DOUGLAS REPORT DAC-59196

FINAL REPORT
APPLICATION OF TRANSMISSION CONCEPTS TO FLEXIBLE LAUNCH VEHICLE DYNAMICS

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Prepared for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
4 January 1967

CONTRACT NAS 8-20292

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PREFACE

This report was prepared by the Douglas Aircraft Company, Inc., Missile and Space Systems Division, under NASA contract NAS 8-20292. This study, performed for the Marshall Space Flight Center, Huntsville, Alabama, is an investigation of bending vibration for flexible aerospace vehicles in terms of distributed-parameter or wave-transmission concepts. The Study Director at NASA-MSFC was Dr. George McDonough, R-AERO-D.

The report covers the work period from 4 January 1966 through 4 January 1967. It is submitted to fulfill this contract and is catalogued by Douglas as Report No. DAC-59196.

At Douglas, the study was administered by R. W. Hallet, Jr., and (later) J. L. Waisman, Director of Research and Development, Missile and Space Systems Division. C. C. Ward was the Study Director and W. C. Nowak was the Principal Investigator. Dr. D. R. Vaughan acted as Program Consultant.

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ACKNOWLEDGEMENT

The author acknowledges with appreciation the work of C. C. Ward, Program Director and of D. R. Vaughan, Program Consultant who contributed substantially to the study and this report.

ABSTRACT

The report describes a study of the analysis of bending vibration in flexible aerospace vehicles in terms of distributed-parameter or wave-transmission concepts, rather than in terms of the normal-mode approach. A relationship is developed between the vibration equation of motion and the propagating-wave equation. Transmission models and matrixes are developed for uniform beam segments, including the effects of shear compliance and rotary inertia; axial loading; longitudinal displacement because of axial loading; and distributed, uniform, lateral loading. Through use of a transformation technique, the local state variables are related to the characteristic variables of the beam permitting factorization of the solution into propagation and end-effect matrixes. The technique is applied to a non-uniform beam approximated by a cascaded or step beam structure combined with a damped spring-mass sloshing model. The advantages and disadvantages of the distributed-parameter/transmission matrix approach are investigated through a few examples and the use of this approach for the dynamic analysis of aerospace vehicles is evaluated.

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CONTENTS

FIGURES

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Symbols

A Cross-section area of beam, in.^{2} Also parameter state matrix (4×4).

A_i Coefficients in the general solution.

A_s AK.

a $\sqrt{EI/\rho A}$, in.^{2}/sec.

B Matrix that carries P to P* by the relation $P^* = BPB^{-1}$, (4×4).

B_i Coefficients in the general solution.

C_i Constants for the exponential form of the transmission matrix.

c Damping for sloshing model, lb-sec/in.

D (4×1) disturbance array.

E Young's modulus of elasticity, lb/in.^{2}

e Exponential symbol.

G Shear modulus of elasticity, lb/in.^{2}

I (4×4) identity matrix; also area moment of inertia of beam cross-section, in.^{4}

j $\sqrt{-1}$

K Shape factor for shear.

K_j Terms of general A matrix, $j = 1, \ldots, 5$.

k Spring constant for sloshing model, lb/in.

L Left, lateral end-effect matrix for beam model.

l, ℓ Length of finite beam, in. Also longitudinal displacement when dl/dx.

M External moment, in.-lb.

M_o Bulk fluid mass, lb-sec^2/in.

M_1 First sloshing mass, lb-sec^2/in.

Symbols (Continued)

m	M/EI.
P	Diagonal Eigenvalue matrix 4 x 4; axial force, lb.
P*	Matrix of real variables, diagonal in the 2 x 2 partitioned sense for thin, uniform beam.
Pr	Propagation matrix, 2 x 2.
Q	External Transverse force, lb.
q	Q/EI.
R	Right, lateral end-effect matrix for beam model.
r	radius of gyration.
S,s	Laplace transform operator, sec^{-1}.
T	Transmission matrix, 4 x 4.
T_{ij}	2 x 2 partition of the T matrix.
t	Time (sec).
U	Eigenvector matrix, 4 x 4.
U*	UB^{-1}, 4 x 4.
V	$(u^-, v^-, v^+, u^+)^T$, set of characteristic variables for transverse bending vibration.
w	External uniform, constant lateral loading, lb/in.
x	Location coordinate along beam, in.
Y	Column vector defining either, $\dot{y}$, $\dot{\theta}$, M, Q or y, θ, m, q at a point.
y	Transverse beam displacement, in.
$\dot{y}$	Transverse beam velocity, in./sec.
α, β, γ, δ, ζ, ϕ, ψ	Terms of the parameter state matrix and associated Eigenvalues varying from case to case and defined within the text.

θ Angle of rotation, radians.

$\dot{\theta}$ Angular velocity, radians/sec.

λ_i Eigenvalues.

ρ Density of beam material, lb-sec^2/in.^{4}

Subscripts:

aL Value at left face of beam a.

aR Value at right face of beam a.

aR,aL Denotes transfer or transmission matrix from point aR to aL.

Superscripts:

T Denotes transpose.

-1 Denotes inverse.

(-) Denotes travel in negative x coordinate.

(+) Denotes travel in positive x coordinate.

Operations:

$\mathcal{L}(\ )$ Laplace transform of.

$\mathcal{L}^{-1}(\ )$ Inverse Laplace transform of.

$(\dot{\ })$ Denotes d( )/dt.

Section 1
INTRODUCTION

The transverse vibration analysis of flexible structures has been treated mainly through a lumped parameter approximation whereby the continuum is replaced by a finite degree of freedom system composed of lumped elements. This analysis method was first applied by Lagrange (Reference 1) and Rayleigh (Reference 2) in studying the vibrating string. More recently, dynamics and controls analysis of flexible vehicles in terms of the natural modes has been studied by Bisplinghoff (Reference 3) and others (References 4 and 5).

Transmission matrixes, which have been applied only recently to vibration analyses, can yield either an exact or approximate approach for describing continuous systems. The first use of matrixes was for describing four terminal electrical networks. Later, this was applied to acoustical, mechanical, and electromechanical vibration problems (Reference 6). Treatment of bending vibrations through matrix methods for various structures has been notably accomplished by Pestel and Leckie (Reference 7), Brown (Reference 8), and Pipes (Reference 9).

Vaughan (Reference 10) treated bending vibrations through an analogy developed between propagation and reflection in the Bernoulli-Euler and the wave equations. Though restricted to uniform, thin beams, it provided a novel analysis method. Essentially, this study applies this method to beams with secondary dynamic effects, external loadings, slosh dynamics, and nonuniform structures. Transmission matrixes and models are developed for the various cases.

1.1 SCOPE OF WORK

This report summarizes the work performed under the National Aeronautics and Space Administration Contract NAS 8-20292, Application of Transmission Concepts to Flexible Launch Vehicle Dynamics. Fundamental objectives of this study were:

1. Extension of the transmission matrix concept to simple nonuniform structures composed of uniform thin beam segments (Bernoulli-Euler beam).

2. Investigation of methods of developing a transmission matrix for a beam with secondary dynamic effects included.

3. Attempt to develop the transmission matrix for the beam in Item 2 when:

    A. Subjected to an axial load.

    B. Subjected to an axial load and considering the resulting longitudinal displacement.

    C. Subjected to a distributed uniform lateral load.

4. Consider the incorporation of slosh dynamics into the beam and determine the slosh dynamics matrix.

1.2 REPORT FORMAT

Section 3 briefly defines the matrixes which are developed in the remainder of this report. Its primary purpose is to familiarize readers not familiar with the use of matrixes.

Sections 4 and 5 develop a general transmission matrix in two different ways. Section 4 develops the matrix from the solution to a fourth-order differential equation developed from four first-order differential equations. Section 5 develops the same transmission matrix directly from the four first-order differential equations. Specific beam cases are then presented in Appendixes A and B for avoidance of repetition and loss of understanding of the technique involved.

Sections 6 and 7 present the slosh dynamics matrix and the nonuniform structure's transmission matrix development, respectively.

Sample solutions are then presented in Section 8 and conclusions are given in Section 9.

If the reader is only interested in techniques, he can read Sections 4, 5, 6, and 7 without loss of understanding.

Section 2
SUMMARY

The dynamics analysis of bending vibrations of flexible vehicles is approached through a wave-transmission concept as opposed to the traditional normal-mode method. The system to be modeled is governed by partial and/or integral equations resulting in the characteristic feature of a widely dispersed spatial energy distribution. The parameters of the structure are distributed in spatial coordinates as they are in actuality. The current method of analysis meanwhile is the lumped-parameter type and only approximates the behavior of the structure with ordinary differential equations. The mass and inertia are lumped at stations along the beam and the normal modes are then calculated. computation times are greater and resulting accuracy is degraded relative to that achieved by the wave-transmission or distributed-parameter method.

The distributed parameter method involved the determination of a transmission matrix relating the state variables at one end of a uniform beam segment to those at the other end. Then, a transformation technique is applied relating the local state variables to the characteristic variables of the beam and resulting in a transmission line analog model. This model is composed of two end-effect matrixes and two propagation matrixes. The transmission matrix and associated matrix are first obtained in general terms and then four uniform beam cases are considered: (1) a free-free beam with shear compliance and rotary inertia; (2) application of an axial force; (3) consideration of the resultant longitudinal displacement from the axial force; and (4) application of a uniform, distributed lateral loading. Next, using a thin, uniform beam segment transmission matrix, a nonuniform structure is formed. The natural frequencies are obtained through the modal technique and the distributed-parameter technique. These techniques are compared and some of the advantages as well as difficulties associated with the distributed parameter method are described.

Because launch vehicles may be simulated with as many beam segments as required for accuracy, this simple example constitutes the first step in the extension

of the wave-transmission concept for the analysis of launch vehicles. Inclusion of the effects of the launch vehicle propellant required a special transmission matrix to be developed to represent the propellant's dynamic characteristics. These matrixes may then be incorporated into the beam matrix at positions corresponding to the hydrodynamic forces.

The salient characteristics of the transmission-concept technique are that:

1. It includes both the spatial configuration and the propagating time for the wave to travel along the structure.

2. It permits investigation of the reflection of flexural waves as they encounter discontinuities along the beam.

3. It yields more accurate computation of response than the normal mode techniques.

4. Incorporation of the sloshing masses into the bending equation is direct.

5. Manipulation of the matrix to form transfer functions is straight-forward and simple but computer simulation if difficult.

Section 3

DEFINITION OF MATRIX FORMS

Section 3 defines the matrix forms which will be used in the following sections of this report. For additional discussion, refer to Reference 7 and 11.

3.1 TRANSMISSION MATRIX

A transmission matrix is generally defined as a matrix which relates input states such as forces and velocities to those states at the output terminals. A wide variety of transmission matrixes exist for any given system depending upon the states defined as inputs and outputs. A transmssion matrix, when so defined, can be conveniently used in describing the performance of a series of structures connected in tandem. For example, non-uniform beams may be approximated to any degree of accuracy desired by a series of describing transmission matrixes, with parameter variation along the beam taken into consideration. The output states from one uniform "building-block" transmission matrix become the input states for the next dissimilar but uniform "building-block" transmission matrix until the overall beam is finally represented.

The three basic methods of obtaining the uniform transmission matrix are: (1) a fourth-order differential equation approach; (2) a state variable approach; and (3) an equation approach for lumped elements. The first two methods are for distributed structures while the last is for a lumped element such as a sloshing mode. The non-uniform beam transmission matrix can now be found by merely multiplying the uniform transmission matrixes together or by another method of factored forms which takes into consideration gradual parameter variation along the beam. These methods will be further enumerated upon in the remainder of this report.

3.2 END-EFFECT MATRIXES

The fourth-order differential equation and state variable approaches both yield an analog to transmission lines. That is, the resulting solution steps and transmission matrixes are factored to form a "building-block" diagram composed of two (4x4) matrixes and two (2x2) matrixes. The (4x4) matrixes are referred to as the end-effect matrixes. These matrixes relate the states at the end of a uniform beam segment to a set of characteristic states yet to be defined.

3.3 PROPAGATION MATRIXES

In the previously mentioned analog transmission-line model, the two (2x2) matrixes are referred to as propagation matrixes. These matrixes relate the characteristic states or propagating states of the wave moving down and back along the uniform beam, and are composed in general of a time delay and phase. The characteristic variables are composed of combinations of the system states.

3.4 CHARACTERISTIC ADMITTANCE AND INPUT ADMITTANCE MATRIXES

The characteristic admittance matrix, by definition, is the matrix relating the velocities to the forces at the same end for a semi-infinite beam. A further property of the characteristic admittance matrix is that termination of the beam in it cancels all reflections. For a beam of length l, it is referred to as the input admittance matrix for the beam. Of course, in both instances, the inversion relation is referred to as the impedance matrix.

Section 4

GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION APPROACH

In Section 4, the distributed-parameter approach to transverse bending of a free-free beam with secondardy dynamic effects considered is applied to a general system model. Specific cases are described in Appendix A.

4.1 GENERAL SYSTEM AND TRANSMISSION MATRIX

The general system matrix and its relation to the state variables are derived from four first-order differential equations for the model. These are (1) summation of forces in the transverse coordinate; (2) static beam flexure; (3) moment balance; and (4) geometry considerations. This results in the general matrix equation:

$$\frac{d}{dx}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{Bmatrix} 0 & K_1 & 0 & K_2 \\ 0 & 0 & 1 & 0 \\ 0 & K_3 & 0 & K_4 \\ K_5 & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} \qquad (4\text{-}1)$$

or written in state vector form:

$$\frac{d}{dx} Y = AY \qquad (4\text{-}2)$$

The four equations may be manipulated to obtain the desired fourth-order equation of the form:

$$\left(\frac{d^4 y}{dx^4} + \frac{\phi}{\ell^2}\frac{d^2 y}{dx^2} - \frac{\psi}{\ell^4} y\right) = 0 \qquad (4\text{-}3)$$

The procedure now is to obtain the equation's characteristic roots or Eigenvalues from its solution and finally obtain the general transmission matrix.

The general solution of Equation 4-3 is of the form:

$$y = Ce^{\lambda x/\ell} \qquad (4\text{-}4)$$

The Eigenvalues may be obtained either through the standard method of solving the matrix equation $(A-I\lambda) = 0$, or by substitution of the solution into Equation 4-3. Choosing the latter approach, though both yield identical answers, results in the equation:

$$(\lambda^4 + \phi\lambda^2 - \psi) = 0 \qquad (4\text{-}5)$$

The characteristic roots are found to be:

$$\lambda_1 = - \sqrt{\sqrt{\frac{\phi^2}{4} + \psi} - \frac{\phi}{2}} = - R_1 \qquad (4\text{-}6a)$$

$$\lambda_2 = - j\sqrt{\sqrt{\frac{\phi^2}{4} + \psi} + \frac{\phi}{2}} = - jR_2 \qquad (4\text{-}6b)$$

$$\lambda_3 = + j\sqrt{\sqrt{\frac{\phi^2}{4} + \psi} + \frac{\phi}{2}} = jR_2 \qquad (4\text{-}6c)$$

$$\lambda_4 = + \sqrt{\sqrt{\frac{\phi^2}{4} + \psi} - \frac{\phi}{2}} = + R_1 \qquad (4\text{-}6d)$$

or in matrix form:

$$p = \begin{Bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_1 \end{Bmatrix} \qquad (4\text{-}7)$$

Because the Eigenvalues are now known, the solution of Equation 4-3 may be now written in a more convenient form.

$$y = \{B_1\cosh(\lambda_1 \tfrac{x}{\ell}) + B_2\sinh(\lambda_1 \tfrac{x}{\ell}) + B_3\cos(\lambda_2 \tfrac{x}{\ell}) + B_4\sin(\lambda_2 \tfrac{x}{\ell})\} \qquad (4\text{-}8)$$

Furthermore, because all four dependent variables have essentially the same solution form, for convenience the solution for q will be used.

$$q = [A_1\cosh(\lambda_1 \frac{x}{\ell}) + A_2\sinh(\lambda_1 \frac{x}{\ell}) + A_3\cos(\lambda_2 \frac{x}{\ell}) + A_4\sin(\lambda_2 \frac{x}{\ell})] \qquad (4\text{-}9)$$

Referring to matrix Equation 4-1 and performing the operations indicated by the following three equations:

$$y = -\frac{1}{K_5}\frac{dq}{dx} \qquad (4\text{-}10a)$$

$$\theta = (\frac{-1}{K_1}\frac{dy}{dx} - q\frac{K_2}{K_1}) \qquad (4\text{-}10b)$$

$$m = \frac{d\theta}{dx} \qquad (4\text{-}10c)$$

the following matrix equation is written.

$$\begin{Bmatrix} y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{Bmatrix} \frac{-\lambda_1}{K_5\ell}sh(1) & \frac{-\lambda_1}{K_5\ell}ch(1) & \frac{\lambda 2}{K_5\ell}S(2) & \frac{-\lambda_2}{K_5\ell}C(2) \\ K_6ch(1) & K_6sh(1) & K_7C(2) & K_7S(2) \\ \frac{\lambda_1}{\ell}K_6sh(1) & \frac{\lambda_1}{\ell}K_6ch(1) & \frac{-\lambda_2}{\ell}K_7S(2) & \frac{\lambda_2}{\ell}K_7C(2) \\ ch(1) & sh(1) & C(2) & S(2) \end{Bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \qquad (4\text{-}11)$$

or

$$Y(x) = W(x)A \qquad (4\text{-}12)$$

where:

$$K_6 = \frac{1}{K_1}\left[\frac{\lambda_1^2}{\ell^2 K_5} - K_2\right] \qquad (4\text{-}13a)$$

$$K_7 = \frac{1}{K_1}\left[\frac{-\lambda_2^2}{\ell^2 K_5} - K_2\right] \qquad (4\text{-}13b)$$

The transmission matrix relating the system variables at end x to those at end x=0 is now obtained.

$$Y(0) = W(0)A \tag{4-14}$$

$$A = W^{-1}(0)Y(0) \tag{4-15}$$

$$Y(x) = \underbrace{[W(x)W^{-1}(0)]}_{\text{Transmission Matrix}}Y(0) \tag{4-16}$$

where:

$$W(0) = W(x)\Big|_{x=0} = \begin{Bmatrix} 0 & \dfrac{-\lambda_1}{K_5\ell} & 0 & -\dfrac{\lambda_2}{K_5\ell} \\ K_6 & 0 & K_7 & 0 \\ 0 & \dfrac{\lambda_1 K_6}{\ell} & 0 & \dfrac{\lambda_2 K_7}{\ell} \\ 1 & 0 & 1 & 0 \end{Bmatrix} \tag{4-17}$$

and

$$W^{-1}(0) = \frac{1}{(K_7-K_6)} \begin{Bmatrix} 0 & -1 & 0 & K_7 \\ \dfrac{-K_5K_7\ell}{\lambda_1} & 0 & -\ell/\lambda_1 & 0 \\ 0 & 1 & 0 & -K_6 \\ \dfrac{K_5K_6\ell}{\lambda_2} & 0 & \ell/\lambda_2 & 0 \end{Bmatrix} \tag{4-18}$$

Performing the matrix operation $W(x)W^{-1}(0)$ results in the general transmission matrix T.

$$
T = \frac{1}{\Delta}\left\{\begin{array}{ll}
[K_7 ch(1) - K_6 C(2)] & [\gamma_1/K_5\ell]\ldots \\[2ex]
[-\ell K_5 K_6 K_7 \gamma_3] & [-K_6 ch(1) + K_7 C(2)]\ldots \\[2ex]
(-K_5 K_6 K_7 \gamma_2) & [\frac{-1}{\ell}\left(\lambda_1 K_6 sh(1) + \lambda_2 K_7 S(2)\right)]\ldots \\[2ex]
[-K_5\ell\left(\frac{K_7}{\lambda_1}\, sh(1) - \frac{K_6}{\lambda_2}\, S(2)\right)] & [-\gamma_2]\ldots
\end{array}\right.
$$

$$
\left.\begin{array}{ll}
\ldots[\gamma_2/K_5] & [\frac{-1}{K_5\ell}\left(\lambda_1 K_7 sh(1) - \lambda_2 K_6 S(2)\right)] \\[2ex]
\ldots[-\ell\left(\frac{K_6}{\lambda_1}\, sh(1) - \frac{K_7}{\lambda_2}\, S(2)\right)] & [K_6 K_7 \gamma_2] \\[2ex]
\ldots[-K_6 ch(1) + K_7 C(2)] & [\frac{K_6 K_7}{\ell}\,\gamma_1] \\[2ex]
\ldots[-\ell\gamma_3] & [K_7 ch(1) - K_6 C(2)]
\end{array}\right\} \qquad (4\text{-}19)
$$

Where:

$$
\begin{aligned}
ch(1) &= \cosh(\lambda_1 \tfrac{x}{\ell}) \\
sh(1) &= \sinh(\lambda_1 \tfrac{x}{\ell}) \\
C(2) &= \cos\ (\lambda_2 \tfrac{x}{\ell}) \\
S(2) &= \sin\ (\lambda_2 \tfrac{x}{\ell})
\end{aligned} \qquad (4\text{-}20)
$$

$$
\begin{aligned}
\gamma_1 &= \left(\lambda_1 sh(1) + \lambda_2 S(2)\right) \\
\gamma_2 &= \left(ch(1) - C(2)\right) \\
\gamma_3 &= \left(\frac{1}{\lambda_1}\, sh(1) - \frac{1}{\lambda_2}\, S(2)\right) \\
\Delta &= (K_7 - K_6)
\end{aligned} \qquad (4\text{-}21)
$$

4.2 GENERAL EIGENVECTOR MATRIX

The solutions of the homogeneous set of equations defined by the matrix Equation 4-1 are now the associated Eigenvectors.

$$(A - I\lambda)\ U = 0 \tag{4-22}$$

$U_i = \begin{Bmatrix} U_{1i} \\ U_{2i} \\ U_{3i} \\ U_{4i} \end{Bmatrix}$, where i refers to the set solution corresponding to a particular Eigenvalue.

Now the i^{th} Eigenvector corresponding to the i^{th} Eigenvalue λ_i is determined from the following four equations.

$$(K_1\ U_{2i} + K_2\ U_{4i}) = \lambda_i\ U_{1i} \tag{4-23a}$$

$$U_{3i} = \lambda_i\ U_{2i} \tag{4-23b}$$

$$(K_3\ U_{2i} + K_4\ U_{4i}) = \lambda_i\ U_{3i} \tag{4-23c}$$

$$K_5\ U_{1i} = \lambda_i\ U_{4i} \tag{4-23d}$$

Because Eigenvectors are uniquely determined in their directions and only their lengths are arbitrary, the Eigenvector can be normalized in any manner desired. Choosing $U_{2i} = \lambda_i$:

$$U_i = \begin{Bmatrix} U_{1i} \\ U_{2i} \\ U_{3i} \\ U_{4i} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\lambda_i}{K_5}\ \dfrac{\lambda_i 3 - K_3\lambda_i}{K_4} \\ \lambda_i \\ \lambda_i^{\,2} \\ \dfrac{\lambda_i 3 - K_3\lambda_i}{K_4} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\lambda_i 4 - K_3\lambda_i^{\,2}}{K_4 K_5} \\ \lambda_i \\ \lambda_i^{\,2} \\ \dfrac{\lambda_i 3 - K_3\lambda_i}{K_4} \end{Bmatrix} \tag{4-24}$$

The specific Eigenvector matrix is:

$$U = \begin{Bmatrix} \left(\frac{\lambda_1^4 - K_3\lambda_1^2}{K_4K_5}\right) & \left(\frac{\lambda_2^4 + K_3\lambda_2^2}{K_4K_5}\right) & \left(\frac{\lambda_2^4 + K_3\lambda_2^2}{K_4K_5}\right) & \left(\frac{\lambda_1^4 - K_3\lambda_1^2}{K_4K_5}\right) \\ -\lambda_1 & -j\lambda_2 & j\lambda_2 & \lambda_1 \\ +\lambda_1^2 & -\lambda_2^2 & -\lambda_2^2 & \lambda_1^2 \\ \left(\frac{-\lambda_1^3 + K_3\lambda_1}{K_4}\right) & j\left(\frac{\lambda_2^3 + K_3\lambda_2}{K_4}\right) & j\left(\frac{-\lambda_2^3 - K_3\lambda_2}{K_4}\right) & \left(\frac{\lambda_1^3 - K_3\lambda_1}{K_4}\right) \end{Bmatrix} \quad (4\text{-}25)$$

4.3 GENERAL END-EFFECT MATRIXES

The spatial variables, $Y = (-y, \theta, m, q)^T$, are now related to a set of characteristic variables of transverse bending, $V = (u^-, v^-, v^+, u^+)^T$, through the Eigenvector matrix U:

$$Y = UV \quad (4\text{-}26)$$

Rewriting Equation 4-26 in a (2x2) partitioned form results in:

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{Bmatrix} \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} \quad (4\text{-}27)$$

$$Y_1 = (-y, \theta,)^T \quad (4\text{-}28a)$$

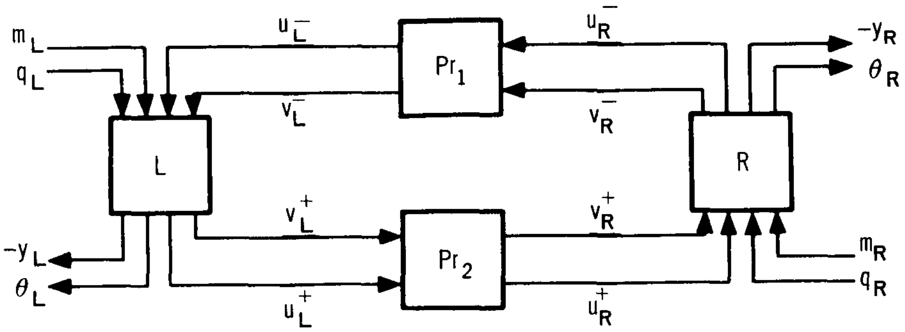
$$Y_2 = (m,q)^T \quad (4\text{-}28b)$$

$$V_1 = (u^-, v^-)^T \quad (4\text{-}28c)$$

$$V_2 = (v^+, u^+)^T \quad (4\text{-}28d)$$

$U_{11}, U_{12}, U_{21}, U_{22}$ = (2x2) array of the (4x4) Eigenvector matrix.

Suitably manipulating the above matrix equation results in the block diagram for the beam with secondary dynamic effects (Figure 4-1).



$$L = \text{LEFT END EFFECT MATRIX} = \left\{ \begin{array}{c|c} (U_{12}U_{22}^{-1}) & (U_{11}-U_{12}U_{22}^{-1}U_{21}) \\ \hline (U_{22}^{-1}) & (-U_{22}^{-1}U_{21}) \end{array} \right\} \quad (4\text{-}30a)$$

$$R = \text{RIGHT END EFFECT MATRIX} = \left\{ \begin{array}{c|c} (U_{21}^{-1}) & (-U_{21}^{-1}U_{22}) \\ \hline (U_{11}U_{21}^{-1}) & (U_{12}-U_{11}U_{21}^{-1}U_{22}) \end{array} \right\} \quad (4\text{-}30b)$$

$Pr_{1,2}$ = PROPAGATION MATRIXES

Figure 4-1. Beam Block Diagram

4.4 GENERAL PROPAGATION MATRIXES

The general propagation matrix is found through the use of the characteristic vector relation:

$$Y = UV \tag{4-30}$$

where V satisfies the equation:

$$\frac{d}{dx} V = PV \tag{4-31}$$

and has the solution:

$$V_L(s) = e^{\ell P} V_R(s) \tag{4-32}$$

The exponential matrix can be found by means of Laplace transform with respect to x (Reference 12).

Letting:

$$E(x) = e^{Px} \text{ and } E(s) = \mathcal{L}E(x)$$

then:

$$E(s) = (SI - P)^{-1} \text{ and } E(x) = \mathcal{L}^{-1}(sI - P)^{-1}$$

$$(SI-P)^{-1} = \begin{Bmatrix} \frac{1}{(s-\lambda_1)} & 0 & 0 & 0 \\ 0 & \frac{1}{(s-\lambda_2)} & 0 & 0 \\ 0 & 0 & \frac{1}{(s-\lambda_3)} & 0 \\ 0 & 0 & 0 & \frac{1}{(s-\lambda_4)} \end{Bmatrix} \tag{4-33}$$

therefore:

$$e^{Px} = \mathcal{L}^{-1}(SI-P)^{-1} = \begin{Bmatrix} e^{\lambda_1 \ell} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 \ell} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 \ell} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 \ell} \end{Bmatrix}_{x = \ell} \tag{4-34}$$

and the relation from Equation 4-33 is obtained, that:

$$\begin{Bmatrix} u- \\ v- \\ v+ \\ u+ \end{Bmatrix}_L = \begin{Bmatrix} e^{-R_1 \ell} & 0 & 0 & 0 \\ 0 & e^{-jR_2 \ell} & 0 & 0 \\ 0 & 0 & e^{+jR_2 \ell} & 0 \\ 0 & 0 & 0 & e^{+R_1 \ell} \end{Bmatrix} \begin{Bmatrix} u- \\ v- \\ v+ \\ u+ \end{Bmatrix}_R \tag{4-35}$$

It is evident from matrix Equation 4-35 and Figure 4-1 that:

$$P_{r_1} = \begin{bmatrix} e^{-R_1 \ell} & 0 \\ 0 & e^{-jR_2 \ell} \end{bmatrix} \tag{4-36}$$

and

$$P_{r_2} = \begin{bmatrix} e^{-jR_2 \ell} & 0 \\ 0 & e^{-R_1 \ell} \end{bmatrix} \tag{4-37}$$

4.5 GENERAL CHARACTERISTIC ADMITTANCE MATRIX

Because the Eigenvectors and the relations between the spatial and characteristic variables are known, the conditions for the characteristic termination can be determined. As in transmission-line theory, the characteristic admittance is defined for a semi-infinite line or, in this case, a beam. If the semi-infinite beam extends from the right-end of the model in Figure 4-1 to $-\infty$, then the components u+ and v+ will be zero or $V_2 = 0$ in the matrix Equation 4-27.

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{Bmatrix} \begin{Bmatrix} V_1 \\ 0 \end{Bmatrix} \tag{4-38}$$

This matrix relation yields the desired characteristic admittance at the right-end of the beam:

$$Y_1 = (U_{11} U_{21}^{-1}) Y_2 \tag{4-39}$$

where:

$(U_{11} U_{21}^{-1})$ = Characteristic Admittance at the right face.

Section 5
STATE VARIABLE APPROACH

The transmission matrix developed in Section 4 can also be determined in another manner. In Section 4, a set of four first-order differential equations were reduced to one of fourth order in y and the resulting steps taken to obtain the associated transmission matrix. Section 5 outlines the procedure to obtain the same transmission matrix directly from the four first-order equations.

If four first-order equations with constant coefficients, represented in the matrix equation, are considered:

$$\frac{dY}{dx} = AY \tag{5-1}$$

the solution of Equation 5-1 is known to be:

$$Y_{x = \ell} = e^{A\ell} Y_{x = 0} \tag{5-2}$$

and, hence, immediately, $e^{A\ell}$ is the transmission matrix that is sought. Therefore, it remains to find $e^{A\ell}$. The transmission matrix, $T = e^{A\ell}$, can be represented as an infinite series expansion:

$$T = (I + A\ell + A^2 \frac{\ell^2}{2!} + A^3 \frac{\ell^3}{3!} + \ldots) \tag{5-3}$$

The above expansion is simplified by using the Cayley-Hamilton theorem, which results in:

$$T = (C_o I + C_1 A\ell + C_2 (A\ell)^2 + C_3 (A\ell)^3) \tag{5-4}$$

The Eigenvalues for the system may be now determined from the characteristic equation for $A\ell$, i.e.:

$$|\lambda I - A\ell| = 0 \tag{5-5}$$

Now $A\ell$ may be replaced by its four Eigenvalues and the four constants, C_o, C_1, C_2, and C_3 found. That is,

$$e^{\lambda_i \ell} = (C_o I + C_1 \lambda_i \ell + C_2 \lambda_i^2 \ell^2 + C_3 \lambda_i^3 \ell^3) \qquad (5\text{-}6)$$

where $i = 1, \ldots, 4$.

Now, the transmission matrix as defined by Equation 5-4 is formed. The details and results are not presented because they are straightforward and the results are identical to those obtained in Section 4.

Section 6

FUEL-SLOSHING TRANSMISSION MATRIX

Section 6 describes two methods of including the fuel-slosh dynamics in the distributed-parameter approach to flexible beams. One method involves a lumped spring-mass system combined with the transmission matrix for each uniform beam section. The other method utilizes a distributed fluid mass incorporated with the transmission matrix for the beam while still retaining the spring-mass system for the first sloshing mode.

The interaction between the tank wall and the fluid must satisfy a force and deflection relationship. A matrix representation of this interaction along the length of the tank appears quite complex and has been deferred for future study. The method of distributing the nonsloshing liquid along the tank or uniform beam corresponds to changing the density per unit length of the beam. Because no stiffness can be attributed to the fluid, the other characteristics of the beam matrix do not change. The beam with added density will have a slower response, as would be expected for a tank of fluid.

Cylindrical tanks are considered and only the first sloshing mass is used. This is justified because higher sloshing modes have practically no influence; they have greater damping ratios, and limited peaking (amplitudes). Furthermore, a great deal of turbulent mixing accompanies the higher frequency tank oscillations. In a circular cylindrical tank, (Reference 13), the ratio of the sloshing mass to the propellant mass for the second sloshing mode is very small compared to the first mode.

For flexible missile (beam) stuides, it is desired that the forces and moments derived from the mechanical analogy act at points on the beam, corresponding to those of the forces and moments as derived from a hydrodynamic solution. Although exact solutions to the flexible-wall problem do exist, they are complicated and application is anything but direct. As will be seen, the wave transmission and the associated transmission matrix are relatively simple and straightforward when applied to the flexible-wall problems.

The wave-transmission method has a distinct advantage over the standard modal approach for incorporating sloshing masses. In the transmission-matrix approach, the sloshing masses are included directly into the bending equation with use of a simple matrix multiplication. In the modal approach, the natural frequencies and modal masses are determined without the sloshing masses. Later, when the equations are in generalized coordinates or transfer function forms, the sloshing masses are included. The deflection at the sloshing mass is a summation of the mode deflections and cannot be as accurate as the deflection used in the transmission approach.

6.1 LUMPED BULK MASS AND MODE SPRING-MASS APPROACH

This approach first lumps the fluid at rest (bulk mass) and the first sloshing mode and then interconnects these through uniform beam segments as depicted in Figure 6-1.

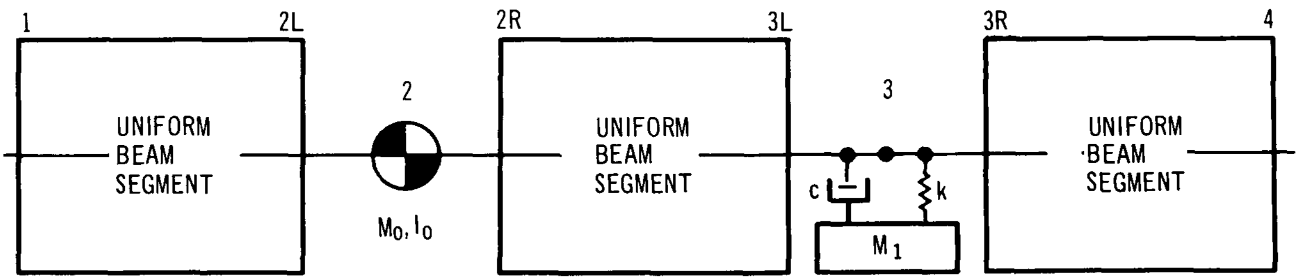


Figure 6-1. Incorporated Sloshing Dynamics and Beam Model

Transmission matrixes are found for each of the lumped elements and then the overall transmission matrix is presented in general matrix form. Next, the transmission matrixes for the lumped elements will be found. Therefore, consider the model for the first sloshing mode in Figure 6-2.

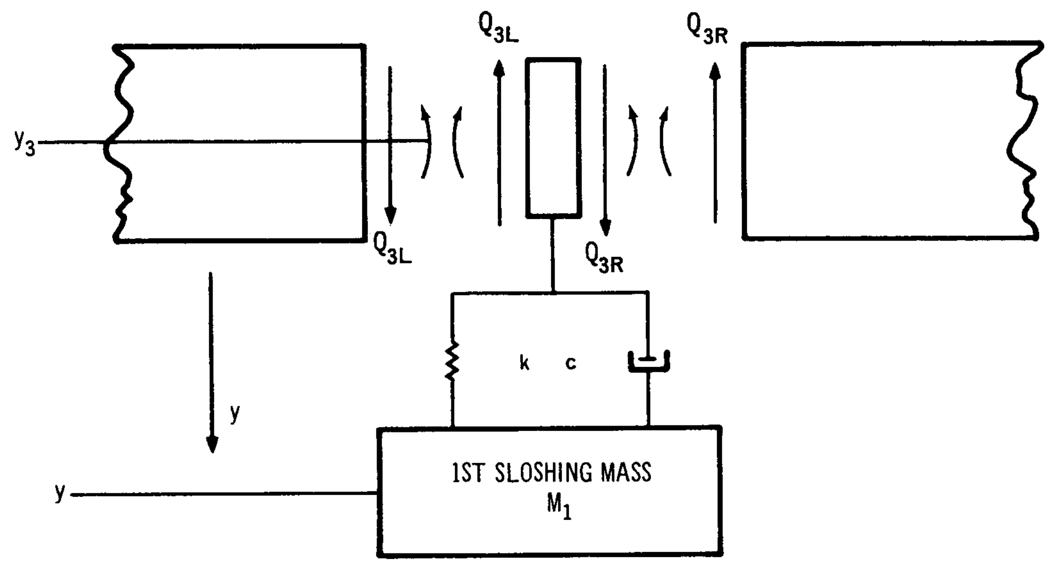


Figure 6-2. Sloshing Model

What is desired is obtaining a transmission matrix relating 3R to 3L. Writing the governing equations:

$$(y - y_3)k + (\dot{y} - \dot{y}_3)c + \ddot{y}M_1 = 0 \tag{6-1}$$

$$-(y - y_3)k - (\dot{y} - \dot{y}_3)c = Q_3 \tag{6-2}$$

in Laplace notation:

$$(y - y_3)k + s(y - y_3)c + s^2 yM_1 = 0 \tag{6-3}$$

$$-(y - y_3)k - s(y - y_3)c = Q_3 \tag{6-4}$$

$$Q_3 = s^2 yM_1 \tag{6-5}$$

$$y = \left(\frac{k + sc}{k + sc + s^2 M_1}\right) y_3 \tag{6-6}$$

and therefore:

$$Q_3 = \frac{s^2 M_1 (k + sc)}{s^2 M_1 + sc + k} y_3 \tag{6-7}$$

because:

$$Q_3 = (Q_{3R} - Q_{3L}) \tag{6-8}$$

$$Q_{3R} = (Q_3 + Q_{3L}) \tag{6-9}$$

The resulting transmission matrix is:

$$\begin{Bmatrix} y_{3R} \\ \theta_{3R} \\ M_{3R} \\ Q_{3R} \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ G & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} y_{3L} \\ \theta_{3L} \\ M_{3L} \\ Q_{3L} \end{Bmatrix} \tag{6-10}$$

where:

$$G = \frac{s^2 (k + sc) M_1}{s^2 M_1 + sc + k} \tag{6-11}$$

while the transmission matrix for the undisturbed or fluid at rest, at point 2 is:

$$\begin{Bmatrix} y_{2R} \\ \theta_{2R} \\ M_{2R} \\ Q_{2R} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & I_o s^2 & 1 & 0 \\ M_o s^2 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_{2L} \\ \theta_{2L} \\ M_{2L} \\ Q_{2L} \end{Bmatrix} \qquad (6\text{-}12)$$

The overall transmission matrix T_{41} is developed with use of the following relationships:

$$Y_{2L} = T_{2L,1} Y_1$$

$$Y_{2R} = T_2 Y_{2L}$$

$$Y_{3L} = T_{3L,2R} Y_{2R} \qquad Y_4 = \underbrace{(T_{4,3R} \cdot T_3 \cdot T_{3L,2R} \cdot T_2 \cdot T_{2L,1})}_{T_{4,1}} Y_1 \qquad (6\text{-}13)$$

$$Y_{3R} = T_3 Y_{3L}$$

$$Y_4 = T_{4,3R} Y_{3R}$$

where $T_{2L,1}$, $T_{3L,2R}$, and $T_{4,3R}$ are the uniform beam transmission matrixes.

6.2 DISTRIBUTED BULK MASS AND LUMPED MODE SPRING-MASS APPROACH

The second model considered has a distributed bulk mass, while the first sloshing mode remains lumped as seen in Figure 6-3.

As before, we see immediately that the overall transmission matrix is given by:

$$Y_3 = (T_{3,2} T_{2R,2L} T_{2,1})\ Y_1 \qquad (6\text{-}14)$$

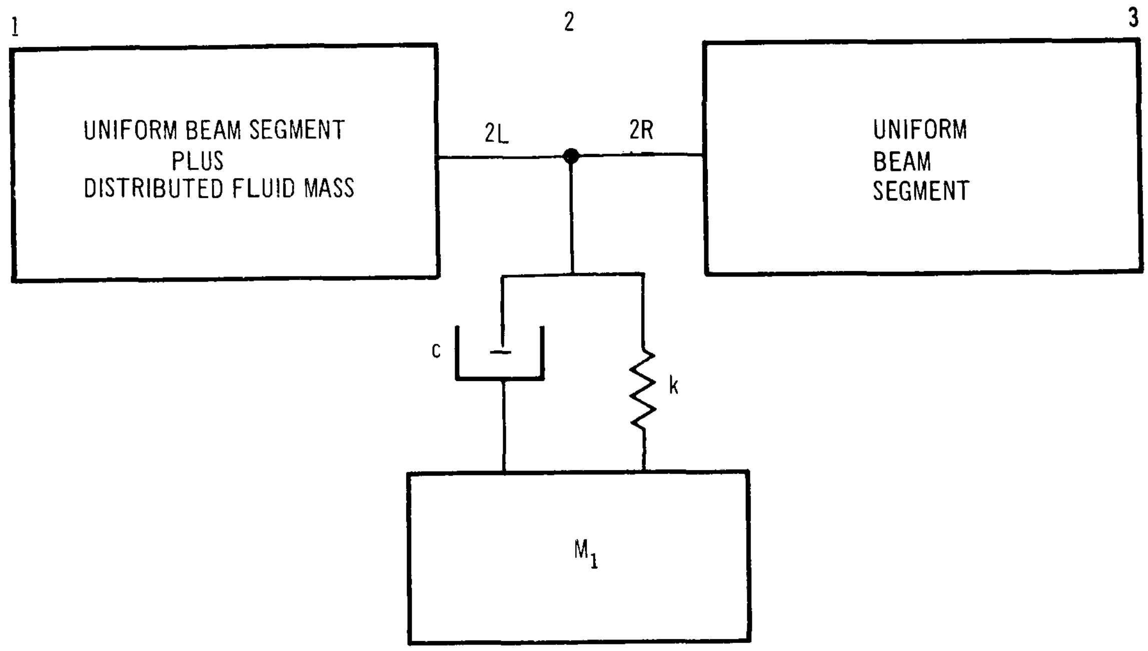


Figure 6-3. Distributed Bulk Mass and Lumped Spring-Mass Model

where: $T_{3,2}$ is a uniform beam transmission matrix.

$T_{2,1}$ is a uniform beam transmission matrix with the addition of the nonsloshing fluid's density to the density of the tank, i.e., the density increases and the radius of gyration is adjusted.

$T_{2R,2L}$ is, of course, the first sloshing mode's transmission matrix defined by Equation 6-10.

Section 7

TRANSMISSION MATRIX FOR NON-UNIFORM STRUCTURE

7.1 CASCADING METHOD

The distributed parameter concepts of propagation and reflection are applied to a thin, uniform beam, i.e., secondary dynamic effects are neglected. In a somewhat similar manner as in Section 4.1, the transmission matrix for this case will be developed first. Transverse vibrations are approached in an analogous method to propagation and reflection treatment in the wave equation. For further details of this method of obtaining the transmission matrix for a thin-uniform beam, the reader is referred to Reference 10 and to Appendix B.

Figure 7-1 shows the nomenclature and sign conventions for shear forces, Q, moments, M, angular displacements, θ, and transverse displacements, Y, of a thin, uniform beam and microelement.

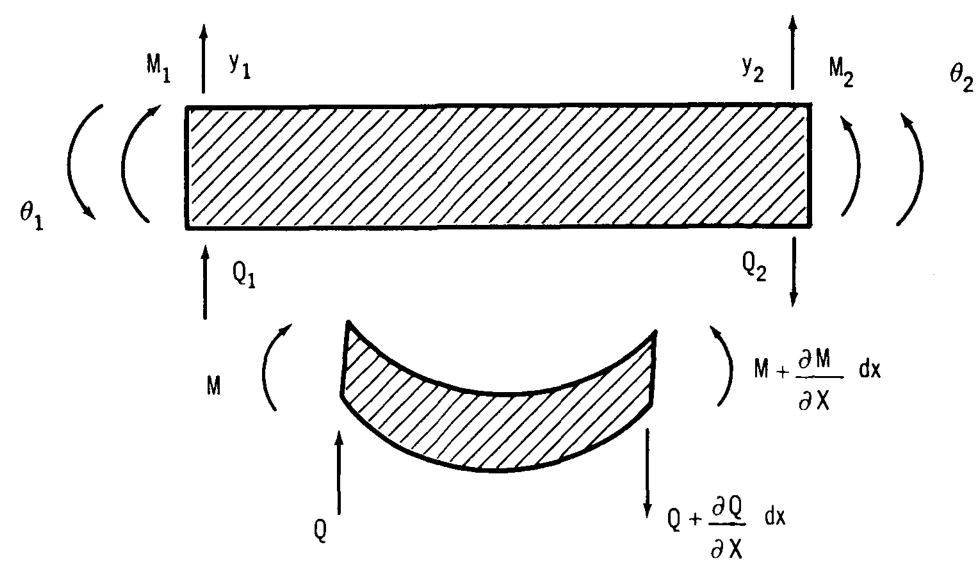


Figure 7-1. Beam and Sign Convention Model

The four first-order differential equations describing the model are reduced to a canonical form and written in compact state vector form as:

$$\frac{dY}{dx} = AY \tag{7-1}$$

where

$$Y = (\dot{y}, \dot{\theta}, M, Q)^T \tag{7-2}$$

$$A = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{s}{EI} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{EI}{a^2} s & 0 & 0 & 0 \end{Bmatrix} \tag{7-3}$$

$$a^2 = (EI/\rho A) \tag{7-4}$$

Finding the Eigenvalue matrix P and the Eigenvector matrix U of matrix A in the usual manner, and obtaining a transformation matrix B which operates upon the complex conjugate pairs in P to form only real terms, P* may be written and defined as:

$$P^* = (BPB^{-1}) \tag{7-5}$$

and

$$A = (UP^{-1}U^{-1}) \tag{7-6}$$

The transformation matrix U* that provides the transformation between the A matrix and the quasidiagonal P* matrix, is found as follows:

$$P = (B^{-1}P^*B) \tag{7-7}$$

and through substitution into Equation 7-6,

$$A = (UB^{-1})\ P^*\ (BU^{-1}) \tag{7-8}$$

or that

$$U^* = (UB^{-1}) \tag{7-9}$$

a new state vector V is now chosen so that:

$$Y = U^*V \tag{7-10}$$

and which satisfies:

$$\frac{dV}{dx} = P^*V \tag{7-11}$$

whose solution is

$$V_2\ (s) = e^{\ell P^*} V_1\ (s) \tag{7-12}$$

The matrix $e^{\ell P^*}$ is the propagation matrix and V_1 and V_2 are the characteristic vector values at stations 1 and 2 and ℓ is the length of the beam. Equation 7-10 can now be written as:

$$Y_2 = U^*V_2 = (U^*e^{\ell P^*})\ V_1 \tag{7-13}$$

and

$$Y_1 = U^*V_1, \tag{7-14}$$

substitution of Equation 7-14 into 7-13, results in Equations 7-15 and 7-16.

$$Y_2 = (U^*e^{\ell P^*}U^{*-1})\ Y_1 \tag{7-15}$$

$$Y_2 = T_{2,1}\ Y_1 \tag{7-16}$$

The (4 x 4) $T_{2,1}$ matrix is the transmission matrix for a thin, uniform beam relating the states at end 2 to those at end 1.

End-effect matrixes have the same form as those found in Section A.3 and the specific values for the case are given in Appendix B.

The non-uniform beam can now be formed or approximated by a series of uniform beam elements because the output of one uniform segment becomes the input for the next until the overall structure is completed. This "building-block" approach is equally applicable for the uniform transmission matrixes developed for a beam with secondary dynamic effects considered. But for the sake of simplicity, the method will be developed for the thin, uniform beam segments specifically.

If, for example, the structure is adequately approximated by three thin, uniform beam segments as depicted in Figure 7-2, than the overall transmission matrix is formed from the three segment transmission matrixes and relations as:

$$Y_2 = T_{2,1} Y_1 \tag{7-17}$$

$$Y_3 = T_{3,2} Y_2 \tag{7-18}$$

$$Y_4 = T_{4,3} Y_3 \tag{7-19}$$

$$Y_4 = (T_{4,3} T_{3,2} T_{2,1}) Y_1 \tag{7-20}$$

or

$$Y_4 = T_{4,1} Y_1 \tag{7-21}$$

Therefore, in general, for a beam of n uniform segments, the overall transmission matrix and state vector relation becomes:

$$Y_{n+1} = (T_{n+1,n} T_{n,n-1} \cdots T_{2,1}) Y_1 \tag{7-22}$$

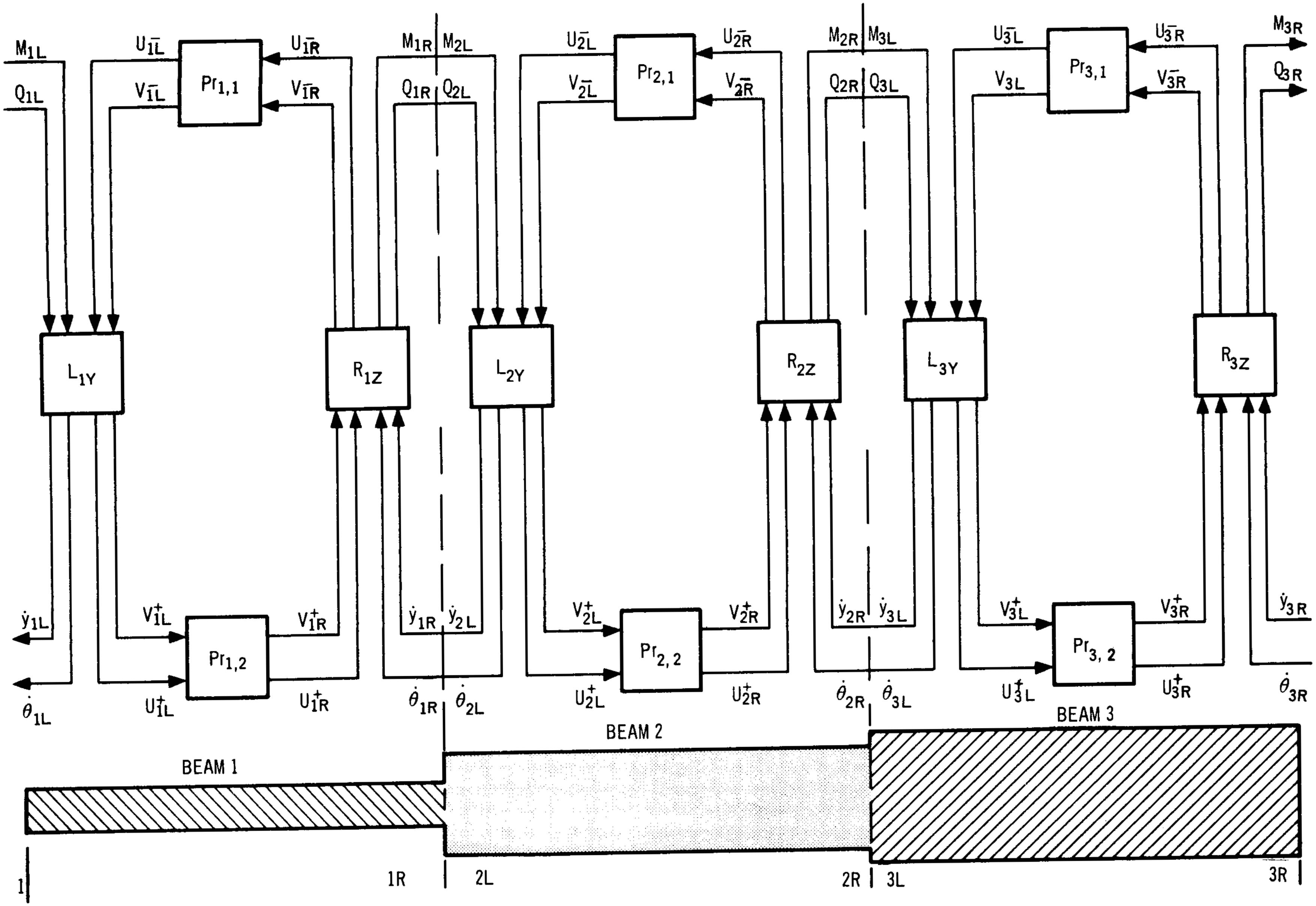


Figure 7-2. Approximated Nonuniform Structure

7.2 MODIFIED CASCADING METHOD

Consider now once again the model of three beams to form a non-uniform structure as in Figure 7-2. If we assume that some parameter varies only slightly from the left-end of beam 1 (1L) to the right-end of beam 3 (3R), then a somewhat different approach may be taken. This approach is actually a slight variation of the approach given in Section 7.1.

At the interface between two uniform beams, the characteristic variables are related according to:

$$V_{2L} = (U_2^{-1} U_1) V_{1R} \tag{7-23}$$

and the overall transmission matrix relation equation has the form:

$$Y_{3R} = (U_3 e^{\ell_3 P_3} U_3^{-1} U_2 e^{\ell_2 P_2} U_2^{-1} U_1 e^{\ell_1 P_1} U_1^{-1}) Y_{1L} \tag{7-24}$$

Now, if the cascade of uniform beams has a slight variation of a_i's which is not drastically different, then

$$U_2^{-1} \; U_1 = \begin{Bmatrix} I + \xi_{21} & \delta_{21} \\ \delta_{21} & I + \xi_{21} \end{Bmatrix} \tag{7-25}$$

where ξ_{21} and δ_{21} are matrixes with small elements because $a_2 \cong a_1$ and a similar relation holds for $U_3^{-1} U_2$. Then Equation 7-24 can be written as:

$$Y_4 = U_3 \begin{pmatrix} G_3 & 0 \\ 0 & H_3 \end{pmatrix} \begin{pmatrix} I+\xi_{32} & \delta_{32} \\ \delta_{32} & I+\xi_{32} \end{pmatrix} \begin{pmatrix} G_2 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I+\xi_{21} & \delta_{21} \\ \delta_{21} & I+\xi_{21} \end{pmatrix} \begin{pmatrix} G_1 & 0 \\ 0 & H_1 \end{pmatrix} U_1^{-1} Y_1 \tag{7-26}$$

Assuming all matrix products of two or more ξ or δ matrixes negligible in the limit compared to only one small ξ or δ,

$$Y_4 = U_3 \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} U_1^{-1} Y_1 \tag{7-27}$$

where:

$$N_{11} = (G_3G_2G_1 + G_3\xi_{32}G_2G_1 + G_3G_2\xi_{21}G_1) \tag{7-28}$$

$$N_{12} = (G_3G_2\delta_{21}H_1 + G_3\delta_{32}H_2H_1) \tag{7-29}$$

$$N_{21} = (H_3H_2\delta_{21}G_1 + H_3\delta_{32}G_2G_1) \tag{7-30}$$

$$N_{22} = (H_3H_2H_1 + H_3\xi_{32}H_2H_1 + H_3H_2\xi_{21}H_1) \tag{7-31}$$

When the interstage transition is very gradual ($a_{i+1} \cong a_i$), then the ξ and δ matrixes might be completely neglected resulting in:

$$N_{11} = G_3G_2G_1 \tag{7-32}$$

$$N_{12} = N_{21} = 0 \tag{7-33}$$

$$N_{22} = H_3H_2H_1 \tag{7-34}$$

Even though $a_{i+1} \neq a_i$, the relations for N_{11} and N_{22} take on a particularly simple form if accuracy requirements are relaxed. For the simple, thin-uniform beam:

$$N_{11} = e^{(\ell_3P_3^* + \ell_2P_2^* + \ell_1P_1^*)} \tag{7-35}$$

or

$$N_{11} = e^{\sqrt{T_t}s} \begin{bmatrix} \cos\sqrt{T_t}s & \sin\sqrt{T_t}s \\ -\sin\sqrt{T_t}s & \cos\sqrt{T_t}s \end{bmatrix} \tag{7-36}$$

where:

$$T_t^{1/2} = \frac{\ell_1}{\sqrt{2a_1}} + \frac{\ell_2}{\sqrt{2a_2}} + \frac{\ell_3}{\sqrt{2a_3}} \tag{7-37}$$

and:

$$N_{22} = e^{-\sqrt{T_t}s} \begin{bmatrix} \cos\sqrt{T_t}s & \sin\sqrt{T_t}s \\ -\sin\sqrt{T_t}s & \cos\sqrt{T_t}s \end{bmatrix} \tag{7-38}$$

Therefore, when these approximations are valid, the overall transmission between the characteristic states is the same as for a uniform beam except T_t is used. If the beam is broken into a much larger number of segments or sections, the gradual variation in parameter a, from end to end, could accumulate a large difference from the first to last segments. In this case, the approximations, become less valid and the overall beam approximation becomes inaccurate.

Section 8

SIMULATIONS AND SAMPLE SOLUTIONS

Section 8 presents some of the simulation results, solutions, program discussions, and difficulties encountered when using the distributed-parameter or wave-transmission approach to transverse vibrations of flexible beams.

8.1 NATURAL FREQUENCY COMPARISON

The natural frequencies for a cascade of three thin, uniform beams were first determined using the Myklestad method (Reference 14). In this method, the structure is approximated by one in which the mass is concentrated at discrete points while the flexural stiffness of distributed massless elements is used. Generally, uniform massless elastic fields are assumed between the lumped mass points. However, the wave-transmission-matrix method has the advantage that the spatial distribution of mass is included.

A comparative study was accomplished into the advantages and disadvantages of the two methods of determining the natural frequencies of the composite, three-beam, cascaded structure. Theoretically, because of the spatial distributed nature of the transmission matrix method, one would expect to obtain the exact natural frequencies of the beam. The lumped model can yield only approximations to the natural frequencies. Of course, lumping the mass at a large number of stations will result in more accurate estimates of the natural frequencies, but at the expense of more computation. Use of the wave-transmission method, in practice, requires a fairly comprehensive coverage of the frequency spectrum for accurate estimates of the natural frequencies. The most efficient procedure (wave transmission or Myklestad) might be defined as the one which yields the desired number of natural frequencies with acceptable accuracy with the least computation and engineering cost.

The wave-transmission approach to obtain the natural frequencies of the non-uniform, three-beam cascade (Figure 7-2) requires that proper boundary conditions be applied to the overall transmission matrix relations. If the model was representative of a launch vehicle on a launch pad, the specified end conditions would be:

$$\dot{y}_{3R} = 0 \quad M_{1L} = 0$$
$$\dot{\theta}_{3R} = 0 \quad Q_{1L} = 0 \tag{8-1}$$

However, for the free-free condition as in this study, moments and forces at both ends are zero. Therefore, for example, if the overall transmission matrix were composed of a cascade of three, thin uniform beam* transmission matrixes, then the overall transmission matrix written in a (2 x 2) partitioned sense would relate:

$$\begin{pmatrix} \dot{x} \\ f \end{pmatrix}_{3R} = \begin{pmatrix} T_{11} T_{12} \\ T_{21} T_{22} \end{pmatrix}_{3R,1L} \begin{pmatrix} \dot{x} \\ f \end{pmatrix}_{1L} \tag{8-2}$$

where:

$$\dot{x} = (\dot{y},\dot{\theta})^T \tag{8-3a}$$

$$f = (M,Q)^T \tag{8-3b}$$

and the frequencies which make $|(T_{21})|_{3R,1L} = 0$ are the natural frequencies of the composite free-free beam. This procedure of finding the natural frequencies of a system is analogous to that of a system composed of concentrated masses and springs, namely, application of proper boundary conditions, $f_{3R} = 0$, and obtaining the resultant determinant.

---

* Equivalent to σ and τ being zero in Appendix A.2 or using the results in Section 7.1 and the transmission matrix in Appendix B.

For simplification, the determinant of the (2 x 2) T_{21} array can be directly calculated from the relation:

$$(T_{21})_{3R,1L} = (T_{21} \quad T_{22})_{3R,3L} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}_{2R,2L} \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}_{1R,1L} \qquad (8\text{-}4)$$

Once the final (2 x 2) matrix $(T_{21})_{3R,1L}$ is formed, the associated determinate is evaluated much in the same manner as suggested by Pestel and Leckie (Reference 7). Namely, various values for ω are substituted until the cross-over points (points where the function's magnitude equals zero) are obtained. The main difference between the one used and Pestel and Leckies' was that a frequency response of the function was obtained. This was done because of the ease of computation through available programs. The result is shown in Figure 8-1. Here, the troughs or "anti-resonance" points indicate the natural frequency points. This is correct because one would expect such response as a result of the relationship between zero magnitude points of the function and its frequency response plot (dB). Of course, with an actual transfer function, resonance points or peaks in the plot at the natural frequencies would normally be expected. Table 8-1 lists the natural frequencies obtained from a Myklestad run using 30 stations and the wave-transmission method.

Table 8-1

TABLE OF NATURAL FREQUENCIES

Mode	Myklestad Method 30 Stations (cps)	Transmission Matrix Method (cps)	Difference Between the Two Methods
1	0.2165	0.2165	0
2	0.593	0.589	-0.004
3	1.266	1.273	+0.007
4	2.029	2.037	+0.008
5	2.967	2.976	+0.009

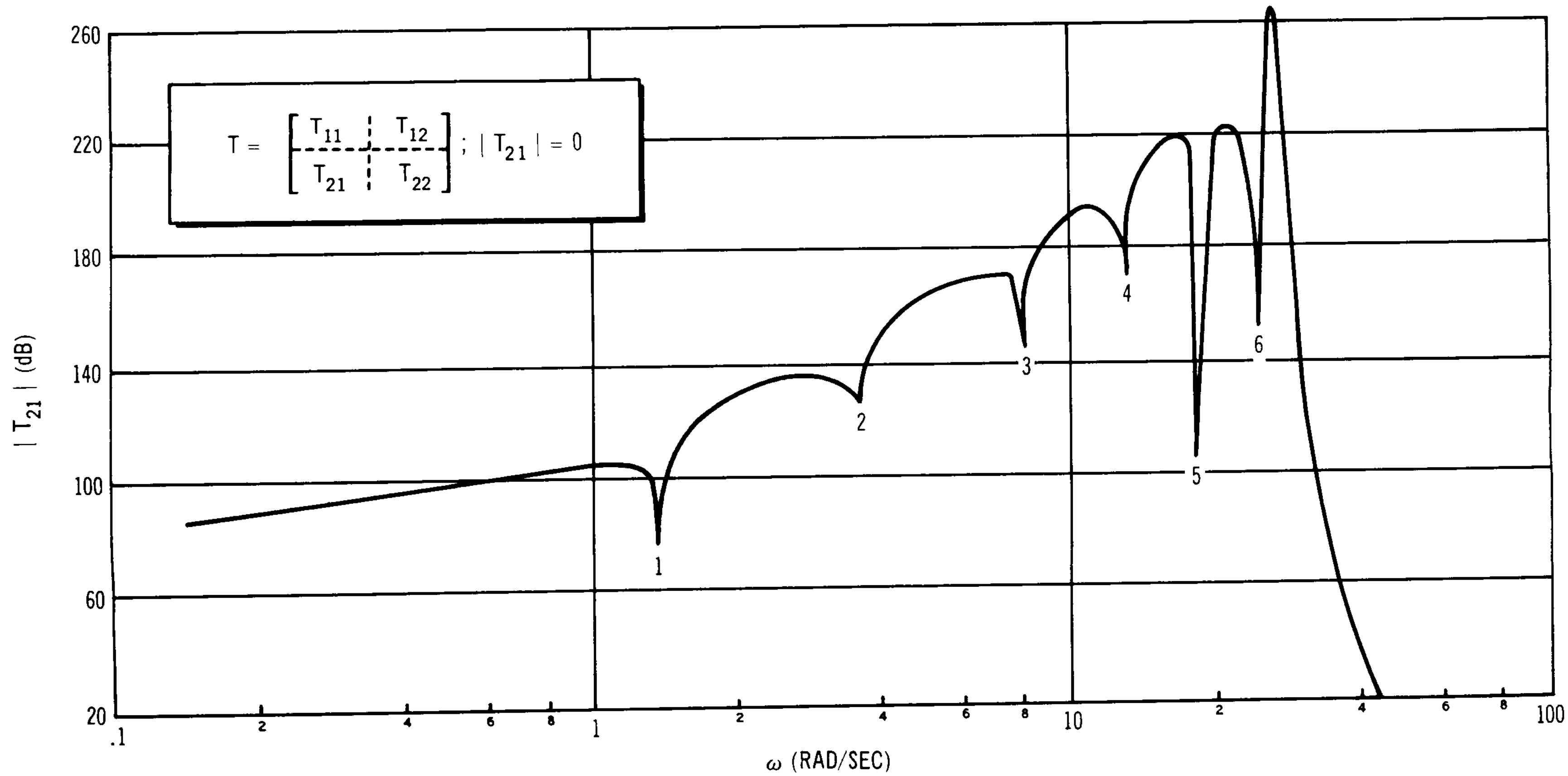


Figure 8-1. Frequency Plot

It is apparent that both methods result in fairly accurate data, with the wave-transmission method being computationally slightly faster, and somewhat more accurate. Therefore, the wave-transmission method has a slight advantage over the Myklestad method.

The physical parameters for the three, thin, uniform beam segments used in this specific case are listed in Table 8-2.

Table 8-2

PHYSICAL BEAM PARAMETERS*

Parameter	Beam 1	Beam 2	Beam 3	Dimensions
ℓ	500	500	500	in.
A	3.1416	6.2832	12.5664	in.^{2}
I	0.7854	3.1416	12.5664	in.^{4}
EI	7.854×10^6	31.416×10^6	125.664×10^6	lb/in.^{2}
a	0.995×10^5	1.412×10^5	1.995×10^5	in.^{2}/sec
wt	0.3063	0.6126	1.2252	lb/in.

* Material: aluminum

Furthermore, the lowest natural frequency could be obtained approximately from the modified cascading method of Section 7.2 With use of the above parameters,

$$T_t^{1/2} = 2.86 \tag{8-5}$$

If end effects are disregarded, the approximating equations previously developed are similar to those of a uniform beam. Therefore, the natural frequencies of the approximating system must be the same as for a uniform beam with same T_t. For a uniform beam, the period of the lowest natural vibration is given by:

$$T_1 = 0.563 \frac{\ell^2}{2a} = 0.563T_t \tag{8-6}$$

For the composite beam with $T_t = 8.15$, $T_1 = 4.58$, and its reciprocal or first bending mode becomes:

$$f_1 = \frac{1}{T_1} = 0.218 \text{ cps} \tag{8-7}$$

Hence, for estimates of the lowest bending mode this estimation procedure appears attractively simple.

8.2 TRANSIENT RESPONSE

The main obstacle in using the distributed parameter approach to analyze the transverse bending dynamics of a structure is in obtaining time response solutions. The problem is twofold, and exists as a result of the computers and the computer techniques currently available. First, the terms involved in most transfer functions are nonanalytical, resulting in standard time response methods being inadequate. Some of these methods may be altered, such as adding damping to the system, but are approximate at best. Secondly, the hyperbolic functions involved and the lack of enough significant digits in computer programs often cause erroneous results. This limitation occurs in differences of very large numbers being so small in the transfer function that the computer assumes them to be zero (Reference 7). This problem will be seen to occur as the frequency increases and is first evident at the second or third natural frequency. Furthermore, this difficulty becomes compounded as more uniform beam segments are used to approximate a non-uniform structure.

As an example, the thin, uniform beam transmission matrix given in Appendix B was used with 1% damping added. The bending poles of the system originally were distributed along the imaginary axis. The addition of the 1% damping displaced them slightly into the left half plane, off the $j\omega$-axis. The resulting transient ($\ddot{y}_2$ to a Q_1 step) still exhibits an oscillation at the first bending-mode frequency of 5.1 cps as see in Figure 8-2. This is caused by the first mode's dominance over the other modes as seen in the transfer

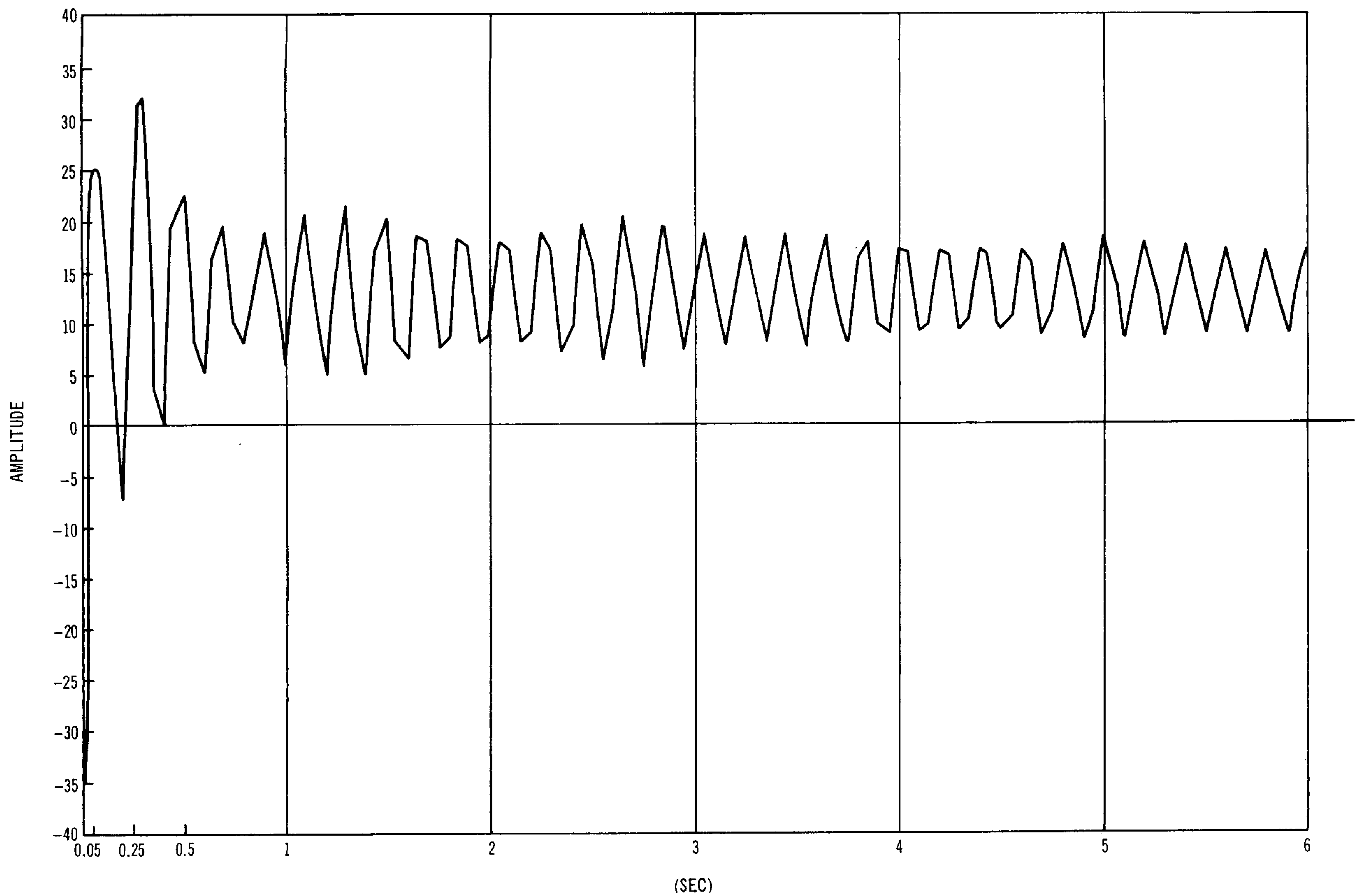


Figure 8-2. ($\ddot{y}_2/Q_1$) Transient for Unit Step Input

function's frequency response (Figure 8-3). Also observed is the function's breakdown occurring at the third and higher bending modes. The beam parameters in this case are $\lambda = 200$ in., $A = 1$ in.^{2}, $E = 30 \times 10^6$ lb/in.^{2}, $\rho = 0.725 \times 10^{-3}$ lb-sec^2/in.^{4}, $I = 0.0796$ in.^{4}, and $a = 57.5 \times 10^3$ in.^{2}/sec and the five bending-mode frequencies in radians per second are first, 32; second, 89; third, 174; fourth, 288; and fifth, 433.

Also, the time response of a cascade of three dissimilar, thin, uniform beams whose parameters are listed in Section 8.1 was obtained and presented in Figure 8-4. Here the linear velocity at stations along the beam's length for a unit step applied at the small end ($x = 1{,}500$ in.) is obtained through a mode summation technique.

This check case indicates that the disturbance propagates instantaneously down the beam. But, obviously, the results obtained from the transmission technique would not agree because a delay occurs in the characteristic variable's path (see Figure 4-1).

8.3 FORMAC (FORMULA MANIPULATION COMPILER) PROGRAM

FORMAC provides a capability for formal manipulation of mathematical expressions as well as analytical differentiation. Operations may be performed on equations or, in this case, matrixes, which are combinations in numeric and symbolic form. However, if numerical values are required, the Fortran IV capability is available as a subset. Detailed information and example programs on FORMAC are available in References 15, 16, 17, and 18.

The advantages associated with such a system are obvious. The most important one is the time savings. Without FORMAC, much time would be spent symbolically multiplying the transmission matrixes manually and, with the addition of second-order effects (shear compliance and rotary inertia), the time spent performing these tedious operations would greatly increase. The other advantage is the reduced likelihood of error. Therefore, by automating the algebraic computations, more time is available for creative work.

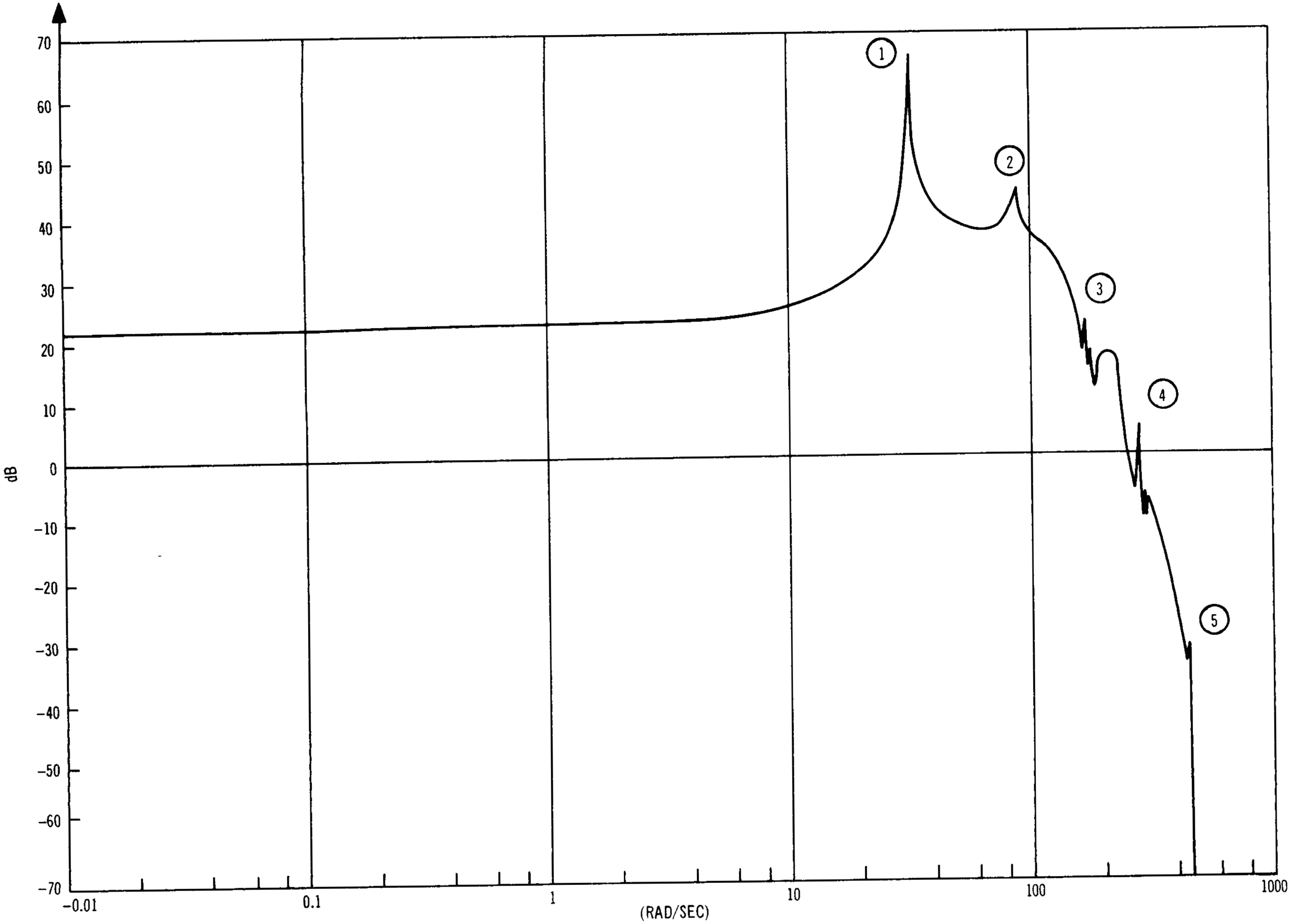


Figure 8-3. $(\ddot{y}_2/Q_1)$ Transfer Function Frequency Response

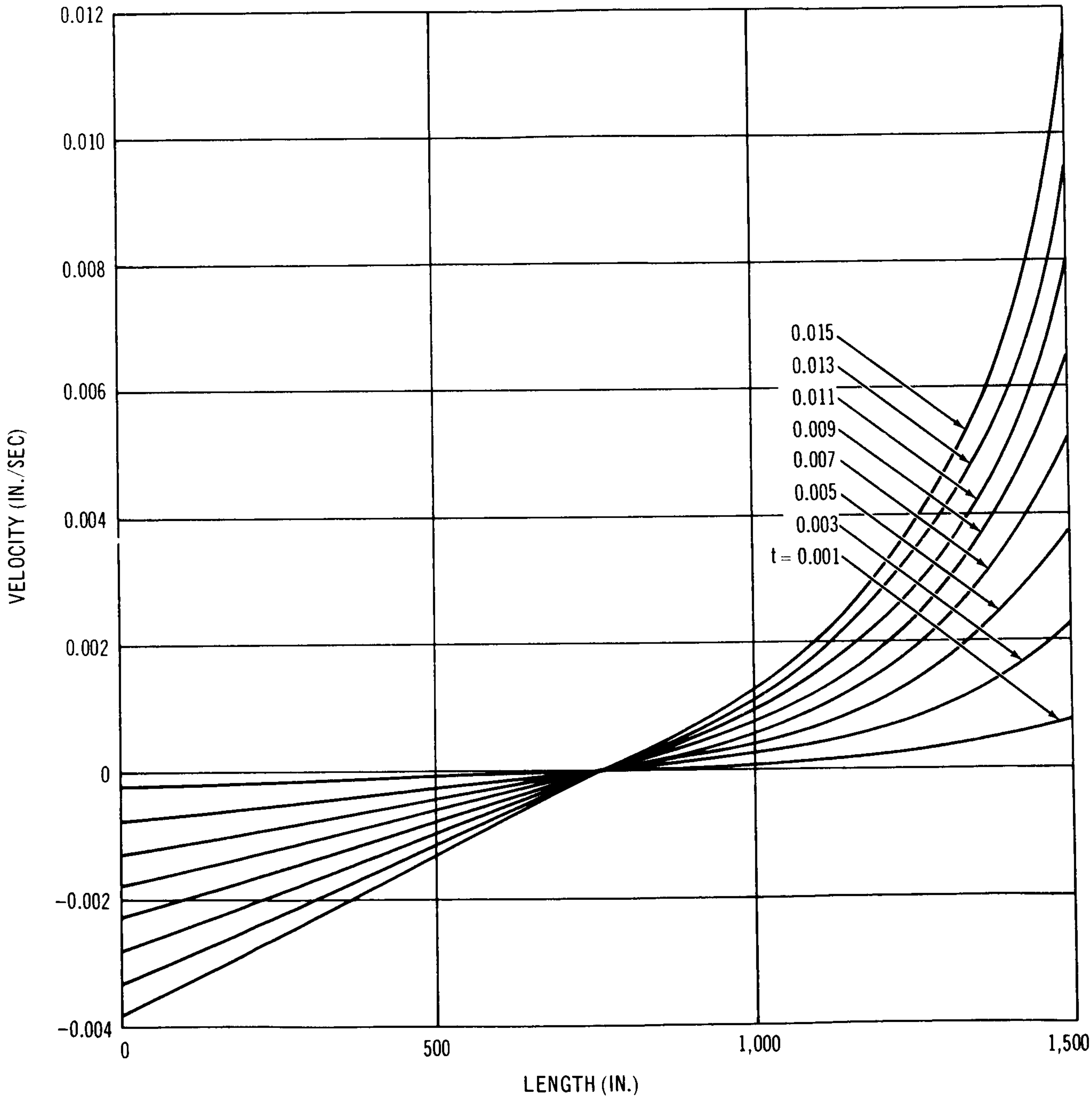


Figure 8-4. Transient Response of Three Beam Cascade

Because FORMAC is a new system, it has certain disadvantages. First, the documentation is somewhat incomplete. A better user's manual is needed. The lag in documentation is understandable, considering the many variations and uses of this symbolic system.

Second, because FORMAC is still an experimental IBM program, errors which the engineer may make in using it become farily difficult to locate.

Troubleshooting must be done either by the engineer or by the IBM representative. Furthermore, running a FORMAC program on the IBM 7094 computer is more difficult because of the special IBM tapes required. However, the FORMAC program has been used with some success in this study.

Section 9

CONCLUSIONS

The authenticity of the general transmission matrix presented in Section 4 and the specific transmission matrixes of Appendixes A and B was established by a series of tests. First, the general transmission matrix was obtained by use of the following methods: (1) the solution of a fourth-order differential equation, and (2) the exponential representation obtained from the canonical form of the state vector equation. By neglect of the shear compliance, rotary inertia, and axial and lateral loads, the transmission matrixes as well as Eigenvalues simplify with slight manipulation to those for a thin, uniform beam (Reference 10). Furthermore, changing the state vector and assuming harmonic motion, the transmission matrix converts to the form of Reference 11.

The combination of transmission matrixes was checked by considering a cascade of three identical beams of length ℓ. The product of the three transmission matrixes reduced to a transmission matrix for a uniform beam of length 3ℓ. An additional check and accuracy test of the transmission matrix formed and transmission technique was demonstrated by comparison of the natural frequencies for the cascaded beam. In this case, the transmission method proved to be faster in computer time and easier to code than the modal method.

The transmission method has a distinct advantage over the standard modal approach for incorporating sloshing masses. In the transmission matrix approach, the sloshing masses are included directly into the bending equation with use of a simple matrix multiplication. In the modal approach, the natural frequencies and modal masses are determined without the sloshing masses. Later, when the equations are in generalized coordinates or transfer function form, the sloshing masses are included. The deflection at the sloshing mass is a summation of the mode deflections and cannot be as accurate as the deflection used in the transmission approach.

The wave-transmission technique's principal difficulty lies in the fractional powers of s and general complexity of matrix forms and resulting transfer functions. The FORMAC program was used with some degree of success to overcome this, but simplification of forms was still left to the engineer. Furthermore, complexity of the terms increases as a function of the non-uniformity of the structure. The technique at present works best on simple structures. Direct application to the analysis of a complete launch vehicle would require either gross simplifying assumptions or development of a digital computer program. Because time solutions were not a goal of this study, no major effort was expended to develop such a program. The matrix operations would lend themselves very well to digital computations but no current technique is available for performing the inverse Laplace transformation of the long algebraic expressions having fractional powers of s. It is recommended that a program with these capabilities be written. Once this is accomplished, the wave-transmission technique should prove itself a valuable analysis technique.

The wave-transmission technique could be very useful in studying the response of structures subjected to loads that are suddenly applied (for example, the study of separation shocks and their effect on the local structure). For problems in which the structure can be represented by a uniform element and the propagating time is of prime importance, this method has a distinct advantage because it can provide information which is unobtainable with other methods.

Appendix A

SPECIFIC BEAM CASES

Appendix A is divided into five sections:

1. Section A.1 -- Presents the micro-element model used in the subsequent sections.

2. Section A.2 -- Presents the specific case of a free-free beam with secondary dynamic effects. Also, it presents a more thorough treatment of the transmission matrix to serve as an example for the cases in Sections A.3 through A.5.

3. Section A.3 -- Presents the equations of interest, defines the general terms of the transmission matrix, the system matrix, the Eigenvector matrix, and lists the specific Eigenvalues for the free-free beam with secondary dynamic effects plus a constant axial force being applied.

4. Section A.4 -- Presents the same as Section A.3 but takes into consideration the longitudinal displacement's effect caused by the axial force upon the transverse bending.

5. Section A.5 -- Presents the same as Section A.3, but for a distributed constant lateral force being applied and the axial force being zero. Also, the additional development techniques required for augmenting the transmission matrix are presented.

A.1 GENERAL MODEL AND RELATIONS

Figure A-1 represents the micro-element model of the beam with lateral inertia, rotary inertia, and lateral deflections caused by bending and shearing strain. This model is under constant axial force with resulting longitudinal displacement and is under distributed, constant, lateral loading. In the following sections, this model will altered to suit the cases considered.

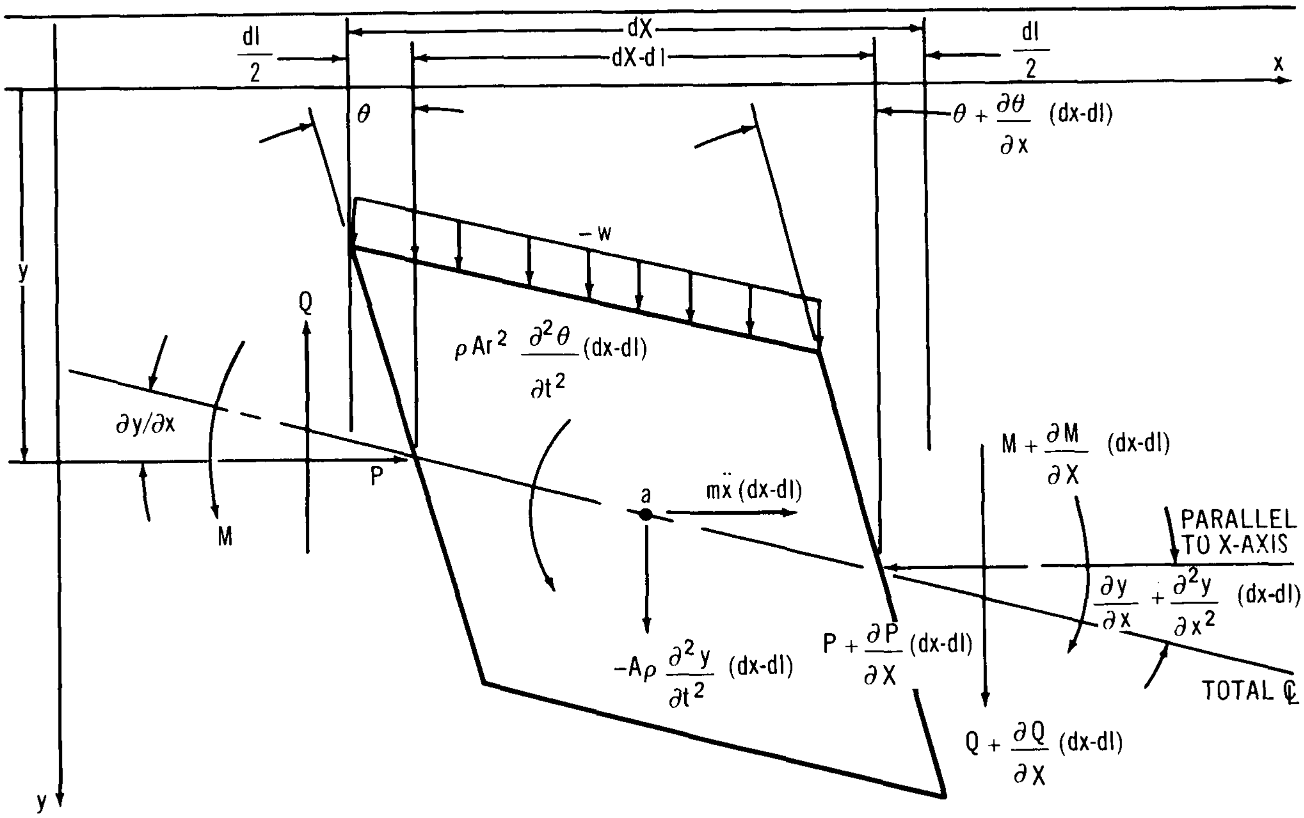


Figure A-1. Microelement Model of Beam

Here the shear force, Q, is assumed to remain parallel to the y-axis and rotates the segment's centerline through an angle $\partial y_s/\partial x$ while the bending moment, M, adds an additional centerline and vertical face rotation of angle θ. The axial force, P, is assumed to remain parallel to the x-axis and the constant distributed lateral force, w, parallel to the y-axis.

A.2 TRANSVERSE BENDING OF A FREE-FACE BEAM WITH SECONDARY DYNAMIC EFFECTS

Consider the micro-element in Figure A-1 with the effects of lateral inertia, rotary inertia, lateral deflections caused by bending strain, and lateral deflections caused by shearing strain ($P = 0$, $w = 0$, $d\ell = 0$, $\ddot{x} = 0$).

The associated four describing equations are:

$$\frac{dy}{dx} = \left(\frac{Q}{GA_s} - \theta\right) \quad \text{or} \quad \frac{dy}{dx} = \left[q\left(\frac{EI}{GA_s}\right) - \theta\right] \tag{A-1}$$

$$\frac{d\theta}{dx} = (M/EI) \quad \text{or} \quad \frac{d\theta}{dx} = [m] \tag{A-2}$$

$$\frac{dM}{dx} = \left(\rho A r^2 \frac{\partial^2\theta}{\partial t^2} - Q\right) \quad \text{or} \quad \frac{dm}{dx} = \left[\frac{\rho A r^2}{EI}\frac{\partial^2\theta}{\partial t^2} - q\right] \tag{A-3}$$

$$\frac{dQ}{dx} = (A\rho s^2 y) \quad \text{or} \quad \frac{dq}{dx} = \left[\frac{A\rho s^2}{EI}\, y\right] \tag{A-4}$$

Defining:

$$\sigma = \left(\frac{-\rho A s^2 \ell^2}{GA_s}\right) \tag{A-5a}$$

$$\tau = \left(\frac{-\rho A r^2 s^2 \ell^2}{EI}\right) \tag{A-5b}$$

$$\beta^4 = \left(\frac{-\rho A s^2 \ell^4}{EI}\right) \tag{A-5c}$$

The specific system matrix becomes:

$$\frac{d}{dx}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & (-EI/GA_s) \\ 0 & 0 & 1 & 0 \\ 0 & (\rho Ar^2 s^2/EI) & 0 & -1 \\ (-A\rho s^2/EI) & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} \tag{A-6}$$

or noting that:

$$(-\sigma \ell^2/\beta^4) = (-EI/GA_s) \tag{A-7a}$$

$$(-\tau/\ell^2) = (\rho Ar^2 s^2/EI) \tag{A-7b}$$

$$(\beta^4/\ell^4) = (\frac{-\rho As^2 \ell^4}{EI}) \tag{A-7c}$$

results in:

$$\frac{d}{dx}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & (-\sigma \ell^2/\beta^4) \\ 0 & 0 & 1 & 0 \\ 0 & (-\tau/\ell^2) & 0 & -1 \\ (\beta^4/\ell^4) & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} \tag{A-8}$$

By manipulating the four equations in the matrix relation above the specific fourth-order differential equation in y evolves as:

$$\{\frac{d^4 y}{dx^4} + (\frac{\sigma+\tau}{\ell^2}) \frac{d^2 y}{dx^2} - (\frac{\beta^4 - \sigma\tau}{\ell^4}) y\} = 0 \tag{A-9}$$

Therefore, $\phi = (\sigma+\tau)$ and $\psi = (\beta^4 - \sigma\tau)$, and the Eigenvalues are:

$$\lambda_{1,2} = \sqrt{\sqrt{\beta^4 + \frac{1}{4}(\sigma-\tau)^2} \mp \frac{1}{2}(\sigma+\tau)} \tag{A-10}$$

and the specific transmission matrix with:

$$K_1 = 1 \tag{A-11a}$$

$$K_2 = (-\sigma\ell^2/\beta^4) \tag{A-11b}$$

$$K_3 = (-\tau/\ell^2) \tag{A-11c}$$

$$K_4 = -1 \tag{A-11d}$$

$$K_5 = (-\beta^4/\ell^4) \tag{A-11e}$$

$$K_6 = \frac{\ell^2}{\beta^4}\,(\lambda_1^2+\sigma) \tag{A-11f}$$

$$K_7 = \frac{-\ell^2}{\beta^4}\,(\lambda_2^2-\sigma) \tag{A-11g}$$

$$\frac{1}{\Delta} = \frac{-\beta^4}{\ell^2}\,\left(\frac{1}{\lambda_1^2+\lambda_2^2}\right) \tag{A-11h}$$

becomes:

$$T = \left\{\begin{array}{cccc}
[\Lambda_2 ch(1) + \Lambda_1 C(2) & [+\ell\Lambda\gamma_1] & [+\ell^2\Lambda\gamma_2] & [\frac{\ell^3}{\beta^4}(+\Lambda_2\lambda_1 sh(1) - \Lambda_1\lambda_2 S(2))] \\
[+\frac{\Lambda\beta^4}{\ell}\gamma_3] & [\Lambda_1 ch(1) + \Lambda_2 C(2)] & [\ell(\frac{\Lambda_1}{\lambda_1} sh(1) + \frac{\Lambda_2}{\lambda_2} S(2))] & [\ell^2\Lambda\gamma_2] \\
[+\frac{\beta^4\Lambda}{\ell^2}\gamma_2] & [\frac{1}{\ell}(\lambda_1\Lambda_1 sh(1) - \lambda_2\Lambda_2 S(2))] & [\Lambda_1 ch(1) + \Lambda_2 C(2)] & [\ell\Lambda\gamma_1] \\
[\frac{-1}{\ell^3}(\frac{\Lambda_2}{\lambda_1} sh(1) + \frac{\Lambda_1}{\lambda_2} S(2))] & [\frac{\beta^4\Lambda}{\ell^2}\gamma_2] & [\frac{\beta^4\Lambda}{\ell}\gamma_3] & [\Lambda_2 ch(1) + \Lambda_1 C(2)]
\end{array}\right\} \tag{A-12}$$

where:

$$\Lambda = \left(\frac{1}{\lambda_1^2 + \lambda_2^2}\right) \qquad \text{(A-13a)}$$

$$\Lambda_1 = \left(\frac{\sigma + \lambda_1^2}{\lambda_1^2 + \lambda_2^2}\right) \qquad \text{(A-13b)}$$

$$\Lambda_2 = \left(\frac{\lambda_2^2 - \sigma}{\lambda_1^2 + \lambda_2^2}\right) \qquad \text{(A-13c)}$$

A.3 TRANSVERSE BENDING OF A FREE-FACE BEAM WITH SECONDARY DYNAMIC EFFECTS SUBJECTED TO AN AXIAL FORCE, P

In this case, $w = 0$ and $d\ell = 0$ when referring to Figure A-1. Two approaches to writing the equilibrium equations of the beam segment model could now be applied. The first approach would treat the beam similarly to that of a catenary cord of uniform weight suspended freely between two points. As a result of equilibrium considerations, an axial centerline force would exist greater than the force P. Here P would be the axial centerline force's component in the x-direction and there also would exist a component in the y-direction. Therefore, the axial force would contribute to the shear on the segment through the equilibrium equations. In this case, the basic shear relation, $Q = GA_s\ \partial y_s/\partial x$ would remain as written. This approach has the disadvantage of complicating the equations of equilibrium for the segment.

The other approach considers the P force parallel to the x-axis only when writing the equation of motion and incorporates the P force's shear influence by modifying the shear relation equation: $(Q + P\theta) = GA_s\ \partial y_s/\partial x$. Because this approach is more convenient than the first approach and because both methods result in identical solution, this is the approach utilized.

The four describing equations in this case when written in matrix form are:

$$\frac{d}{dx}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{Bmatrix} 0 & (1- P/GA_s) & 0 & (-\sigma\ell^2/\beta^4) \\ 0 & 0 & 1 & 0 \\ 0 & (-\frac{\tau}{\ell^2} -P- \frac{P^2}{GA_s}) & 0 & -1 \\ \beta^4/\ell^4 & 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} \quad \text{(A-14)}$$

The resultant forth-order differential equation in y becomes:

$$\{\frac{d^4y}{dx^4} + [(\frac{\sigma+\tau}{\ell^2}) + \frac{P(P+GA_s)}{EIGA_s}]\frac{d^2y}{dx^2} - [\frac{\beta^4(P+GA_s)}{GA_s\ell^4} - \frac{\sigma\tau}{\ell^4}]\, y\} = 0 \quad \text{(A-15)}$$

Therefore, the general terms for the specific Eigenvalues become:

$$\phi = \ell^2(\frac{\sigma+\tau}{\ell^2} + \frac{P(P+GA_s)}{EIGA_s}) \quad \text{(A-16)}$$

$$\psi = [\beta^4(P+GA_s) - \sigma\tau) \quad \text{(A-17)}$$

And from the system matrix, the general terms become:

$$K_1 = (1 - \frac{P}{GA_s}) \quad \text{(A-18a)}$$

$$K_2 = (-\sigma\ell^2/\beta^4) \quad \text{(A-18b)}$$

$$K_3 = (\frac{-\tau}{\ell^2} -P- \frac{P^2}{GA_s}) \quad \text{(A-18c)}$$

$$K_4 = -1 \quad \text{(A-18d)}$$

$$K_5 = (-\beta^4/\ell^4) \quad \text{(A-18e)}$$

$$K_6 = \frac{\ell^2}{\beta^4}\left(\frac{\lambda_1^2 + \sigma}{1-P/GA_s}\right) \quad \text{(A-18f)}$$

$$K_7 = \frac{\ell^2}{\beta^4}\left(\frac{-\lambda_2^2 + \sigma}{1-P/GA_s}\right) \quad \text{(A-18g)}$$

$$\Delta = \frac{-\ell^2}{\beta^4}\left(\frac{\lambda_1^2 + \lambda_2^2}{1-P/GA_s}\right) \quad \text{(A-18h)}$$

The above terms may now be substituted in the general transmission, Eignvector, end-effect, and characteristic matrixes to obtain the specific matrixes for this case.

A.4 TRANSVERSE BENDING OF A FREE-FREE BEAM WITH SECONDARY DYNAMIC EFFECTS SUBJECTED TO AN AXIAL FORCE, P, AND CONSIDERING THE RESULTANT LONGITUDINAL DISPLACEMENT.

Referring to Figure A-1, only w = 0 and the same assumptions and definitions as in Section A.3 apply in this case. Since the compressive influence of the axial force is considered, this case is a more accurate or correct model than that presented in Section A.3.

From the equilibrium and angle relation equations, the system matrix formed is

$$\frac{d}{dx}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{bmatrix} 0 & (1-P/GA_s) & 0 & -\sigma\ell^2/\beta^4 \\ 0 & 0 & 1 & 0 \\ 0 & \left[\frac{-\tau}{\ell^2} - \left(\frac{P+P^2/GA_s}{1-P/EA}\right)\right] & 0 & \left(-1+\frac{P/EA}{(1-P/EA)}\right) \\ \beta^4/\ell^4 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} \quad \text{(A-19)}$$

The $d\ell/dx$ which would otherwise appear in the above matrix equation has been replaced by $(\frac{P/A}{E})$. This expression was obtained by considering the reaction force exerted back by the material being similar to that of a restoring spring force,

$$P = EA \frac{d\ell}{dx} \tag{A-20}$$

That is, proportional to the cross-sectional area A and inversely proportional to the length dx. This is the same as the results one obtains in considering the longitudinal wave-transmission case.

The resultant fourth-order differential equation in this case becomes:

$$\{\frac{d^4y}{dx^4} + [\frac{\sigma+\tau}{\ell^2} + \frac{PEA(P+GA_s)}{EIGA_s(EA-P)}] \frac{d^2y}{dx^2} - [\frac{\beta^4}{\ell^4} (\frac{P+GA_s}{GA_s}) - \frac{\sigma\tau}{\ell^4}] y\} = 0 \tag{A-21}$$

The terms for the Eigenvalues now become:

$$\phi = [\frac{\sigma+\tau}{\ell^2} + \frac{PEA\ (P+GA_s)}{EIGA_s(EA-P)}]\ \ell^2 \tag{A-22}$$

$$\psi = [\ -\sigma\tau + \beta^4\ (\frac{P\ +\ GA_s}{GA_s})] \tag{A-23}$$

and from the system matrix, the K_i's are found to be for the transmission, end-effect, Eigenvector, and characteristic matrixes:

$$K_1 = (1 - P/GA_s) \tag{A-24a}$$

$$K_2 = (-\sigma\ell^2/\beta^4) \tag{A-24b}$$

$$K_3 = [\frac{-\tau}{\ell^2} - (\frac{P+P^2/GA_s}{1 - P/EA})] \tag{A-24c}$$

$$K_4 = [-1 + \frac{P/EA}{(1 - P/EA)}] \qquad \text{(A-24d)}$$

$$K_5 = (-\beta^4/\ell^4) \qquad \text{(A-24e)}$$

$$K_6 = \frac{\ell^2}{\beta^4} [\frac{\lambda_1^2 + \sigma}{1 - P/GA_s}] \qquad \text{(A-24f)}$$

$$K_7 = \frac{-\ell^2}{\beta^4} [\frac{-\lambda_2^2 + \sigma}{1 - P/GA_s}] \qquad \text{(A-24g)}$$

$$\Delta = \frac{-\ell^2}{\beta^4} [\frac{\lambda_1^2 + \lambda_2^2}{1 - P/GA_s}] \qquad \text{(A-24h)}$$

A.5 TRANSVERSE BENDING OF A FREE-FREE BEAM WITH SECONDARY DYNAMIC EFFECTS SUBJECTED TO A DISTRIBUTED CONSTANT LATERAL LOAD

In this case P, $\ddot{x}$, and $d\ell$ are zero; w, the external loading per unit length, is assumed invariant in time and length. The four describing differential equations become:

$$\frac{dy}{dx} = (\frac{Q}{GA_s} - \theta) \qquad \text{(A-25a)}$$

$$\frac{d\theta}{dx} = M/EI \qquad \text{(A-25b)}$$

$$\frac{dM}{dx} = (\rho A r^2 s^2 \theta - Q) \qquad \text{(A-25c)}$$

$$\frac{dQ}{dx} = (A\rho s^2 y + w) \qquad \text{(A-25d)}$$

Using the relations of Equations A-7 and A-25, the fourth-order differential equation in y becomes:

$$\{\frac{d^4y}{dx^4} + (\frac{\sigma+\tau}{\ell^2}) \frac{d^2y}{dx^2} - (\frac{\beta^4-\sigma\tau}{\ell^4}) y\} = (\frac{-w}{EI}) \qquad \text{(A-26)}$$

The Eigenvalues and associated Eigenvectors are identical to those for the model in Section A.2. The system matrix and disturbance matrix are related as:

$$\frac{d}{dx}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & -\sigma\ell^2/\beta^4 \\ 0 & 0 & 1 & 0 \\ 0 & -\tau/\ell^2 & 0 & -1 \\ \beta^4/\ell^4 & 0 & 0 & 0 \end{Bmatrix}\begin{Bmatrix} -y \\ \theta \\ m \\ q \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ w/EI \end{Bmatrix} \tag{A-27}$$

or in canonical matrix form:

$$\frac{d}{dx} Y = [AY + \bar{w}] \tag{A-28}$$

Because all the elements of the A-matrix are constant, the solution of Equation A-28 is:

$$Y(x) = \{e^{Ax} Y_o + e^{Ax} \int_o^x e^{-A\sigma} \bar{w}(\sigma)d\sigma\} \tag{A-29}$$

where:

Y_o is the initial state vector at the point $x = 0$.

e^{Ax} is the transmission matrix T as defined in Section A.2.

$e^{-A\sigma}$ is $T^{-1}(\sigma)$ with σ a dummy integration variable.

$T^{-1}(\sigma)$ in this case is $T^{-1}(x)$ and is found through the relation $T^{-1}(x) = T(-x)$; this property holds whenever the A-matrix is independent of the space coordinate. Therefore, Equation A-29 may be written as:

$$Y(x) = \{T(x)\, Y_o + T(x) \int_o^x T(-\sigma)\bar{w}(\sigma)d\sigma\} \tag{A-30}$$

The integration in Equation A-30 results in the (4x1) matrix:

$$\int_0^x T(-\sigma)\overline{w}(\sigma)d\sigma = \begin{Bmatrix} \frac{-w\ell^4}{\beta^4}[\Lambda_2 ch(1) + \Lambda_1 C(2) - 1] \\ w\ell^3\Lambda\gamma_3 \\ -w\ell^2\Lambda\gamma_2 \\ w\ell\,[\frac{\Lambda_2}{\lambda_1} sh(1) + \frac{\Lambda_1}{\lambda_2} S\ (2)] \end{Bmatrix} \qquad \text{(A-31)}$$

Finally, after performing the premultiplication of the above column matrix by the transmission matrix T(x), the augmented, final transmission matrix has the form:

$$\begin{Bmatrix} Y\ (x) \\ 1 \end{Bmatrix} = \begin{Bmatrix} T\ (x) & D\ (x) \\ 0 & 1 \end{Bmatrix} \begin{Bmatrix} Y\ (o) \\ 1 \end{Bmatrix} \qquad \text{(A-32)}$$

where D(x) is the (4x1) disturbance array influencing the states of the system and whose elements are:

$$(1,1) = \frac{w\ell^4}{\beta^4}[(\Lambda_2 ch(1) + \Lambda_1 C(2))(-\Lambda_2 ch(1) - \Lambda_1 C(2) + 1) + \\ +(\frac{\Lambda_2}{\lambda_1} sh(1) + \frac{\Lambda_1}{\lambda_2} S(2))(\Lambda_2\lambda_1 sh(1) - \Lambda_1\lambda_2 S(2)) + \\ + w\ell^4\Lambda^2\ (\gamma_1\gamma_3 - \gamma_2^2)] \qquad \text{(A-33a)}$$

$$(2,1) = [w\ell^3\Lambda\gamma_2\gamma_3] \qquad \text{(A-33b)}$$

$$(3,1) = w\ell^2\Lambda[-\gamma_2(ch(1) + C(2))(\Lambda_1 + \Lambda_2) + \gamma_2 + \gamma_3(\lambda_1\Lambda_1 sh(1) \\ - \lambda_2\Lambda_2 S(2)) + \lambda_1\Lambda_2(\frac{1}{\lambda_1} sh(1) + \frac{1}{\lambda_2} S(2)] \qquad \text{(A-33c)}$$

$$(4,1) = \frac{w\ell}{\beta^4}\ (\frac{\Lambda_2}{\lambda_1} sh(1) + \frac{\Lambda_1}{\lambda_2} S(2))[(\Lambda_2 ch(1) + \Lambda_1 C(2))(1 + \beta^4) - 1] \qquad \text{(A-33d)}$$

APPENDIX B

TRANSMISSION AND PROPAGATION MATRIXES FOR THE THIN, UNIFORM BEAM

Appendix B contains the specific matrixes for the thin, uniform beam (sometimes referred to as the Bernoulli-Euler beam). For additional information, see Reference 10.

The transmission matrix for a uniform beam is given as:

$$\left\{\begin{matrix} \alpha & \frac{a}{2s} B & \frac{a}{EI} \gamma & \frac{a}{EI} \frac{a}{2s} \delta \\ -\frac{s}{2a} \delta & \alpha & \frac{a}{EI} \frac{s}{2a} B & \frac{a}{EI} \gamma \\ -\frac{EI}{a} \gamma & -\frac{EI}{a} \frac{a}{2s} \delta & \alpha & \frac{a}{2s} B \\ -\frac{EI}{a} \frac{s}{2a} B & -\frac{EI}{a} \gamma & -\frac{s}{2a} \delta & \alpha \end{matrix}\right\} \qquad \text{(B-1)}$$

where:

$$\alpha = \cos \sqrt{Ts} \cosh \sqrt{Ts} \qquad \text{(B-2a)}$$

$$B = (\sin \sqrt{Ts} \cosh \sqrt{Ts} + \cos \sqrt{Ts} \sinh \sqrt{Ts}) \qquad \text{(B-2b)}$$

$$\gamma = \sin \sqrt{Ts} \sinh \sqrt{Ts} \qquad \text{(B-2c)}$$

$$\delta = (\sin \sqrt{Ts} \cosh \sqrt{Ts} - \cos \sqrt{Ts} \sinh \sqrt{Ts}) \qquad \text{(B-2d)}$$

$$a = \frac{EI}{\rho A} \qquad \text{(B-2e)}$$

$$T = \frac{\ell^2}{2a} \qquad \text{(B-2f)}$$

And the two propagation matrixes Pr_1 and Pr_2 are:

$$Pr_1 = Pr_2 = \left\{\begin{matrix} e^{-\sqrt{Ts}} \cos \sqrt{Ts} \\ e^{-\sqrt{Ts}} \sin \sqrt{Ts} \end{matrix}\right\} = \left\{\begin{matrix} -e^{-\sqrt{Ts}} \sin \sqrt{Ts} \\ e^{-\sqrt{Ts}} \cos \sqrt{Ts} \end{matrix}\right\} \qquad \text{(B-3)}$$

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