_k dx_k \equiv P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \ldots P_n dx_n = 0. \tag{D-1}$$

Its solution, if it exists, consists of finding one of the x's as a function of all the others. The P_k are, in general, functions of the x_k. If a solution exists for D-1, D-1 is said to be integrable. Solutions to total differential equations are usually found by finding an integration factor $\lambda(x_1, x_2 \ldots)$ such that $d\phi = \sum_k \lambda P_k dx_k$ is a perfect differential, that is, finding a λ such that

$$\frac{\partial\phi(x_1,x_2 . . .)}{\partial x_k} = \lambda P_k, \quad k = 1, 2 . . . n.$$

The general solution of the total differential equation is then given by the quadrature,

$$\phi(x_1,x_2 . . .) = \int \sum_k \lambda P_k dx_k = \text{constant}.$$

It can be shown that D-1 has a solution and thus an integration factor if and only if

$$\text{Anti}_{[kms]}\, P_k \frac{\partial P_s}{\partial x_m} = 0$$

where the operator $\text{Anti}_{[kms]}$ means that the following term is antisymmetric with respect the indices k,m,s. (See Appendix III.)

Transformation groups are used in solving total differential equations through the following theorem:

If the total differential equation $\sum_{k=1}^{n} P_k dx_k = 0$ is invariant with respect to a transformation group characterized by

$$U \equiv \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} \equiv \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + . . . \alpha_n \frac{\partial}{\partial x_n}$$

then an integrating factor λ is given by $\lambda = (\sum_{k=1}^{n} \alpha_k P_k)^{-1}$, provided this reciprocal is not identically zero. The α_k are in general functions of the x's. Appendix II gives a proof of this.

This theorem is a generalization of the one given for ordinary differential equations (in paragraph C) and reduces to it in the case n = 2. It is also a generalization of Brand's theorem [4].

Example 1 [4]: The total differential equation

$$2xyzdx + z(1 - yz^2)dy + y(3 - 2yz^2)dz = 0$$

is invariant with respect to the transformation

$$\bar{x} = a^0 x$$

$$\bar{y} = a^2 y$$

$$\bar{z} = a^{-1} z$$

which is characterized by

$$U = 2y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} .$$

The integration factor is

$$\lambda = \frac{1}{2yz(1 - yz^2) - zy(3 - 2yz^2)} = - \frac{1}{yz} .$$

The quadrature problem then is

$$\int \left(2xdx + (\frac{1}{y} - z^2) dy + (\frac{3}{z} - 2yz)dz\right) = const$$

and can be integrated along the path (0,1,1) to (0,1,z) to (0,y,z) to (x,y,z) or

$$\int_0^x 2tdt + \int_1^y (\frac{1}{t} + z^2)dt + \int_1^z (\frac{3}{t} - 2zy)dt.$$

Thus $\ln(yz^3) + x^2 - yz^2 = const$ is the general solution.

Example 2: The total differential equation

$$-ydx + xdy + (x^2 + y^2)dz = 0$$

is invariant with respect to a rotation about the z axis, that is,

$$\bar{x} = x \cos a - y \sin a$$

$$\bar{y} = x \sin a + y \cos a$$

$$\bar{z} = z$$

which is characterized by

$$U = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} .$$

The multiplier is

$$\lambda = \frac{1}{x^2 + y^2} .$$

The quadrature problem is

$$\int \left(\frac{-ydx}{x^2 + y^2} + \frac{xdy}{x^2 + y^2} + dz \right) = \text{const}$$

and is most easily integrated along the path

(0,1,0) to (0,1,z) to (0,y,z) to (x,y,z)

or

$$\int_0^x \frac{-ydt}{t^2 + y^2} + \int_1^y \frac{(0)dt}{t^2} + \int_0^z dt = \text{const}$$

$$= - \tan^{-1}\left(\frac{x}{y}\right) + z = \text{const} .$$

As with ordinary differential equations, if two integration factors can be found, say λ and μ, for a total differential equation then, provided that the ratio of λ to μ is not a constant, the equation λ/μ = constant is a solution to the total differential equation. Thus, if two distinct transformation groups can be found such that the total differential equation is invariant under both, the solution can be given directly. But as mentioned in the section on ordinary differential equations, it seldom happens that two integration factors can easily be found from group considerations alone. The usual situation is that one integration factor can be found by transformation invariance while a second is found by other means.

Example 3 [4]: The equation

$$x^2 dw + (2x + y^2 + 2xw - z)\, dx - 2xydy - xdz = 0$$

is already an exact differential* so that $\mu = 1$ is an integration factor. It is also invariant with respect to the transformation

$$\bar{w} = a^0 w$$

$$\bar{x} = a^2 x$$

$$\bar{y} = a^1 y$$

$$\bar{z} = a^2 z$$

so that

$$\lambda = \frac{1}{4(x^2 + xy^2 + x^2 w - xz)}$$

is an integration factor. Therefore μ/λ = constant is a solution or

$$x^2 + xy^2 + x^2 w - xz = \text{const}$$

is a solution.

*It is exact since $\dfrac{\partial P_r}{\partial x_m} - \dfrac{\partial P_m}{\partial x_r} = 0$ for all r and m.

E. Systems of total differential equations

The transformation group approach can also be applied to systems of total differential equations. Superficially it might appear that nothing new is necessary when dealing with systems of total differential equations, and that all that is necessary is to integrate each individual equation without regard to the others. This will not suffice since we are looking for solutions that have a common intersection and the individual solutions need not have a common intersection.

Consider the system of J total differential equations

$$\sum_{m=1}^{M} P_\beta^m \, dx_m = 0, \quad \beta = 1, 2 \ldots J(\leq M) \tag{E-1}$$

Here each of the P_β^m can be a function of the x_m. By an integration factor to this system of equations we mean a matrix function, $\lambda_\gamma^\beta \, (x_1, x_2 \ldots)$ such that

$$\frac{\partial \phi_\gamma}{\partial x_m} = \sum_{\beta=1}^{J} \lambda_\gamma^\beta \, P_\beta{}^m \, , \qquad \begin{cases} \gamma = 1, 2, \ldots J, \\ m = 1, 2, \ldots M \end{cases}$$

for some set of functions $\phi_\gamma(x_1, x_2 \ldots)$. The general solution to the system E-1 then is the system $\phi_\gamma(x_1, x_2 \ldots) = C_\gamma$, $\gamma = 1, 2 \ldots J$, where the C_γ are constants.

The principal theorem for use of transformation groups with systems of total differential equations is analogous to the theorems of sections C and D above, and is stated as follows:

If the system of total differential equations $\sum_{m=1}^{M} P_\beta^m \, dx_m = 0$, $\beta = 1, 2 \ldots J(\leq M)$ has solutions $\phi_\gamma (\gamma = 1, 2 \ldots J)$ which are invariant as a family with respect to the transformations

$$U^\beta = \sum_{m=1}^{M} \alpha_m^\beta \frac{\partial}{\partial x_m}, \beta = 1, 2 \; . \; . \; . J$$

$$\left(\alpha_m^\beta = \alpha_m^\beta (x_1 x_2 . \; . \; .) \right)$$

then an integration factor λ_γ^β is given by

$$\lambda_\gamma^\beta = \left((P\alpha^T)^{-1} \right)_\gamma^\beta$$

where $(P\alpha^T)^{-1}$ is the inverse of the matrix product $P\alpha^T$, $(P\alpha^T)_\eta^\gamma = \sum_m P_\eta^m \alpha_m^\gamma$. Proof of this theorem is given in Appendix IV.

Before giving examples of the use of this theorem it will be noted that to find one-parameter transformations U^β such that every one of the equations is invariant with respect to all the transformations is usually rather difficult. But often it happens that it is possible to transform the original equations to a new set that has the same solution. That is, if the original equations

$$\sum_m P_\gamma^m \, dx_m = 0, \gamma = 1 \; . \; . \; . J$$

have solutions $\phi_\beta = C_\beta$ then if one introduces $P_\beta'^m = \sum_{\gamma=1}^{J} T_\beta^\gamma P_\gamma^m$ where $T_\beta^\gamma = T_\beta^\gamma(x_1, x_2 . \; . \; .)$ then

$$\sum_m P_\gamma' \, dx_m = 0$$

has solutions $\phi_\beta' = C_\beta'$. While the ϕ_β and ϕ_β' are different, their intersection is the same, and it is the intersection that is the "solution" to the system.

Example 1: Consider the two total differential equations

$$x dx + y\, dy - z dz = 0$$

$$x dx + y\, dx + z dz = 0$$

In matrix form:

$$\begin{bmatrix} x & y & -z \\ x & y & +z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This can be transformed to

$$\begin{bmatrix} x & y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by the matrix transformation T,

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1/z \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} .$$

These two transformed equations are invariant with respect to the two transformations

$$U^1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$$ (uniform magnification along x, y and z axes)

and

$$U^2 = \frac{\partial}{\partial z}$$ (translation along z axis)

so that $\alpha = \begin{bmatrix} x & y & z \\ 0 & 0 & 1 \end{bmatrix}$.

The product $P\alpha^T$ is

$$P\alpha^T = \begin{bmatrix} x & y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ y & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} x^2 + y^2 & 0 \\ z & 1 \end{bmatrix}$$

and

$$(P\alpha^T)^{-1} = \frac{1}{(x^2 + y^2)} \begin{bmatrix} 1 & 0 \\ -z & x^2 + y^2 \end{bmatrix} .$$

The integration factor λ in matrix form is

$$\lambda = \begin{bmatrix} \frac{1}{x^2 + y^2} & 0 \\ -\frac{z}{x^2 + y^2} & 1 \end{bmatrix}$$

and multiplying by λ gives the system of perfect differentials

$$\frac{xdx}{x^2 + y^2} + \frac{y}{x^2 + y^2} dy = 0$$

$$-\frac{zxdx}{x^2 + y^2} - \frac{zydy}{x^2 + y^2} + dz = 0.$$

Integrating along the path

$$(0,1,0) \text{ to } (0,1,z) \text{ to } (0,y,z) \text{ to } (x,y,z)$$

gives

$$\ln(x^2 + y^2) = C_1$$

$$-z \ln(x^2 + y^2) + z = C_2$$

or

$$x^2 + y^2 = C_1'$$

$$z = C_2'$$

whose intersection is a family of circles in planes parallel to the x, y plane with origin on the z axis.

Example 2: Consider the equations

$$-y\,dx + x dy + (x^2 + y^2)\,dz = 0$$

$$\frac{dx}{y} + \frac{dy}{x} = 0 \ .$$

Each of these equations is invariant with respect to a rotation about the z axis ($U^1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$) and a uniform magnification along the x and y axis ($U^2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$).

Thus

$$P\alpha^T = \begin{bmatrix} -y & x & x^2+z^2 \\ 1/y & 1/x & 0 \end{bmatrix} \begin{bmatrix} -y & x \\ x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & \frac{x}{y} + \frac{y}{x} \end{bmatrix}$$

and

$$\lambda = \begin{bmatrix} \frac{1}{x^2 + y^2} & 0 \\ 0 & (\frac{y}{x} + \frac{x}{y})^{-1} \end{bmatrix} .$$

Multiplying by λ gives the two equations

$$-\frac{y\,dx}{x^2 + y^2} + \frac{x dy}{x^2 + y^2} + dz = 0$$

$$\frac{x dx}{x^2 + y^2} + \frac{y dy}{x^2 + y^2} = 0 \ .$$

Integrating along the path

$$(0,1,0) \text{ to } (0,1,z) \text{ to } (0,y,z) \text{ to } (x,y,z)$$

gives the results

$$x^2 + y^2 = C_1$$

$$-\tan^{-1}(x/y) + z = C_2 \quad .$$

The intersection of these two surfaces is a 45° helix whose axis coincides with the z axis.

III. PARTIAL DIFFERENTIAL EQUATIONS

In 1948 Birkhoff discussed the application of dimensional analysis to the solution of partial differential equations. He showed how it was possible to reduce by one the number of independent variables in a partial differential equation if that equation was invariant with respect to one of the transformation groups of dimensional analysis (magnifications). Later Morgan generalized this procedure to include all one-parameter transformation groups. Morgan's theorems represent the most progress to date in the application of groups to partial differential equations, and it is this work that will be discussed next.

A. Morgan's Theorems

We are concerned here with the sets of variables $x_1, x_2 \ldots x_m$, $y_1, y_2 \ldots y_n$ and the one-parameter group of transformations

$$\bar{x}_k = f_k\,(x_1, x_2 \;.\;.\;.\; x_m, a), \;\; k = 1, 2 \;.\;.\;.\; m$$

$$\bar{y}_\beta = f_\beta\,(y_1, y_2 \;.\;.\;.\; y_m, a), \;\; \beta = 1, 2 \;.\;.\;.\; n$$

where the functions f_k and f_β are differentiable with respect to the parameter a. The y's in turn are considered to be differentiable (to any required order) functions of the x's. If the transformations of the partial derivatives of the y's with respect to the x's are appended to the above transformations the resulting set of transformations also form a continuous one-parameter group called the enlargements of the group or the extended group. When considered as a function of the m + n independent variables $x_1, x_2 \ldots x_m$, $y_1, y_2, \ldots y_n$, the group has m + n - 1 functionally independent absolute invariants.* Call the absolute invariants $\eta_1, \eta_2, \ldots \eta_{m-1}$ and $g_1, g_2 \ldots$

*See L. P. Eisenhart [8].

where $\eta_k = \eta_k\,(x_1, x_2 \,.\,.\,.\, x_m)$ and $g_\beta = g_\beta\,(y_1, y_2 \,.\,.\,.\, y_n, x_1, x_2 \,.\,.\,.\, x_m)$.

Morgan's first theorem then states the following:

If the y and $\bar{y}$ are defined explicitly as functions of the x's and $\bar{x}$'s respectively by the relations

$$z_\beta(x_1, x_2 \,.\,.\,.\, x_m) = g_\beta(y_1, y_2 \,.\,.\,.\, y_n, x_1, x_2 \,.\,.\,.\, x_m)$$

$$\bar{z}_\beta(\bar{x}_1, \bar{x}_2 \,.\,.\,.\, \bar{x}_m) = g_\beta(\bar{y}_1, \bar{y}_2 \,.\,.\,.\, \bar{y}_n, \bar{x}_1, \bar{x}_2 \,.\,.\,.\, \bar{x}_m)$$

then a necessary and sufficient condition that the y be exactly the same functions of the x's as the $\bar{y}$ are of the $\bar{x}$'s is that

$$z_\beta(x_1, x_2 \,.\,.\,.\, x_m) = \bar{z}_\beta(\bar{x}_1, \bar{x}_2 \,.\,.\,.\, \bar{x}_m)$$

$$= z_\beta(\bar{x}_1, \bar{x}_2, \,.\,.\,.\, \bar{x}_m) = F_\beta(\eta_1, \eta_2 \,.\,.\,.\, \eta_{m-1}).$$

The η's are the invariants of the subgroup

$$\bar{x}_k = f_k\,(x_1, x_2 \,.\,.\,.\, x_m, a).$$

This theorem will not be proved here; the reader is referred to Morgan [20].

Several definitions are as follows:

By an invariant solution of a system of partial differential equations is meant that class of solutions which has the property that the y_β are exactly the same functions of the x_k as the $\bar{y}_\beta$ are of the $\bar{x}_k$ where the x's and y's are related by some one-parameter transformation group.

By a differential form of the k-th order in m independent variables is meant a function of the form

$$\Phi\left(x_1, x_2 \,.\,.\,.\, x_m, y_1, y_2 \,.\,.\,.\, y_n, \,.\,.\,.\, \frac{\partial^k y_1}{\partial (x_1)^k} \,.\,.\,.\,, \frac{\partial^k y_n}{\partial (x_m)^k}\right) \quad .$$

It has as arguments the x's, the y's and all partial derivatives of y's with respect to x's up to order k. The partial derivatives of Φ with respect to all its arguments are assumed to exist.

A differential form Φ is said to be conformally invariant under a one-parameter transformation group if under that group it satisfies

$$\Phi(\bar{z}_1, \bar{z}_2 \ldots \bar{z}_p) = f(z_1, z_2 \ldots z_p, a)\, \Phi(z_1, z_2, \ldots z_p).$$

If f is a function of a only, then Φ is said to be constant conformally invariant and if f = 1 then absolutely invariant.

Morgan's second theorem then states the following:

A necessary and sufficient condition for Φ to be conformally invariant under a continuous one-parameter group is that

$$U\Phi = \omega(z_1, z_2, \ldots z_p)\, \Phi(z_1, z_2 \ldots z_p)$$

for some $\omega(z_1, z_2 \ldots z_p)$. Here U is the operator characterizing the infinitesimal transformation

$$U = \alpha_1 \frac{\partial}{\partial z_1} + \alpha_2 \frac{\partial}{\partial z_2} \ldots \alpha_p \frac{\partial}{\partial z_p}$$

$\alpha = \alpha(z_1, z_2, \ldots z_p)$ (see earlier chapters of this report). Again the reader is referred to Morgan for a proof of this theorem [20].

A system of k-th order partial differential equations $\Phi_\beta = 0$ is said to be invariant under a continuous one-parameter group of transformations if each of the Φ_β is conformally invariant under the enlargements of that group.

Morgan's principal theorem then is the following statement:

If each of the differential forms Φ_β of the form

$$\Phi_\beta\left(x_1, x_2 \ldots x_m, y_1, y_2, \ldots y_n, \ldots \frac{\partial^k y_1}{\partial(x_1)^k} \ldots \frac{\partial^k y_n}{\partial(x_m)^k}\right) = 0$$

is conformally invariant under the k-th enlargement of a transformation group, then the invariant solutions can be expressed as the system

$$A_\beta\left(\eta_1, \eta_2 \ldots \eta_{m-1}, F_1, F_2 \ldots F_n, \ldots \frac{\partial^k F_n}{\partial(\eta_{m-1})^k}\right) = 0$$

a system of k-th order partial differential equations in m-1 independent variables. Here the η are the absolute invariants of the (sub) group of transformations on the x's, and the F's are the other invariants,

$$F_\beta(\eta_1, \eta_2 \ldots \eta_{m-1}) = g_\beta(y_1, y_2 \ldots y_n, x_1, x_2, \ldots x_m).$$

The proof of this theorem is given in Morgan's paper [20].

This theorem is exceedingly powerful and useful. The reduction of the number of independent variables by one in a system of differential equations can greatly aid in obtaining a solution. A partial differential equation in two variables will be reduced to an ordinary differential equation which can be much more quickly solved by numerical methods than the original equations. In the case of equations of three or more independent variables it may be possible to apply Morgan's theorem several times in succession, reducing the number of variables by one each time.

On the other hand, no account is taken in Morgan's theorem of the boundary conditions associated with a specific problem. The invariant solution found may or may not comply with the boundary conditions. The invariant solutions are a smaller set than the total set of solutions. The solutions of the reduced equations are not as general as the original equations. In this sense Morgan's prescription does not give general solutions to the differential equation.

B. Applications of Morgan's Theorems

Example 1:

Consider the partial differential equation of the one dimensional homogeneous heat flow equation:

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0 \tag{B-1}$$

This equation is constant conformally invariant with respect to the non-uniform magnification transformation

$$\bar{y} = a^k y$$

$$\bar{x} = a^s x$$

$$\bar{t} = a^m t,$$

if $k-m = k-2s$. One possibility is $s = \frac{1}{2}$, $m = 1$, $k = 0$. For this transformation the invariant independent variable is

$$\eta = x/t^{\frac{1}{2}}$$

and the invariant dependent variable is

$$y = g = g(\eta) \quad .$$

Working out the partial differential operations in terms of the new variable we have

$$\frac{\partial y}{\partial t} = \frac{dg}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{x}{t^{3/2}}\frac{dg}{d\eta} = -\frac{1}{2}\frac{\eta}{t}\frac{dg}{d\eta}$$

$$\frac{\partial y}{dx} = \frac{dg}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{t^{\frac{1}{2}}}\frac{dg}{d\eta}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{t}\frac{d^2 g}{d\eta^2} .$$

The reduced equation is then the ordinary differential equation

$$\frac{d^2 g}{d\eta^2} + \frac{\eta}{2}\frac{dg}{d\eta} = 0. \qquad \text{(B-2)}$$

A general solution to this particular ordinary differential equation is

$$g(\eta) = A\int_0^{\eta} e^{-\eta^2/4}\, d\eta + B$$

or

$$y(x,t) = A\int_0^{x/t^{1/2}} e^{-\eta^2/4}\, d\eta + B \qquad \text{(B-3)}$$

where A and B are the constants of integration.

It is noted that the solution B-3 may or may not be compatible with the boundary conditions of the original equation, B-1.

Also we note that this is not the only reduction possible. Equation B-1 is also invariant with respect to the transformation

$$\bar{y} = y + \ln a$$

$$\bar{x} = a^{\frac{1}{2}}x$$

$$\bar{t} = at$$

so that invariants

$$g = y - \ln t$$
$$\eta = x/\sqrt{t}$$

are possible. The partial derivatives are

$$\frac{\partial y}{\partial t} = \frac{1}{t} - \frac{1}{2}\frac{\eta}{t}\frac{dg}{d\eta}$$

$$\frac{\partial y}{\partial x} = \frac{1}{t^{\frac{1}{2}}}\frac{dg}{d\eta}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{t}\frac{d^2 g}{d\eta^2}$$

and the ordinary differential equation for g in η becomes

$$\frac{d^2}{d\eta^2} g + \frac{\eta}{2}\frac{dg}{d\eta} - 1 = 0$$

which has a different solution from B-2.

Equation B-1 is invariant under a very large variety of transformations leading to different ordinary differential equations. Each set of boundary conditions must be considered separately and transformations compatible with the boundary conditions sought.

Example 2*: The system of partial differential equations of the classical boundary-layer theory of Blasius is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - b\frac{\partial^2 u}{\partial y^2} = 0$$

*Taken from Morgan [20].

These are constant conformally invariant under the transformation

$$\bar{u} = u$$
$$\bar{v} = a^{-1} v$$
$$\bar{y} = ay$$
$$\bar{x} = a^2 x,$$

a nonuniform magnification. A set of absolute invariants then are

$$g_1 = u$$
$$g_2 = v x^{\frac{1}{2}}$$
$$\eta = y / x^{\frac{1}{2}} \quad .$$

The derivatives are

$$\frac{\partial v}{\partial y} = \frac{1}{x} \frac{dg_2}{d\eta}$$

$$\frac{\partial u}{\partial x} = - \frac{\eta}{2x} \frac{dg_1}{d\eta}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^{\frac{1}{2}}} \frac{dg_1}{d\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x} \frac{d^2 g_1}{d\eta^2}$$

giving the pair of ordinary differential equations

$$- \tfrac{1}{2}\eta \frac{dg_1}{d\eta} + \frac{dg_2}{d\eta} = 0$$

$$- \tfrac{1}{2}\eta\, g_1 \frac{dg_1}{d\eta} + g_2 \frac{dg_1}{d\eta} - b \frac{d^2 g_1}{d\eta^2} = 0 \quad .$$

Example 3: The wave equation in one dimension is

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \ . \qquad \text{(B-4)}$$

This equation is absolutely invariant with respect to the transformation

$$\bar{y} = y$$
$$\bar{x} = x + a$$
$$\bar{t} = t + a,$$

a shift in the origin of x and t coordinates. The invariants are $g = y$, $\eta = x\text{-}t$. The derivatives are

$$\frac{\partial y}{\partial t} = -g'$$
$$\frac{d^2 y}{dt^2} = g''$$
$$\frac{\partial y}{\partial x} = g'$$
$$\frac{\partial^2 y}{\partial x^2} = g''$$

where the primes indicate derivatives with respect to η. The original equation becomes then the identity

$$0 = 0$$

and indicates that there are no restrictions on the function $g(\eta)$. That is every function of η, (at least every twice differentiable function) is a solution to the wave equation, or

$$g(\eta) = y(x\text{-}t)$$

is a solution for every y. This of course is easily verified, and it is well known that a pulse of arbitrary shape propagates with uniform velocity up (or down) the x axis without changing shape if the medium is governed by equation B-4.

C. Linear Equations

In the case of linear equations it is possible to use the transformation groups to derive kernels for use in closed form integral solutions to initial valve problems and for deriving Green's functions. As an example let us look again at the one dimensional linear heat flow equation:

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0. \qquad \text{(C-1)}$$

It is of interest here to find solutions that satisfy the boundary conditions

$$y = 0 \text{ at } t = 0 \text{ and } x \neq 0$$

and

$$\lim_{t \to 0} \int_{-\infty}^{+\infty} y\ (x,t)\ dx = 1.$$

These conditions are sometimes stated as

$$y = \delta(x) \text{ at } t = 0$$

where $\delta(x)$ is the Dirac delta, defined to be zero if $x \neq 0$ but $\int \delta(x) f(x)\, dx = f(0)$ if the range of integration contains the origin and $f(x)$ is reasonably well behaved.

The equation C-1 is constant conformally invariant with respect to the transformation group

$$\bar{y} = ay$$
$$\bar{x} = a^{-1}x$$
$$\bar{t} = a^{-2}t$$

for which invariant coordinates are $g = y\sqrt{t}$, $\eta = x/\sqrt{t}$. The boundary conditions in terms of the invariant coordinates become

$$g = 0 \text{ at } \eta = \pm \infty,$$

and

$$\int_{-\infty}^{+\infty} g\,(\eta)\, d\eta = 1$$

while the differential equation for $g\,(\eta)$ is

$$g'' + \frac{\eta}{2} g' + \frac{g}{2} = 0.$$

The solution satisfying both the equation and boundary conditions is

$$g\,(\eta) = \frac{e^{-\eta^2/4}}{2\sqrt{\pi}} .$$

In terms of x, t and y this gives

$$y\,(x,\, t) = \frac{e^{-(x^2/4t)}}{2\sqrt{\pi t}} , \quad t \geq 0.$$

It is easily verified that this $y\,(x,\, t)$ is the desired solution by back substitution into both the original equation and the boundary conditions.

One now notes that the original differential equation is invariant with respect to arbitrary translations along the x and t axes so that another solution to the equation C-1 is

$$y_1\ (x,\ t) = \frac{e^{-(x-x_1)^2/4(t-t_1)}}{2\sqrt{\pi(t-t_1)}}\ , \qquad t \geq t_1$$

for all x_1 and t_1. y_1 does not satisfy the original boundary conditions but satisfies the boundary condition

$$y_1 = 0 \text{ at } t = t_1 \text{ and } x \neq x_1$$

and

$$\lim_{t \to t_1} \int_{-\infty}^{+\infty} y_1(x,\ t)\ dx = 1\ .$$

Now since the original equation C-1 is <u>linear</u>, any linear superposition of solutions like $y_1\ (x,t)$ is also a solution. Thus

$$y_A\ (x,\ t) = \int_{-\infty}^{+\infty} A\ (x_1)\ \frac{e^{-(x-x_1)^2/4(t-t_1)}}{2\sqrt{\pi(t-t_1)}}\ dx_1\ ,\ t \geq t_1$$

is a solution for arbitrary $A(x_1)$ (provided this integral exists).

The quantity

$$K(x-x_1,\ t-t_1) = \frac{e^{-(x-x_1)^2/4(t-t_1)}}{2\sqrt{\pi(t-t_1)}}$$

is called the kernel (or propagator) of the operator $\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ and has, as noted, the property

$$K(x-x_1,\ t-t_1) = 0 \text{ at } t = t_1,\ x \neq x_1$$

and

$$\lim_{t \to t_1} \int_{-\infty}^{+\infty} K(x-x_1,\ t-t_1)\ dx_1 = 1$$

that is,

$$K(x-x_1,\ 0) = \delta(x-x_1).$$

This allows one to solve for $A(x)$ giving

$$y_A(x,t_1) = A(x).$$

A solution to the initial value problem (that is, given $y_o(x)$ at t_o and the differential equation C-1, find $y(x,t)$ for $t \geq t_o$) then is

$$y(x,t) = \int_{-\infty}^{+\infty} y(x_o,t_o)\ \frac{e^{-(x-x_o)/4(t-t_o)}}{2\sqrt{\pi(t-t_o)}}\ dx_o \quad t \geq t_o . \qquad \text{(C-2)}$$

Equation C-2 is well-known and is derived in most elementary text books on heat flow or applied mathematics. Here the derivation of the kernel is given by use of Morgan's theorem and the transformation group of nonuniform magnifications.

By a similar technique kernels can be derived for other linear partial differential operators such as the wave operator, $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$, and the operator describing transverse vibrations in a rod*, $\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}$. In each case an ordinary linear differential equation is obtained whose solution can be used to give the kernel.

One can use the kernels further to obtain Green's functions for use in solving the inhomogeneous equation but this will not be covered here.

---

*See page 64 of Hansen [12]

D. Morgan's theorem for ordinary differential equations

In giving his proof of the basic theorems for reducing the number of independent variables by one, Morgan was careful to specify that there must be at least two independent variables. But in examining his proof it is clear that no use is made of this condition except in the terminology. This suggests that Morgan's process can be applied to ordinary differential equations and that in doing so a solution is obtained. Since the reduction by one of the number of independent variables in an ordinary differential equation gives no independent variable, the reduction is to an ordinary equation in the remaining absolute invariants.

The solutions so obtained have no arbitrary constants and are particular solutions to the system of differential equations. As such, they are not as general or as useful as the solutions obtained by the methods in Chapter II of this report. On the other hand, there are ordinary differential equations for which the methods of Chapter II do not work but for which Morgan's procedure will give particular solutions.

Example 1: In the example in the boundary layer problem, the partial differential equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(D-1a)}$$

$$\frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} - \frac{b\partial^2 u}{\partial y^2} = 0 \qquad \text{(D-1b)}$$

were reduced to the pair of ordinary differential equation

$$-\tfrac{1}{2}\,\eta\,\frac{dg_1}{d\eta} + \frac{dg_2}{d\eta} = 0 \qquad \text{(D-2a)}$$

$$-\tfrac{1}{2}\,\eta g_1\,\frac{dg_1}{d\eta} + g_2\,\frac{dg_1}{d\eta} - \frac{bd^2g_1}{d\eta^2} = 0 \qquad \text{(D-2b)}$$

by the transformations

$$\eta = y/\sqrt{x}$$

$$g_1 = u, \; g_2 = v\sqrt{x} \,.$$

The equations D-2 are invariant with respect to the transformations

$$\bar{\eta} = a^{-1}\eta$$

$$\bar{g}_1 = a^2 g_1$$

$$\bar{g}_2 = a\, g_2 .$$

Two new invariants G_1 and G_2 can be introduced as

$$G_1 = g_1\eta^2$$

$$G_2 = g_2\eta$$

where G_1 and G_2 now are independent of η, that is, constants. Substituting for g_1 and g_2 in equations D-2 gives

$$-\tfrac{1}{2}\,\eta\,(-2G_1\eta^{-3}) + (-\frac{G_2}{\eta^2}) = 0$$

and

$$-\tfrac{1}{2}\,\eta\, G_1\eta^{-2}\,(-2G_1\eta^{-3}) + G_2\eta^{-1}\,(-2\frac{G_1}{\eta^3}) - b\,(6G_1\eta^{-4}) = 0$$

or

$$G_1 - G_2 = 0 \text{ and } G_1^{\,2} - 2G_1G_2 - 6bG_1 = 0$$

or

$$G_1 = G_2 = -6b .$$

This gives the particular solutions

$$g_1 = \frac{-6b}{\eta^2}, \quad g_2 = \frac{-6b}{\eta}$$

for equation D-2 or

$$u = \frac{-6bx}{y^2}, \quad v = \frac{-6b}{y}$$

as solutions to equations D-1, which is easily verified by back substitution.

This solution is, however, of doubtful value. It contains no arbitrary constants and can therefore be made to satisfy only very special boundary conditions. It cannot, for example, satisfy the usual boundary conditions associated with the boundary layer flow over a flat plate*.

*See [12] page 12.

IV. HAMILTONIAN AND EULER-LAGRANGE EQUATIONS

Most of the ordinary and partial differential equations arising in physical problems can be derived from a variational principle. Group theory can be applied to the study of these equations and their solutions through the study of the Lie algebras and corresponding Lie groups*. This section gives a brief outline of how this connection arises.

Consider a class of partial differential equations that are derivable from the variation of an action integral. These equations, which arise in many physical problems, are referred to as Euler-Lagrange equations.

(The notation used here is the following:

x_ℓ is a set of Cartesian coordinates x_1, x_2, x_3 . . ., referred to as the spatial coordinates;

t is referred to as the temporal coordinate;

dx is the differential volume element $dx_1\, dx_2\, dx_3$ . . .;

ψ is a function of the coordinates x_ℓ and t, and will be referred to as the field variable;

the summation convention is used for repeated indices, i.e., $A_\ell B_\ell \equiv A_1B_1 + A_2B_2 + A_3B_3 + \cdots$;

definitions $\psi_{,\ell} \equiv \frac{\partial\psi}{\partial x_\ell}$ and $\dot{\psi} \equiv \frac{\partial\psi}{\partial t}$ are used.)

The integral I, called the action, is defined as

$$I = \int_{t_1}^{t_2} \int_R \mathcal{L}(x_\ell|t|\psi(x_\ell|t)|\psi_{,\ell}|\dot{\psi})\, dx\, dt ,$$

where the integration is over some region R of the spatial coordinates and the

*See Appendix V for definition of a Lie algebra and Lie group.

temporal interval t_1 to t_2. I then is a functional of ψ and the problem consists in finding ψ such that I is an extremum. By letting ψ be replaced by $\psi + Kv$, where K is in some sense small and $v(x,t)$ an arbitrary function of x and t that vanishes on the boundary of the region of integration, and setting

$$\left.\frac{dI(K)}{dK}\right|_{K=0} = 0$$

one obtains in a lengthy but straightforward manner the partial differential equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\ell}}\right)_{,\ell} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = 0 \ .$$

This is called the Euler-Lagrange equation or just the Lagrange equation. $\mathcal{L}$ will be called the Lagrangian density and $L = \int_R \mathcal{L}\, dx$, the Lagrangian. Further notations adopted here are that capital script letters will be used to indicate a density and the corresponding Latin capital will indicate the spatial integral of that density over the region R, i.e.,

$$A(t) \equiv \int_R \mathcal{A}(x_\ell | t | \ \psi(x|t) \ | \ \psi_{,\ell} | \dot{\psi} \ . \ . \ . \ ) \, dx \ ,$$

$$L \equiv \int_R \mathcal{L}\, dx \ , \text{ etc.}$$

Also define

$$\frac{\delta A}{\delta \psi} \equiv \frac{\delta \mathcal{A}}{\delta \psi} \equiv \frac{\partial \mathcal{A}}{\partial \psi} - \left(\frac{\partial \mathcal{A}}{\partial \psi_{,\ell}}\right)_{,\ell} + \left(\frac{\partial \mathcal{A}}{\partial \psi_{,\ell,m}}\right)_{,\ell,m} - \ . \ . \ . \ ;$$

this will be called the functional derivative of A (or $\mathcal{A}$). Thus Lagrange's equation can be written

$$\frac{\delta L}{\delta \psi} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) \quad \text{or} \quad \frac{\delta \mathcal{L}}{\delta \psi} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) .$$

The following definitions and terminology are introduced:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \qquad \text{(conjugate momentum)}$$

$$\mathcal{H} \equiv \pi\dot{\psi} - \mathcal{L} \qquad \text{(Hamiltonian or energy density)}$$

$$\mathcal{S}_\ell \equiv \dot{\psi}\,\frac{\partial \mathcal{L}}{\partial \psi_{,\ell}} \qquad \text{(energy flux density)}$$

$$\mathcal{G}_\ell \equiv \psi_{,\ell}\,\frac{\partial \mathcal{L}}{\partial \dot{\psi}} \qquad \text{(momentum density)}$$

$$\mathcal{T}_{\ell m} \equiv \psi_{,m}\,\frac{\partial \mathcal{L}}{\partial \psi_{,\ell}} - \delta_{\ell m}\mathcal{L} \qquad \text{(stress tensor) .}$$

While these names are suggestive of certain physical quantities, they need not in fact correspond to the usual physical concept suggested and can be considered as merely convenient conventional names.

The following relationships exist:

$$\frac{\delta H}{\delta \psi} \equiv \frac{\delta \mathcal{H}}{\delta \psi} = -\dot{\pi} \text{ and } \frac{\delta H}{\delta \pi} \equiv \frac{\delta \mathcal{H}}{\delta \pi} = \frac{\partial \mathcal{H}}{\partial \pi} = \dot{\psi} \qquad \text{(Hamilton's Equations)}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\mathcal{S}_{\ell,\ell} + \frac{\partial \mathcal{L}}{\partial (t)} \text{ and } \frac{\partial \mathcal{G}_m}{\partial t} = -\mathcal{T}_{\ell m,\ell} + \frac{\partial \mathcal{L}}{\partial (x^m)} .$$

The operation

$$\frac{\partial \mathcal{L}}{\partial (t)}$$

is meant to indicate the derivative with respect to the explicit dependence of $\mathcal{L}$ on t, i.e.,

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{L}}{\partial (t)} + \frac{\partial \mathcal{L}}{\partial \psi}\dot{\psi} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\ddot{\psi} + \frac{\partial \mathcal{L}}{\partial \psi_{,\ell}}\dot{\psi}_{,\ell}$$

and similarly for $\frac{\partial \mathcal{L}}{\partial (x_m)}$. The function $\dot{\psi}$ is considered to be a function of ψ, π, and $\psi_{,\ell}$, i.e., $\dot{\psi} = \dot{\psi}\,(\psi|\pi|\psi_{,\ell})$.

Spatial integrals of functions depending on x, t, ψ, π, and any derivatives of ψ and π will be called dynamic variables. The quantity

$$[A,\ B] \equiv \int_R \left(\frac{\delta A}{\delta \psi}\frac{\delta B}{\delta \pi} - \frac{\delta B}{\delta \psi}\frac{\delta A}{\delta \pi}\right) dx$$

is called the Poisson bracket of A and B and is also a dynamic variable. The dynamic variables form a Lie algebra under the Poisson bracket operation, and this algebra has an associated Lie group. A vast amount of literature is available, and much is known about Lie groups* which can be applied to the study of the set of dynamic variables of which $\psi(x,t)$ itself is a member.

The Lie algebra of the complete set of dynamic variables, however, is not of great use, but there are subalgebras that are of interest. For example, the conserved dynamic variables are of interest. (If $\frac{dA}{dt} = 0$ then A is said to be conserved or a constant of the motion.) It can be shown that for conserved dynamic variables A and B,

$$\frac{d}{dt}\left[A, B\right] = \left[\frac{dA}{dt}\ ,\ B\right] + \left[A,\ \frac{dB}{dt}\right] = 0$$

so that the conserved dynamic variables form subalgebras of the algebra of the total set of dynamic variables.

The present day uses of Lie groups in physical theories, especially in particle physics, however, are not for solving the differential equations,

*See reference [1] for an excellent review of Lie groups and applications.

but rather for discovering the form of the Hamiltonian when the symmetries of the system are known. Lipkin's book [18] is suggested as one of the best elementary introductions to this use of Lie groups, and the subject will not be pursued further here.

V. RESULTS AND CONCLUSIONS

A. Results

1. Introduction

The results of this investigation can be divided into two categories. The first are those theorems and methods in transformation groups useful in solving ordinary and partial differential equations which are generally known by mathematicians working in the field but not generally known or in common use by physicists, engineers, or others who are concerned with practical problems. This would include Lie's basic results and Morgan's theorems. They are quite powerful yet not well known or as widely exploited as they might be, probably because most people have so little background in group theory. This is unfortunate since the group theory needed to exploit the results of Lie and Morgan in practical problems is quite simple, considerably less than what is needed to derive their results.

The other category of results in this report are those results which appear to be original as far as can be seen from the literature survey. The first of these is the application of the one-parameter transformation groups to finding integration factors for a total differential equation. This was a simple extension of Lie's basic theorem. A further extension that appears to be original here is to systems of total differential equations. An integration factor can be found for a system of total differential equations if a sufficient number of independent invariance groups can be found for the system.

In this report it was also possible to show that Morgan's theorem for partial differential equations can also be extended "backwards" to ordinary differential equations. But in this case it is not nearly so powerful or useful, giving only particular solutions.

2. Summary of results

The basic results of group theory useful in solving differential equations then are as follows:

a) Lie's theorem. If a differential equation of the form

$$P(x,y)\ dx + Q(x,y)\ dy = 0$$

has solutions that are invariant as a family with respect to the transformation

$$U = \alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y}$$

then an integration factor is

$$\lambda = 1 / (\alpha P + \beta Q) ,$$

provided the denominator is not zero.

b) Extension of Lie's theorem to total differential equations. If a total differential equation of the form

$$\sum_{k=1}^{n} P_k\ (x_1, x_2 . . . x_n) dx_k = 0$$

has solutions that are invariant as a family with respect to the transformation

$$U = \sum_{k=1}^{n} \alpha_k (x_1, x_2 . . . x_n) \frac{\partial}{\partial x_k}$$

then an integration factor is

$$\lambda = 1 / \left(\sum_{k=1}^{n} \alpha_k P_k \right)$$

provided the sum in the denominator is not zero.

c) Extension of Lie's theorem to systems of total differential equations. If a system of total differential equations of the form

$$\sum_{k=1}^{n} P_{\gamma}^{k} (x_1, x_2 \ldots x_n)\, dx_k = 0, \quad \gamma = 1, 2 \ldots M$$

has solutions which as a family are each invariant with respect to all of the transformations

$$U^{\beta} = \sum_{k=1}^{n} \alpha_k^{\beta} (x_1, x_2 \ldots x_n) \frac{\partial}{\partial x_k} \qquad , \beta = 1, 2 \ldots M,$$

then an integration matrix is given by

$$\lambda_{\gamma}^{\beta} = \left((P\alpha^T)^{-1} \right)_{\gamma}^{\beta} \quad .$$

Here $P\alpha^T$ stands for the matrix product of P with α^T(i.e.,$\sum_k \alpha_k^{\beta} P_{\gamma}^{k}$) and $(P\alpha^T)^{-1}$ is the inverse of this matrix. λ_{γ}^{β} is an integration factor only if the matrix $P\alpha^T$ has an inverse.

d) Morgan's theorem. If each of a set of partial differential equations of the form

$$\Phi_{\gamma} \left(x_1, \ldots x_m \ldots y_1, \ldots y_n, \ldots \frac{\partial^k y_1}{\partial (x_1)^k} \ldots \frac{\partial^k y_n}{\partial (x_m)^k} \right) = 0, \ \gamma = 1, 2 \ldots$$

is conformally invariant with respect to some one-parameter group of continuous transformations then the set of equations can be reduced to a new set of the form

$$A_{\gamma} \left(\eta_1, \ldots \eta_{m-1}, F_1 \ldots F_n, \ldots, \frac{\partial^k F_1}{\partial (\eta_1)^k} \ldots \frac{\partial F_n}{\partial (\eta_{m-1})^k} \right) = 0$$

where the η's and F's are the invariants of the transformation group. Note this reduces by one the number of independent variables in the system of partial differential equations.

f) Morgan's theorem for ordinary differential equations. Morgan's theorem can be applied even when there is only one independent variable. In this case it reduces a system of ordinary differential equations to ordinary equations in the invariant F's.

B. Conclusions

Some conclusions can be drawn from a practical application of the above techniques for solving differential equations.

Lie's original theorem for partial differential equations and its extension to total differential equations is quite effective and practical in solving ordinary and total differential equations. But it depends on finding an invariance group without giving a straightforward prescription as to how to look for such a group. Thus it is a trial and error method that depends for its effectiveness on the skill of the user and is not a straight forward prescription for solving all equations.

Lie's theorem extended to systems of total differential equations is much more difficult to use in practice. It requires finding many invariance groups such that each and every one of the total differential equations is invariant with respect to every group. Furthermore, the groups must be independent. In the present form this theorem is of doubtful practical use.

Morgan's theorem is extremely powerful and useful in practice. Its use however has the same drawback as Lie's theorem, namely that an invariance group must be found and the theorems give no hint about how to search for such groups. But a skilled user can often use physical reasoning to great advantage in practical problems and quickly discover the needed group.

A more serious difficulty with Morgan's results is that no considerations of the boundary conditions enter the theorems. Thus while one may discover an invariance group for the differential equations, this transformation may not be compatible with the boundary conditions and thus be of no use. Since the boundary conditions are different for different problems, each individual problem must be attacked separately.

C. Recommendations for further study

The treatment of systems of total differential equations by group theory outlined in this report appears to be useful. However, modification of this technique may be possible and desirable to make it more flexible. In the present form, if there are M equations then M different invariance groups are needed to find an integration factor (matrix), and this is a rather stringent requirement. Hopefully further investigation could show that these conditions could be relaxed to a single invariance group.

A second extension that would be useful is to extend the technique used for systems of equations to partial differential equations. Such an extension would not be easy but ought to be possible at least in principle since a partial differential equation can be viewed as a continuously infinite system of ordinary differential equations.

Another extension would be to try to apply the method for systems of differential equations to the discrete approximation of a partial differential equation. This would give only approximate solutions to the partial differential equation. But these approximations could be arbitrarily close to the exact solutions, or exact solutions might be obtained by a limiting process. The limiting process might also be effective in obtaining the needed theorems mentioned in the previous paragraph.

Respectfully submitted,

L. J. Gallaher
Project Director

APPENDIX I

Definition and Examples of a Group

A set of elements are said to form a group under an associative operation (called product) if the following conditions are satisfied:

1. The product of any two elements in the group is in the group.

2. There is a unique identity element in the group such that its product with every element leaves that element unchanged.

3. A unique inverse of every element is in the group such that the product of the element with its inverse is the identity.

Some examples of groups are the following:

The numbers +1 and -1 form a two element group under multiplication.

The positive and negative integers with zero (as the identity) form a group under the operation of addition.

The positive rational numbers form a group under multiplication with 1 as the identity.

The complex numbers of unit magnitude form a group under multiplication.

The real numbers form a group under addition.

The set of all one-to-one transformations on any space is a group.

APPENDIX II

Lie's Theorem for Total Differential Equations

In this appendix it is shown that if the solution of a total differential equation is invariant as a family under a one-parameter group of transformations, an integrating factor can be given.

Consider the total differential equation in n variables ($n \geq 2$):

$$P_1 dx_1 + dx_2 + \cdots P_n dx_n \equiv \sum_{k=1}^{m} P_k\ (x_1,\ x_2\ \ldots\ )\ dx_k = 0 \qquad \text{A II-1}$$

and let

$$\phi\ (x_1,\ x_2,\ \ldots\ ) = c\ \text{(a constant)}$$

be the family of solutions. That is,

$$\frac{\partial\phi}{\partial x_k} \propto P_k \qquad \text{or}$$

$$\frac{\partial\phi}{\partial x_k} = \lambda(x_1,\ x_2,\ \ldots\ )\ P_k\,, \qquad k = 1,\ 2,\ \ldots\ n.$$

where λ is independent of k. Then ϕ is a solution to the set of partial differential equations

$$\frac{1}{P_k}\frac{\partial f}{\partial x_k} - \frac{1}{P_{k+1}}\frac{\partial f}{\partial x_{k+1}} = 0\,, \qquad k = 1,\ 2\ \ldots\ n-1$$

(provided none of the P_k are identically zero).

Assume that as a family, $\phi = c$ is invariant under the group U

$$U \equiv \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \cdots \equiv \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k}$$

(the α's can be functions of the x's), so that

$$U\phi = g(\phi).$$

Let

$$\Phi \equiv \int \frac{d\phi}{g(\phi)}$$

so that $\Phi = C$ is identical with the family $\phi = c$. Then

$$U\Phi = U\phi \frac{d\Phi}{d\phi} = 1$$

and Φ is a solution of the partial differential equations

$$\frac{1}{P_k} \frac{\partial \Phi}{\partial x_k} - \frac{1}{P_{k+1}} \frac{\partial \Phi}{\partial x_{k+1}} = 0 \qquad k = 1, 2 \ldots n-1$$

and

$$\sum_k \alpha_k \frac{\partial \Phi}{\partial x_k} = 1$$

This system of n linear equations can be solved and gives

$$\frac{\partial \Phi}{\partial x_k} = \frac{P_k}{\sum_s \alpha_s P_s}$$

(provided the denominator is not identically zero). Then $d\Phi$ is a perfect differential and

$$d\Phi = \sum_k \frac{\partial \Phi}{\partial x_k} dx_k$$

$$= \sum_k \frac{P_k dx_k}{\sum_s \alpha_s P_s} .$$

Thus if $\sum_s \alpha_s P_s$ is not identically zero,

$$\lambda = \frac{1}{\sum_s \alpha_s P_s}$$

is an integrating factor of the total differential equation

$$\sum_k P_k dx_k = 0$$

and

$$\int \frac{\sum_k P_k dx_k}{\sum_s \alpha_s P_s} = K$$

is the quadrature solution, where K is a constant, and the path of integration in x_k space can be chosen for convenience.

An alternate point of view to the above reasoning can also be given. The equation A II-1 is **invariant** under the group U, if it preserves its form under an infinitesimal transformation. That is, if

$$\sum_k P_k (\bar{x}_1, \bar{x}_2 \ldots) d\bar{x}_k = 0 \qquad \text{A II-2}$$

where

$$\bar{x}_k = x_k + \alpha_k \delta t;$$

δt is a small parameter. To first order in δt we have

$$\sum_k P_k (x_1, x_2 \dots) dx_k =$$

$$\sum_k \left(P_k(\bar{x}) + \sum_s \frac{\partial P_k(\bar{x})}{\partial \bar{x}_s} \bar{\alpha}_s \; \delta t\right) \left(d\bar{x}_k + \sum_s \frac{\partial \bar{\alpha}_k}{\partial x_s} dx_s \delta t\right) =$$

$$\sum_k \left(P_k(\bar{x}) + \delta t \sum_s \left(\frac{\partial P_k}{\partial \bar{x}_s} \bar{\alpha}_s + \frac{\partial \bar{\alpha}_s}{\partial \bar{x}_k} P_s\right)\right) dx_k$$

(here $\bar{\alpha}_k \; (\bar{x}) = -\alpha_k(x)$ to lowest order in δt).

If A II-2 is to hold, then

$$\sum_s \left(\frac{\partial P_k}{\partial \bar{x}_s} \bar{\alpha}_s + \frac{\partial \bar{\alpha}_s}{\partial \bar{x}_k} P_s\right) = w \; (\bar{x}) \; P_k(\bar{x}) \qquad \text{A II-3}$$

where $w(\bar{x})$ does not depend on the index k but may depend on the $\bar{x}$'s. If A II-3 holds, then

$$P_m \sum_s \left(\frac{\partial P_k}{\partial \bar{x}_s} \bar{\alpha}_s + \frac{\partial \bar{\alpha}_s}{\partial \bar{x}_k} P_s\right) - P_k \sum_s \left(\frac{\partial P_m}{\partial \bar{x}_s} \bar{\alpha}_s + \frac{\partial \bar{\alpha}_s}{\partial \bar{x}_m} P_s\right) = 0$$

or (dropping the bars for convenience)

$$\sum_s \alpha_s \left(P_m \frac{\partial P_k}{\partial x_s} - P_k \frac{\partial}{\partial x_s} P_m\right) + \sum_s P_s \left(P_m \frac{\partial}{\partial x_k} - P_k \frac{\partial}{\partial x_m}\right) \alpha_s = 0$$

or

$$\sum_s \alpha_s \left(P_m \frac{\partial P_k}{\partial x_s} - P_m \frac{\partial P_m}{\partial x_s} - \frac{P_m \partial P_s}{\partial x_k} + P_k \frac{\partial P_s}{\partial x_m}\right)$$

$$- \left(P_k \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_k}\right) \sum_s P_s \alpha_s = 0 \qquad \text{A II-4}$$

Now in Appendix III it is shown that a necessary condition for the existance of an integrating factor for A II-1 is that

$$\text{Anti}_{[smk]}\, P_s \frac{\partial}{\partial x_m} P_k = 0 \quad .$$

Thus if an integrating factor does exist, A II-4 can be written

$$\sum_s \alpha_s \left(P_s \frac{\partial}{\partial x_m} P_k - P_s \frac{\partial}{\partial x_k} P_m\right) - \left(P_k \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_k}\right) \sum_s P_s \alpha_s = 0 .$$

Provided $\sum_s P_s \alpha_s$ is not identically zero this gives

$$\frac{\partial}{\partial x_m}\left(\frac{P_k}{\sum_s P_s \alpha_s}\right) - \frac{\partial}{\partial x_k}\left(\frac{P_m}{\sum_s P_s \alpha_s}\right) = 0 \quad . \qquad \text{A II-5}$$

Now expression A II-5 states that the n dimensional "curl" of the vector $P_k/\sum_s P_s \alpha_s$ is zero, and it is a well-known theorem that, if the curl of the vector is zero, this is a necessary and sufficient condition that the vector be expressable as the gradient of some scalar function. Thus it is shown that

$$\frac{P_k}{\sum_s \alpha_s P_s} = \frac{\partial \phi}{\partial x_k}$$

for some ϕ or that $1/\sum_s \alpha_s P_s$ is an integrating factor for the original equation A II-1.

This second form of the proof is informative since it is more closely related to the test for invariance actually used in solving practical problems.

APPENDIX III

The Necessary Conditions for the Existence of an Integration Factor

It is shown here that in integration factor to the differential equation

$$\sum_{k=1}^{n} P_k dx_k = 0 \qquad \text{A III-1}$$

exists only if

$$\text{Anti}_{[kms]}\ P_k \frac{\partial P_s}{\partial x_m} = 0. \qquad \text{A III-2}$$

(Here the notation $\text{Anti}_{[kms]}$ means that the term following is to be anti-symmetric in the indices k, m and s.)

Assume then that there exists a λ ($\neq 0$), such that

$$\frac{\partial}{\partial x_m} \lambda P_k - \frac{\partial}{\partial x_k} \lambda P_m = 0 \qquad \text{A III-3}$$

then

$$P_k \frac{\partial}{\partial x_m} \lambda + \frac{\lambda \partial P_k}{\partial x_m} - P_m \frac{\partial \lambda}{\partial x_k} - \lambda \frac{\partial P_m}{\partial x_k} = 0$$

or

$$\left(P_k \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_k}\right) \ln \lambda + \frac{\partial P_k}{\partial x_m} - \frac{\partial P_m}{\partial x_k} = 0.$$

Multiply this by P_s and antisymmetrize with respect to s and k, and s and m, to give:

$$P_s \left(P_k \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_k}\right) \ln \lambda + P_s \left(\frac{\partial}{\partial x_m} P_k - \frac{\partial P_m}{\partial x_k}\right)$$

$$- P_k \left(P_s \frac{\partial}{\partial x_m} - P_m \frac{\partial}{\partial x_s}\right) \ln \lambda - P_k \left(\frac{\partial P_s}{\partial x_m} - \frac{\partial P_m}{\partial x_s}\right)$$

$$- P_m \left(P_k \frac{\partial}{\partial x_s} - P_s \frac{\partial}{\partial x_k}\right) \ln \lambda - P_m \left(\frac{\partial P_k}{\partial x_s} - \frac{\partial P_s}{\partial x_k}\right) = 0$$

The operator operating on $\ln \lambda$ is identically zero. The remaining terms are $\mathrm{Anti}_{[smk]} P_k \frac{\partial P_s}{\partial x_m}$. Thus if an integrating factor for A III-1 exists A III-2 must hold.

We note here that A III-2 is a necessary condition for the existence of an integrating factor; it is also a sufficient condition, but that was not demonstrated here.

APPENDIX IV

Lie's Theorem for Systems of Total Differential Equations

In this appendix it is shown how one-parameter transformation groups are used to find integrating factors for systems of total differential equations. This is Lie's theorem extended to systems of total differential equations.

Consider the system of M total differential equations in n variables ($n \geq M \geq 2$).

$$\sum_{k=1}^{n} P_{\gamma}^{k}(x_1, x_2, \ldots)\, dx_k = 0 \ , \ \gamma = 1,2 \ldots M. \qquad \text{A IV-1}$$

The convention will be used here that the Latin indices run from 1 to n and the Greek indices run from 1 to M. Let

$$\phi_{\rho}(x_1, x_2, \ldots) = c_{\rho} \text{ (constants)}, \ \rho = 1, 2, \ldots M$$

be the family of solutions to A IV-1. That is

$$\frac{\partial \phi_{\rho}}{\partial x_k} = \sum_{\beta} \lambda_{\rho}^{\beta} \ P_{\beta}^{k} \qquad \text{A IV-2}$$

where the λ_{ρ}^{β} may be functions of the x's but are independent of the index k. (λ is an integration factor and will be an M-by-M matrix here.)

Assume that as a family $\phi_{\rho} = c_{\rho}$ are invariant under the groups U^{γ}

$$U^{\gamma} \equiv \sum_{k=1}^{n} \alpha_{k}^{\gamma}(x_1, x_2, \ldots) \frac{\partial}{\partial x_k} \qquad , \ \gamma = 1, 2 \ldots M$$

so that

$$U^{\gamma}\phi_{\rho} = g^{\gamma}_{\rho}(\phi_{\rho}). \qquad \text{A IV-3}$$

Introduce Φ_{η} defined as

$$\Phi_{\eta} \equiv \sum_{\beta} \int (g^{-1})^{\beta}_{\eta}\, d\phi_{\beta} \qquad \text{A IV-4}$$

so that $\Phi_{\eta} = C_{\eta}$ (a constant) is identical with the family, $\phi_{\rho} = c_{\rho}$. The notation here is that $(g^{-1})^{\beta}_{\eta}$ is the β, η component of the inverse of the matrix g^{β}_{η} (it being assumed here that the inverse of g exists).

Then

$$U^{\gamma}\Phi_{\eta} = \sum_{\beta} U^{\gamma}\phi_{\beta} \frac{\partial \Phi_{\eta}}{\partial \phi_{\beta}} = \sum_{\beta} g^{\gamma}_{\beta} (g^{-1})^{\beta}_{\eta} = \delta^{\gamma}_{\eta} \quad , \qquad \text{A IV-5}$$

where $\delta^{\gamma}_{\eta} = 1$ if $\gamma = \eta$, or 0 if $\gamma \neq \eta$.

We note also that

$$\frac{\partial \Phi_{\eta}}{\partial x_k} = \sum_{\beta} \frac{\partial \phi_{\beta}}{\partial x_k} \frac{\partial \Phi_{\eta}}{\partial \phi_{\beta}}$$

$$= \sum_{\beta\gamma} (g^{-1})^{\beta}_{\eta}\, \lambda^{\gamma}_{\beta}\, P^{k}_{\gamma} \qquad \text{A IV-6}$$

$$= \sum_{\gamma} \Lambda^{\gamma}_{\eta}\, P^{k}_{\gamma}$$

where $\Lambda^{\gamma}_{\eta} \equiv \sum_{\beta} (g^{-1})^{\beta}_{\eta} \lambda^{\gamma}_{\beta}$. ($\Lambda$ may be a function of the x's but does not depend on the index k.)

Thus if

$$d\Phi_\eta = \sum_k \frac{\partial \Phi_\eta}{\partial x_k} dx_k$$

A IV -7

$$= \sum_{\gamma k} \Lambda_\eta^\gamma P_\gamma^k dx_k$$

is to be a perfect differential, Λ_η^γ is also an integration factor for the original equation A IV-1.

But in order that A IV-5 (equivalent to A III-3) be satisfied it is necessary that

$$\sum_{\gamma k} \Lambda_\eta^\gamma P_\gamma^k \alpha_k^\beta = \delta_\eta^\beta \qquad \text{A IV-8}$$

or that $\Lambda_\eta^\gamma = \left((P\alpha^T)^{-1}\right)_\eta^\gamma$. Here $(P\alpha^T)^{-1}$ is the inverse of the matrix product $P\alpha^T$; that is, $\left((P\alpha^T)^{-1}\right)_\eta^\gamma$ is the γ, η component of the inverse of $\sum_k P_\eta^k \alpha_k^\gamma$. (Again one makes the assumption that the needed inverse does in fact exist.)

The quadrature solution of the system A IV-1 is then given by

$$\int \sum_{k\gamma} \left((P\alpha^T)^{-1}\right)_\eta^\gamma P_\gamma^k dx_k = K_\eta \qquad \text{A IV-9}$$

where the K_η are the M arbitrary constants of the system and the path of integration is chosen for convenience.

APPENDIX V

Definition of Lie Algebras and Lie Groups

A Lie algebra and corresponding Lie group are defined as follows. A set of vectors a, b, c . . . is said to form a Lie algebra under an operation (denoted by [,]) if the following conditions are satisfied:

1) the result of the operation [a,b] is a member of the set for all a and b in the set (closed).

2) $[a + c, b] = [a,b] + [c, b]$ (linearity).

3) $[a, b] + [b, a] = 0$ (antisymmetric).

4) $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ (Jacobi identity).

(The entities a, b, c . . are vectors in some vector space in the sense that multiplication by a constant and vector addition are defined in the usual way. A norm may or may not be defined.)

Associated with every Lie algebra will be a Lie group. A group is obtained by putting the elements of the algebra a, b, c . . . in a one-to-one correspondence with a set of operators A, B, C . . . such that for all a, b, c, A, B, C, if $c = [a,b]$ then $C = [A, B]$ where $[A,B] \equiv AB - BA$ is the commutator of A and B. The operators are to have quantities, ψ, to operate on, and AB means operate first with B and then with A. The operators e^A, e^B, e^C . . . are then transformations on the ψ that form a corresponding Lie group, where $e^A \equiv I + A + AA/2! + AAA/3! + \ldots$ and I is the identity operator.

Since a Lie algebra or group is defined for a set of vectors, it will have a set of basis vectors, x_m, such that any member of the set can be given as a linear combination of the x_m. (The index m may take on discrete values, either finite or infinite, or a continuous set of values, or have both a discrete range and a continuous range of values.) In terms of the basis vectors, condition (1) above can then be replaced by the condition

1') $$[x_k, x_j] = \int_m C^m_{kj}\, x_m$$

where the C^m_{kj} are constants and $\int_m$ indicates a sum over the discrete plus an integral over the continuous range of m. These C^m_{kj} are the structure constants of the Lie algebra or group and completely define it.

The most familiar example of a Lie algebra is formed by the vector cross product operation in three dimensional space. Here we define the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, and their cross products, so that

$$[\hat{i},\hat{j}] = \hat{k}$$

$$[\hat{j},\hat{k}] = \hat{i}$$

$$[\hat{k},\hat{i}] = \hat{j}$$

The cross product is also antisymmetric and satisfies the Jacobi identity so this system is the basis of a Lie algebra. The corresponding Lie group is formed by putting the rotation operators in a one-to-one correspondence with $\hat{i}$, $\hat{j}$, and $\hat{k}$ and then letting the commutator correspond to the cross product

$$\hat{i} \leftrightarrow L_x \equiv y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$\hat{j} \leftrightarrow L_y \equiv z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\hat{k} \leftrightarrow L_z \equiv x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} .$$

Then

$$[L_x, L_y] \equiv (L_x L_y - L_y L_x) = L_z \qquad \text{etc.}$$

The transformation operators e^{L_x}, e^{L_y}, e^{L_z} form the basis of the corresponding Lie group. This is the proper orthogonal or rotation group in three dimensions, usually called for short SO(3).

APPENDIX VI

A Discussion of Morgan's Theorems for Systems of Ordinary Differential Equations

In this appendix, the outline of a proof of Morgan's theorems for ordinary differential equations is given. The notation used here is the same as that in Morgan's paper [20] to which the reader is referred.

If G_1 is a continuous one-parameter group of transformations of the independent variable x and the dependent variables $y_1, \ldots, y_n$ of a system of differential equations of the form $\Phi_\delta = 0$, $\delta = 1, \ldots, n$, then a transformation in G_1 is of the form

$$f: \begin{cases} \bar{x} = f_0\,(x;a) \\ \bar{y}_\delta = f_\delta\,(y_\delta;a), \quad \delta = 1, \ldots, n \end{cases}$$

where a is a numerical parameter and the transformations $x \rightarrow \bar{x}$ form a subgroup S_{G_1} of G_1. Let $G_1^E k$ denote the enlargement of G_1 formed by adding successively to G_1 the transformations among the first, second,. . ., and k-th derivatives of the y_δ.

Now consider x, $y_1, \ldots, y_n$ to be independent variables. As Morgan indicated, G_1 has n functionally independent absolute invariants $g_1(y_1, \ldots, y_n, x), \ldots, g_n\,(y_1, \ldots, y_n, x)$ such that $\dfrac{\partial(g_1, \ldots, g_n)}{\partial(y_1, \ldots, y_n)} \neq 0$.

Now, consider the y_δ and $\bar{y}_\delta$ to be implicitly defined as functions of x and $\bar{x}$, respectively, by the equations

$$Z_\delta(x) = g_\delta\,(y_1, \ldots, y_n, x)$$

and

$$Z_\delta(\bar{x}) = g_\delta\,(\bar{y}_1, \ldots, \bar{y}_n, \bar{x})$$

where the g_δ are the absolute invariants of G_1.

Then, with m = 1, Morgan's Th 1 takes the form:

Th 1: A necessary and sufficient condition for the y_δ, implicitly defined as functions of x by the equations $Z_\delta(x) = g_\delta\ (y_1, \ldots, y_n, x)$ to be exactly the same functions of x as the $\bar{y}_\delta$, implicitly defined as functions of $\bar{x}$ by $Z_\delta(\bar{x}) = g_\delta(\bar{y}_1, \ldots, \bar{y}_n, \bar{x})$, are of $\bar{x}$ is that

$$Z(x) = \bar{Z}\ (\bar{x}) = Z(\bar{x})\ ,$$

or, equivalently, that Z is a constant function.

The proof is analogous to that given by Morgan.

Then, considering x and the y_δ to be the independent variable and the dependent variables, respectively, of a system of differential equations, we define:

Def 1: By invariant solutions of a system of differential equations is meant that class of solutions of a system of differential equations which have the property that the y_δ are exactly the same functions of x as the $\bar{y}_\delta$ are of $\bar{x}$.

Theorem 1 makes it possible to reduce the problem of finding invariant solutions of a system of differential equations to one of finding solutions which satisfy relations of the form

$$Z_\delta(x) = g_\delta\ (y_1, \ldots, y_n, x)$$

where

$$Z_\delta(x) \text{ is constant.}$$

Then, since the conditions of the implicit function theorem are satisfied, the y_δ may be written in terms of x and these constants.

In the case where there is one independent variable, Morgan's Def 2 takes the form:

Def 2: By a differential form of the k-th order is meant a function of the form,

$$\Phi(x, y_1, \ldots y_n, \frac{dy_1}{dx}, \ldots, \frac{dy_n}{dx}, \ldots, \frac{d^k y_1}{dx^k}, \ldots, \frac{d^k y_n}{dx^k})$$

whose arguments are the independent variable x, the functions $y_1, \ldots, y_n$ dependent on x and the derivatives of the y_δ up to the k-th order.

If each of these arguments transforms under the transformation laws of a continuous one-parameter group with symbol V and numerical parameter a, then the arguments may be considered as independent variables of the group with symbol V and called $Z_1, Z_2, \ldots, Z_p$, where $p = (k + 1)n + 1$.

Def 3: A differential form Φ will be said to be conformally invariant under a one-parameter group G_1 if, under the transformations of the group, it satisfies the relation.

$\Phi(\bar{Z}_1, \ldots, \bar{Z}_p) = F(Z_1, \ldots, Z_p; a)\ \Phi(Z_1, \ldots, Z_p)$, where Φ is exactly the same function of the Z's as it is of the $\bar{Z}$'s and F is some function of the x's and the parameter a.

If Φ satisfies the above relation with F a function of a only, Φ is said to be constant conformally invariant; if the relation is satisfied with F identically equal to one, then Φ is said to be absolutely invariant.

Th 2: If Φ is a differential form of the k-th order and is at least in class $C^{(1)}$ with respect to each of its arguments, then a necessary and sufficient condition for Φ to be conformally invariant under a one-parameter group of transformations with symbol V is that

$$V\Phi = \omega(Z_1, \ldots, Z_p)\ \Phi(Z_1, \ldots, Z_p),$$

for some $\omega(Z_1, \ldots, Z_p)$, or equivalently, that

$$\Phi(Z_1, \ldots, Z_p) = e^{\zeta(Z_1, \ldots, Z_p)}$$

$$\Phi_o(Z_1, \ldots, Z_p),$$

where Φ_0 is a general absolute invariant of V and $\zeta(Z_1, \ldots, Z_p)$ is a determinable function of $Z_1, \ldots, Z_p$.

The proof of Th 2 is as indicated by Morgan.

Def 4: It is said that a system of k-th order differential equations $\Phi_\delta = 0$ is invariant under a continuous one-parameter group of transformations G_1 if each of the differential forms Φ_δ is conformally invariant under the transformations of $G_1^{E_k}$.

Th 3: If each of the k-th order differential equations $\Phi_1, \ldots, \Phi_n$ in a system of differential equations, with independent variable x and dependent variables $y_1, \ldots, y_n$, is conformally invariant under the k-th enlargement of a continuous one-parameter group G_1 of transformations, then the invariant solutions of the system can be expressed in terms of x and the constants Z where $Z_\delta(x) = g_\delta(y_1, \ldots, y_n, k)$, the g_δ being functionally independent absolute invariants of G_1 (considering x, $y_1, \ldots, y_n$ as the independent variables).

The proof follows directly from theorem 1 and definition 1.

VI. REFERENCES AND BIBLIOGRAPHY

1. Belinfante, J. G., B. Kolman and H. A. Smith, "An Introduction to Lie Groups and Lie Algebras with Applications," SIAM Review 8 (January, 1966) 11.

2. Birkhoff, G., Hydrodynamics, A Study in Logic, Fact, and Similitude, Princeton University Press, Princeton, N. J., 1960.

3. Birkhoff, G., "Dimensional Analysis of Partial Differential Equations," Electrical Engineering, 67 (1948) 1185.

4. Brand, L., "Total Differential Equations in the Light of Dimensional Analysis," American Mathematics Monthly, 69 (September 1962) 618.

5. Chevallez, C., Theory of Lie Groups I, Princeton University Press, Princeton (1946).

6. Ehrenfest-Afanassjewa, Mrs. T., "Dimensionsbegriff und Bau Phsikalisher Geichungen," Math. Annalen 77 (1915) 259.

7. Ehrenfest-Afanassjewa, Mrs. T., "Dimensional Analysis . . . Theory of Similitudes," Phil. Mag. 1 (1926) 257.

8. Eisenhart, L. P., Continuous Groups of Transformations, Princeton (1933).

9. Ford, L. R., Differential Equations, McGraw-Hill Book Company, Inc., New York, N.Y. (1933).

10. Goursat, E., Differential Equations, Ginn and Company, Boston, Mass.(1917).

11. Gukhman, A. A., Introduction to the Theory of Similarity, Academic Press, New York (1965).

12. Hansen, A. G., Similarity Analysis of Boundary Value Problems in Engineering, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1964).

13. Hermann, R., Lie Groups for Physicists, W. A. Benjamin, Inc., New York (1966).

14. Higman, B., Applied Group-Theoretic and Matrix Methods, Oxford University Press, London (1955).

15. Ince, E. L., Ordinary Differential Equations, Dover Publications (1956).

16a. Klein and Lie, Math. Ann. 4 (1871) 80.

16b. Lie, S., Forhand Vid.-Selsk. Christiania, (1875) 1; Math. Ann. 9 (1876) 245; 11 (1877) 464; 24 (1884) 537; 25 (1885) 71.

(Continued)

17. Littlewood, D. E., The Theory of Group Characters, Oxford University Press, London (1940).

18. Lipkin, H. J., "Lie Groups for Pedestrians," North-Holland Publishing Co., Amsterdam (1965).

19. Montgomery, D., Topological Transformation Groups, Interscience Publishers, Inc., New York (1955).

20. Morgan, A. J. A., "The Reduction by One of the Number of Independent Variables in Some Systems of Partial Differential Equations," Quart. J. Math. Oxford (2), 3 (1952) 250.

21. Murnaghan, F. D., The Theory of Group Representations, The Johns Hopkins Press, Baltimore, Md. (1938).

22. Pontrjagin, L., Topological Groups, Princeton University Press, Princeton (19).

23. Tolman, R. C., "The Principle of Similitude," Physical Review 3 (1914) 244.

24. Weyl, H., The Classical Groups, Their Invariants and Representations, Princeton University Press, Princeton (19).

25. Wigner, E. P., Group Theory, Academic Press, New York (1959).