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AN APPLICATION OF THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS TO ORDINARY AND OPTIMAL PROBLEMS IN HYPERVELOCITY FLIGHT DYNAMICS

by

RICHARD ERROL WILLES

B.S. United States Naval Academy 1957
S.M. Massachusetts Institute of Technology 1962
E.A.A. Massachusetts Institute of Technology 1962

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Signature of the Author — Richard E. Willes
Department of Aeronautics and Astronautics,
August 1966

Certified by: — Wallace E. Vander Velde
Chairman, Thesis Committee

Accepted by:
Chairman, Departmental Committee
on Graduate Students

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ABSTRACT

The method of matched asymptotic expansions is introduced as a systematic approach to the problem of analytically describing flight trajectories. Both new and previously known solutions in flight mechanics are produced in an unified procedure that is capable of estimating their region of validity, extending the solution to higher accuracy, and combining the solutions to obtain expressions valid over several regions of interest.

Specifically, all the first approximations to the flight dynamic equations are identified and their region of validity is established. Asymptotic expansions for the solutions of the dynamic equations are produced for a number of regions. The analyses of Sanger[9, 104], Allen and Eggers[8], Chapman[14], Lees[7], Shen[28] are all shown to be systematic approximations within this context. The extent to which Loh's [2, 35] analysis can be considered systematic is demonstrated and its region of validity is identified. A procedure for extending these solutions to higher order and greater accuracy is illustrated. Two of the expansions are matched to produce a composite solution valid for a currently interesting class of lifting trajectories.

Analytical investigation of some optimal flight trajectories is accomplished. Cbservations are made concerning the structure of optimal plane change, minimum velocity lost, maximum range, and minimum heating trajectories. The advantages of uniformly valid analytical solutions for guidance applications are ennumerated and possible implementations in guidance schemes are suggested.

Thesis Committee:

Robert L. Halfman, S.M.
Professor of Aeronautics and Astronautics

Wallace E. Vander Velde, Sc. D.
Professor of Aeronautics and Astronautics

Marten T. Landahl, Tekn. D.
Professor of Aeronautics and Astronautics

Thesis Committee: (continued)

Arthur E. Bryson, Jr., Ph.D.
Hunsaker Professor of Aeronautics Engineering (Visiting)

Michael Athans, Ph.D.
Assistant Professor of Electrical Engineering

James E. Potter, Ph.D.
Assistant Professor of Aeronautics and Astronautics

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"If we see new truths, it is because we have been privileged to stand on the shoulders of giants."

John Milton of Galileo

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LIST OF SYMBOLS

General Notation

A prime will be used to indicate the same variable scaled by a dimensional or nondimensional quantity, i.e., $v^2 = v^{2'}/g_o r_o$, $v^2 = \epsilon v^{2'}$, etc. The order in ϵ of a variable will be indicated by a capital "O", i.e., v^2 is order ϵ is written $v^2 = 0(\epsilon)$. A series expansion in ϵ for a variable will have the form $v_1(t) = \Sigma \epsilon^n v_1^{(n)}(t)$ where the superscript indicates the order of the expansion variable and the subscript indicates that it is the expansion for region one.

A vector will be an underlined lower case letter, i.e., $\underline{v}$. The magnitude of the vector will be the lower case letter without an underline, i.e., v is the magnitude of $\underline{v}$. A matrix will be a capital letter, i.e., Φ, F. The transpose of a vector of matrix is indicated by a superscript T, i.e., Φ^T. The derivative of a variable with respect to an independent variable, not necessarily time, is indicated with a dot, i.e., $\dot{\underline{v}}$. The derivative of a vector function with respect to a vector, i.e., $\partial \underline{f} / \partial \underline{x}$, will be considered a matrix or a second-order tensor. The second derivative of a vector function with respect to a vector, i.e. $\partial^2 \underline{f} / \partial \underline{x} \, \partial \underline{x}$, will be considered a third-order tensor, etc. The statistical expectation of a random variable will be indicated with a capital $\mathcal{E}$, i.e., $\mathcal{E}[\underline{w}(t)] = 0$.

English Symbols

a	semi-major axis
A	reference area
A_A	the set of all possible aerodynamic accelerations
A_T	the set of all possible thrust accelerations
A	the set of all possible thrust and aerodynamic accelerations
c	speed of sound
C_L	lift coefficient
C_D	drag coefficient
C_G	generalized aerodynamic coefficient
D	drag
e	eccentricity

English Symbols (cont.)

$\underline{f}(\ )$	an arbitrary vector function
$\underline{g}(\ )$	an arbitrary vector function
$\underline{g}$	acceleration of gravity
G	universal gravitational constant
G	generalized aerodynamic effect, either load factor or heating
h	altitude (or height) above a reference radius
$\hbar$	nondimensional angular momentum
H	Hamiltonian
J_2	second harmonic coefficient
k	Boltzman constant
ℓ	a reference length
L	planetary latitude
m	vehicle mass
$\overline{m}$	mean molecular weight of the atmospheric gas
M	mass of the planet
M	mach number
n	load factor
p	atmospheric pressure
$\dot{q}$	heating rate
$\underline{r}$	a position vector from the planet's center
R	atmospheric gas constant
$\overline{R}$	universal gas constant
Re	Reynolds number
t	time
T	atmospheric temperature
$\underline{T}$	thrust vector
$\underline{u}$	a general control vector
$\underline{v}$	vehicle velocity vector
$\underline{v}_w$	wind velocity vector

English Symbols (cont.)

V	gravitational potential
$\underline{x}$	a general state vector

Greek Symbols

α	angle of attack
β	inverse atmospheric scale height
γ	flight path angle
Υ	ratio of specific heats
δ	variation or perturbation of the following quantity
ϵ	a general small parameter
ϵ, ϵ_1	ratio of atmospheric scale height to planetary radius
ϵ_2	ratio of orbital period to rotational period
θ	range angle
λ	adjoint variables, costate variables, or Lagrange multipliers
μ	viscosity coefficient
ρ	atmospheric density
ϕ	roll angle - measured about the velocity vector
$\Phi(t_1,t_2)$	a state transition matrix for a linear system from t_2 to t_1
Ω	the planet's rotation rate

Superscripts

T	vector or matrix transpose
-1	matrix inverse
$'$	the same variable scaled by a dimensional or nondimensional quantity
$(\ )$	the order of an expansion variable

Subscripts

o	constant value
o	initial value
f	final value
L	related to lift
D	related to drag
Q	related to heating
$\oplus$	earth value

CHAPTER I

INTRODUCTION

1.1 Scope of Thesis

The method of matched asymptotic expansions has recently emerged as a highly systematic means of treating nonlinear prblems in which a small parameter appears. It has been extensively used in the field of fluid mechanics[1, 4, 12, 57] and recently in a number of problems in celestial mechanics.[64, 66, 67, 70] This thesis is the natural extension of the previous uses of the method into the area of entry dynamics and hypervelocity flight mechanics.

1.2 Analytical Flight Mechanics

In analytical flight mechanics, as in many other problems, one is faced with a set of dynamic equations much too complicated to solve in any generalized sense. The central problem then is one of approximating, or modeling, the more complex system in terms of simpler systems. Hopefully, the simpler system gives insight into the dynamics of the more complex system over some limited region of operation. There are many fine examples of this type of endeavor in both classical flight dynamics[37, 47] and more recently in the dynamics of atmospheric entry or hypervelocity flight.[2, 7, 8, 9, 10, 11, 14, 33, 38]

A common characteristic of all such analyses is that they have a limited range of validity. This has usually been carefully pointed out by the analyst.[7, 8, 9, 10, 11, 33, 38] A further complication is that it is generally not obvious how the original work is extended to some arbitrary degree of accuracy. Also transitioning from one region of flight, where a particular analysis is valid, to another region where another analysis is valid is awkward if at all possible. All of these problems have simple answers, if not always solutions, in the context of the method of matched asymptotic expansions.

1.3 The Method of Matched Asymptotic Expansions

The most naive form of the method of matched asymptotic expansions will serve the purposes of this thesis. For an elaborate and enlighting treatment, the reader is referred to Van Dyke[1]. The approach that will be taken here is to seek a valid first approximation to the dynamic equations of flight. Small neglected terms are then assumed to only cause linear perturbations in a solution to the valid first approximation. So that once the lowest order problem, or first approximation, has been determined, all succeeding corrections are simply linear perturbation problems.

The solution can then, in principle, be extended to any order to include all small effects. The first-order solution and the solution to all the associated linear perturbation problems is called an expansion for the solution of the complete problem.

Some care must be taken in determining the valid first approximation. This approximation is basically different depending on the values of the problem variables that appear. All such first approximations may be systematically ennumerated by considering all possible values of the problem's variables measured in powers of the problem's small parameters. Terms that may be neglected will then appear multiplied by some power of the small parameter. The first, or lowest order, approximation for a particular range of values of the variables is obtained by simply retaining non-negligible terms. The neglected terms are then included as linear perturbations to the lowest order solution. So, small parameters form a scale factor that allows division of the variable, or state, space of the dynamic system into different regions of behavior in which different expansions are valid. After two expansions, valid in two neighboring regimes, have been obtained they may be combined by simply requiring that they match smoothly in the region of their common boundary.

The attempt will be to place previous work in flight mechanics in this framework of systematic approximation. This will allow a careful delineation of the range of validity and accuracy of existing solutions. It will also offer a straightforward method of improving the solutions by extending them to higher order. Solutions uniformly valid over a number of regions, suitable for guidance applications, will be produced by matching the expansions. Finally, the difficult task of analytically modeling the optimal trajectory problems will be accomplished by simply retaining only the lowest order problem in which the control appears. A tracticable analytical problem will often be produced. Thus, many interesting results will be produced with a straight-forward application of a well established perturbation technique in an area in which analytical progress in the past has been difficult.

1.4 The Flight Environment

This thesis will deal with the analytical description of flight paths for a vehicle under the influence of gravity, acceleration, aerodynamic forces and thrust, constrained by limits on gas dynamic heating and vehicle specific force. Prior to initiating this analysis, a brief description of the flight environment is in order.

1.4.1 The Planetary Atmospheres

The atmosphere is considered to be a multicomponent gas of relative uniform composition over the altitudes of interest in flight dynamics. Its motion is predominantly that of rotation with the planet with a small superimposed horizontal wind structure. Its interaction with the planet's gravitational field is conveniently described in terms of a momentum or force balance in the vertical direction (where the wind components are negligible)

$$\frac{\partial p}{\partial \underline{r}} = -\rho \underline{g} \qquad (1.4.1\text{-}1)$$

p and ρ are the atmospheric pressure and density and $\underline{g}$ is the acceleration of gravity in the planet's rotating coordinate system.

The pressure and density are conveniently related to atmospheric temperature, T, by the equation of state of a relatively low density gas

$$p = \rho \frac{\bar{R}}{\bar{m}} T \qquad (1.4.1\text{-}2)$$

where $\bar{R}$ is the universal gas constant and $\bar{m}$ is the molecular weight of the uniform gas. The temperature has small but important variation with both position and altitude; similarly $\underline{g}$ varies with $\underline{r}$. The neglect of these variations for the moment allows Eqs. (1.4.1-1) and (1.4.1-2) to be combined to give an expression for p or ρ as a function of height.

$$p = p_0 e^{-\beta_0 h} \qquad (1.4.1\text{-}3)$$

or

$$\rho = \rho_0 e^{-\beta_0 h} \qquad (1.4.1\text{-}4)$$

where h is altitude above the surface of the planet and β_o is the reciprocal atmospheric scale height defined as

$$\beta_0 = \frac{\bar{m} g_0}{\bar{R} T_0} \qquad (1.4.1\text{-}5)$$

It is observed that the ratio of atmospheric scale height to planetary radius is a small number for all known planetary atmospheres. (See Table I.) This implies that p and ρ vary many orders of magnitude over a height small in comparison with the planetary radius. The atmosphere is thus a relatively thin shell surrounding the planet. The ratio of atmospheric scale height to planetary radius is a small parameter that will be an important scaling factor in the flight dynamic problem. For convenience it will be hereafter referred to as ϵ (or in Chap. IV as ϵ_1).

$$\varepsilon = \varepsilon_1 = \frac{1}{\beta_0 r_0} \qquad (1.4.1\text{-}6)$$

For a more adequate treatment of planetary atmospheres the reader is referred to Appendix E and References (16, 83).

The fundamental physical significance of a small ϵ may be obtained by a slight rearrangement of its definition

$$\varepsilon = \frac{\bar{R}}{\bar{m}} \frac{T_0}{g_0 r_0} = \frac{k T_0}{m g_0 r_0} \qquad (1.4.1\text{-}7)$$

where k is the Boltzmann constant and m is the mass of a typical gas molecule. The quantity kT_o represents the thermal energy of a molecule while $mg_o r_o$ is the energy required for a molecule at the planet's radius to escape the planet's gravitational field. It is therefore necessary for the retention of the atmosphere, that ϵ be small.*

1.4.2 The Gravitational Field

A planet's gravitational field in a rotating coordinate system fixed to the planet is adequately described by the negative gradient of a potential, expressed in terms of spherical harmonics and corrected for planet's rotation, as

$$\underline{g} = -\frac{\partial V}{\partial \underline{r}} \qquad (1.4.2\text{-}1)$$

$$V = -\frac{GM}{r_0}\left[\frac{r_0}{r} - \left[\left(\frac{r_0}{r}\right)^3 \frac{J_2}{2}(3\sin^2 L - 1)\right] + \cdots + \frac{\Omega^2 r_0^3}{GM}\left(\frac{r}{r_0}\right)^2 \cos L\right] \qquad (1.4.2\text{-}2)$$

where $\underline{g}$ is the gravitational acceleration vector, V the potential field, G the universal gravitional constant, M the mass of the planet, $\underline{r}$ a position vector, r_o the equatorial radius, L the planetary latitude, Ω the planet's rotation rate and J_2 the second-harmonic coefficient (see Appendix D). Normally, it will be necessary to retain more than the first term in this series, the spherically symmetric inverse r field, but usually no more than the second term.

*This interesting observation was pointed out to the author by Prof. A. E. Bryson.

This is because atmospheric flight occurs at altitudes small in comparison to the planetary radius where the second term is not necessarily negligible. Also, the planetary oblateness causes a warping of the thin atmospheric shell that is of the same order of magnitude as the thickness of the atmosphere. This implies that the major contribution to variations of atmospheric properties with r is due to the planetary oblateness.

1.4.3 Aerodynamic Forces

By convention, aerodynamic forces are resolved into two components, drag and lift, along and normal to an air mass referenced velocity vector. They are expressed in terms of nondimensional coefficients, C_D, and C_L, as,

$$D = \frac{C_D A}{2} \rho v^2 \qquad L = \frac{C_L A}{2} \rho v^2 \tag{1.4.3-1}$$

where D and L are drag and lift, A is some suitable reference area, and $\underline{v}$ is the vehicle velocity referenced to the air mass. C_L and C_D are functions of the Mach number, M, and Reynolds number, Re,

$$M = \frac{v}{c} \tag{1.4.3-2}$$

$$R_e = \frac{\rho v \ell}{\mu} \tag{1.4.3-3}$$

where c is the speed of sound in the atmospheric gas, μ is the viscosity coefficient, ℓ is some arbitrary reference length. The speed of sound is related to previously defined quantities as

$$c = \sqrt{\gamma R T} \tag{1.4.3-4}$$

where $\overline{\gamma}$ is the ratio of specific heats and R is the gas constant of the atmospheric gas, where $R = \overline{R}/\overline{m}$.

Basically, the Mach number dependence of the aerodynamic coefficient is related to compressibility and the Reynolds number dependence is related to viscosity. The functional dependency of C_L and C_D on M and Re differs markedly depending on the gas dynamic regimes in which flight is being conducted. A description of these regimes follows.

1.4.4 Gas Dynamic Regimes

A vehicle encountering a planetary atmosphere will pass through several gas dynamic regimes depending predominately on the ratio of the vehicle velocity, v, to the speed of sound, c (approximately the mean "thermal" velocity of the gas molecules). This ratio, the Mach number, may be written as

$$M = \frac{v}{\sqrt{\bar{\gamma} R T_0}} \qquad (1.4.4\text{-}1)$$

where $\bar{\gamma}$ is the ratio of specific heats for the atmospheric gas. This may be rewritten in terms of the inverse atmospheric scale height and planetary radius as

$$\begin{aligned} M &= \frac{v}{\sqrt{r_0 g_0}} \sqrt{\frac{\beta_0 r_0}{\bar{\gamma}}} \\ &= \frac{v}{\sqrt{r_0 g_0}} \sqrt{\frac{1}{\bar{\gamma}\epsilon}} \end{aligned} \qquad (1.4.4\text{-}2)$$

If one observes that the quantity $\sqrt{r_0 g_0}$ is orbital velocity and $\bar{\gamma}$ is of order one for all gases, then Eq. (1.4.4-2) indicates that the Mach number is high for vehicles with velocities the order of orbital velocity in any planetary atmosphere. When the ratio of velocity to orbital velocity is of order $\epsilon^{\frac{1}{2}}$, the Mach number is of order one, and when this ratio is of order ϵ, the Mach number is of order $\epsilon^{\frac{1}{2}}$.

It is well known that there are fundamentally different gas dynamic regimes associated with high, intermediate, and low Mach numbers. It will later be shown that the dynamic behavior of the flight vehicle divides into three analogous regimes, also dependent on the value of the Mach number. A cursory description of the gas dynamic regimes follows. For an adequate treatment of the hypersonic flow regime, with which this thesis is mainly concerned, the reader is referred to References (49, 95, 96).

(1) Free Molecular Flow Regime *

At appreciable distances from the planet's surface, and therefore at very low densities, the vehicle's encounters with gas molecules and atoms are relatively infrequent. The molecules are either reflected or accommodated by the vehicle with a resultant exchange of momentum. Then gas particles do not encounter other gas particles at distances of the order of the vehicle's dimension as their mean free path is long. The vehicle thus continuously meets an undisturbed stream of molecules with mean velocity equal to the vehicle velocity relative to the atmosphere. The aerodynamic force, due to the momentum exchange with the molecular stream, is relatively small because of the low encounter rate. But they are large enough, over an appreciable period of time, to cause orbital decay.

(2) Continuum Hypersonic Flow Regime

As the density increases, molecules reflected and emitted from the surface of the vehicle encounter other molecules at distances relatively close to the vehicle. A rather low velocity and consequently, high density cloud of molecules forms on the front side of the vehicle. There is an appreciable momentum transfer to the vehicle that is basically due to the increase in number density of the flow. In both the free molecular flow and the hypersonic flow regimes the aerodynamic forces on the vehicle are dominantly functions of the momentum transport of the free stream, ρv^2. The aerodynamic coefficients C_L and C_D, defined as

$$C_L = \frac{L}{\frac{A}{2}\rho v^2} \qquad C_D = \frac{D}{\frac{A}{2}\rho v^2} \tag{1.4.4-3}$$

vary relatively little with Mach number and Reynolds number.

(3) Supersonic, Transonic and Subsonic Flow Regime

As the vehicle loses kinetic energy, and thus velocity due to aerodynamic drag, the velocity of the vehicle becomes of the same order as the mean thermal speed of the gas molecules. In this velocity range the aerodynamic forces are sensitive to the type of wave pattern being caused by the reflected molecules in the undisturbed stream. Thus C_L and C_D are very sensitive to the vehicle speed relative to the air mass, or the Mach number. A vehicle predominantly

*It should be observed that free molecular flow is associated with long mean free paths and not necessarily with any particular velocity regime.

designed for hypersonic flow, where blunt vehicles are of no particular disadvantage, encounters a particular high wave drag.

(4) Constant Density Flow Regime

When the velocity of the vehicle becomes slow, relative to the thermal speed of the molecules, disturbances are propagated upstream at velocities large in comparison to the velocity of the vehicle. The gas then behaves as though it was a constant density fluid, so C_L and C_D are not M dependent. Drag due to viscosity and separation effects are usually large for hypersonically designed vehicles with a blunt rear end. Further, lift-drag ratios are low because of the usual low-aspect-ratio design of the vehicle. These considerations take on considerable importance if the vehicle is to be landed.

1.4.5 Aerodynamic and Thrust Load Factor

While a vehicle is in free fall under the influence of only gravity there is no acceleration sensed by the pilot. When the vehicle is subjected to either aerodynamic or thrusting acceleration the pilot is accelerated proportionately. To preclude excessive loads on the pilots and/or equipment in a spacecraft, entry trajectories must be constrained to keep this combined aerodynamic and thrust acceleration within certain limits.

For the equipment there is usually some design load factor that must not be exceeded. For a human pilot, there is a more complex limitation. Depending on his orientation, his physical condition, and the task that he has to perform, he can generally accept a given loading for a specified time. The higher the loading the shorter is the duration of time it is acceptable. However, in the present analysis, it will normally be assumed, for analytical convenience, that the trajectory is constrained to keep the load factor below a fixed limit.

1.4.6 Aerodynamic Heating

A vehicle moving at high velocities relative to a gas will experience appreciable transfer of the kinetic energy of the gas molecules to the vehicle. Fortunately, there are a number of mechanisms that preclude the complete transfer of all the energy of the gas stream to the vehicle. Most of the incoming molecules encounter other molecules that are reflected or emitted from the vehicle. These form an intensely dense layer of gas molecules with low mean velocity relative to the vehicle (a "gas cap" in back of a nearly normal shock). The molecules in the gas cap have high thermal velocity, as energy is initially conserved. Energy is dissipated in this gas cap by a number of means. The gas molecules, undergo numerous and violent collisions with other molecules. This excites internal energy modes to the point of disassociating and ionizing the gas (breaking the gas molecules into its constituent atoms and stripping the electrons from the remaining atoms). The hot gas emits photon energy (radiation), unfortunately nearly half of it toward the vehicle, but the other half into free space. Finally, there is a mean flow of the gas in the gas cap around the vehicle and into a wake. This transports the energy imbedded in the heavily excited gas away from the vehicle.

The methods of protecting the vehicle from that portion of the energy that does eventually reach it may be divided into three categories:

(1) Ablation

The exterior of the vehicle that is subjected to the intense heat is covered with an expendable coat of material that either melts or sublimes, absorbing the energy input from the gas, and simultaneously dumping the hot products into the wake.

(2) Reradiation

The exterior of the vehicle is designed to accept some rather high temperature where the energy input can be reradiated into free space.

(3) Heat Sink

The vehicle is designed to accept the energy input which it might attempt to store for a short period or transfer to the wake by dumping coolant.

A given vehicle usually uses all three methods to some extent. But depending on whether it is dominantly reradiative or ablative cooled the trajectory must be designed to either not exceed some heating rate associated with the vehicle's allowable surface temperature or to not exceed some maximum allowable total heat.

For the purpose of estimating energy input to a vehicle, in convective and radiative forms, one can use empirical relations of the following form (See Appendix E)

$$\dot{q} = C_Q \rho^{i} v^{2j} \qquad (1.4.6\text{-}1)$$

where $\dot{q}$ is a heating rate and C_Q is a geometry dependent heating coefficient. The exponents (i and j) are functions of the gas and unfortunately somewhat sensitive to the particular empirical investigation. For analytical purposes here, the exact values of the exponents will be unimportant.

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CHAPTER II

THE DYNAMICS OF TWO-DIMENSIONAL NONTHRUSTING FLIGHT

2.1 Introduction

It is the objective of the present investigation to study the dynamics of flight in a real rotating atmosphere surrounding an oblate planet. To preclude overbearing complexity from the start, it will be convenient to consider a problem commonly treated in the literature of entry dynamics or hypervelocity flight mechanics: Nonthrusting two-dimensional flight in a nonrotating atmosphere surrounding a spherically symmetric planet. This analysis will produce the same result as the more complex problem to lowest and sometimes to next order. Some efforts will be made to pursue higher-order solutions but only to the extent that they will be unefected by the addition of oblateness and rotating atmosphere effects. The justification for this simple approach will follow in Chapter IV.

In the analysis of the two-dimensional problem, the technique will be to identify the lowest-order problems that describe different phases of a flight trajectory. These lowest order problems will be descriptively called regimes of flight. These regimes will be first produced in an ad hoc manner, to both develop solutions that will later be used, and to give insight into a systematic procedure for identifying all such regimes.

Two techniques of combining expansions will be given. One is an intuitive method that comes from considerable familiarity with the expansions. It basically involves finding a solution that reduces to the proper lowest order form in a number of regimes. The other method is analytically straightforward and involves matching expansions. It will be treated in Chapter III.

2.2 The Dynamical Equations for Two Dimensional Nonthrusting Flight

The dynamical equations, which describe two dimensional nonthrusting flight, through a nonrotating atmosphere, surrounding a spherically symmetric planet, can be written in nondimensional form as follows:

$$\frac{dv}{dt} = -\frac{C_D}{2}\frac{\rho v^2}{\varepsilon} - \frac{1}{(1+h)^2}\sin\gamma \qquad (2.2\text{-}1)$$

$$v \frac{d\gamma}{dt} = \frac{C_L}{2}\frac{\rho v^2}{\varepsilon} - \cos\gamma\left(\frac{1}{(1+h)^2} - \frac{v^2}{1+h}\right) \qquad (2.2\text{-}2)$$

$$\frac{d\theta}{dt} = \frac{v\cos\gamma}{1+h} \qquad (2.2\text{-}3)$$

$$\frac{dh}{dt} = v\sin\gamma \qquad (2.2\text{-}4)$$

where

$$\frac{d\rho}{dh} = -\frac{\rho}{\varepsilon(1+h)^2} \qquad (2.2\text{-}5)$$

The following dimensional and nondimensional quantities have been used:

$v = \frac{v'}{(r_0' g_0')^{\frac{1}{2}}}$ - velocity magnitude/reference orbital velocity.

$t = \frac{t'}{\left(\frac{r_0'}{g_0'}\right)^{\frac{1}{2}}}$ - time/orbital period

$h = \frac{h'}{r_0'} = \frac{r' - r_0'}{r_0'}$ - height above reference radius/reference radius

r' - reference radius

$$\frac{g'}{g_0'} = \frac{1}{(1+h)^2}$$ - gravitational acceleration/gravitational acceleration

$$\beta' = \frac{g_0}{RT}$$ - inverse scale height

$$\beta = \frac{\beta'}{\beta_0'}$$ - inverse scale height/reference inverse scale height

$$\varepsilon = \frac{1}{\beta_0' r_0'}$$ - reference scale height/reference radius (a small quantity)

$$\rho = \frac{\rho'}{\frac{m\beta_0'}{A}}$$ - nondimensional density

$$p = \frac{p'}{\frac{mg_0}{A}}$$ - pressure/wing loading

The other quantities are defined in Fig. (2.2-1) and in the List of Symbols.

The first two equations are obtained from an acceleration and force balance along and normal to the flight path. The second two equations are kinematic relation expressing altitude and range rate in terms of velocity and flight path angle. The final equation is the hydrostatic equation relating pressure variations with altitude in the atmosphere.

Pressure and density are related by the equation of state for a low density gas.

$$p' = \rho' RT \qquad (2.2\text{-}6)$$

or in nondimensional form

$$p = \frac{\rho}{\beta} \qquad (2.2\text{-}7)$$

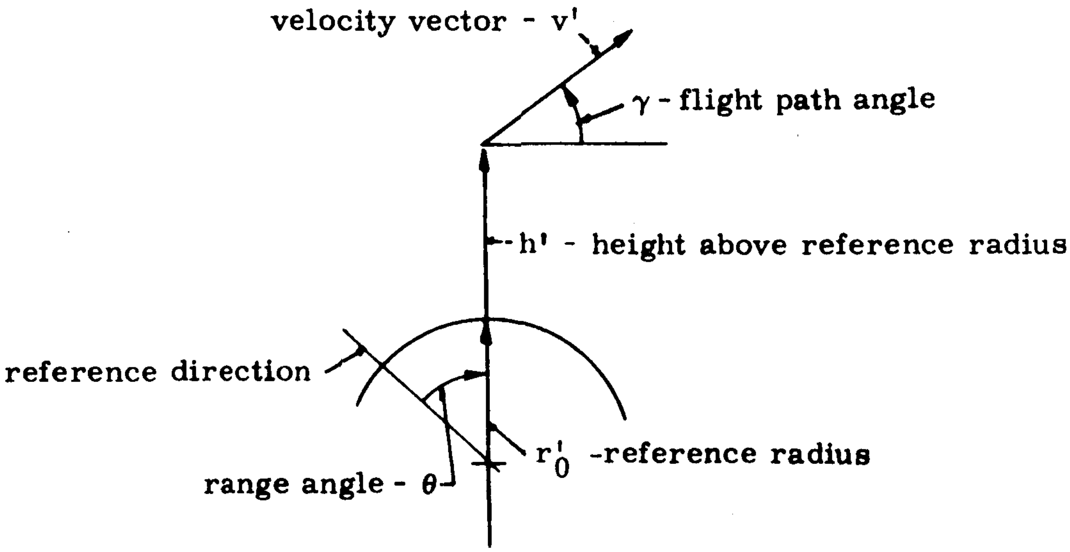


Fig 2.2-1 Geometry of Two Dimensional Flight

As time only appears in the equations in the form of derivatives, it may be conveniently eliminated by dividing Eqs. (2. 2-1- 3) by Eq. (2. 2-4). This gives a new set of dynamic equations in the independent variable h and dependent variables θ, v^2, p, γ as follows:

$$\frac{dv^2}{dh} = -C_D \frac{\rho}{\varepsilon} \frac{v^2}{\sin\gamma} - \frac{2}{(1+h)^2} \tag{2, 2-8}$$

$$\frac{d\cos\gamma}{dh} = -\frac{1}{2} C_L \frac{\rho}{\varepsilon} \cdot \left(\frac{1}{1+h} - \frac{1}{(1+h)^2 v^2} \right) \cos\gamma \tag{2. 2-9}$$

$$\frac{d\theta}{dh} = \frac{\cot\gamma}{1+h} \tag{2. 2-10}$$

$$\frac{dp}{dh} = -\frac{\rho}{\varepsilon(1+h)^2} \tag{2. 2-11}$$

The final equation, Eq. (2. 2-11), is strictly not a dynamic equation by simply a differential equation specifying the variation of p with h. By keeping the relation in this form, it will be possible to obtain results for an atmosphere of arbitrary temperature or variation with h.

2. 3 Scaling the Dynamical Equations

A formalism that is essential to the success of the perturbation method to be applied here is the proper scaling of the variables. Any equation written in nondimensional form must have all its nondimensional variables of order one if the perturbation scheme is to succeed. Thus, Eqs. (2. 2-8 - 11) are satisfactory for flight where: (1) velocities are the order of orbital velocities; (2) flight path angles are the order of one radian; (3) pressures are the order of the wing loading, and (4) heights are the order of the orbital radius. Statements (3) and (4) above are in obvious conflict. It is instructive, however, to proceed naively and see the anomaly that results.

2.4 <u>The Keplerian Regime</u>

It a solution to the dynamic equations, Eqs. (2.2-8 - 11), is sought in the form of a straight forward perturbation expansion, by expression the dependent variables in a power series in ϵ

$$v^2 = \sum \varepsilon^n v^{2(n)} \tag{2.4-1}$$

$$\gamma = \sum \varepsilon^n \gamma^{(n)} \tag{2.4-2}$$

$$\Theta = \sum \varepsilon^n \Theta^{(n)} \tag{2.4-3}$$

$$p = \sum \varepsilon^n p^{(n)} \tag{2,4-4}$$

then, when Eq. (2.4-4) and Eq. (2.2-7) are substituted into Eq. (2.2-11) and terms of equal orders in ϵ are equated, the following sequence results:

$$\varepsilon^0: \qquad 0 = -\frac{\beta p^{(0)}}{(1+h)^2} \tag{2.4-5}$$

$$\varepsilon^1: \qquad \frac{dp^{(0)}}{dh} = -\frac{\beta p^{(1)}}{(1+h)^2} \tag{2.4-6}$$

$$\vdots$$

$$\varepsilon^n: \qquad \frac{dp^{(n-1)}}{dh} = -\frac{\beta p^{(n)}}{(1+h)^2} \tag{2.4-7}$$

Obviously, the only solution to this sequence is $p^{(n)} = 0$ for all orders and therefore the perturbation solution to this problem is $p(h) = 0$. This is a <u>singular</u>

perturbation problem, because a straight forward perturbation expansion to arbitrarily high order can never predict the effect of the small quantity correctly.

As p(h) = 0, substituting Eqs. (2.4-1 - 5) into Eqs. (2.2-8 - 11) gives, to the lowest order in ϵ ,

$$\frac{d v^{2(0)}}{dh} = -\frac{2}{(1+h)^2} \tag{2.4-8}$$

$$\frac{d \cos \gamma^{(0)}}{dh} = -\left(\frac{1}{1+h} - \frac{1}{(1+h)^2 v^{2(0)}}\right) \cos \gamma^{(0)} \tag{2.4-9}$$

$$\frac{d\theta^{(0)}}{dh} = \frac{\cot \gamma^{(0)}}{1+h} \tag{2.4-10}$$

These are clearly the equations for describing motion in Keplerian conics. The solutions given here for future reference are

$$\frac{v^{2(0)}}{2} - \frac{1}{r^{(0)}} = \frac{v_0^{2(0)}}{2} - \frac{1}{r_0^{(0)}} = -\frac{1}{2a} \tag{2.4-11}$$

$$r^{(0)} v^{(0)} \cos \gamma^{(0)} = r_0^{(0)} v_0^{(0)} \cos \gamma_0^{(0)} = \hbar \tag{2.4-12}$$

$$\frac{1 - r^{(0)} v^{2(0)} \cos \gamma^{(0)}}{\cos \theta^{(0)}} = \frac{1 - r_0^{(0)} v_0^{2(0)} \cos \gamma_0^{(0)}}{\cos \theta_0^{(0)}} = e \tag{2.4-13}$$

where the usual constants of integration, the semi-major axis, a, the angular momentum, $\hbar$, and the eccentricity, e, have been introduced.* For convenience, it has been assumed that $\theta = 0$ at $\gamma = 0$. The equations correct to next order in ϵ are

*For convenience, r = 1 + h has also been used.

$$\begin{bmatrix} \dfrac{dv^{2(1)}}{dh} \\[2ex] \dfrac{d\cos\gamma^{(1)}}{dh} \\[2ex] \dfrac{d\theta^{(1)}}{dh} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\[2ex] -\dfrac{\cos\gamma^{(0)}}{(1+h)^2 v^{4(0)}} & \left(\dfrac{1}{1+h} - \dfrac{1}{(1+h)^2 v^{2(0)}}\right)\cos\gamma^{(0)} & 0 \\[2ex] 0 & \dfrac{1}{\cos^2\gamma^{(0)}\sin\gamma^{(0)}} & 0 \end{bmatrix} \begin{bmatrix} v^{2(1)} \\[2ex] \cos\gamma^{(1)} \\[2ex] \theta^{(1)} \end{bmatrix} \qquad (2.4\text{-}14)$$

The integrals to these equations are easily produced in terms of the solutions to the lowest order problem (see Appendix C). Because the small quantity, the aerodynamic force, does not enter in these equations, the solutions are simply the perturbation solutions about Keplerian conics. They are obtained by taking the variation of Eqs. (2.4-11 - 13). Notice that the distinction between the dependent and independent variable in the solutions given in Eqs. (2.4-13-14) has disappeared. In fact, any one of the variables v , γ , θ , or h may be considered to be the independent variable with the remaining three variables considered the dependent variable. This is a common occurrence with solutions of nonlinear equations. To illustrate this point, variations of all variables will be taken.

$$v^{(0)}\delta v^{(0)} + \frac{1}{r^{2(0)}}\delta r^{(0)} = 0 \qquad (2.4\text{-}15)$$

$$\delta v^{(0)} r^{(0)}\cos\gamma^{(0)} + r^{(0)}\cos\gamma^{(0)}\delta r^{(0)} - r^{(0)} v^{(0)}\delta(\cos\gamma^{(0)}) = 0 \qquad (2.4\text{-}16)$$

$$\frac{r^{(0)} v^{2(0)}\,\delta(\cos\gamma^{(0)})}{\cos\theta^{(0)}} + \frac{(1 - r^{(0)} v^{2(0)}\cos\gamma^{(0)})}{\cos^2\theta^{(0)}}\sin\theta^{(0)}\delta\theta^{(0)}$$

$$+ \frac{2 r^{(0)} v^{(0)}\cos\gamma^{(0)}}{\cos\theta^{(0)}}\delta v^{(0)} + \frac{v^{2(0)}\cos\gamma^{(0)}}{\cos\theta^{(0)}}\delta r^{(0)} = 0 \qquad (2.4\text{-}17)$$

Or in matrix form as

$$\begin{bmatrix} v & \frac{1}{r^2} & 0 & 0 \\ r\cos\gamma & v\cos\gamma & -rv & 0 \\ \frac{2rv\cos\gamma}{\cos\theta} & \frac{v^2\cos\gamma}{\cos\theta} & \frac{rv^2}{\cos\theta} & \frac{(1-rv^2\cos\gamma)}{\cos^2\theta}\sin\theta \end{bmatrix} \begin{bmatrix} \delta v \\ \delta r \\ \delta\cos\gamma \\ \delta\theta \end{bmatrix}$$

$$\begin{bmatrix} v_0 & \frac{1}{r_0^2} & 0 & 0 \\ r_0\cos\gamma_0 & v_0\cos\gamma_0 & -r_0 v_0 & 0 \\ \frac{2r_0 v_0\cos\gamma_0}{\cos\theta_0} & \frac{v_0^2\cos\gamma_0}{\cos\theta_0} & \frac{r_0 v_0^2}{\cos\theta_0} & \frac{(1-r_0 v_0^2\cos\gamma_0)}{\cos^2\theta_0}\sin\theta_0 \end{bmatrix} \begin{bmatrix} \delta v_0 \\ \delta r_0 \\ \delta\cos\gamma_0 \\ \delta\theta_0 \end{bmatrix} \qquad (2.4\text{-}18)$$

where superscripts have been dropped for convenience.

To obtain the solution to Eq. (2.4-14), r must be considered the independent variable with δr and δr_0 equal to zero. The above equations can then be inverted to give the perturbation δv, $\delta\cos\gamma$, and $\delta\theta$ in terms of their initial values. It is then finally observed that these perturbations are the solutions to Eq. (2.4-14) Specifically,

$$\begin{aligned} v^{2(1)} &= \delta v^2 \\ \cos\gamma^{(1)} &= \delta\cos\gamma \\ \theta^{(1)} &= \delta\theta \end{aligned} \qquad (2.4\text{-}19)$$

That the solutions to the perturbation equations are related to the solutions of the lowest order problem in this simple manner has been known since the time of Laplace, but is evidently not widely publicized in the engineering literature.

Investigators are continually reporting integrating the perturbation equations, or the adjoint equations (the backward perturbation equations) for Keplerian conics[100, 101].

In future non-singular problems, the small quantity will enter in the higher-order problem. This will make the integration somewhat more difficult in practice, but not different conceptually.

2.5 The Aerodynamically Dominated Regime

A straightforward perturbation expansion failed to produce the effect of aerodynamic forces in the previous section. As has already been indicated, this failure can be directly traced to an improper scaling of one of the variables in the problem. It was suggested that pressures could not be the order of the wing loading at heights that are of the order of the planetary radius. In fact, if one uses an exponential variation of pressure with height (β = const. = 1)

$$p = p_0 e^{-\frac{h}{\varepsilon}} \tag{2.5-1}$$

then it is seen that when h = 1

$$p = p_0 e^{-\frac{1}{\varepsilon}} \tag{2.5-2}$$

which is exponentially small if ϵ is small.

It is therefore required that either pressure or altitude be rescaled. But observe that no amount of rescaling of p will make Eq. (2.5.3) nonsingular

$$\frac{dp}{dh} = - \frac{\beta p}{\varepsilon(1+h)^2} \tag{2.5-3}$$

One is then left only with the possibility of rescaling the altitude.

The definition of a new altitude variable

$$h' = \frac{h}{\varepsilon} \tag{2.5-4}$$

which should be valid for h = $0(\epsilon)$ transforms Eqs. (2.2-8 - 11) into

$$\frac{dv^2}{dh'} = -C_D \frac{\rho v^2}{\sin\gamma} - \epsilon \frac{2}{(1+\epsilon h')^2}$$

$$\frac{d\cos\gamma}{dh'} = -\frac{1}{2} C_L \rho - \epsilon \left(\frac{1}{1+\epsilon h'} - \frac{1}{(1+\epsilon h')^2 v^2} \right) \cos\gamma$$

$$\frac{d\theta}{dh'} = \epsilon \frac{\cot\gamma}{1+\epsilon h'}$$

$$\frac{d\rho}{dh'} = -\frac{\rho}{(1+\epsilon h')^2}$$

(2.5-5)

Formally expanding v^2, p, γ, and θ as

$$v^2 = \sum \epsilon^n v^{2(n)}$$

$$\rho = \sum \epsilon^n \rho^{(n)}$$

$$\gamma = \sum \epsilon^n \gamma^{(n)} \qquad (2.5\text{-}6)$$

$$\theta = \sum \epsilon^n \Theta^{(n)}$$

and equating terms to equal order in ϵ one obtains, to lowest order,

$$\frac{dv^{2(0)}}{dh'} = -\frac{C_D \beta \rho^{(0)} v^{2(0)}}{\sin\gamma^{(0)}}$$

$$\frac{d\cos\gamma^{(0)}}{dh'} = -\frac{1}{2} C_L \beta \rho^{(0)}$$

$$\frac{d\Theta^{(0)}}{dh'} = 0 \qquad (2.5\text{-}7)$$

$$\frac{d\rho^{(0)}}{dh'} = -\beta\rho^{(0)}$$

At this point, it is advantageous to introduce a convention that streamlines notation considerably. Primes and superscripts will be dropped where no confusion will result and an equation will be identified by its order in ϵ , and the the order of the original nondimensional variables that appear. Thus, Eq. (2.5-7) will be called the "zeroth order equation for $v^2 = 0(1)$, $\gamma = 0(1)$, $\theta = 0(1)$, $p = 0(1)$, and $h = 0(\epsilon)$*" or more concisely the "lowest order equations for aero-dominated flight." With the elimination of h and superscripts, Eq. (2.5-7) can be written as

$$\frac{dv^2}{d\rho} = C_D \frac{v^2}{\sin\gamma}$$

$$\frac{d\cos\gamma}{d\rho} = -\frac{C_L}{2} \qquad (2.5\text{-}8)$$

$$\frac{d\Theta}{d\rho} = 0$$

Notice that only aerodynamic forces enter the problem. Further, there is no range gained that is of order of the orbital radius during the maneuver.

For constant C_L and C_D, Eq. (2.5-8) may be integrated to give

*Notice that this h is not the same h that appears in Eq. (2.5-7). No confusion will result after primes have been dropped if the reader recalls that the variables in this labeling statement are always the original nondimensional variables. See Section 2.2.

$$v^2 = v_0^2 \, e^{-\frac{2C_D}{C_L}(\gamma-\gamma_0)}$$

$$\cos\gamma = \cos\gamma_0 + \frac{C_L}{2}(p - p_0) \qquad (2.5\text{-}9)$$

$$\theta = \theta_0$$

These are the "skip" solutions of Allen and Eggers[8, 9]. Notice that they are valid for an arbitrary variation of β with h, p being specified by the last Eq. (2.5-7)

$$p^{(0)} = p_0 \, e^{-\int \beta \, dh} \qquad (2.5\text{-}10)$$

This is of considerable importance especially if the solution is to be computed to higher order. Typical variations of β with h are as much as 20 per cent. This implies that the non-constant β effect of the real atmosphere is as important as the next order correction to the solution. A result equivalent to assuming that β is a constant may be obtained by expanding β in a Taylor series. Specifically,

$$\beta(h) = \beta(h_0) + \varepsilon \frac{\partial\beta}{\partial h}(h-h_0) + \varepsilon^2 \frac{\partial^2\beta}{\partial h^2}(h-h_0)^2 \qquad (2.5\text{-}11)$$

So

$$\int_{h_0}^{h} \beta \, dh = \beta_0 (h-h_0) + \varepsilon \frac{\partial\beta}{\partial h} \frac{(h-h_0)^2}{2} + \cdots \qquad (2.5\text{-}12)$$

or

$$p^{(0)} = p_0 \, e^{-\left\{ \beta_0 (h-h_0) + \varepsilon \frac{\partial\beta}{\partial h} \frac{(h-h_0)^2}{2} + \cdots \right\}} \qquad (2.5\text{-}13)$$

which implies that to lowest order the pressure, or equivalently the density, has a locally exponential variation with height. This is an assumption that is commonly made in entry dynamics.

It should also be observed that Eqs. (2.5- 9) describe a turning maneuver done at the expense of kinetic energy. A turning efficiency factor for the maneuver is seen to be $C_L/2C_D$. Thus a vehicle attempting to turn with minimum loss of kinetic energy should have a high value of C_L/C_D, and the converse. This will be pursued at some length later.

Notice that when $p = p_o$, $\cos \gamma = \cos \gamma_o$ and $\gamma = \pm \gamma_o$. The turn is therefore symmetric about the value, $\gamma = 0$ for $C_L > 0$. Thus, the "entry angle" equals the "exit angle". This trajectory could be matched with a Keplerian conic to give the physically reasonable trajectory that was turned at the bottom by lift and at the top by gravity.

A set of perturbation equations governing small variations from this trajectory may be formed by taking a first variation of these solutions with respect to the initial condition. Again variation with respect to all variables will be taken. The resulting equations are

$$\begin{bmatrix} e^{\frac{2C_D}{C_L}\gamma} & \frac{2C_D}{C_L} e^{\frac{2C_D}{C_L}\gamma} & 0 & 0 \\ 0 & -\sin\gamma & 0 & -\frac{C_L}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta v^2 \\ \delta\gamma \\ \delta\theta \\ \delta p \end{bmatrix} \qquad \begin{bmatrix} e^{\frac{2C_D}{C_L}\gamma_o} & \frac{2C_D}{C_L} e^{\frac{2C_D}{C_L}\gamma_o} & 0 & 0 \\ 0 & -\sin\gamma_o & 0 & -\frac{C_L}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta v^2 \\ \delta\gamma \\ \delta\theta \\ \delta p \end{bmatrix} \tag{2.5-14}$$

If altitude, or equivalently the pressure, is taken as the independent variable, then δp and δp_o are zero and Eq. (2.5-14) may be inverted to give a transition matrix for the perturbations from some initial to final p. Specifically,

$$\begin{bmatrix} \delta v^2(p) \\ \delta\gamma(p) \\ \delta\theta(p) \end{bmatrix} = \begin{bmatrix} e^{\frac{2C_D}{C_L}(\gamma_o-\gamma)} & \left[-\frac{2C_D}{C_L}\left(\frac{\sin\gamma_o}{\sin\gamma}\right) + \frac{2C_D}{C_L} e^{\frac{2C_D}{C_L}(\gamma-\gamma_o)}\right] & 0 \\ 0 & \frac{\sin\gamma_o}{\sin\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta v^2(p_o) \\ \delta\gamma(p_o) \\ \delta\theta(p_o) \end{bmatrix}$$

(2.5-15)

The solution to the next order problem

$$
\begin{bmatrix} \dfrac{dN^{2(1)}}{dh} \\[2ex] \dfrac{d\gamma^{(1)}}{dh} \\[2ex] \dfrac{d\theta^{(1)}}{dh} \end{bmatrix}
=
\begin{bmatrix} -\dfrac{C_D\rho^{(0)}}{\sin\gamma^{(0)}} & +C_D\rho^{(0)}N^{2(0)}\csc\gamma^{(0)}\operatorname{ctn}\gamma^{(0)} & 0 \\[2ex] 0 & +\frac{1}{2}C_L\rho^{(0)}\csc\gamma^{(0)}\operatorname{ctn}\gamma^{(0)} & 0 \\[2ex] 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} N^{2(1)} \\[2ex] \gamma^{(1)} \\[2ex] \theta^{(1)} \end{bmatrix}
+
\begin{bmatrix} -2 \\[1ex] +\left(1-\dfrac{1}{N^{2(0)}}\right)\cot\gamma^{(0)} \\[1ex] \cot\gamma^{(0)} \end{bmatrix}
\tag{2.5-16}
$$

may be written explicitly in terms of this transition matrix as

$$
\begin{bmatrix} N^{2(0)} \\[2ex] \gamma^{(1)} \\[2ex] \theta^{(1)} \end{bmatrix}
=
\int_{h_0}^{h}
\begin{bmatrix} e^{\frac{2C_D}{C_L}(\gamma_0-\gamma^{(0)})} & \left[-\dfrac{2C_D}{C_L}\left(\dfrac{\sin\gamma_0}{\sin\gamma^{(0)}}\right)+\dfrac{2C_D}{C_L}e^{\frac{2C_D}{C_L}(\gamma_0-\gamma^{(0)})}\right] & 0 \\[2ex] 0 & \dfrac{\sin\gamma_0}{\sin\gamma^{(0)}} & 0 \\[2ex] 0 & 0 & 1 \end{bmatrix}
$$

$$
\times
\begin{bmatrix} -2 \\[1ex] +\left(1-\dfrac{1}{N^{2(0)}}\right)\cot\gamma^{(0)} \\[1ex] \cot\gamma^{(0)} \end{bmatrix} dh
\tag{2.5-17}
$$

(See Appendix C.) $v^{2(0)}$ and dh are specified in terms of $\gamma^{(0)}$ in Eqs. (2.5-7) and (2.5-9), but unfortunately Eq. (2.5-11) is not integratable in terms of normally tabulated functions. This is of little consequence as Eq. (2.5-12) serves to define a function that can be numerically tabulated. An expansion for the solution correct to order ϵ is then

$$\begin{aligned} v^2 &= v^{2(0)} + \epsilon v^{2(1)} \\ \gamma &= \gamma^{(0)} + \epsilon \gamma^{(1)} \\ \theta &= \theta^{(0)} + \epsilon \theta^{(1)} \end{aligned} \tag{2.5-18}$$

It has thus been demonstrated how one proceeds to higher order, more accurate, solutions once the lowest order approximation has been established in the framework of an asymptotic expansion.

If C_L is zero, or more correctly, $0(\epsilon)$ Eqs. (2.5-9), are undefined. But Eqs. (2.5-8) may be integrated directly to give

$$\begin{aligned} v^2 &= v_0^2 \, e^{+\frac{C_D(\psi - \psi_0)}{\sin \gamma_0}} \\ \cos \gamma &= \cos \gamma_0 \end{aligned} \tag{2.5-19}$$

These are the ballistic entry solutions of Allen and Eggers [8, 9]. Notice that these equations are also valid for an arbitrary variation of β with h.

Similar to the procedure just performed for the skip equation, the solution to the next order approximation is produced by first obtaining the perturbation equations, associated with these solutions. These perturbation equations are

$$
\begin{bmatrix}
e^{-\frac{C_D \varphi}{\sin\gamma_0}} & -v^2 C_D \frac{\cos\gamma_0}{\sin^2\gamma_0} \varphi e^{-\frac{C_D \varphi}{\sin\gamma_0}} & 0 & -v^2 \frac{C_D}{\sin\gamma_0} e^{-\frac{C_D \varphi}{\sin\gamma_0}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta v^2 \\ \delta\gamma \\ \delta\theta \\ \delta\varphi
\end{bmatrix}
$$

$$
= \begin{bmatrix}
e^{-\frac{C_D \varphi_0}{\sin\gamma_0}} & -v_0^2 C_D \frac{\cos\gamma_0}{\sin^2\gamma_0} \varphi_0 e^{-\frac{C_D \varphi_0}{\sin\gamma_0}} & 0 & -v_0^2 \frac{C_D}{\sin\gamma_0} e^{-\frac{C_D \varphi}{\sin\gamma_0}} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta v_0^2 \\ \delta\gamma_0 \\ \delta\theta_0 \\ \delta\varphi_0
\end{bmatrix}
\qquad (2.5\text{-}20)
$$

The transition matrix associated with Eq. (2.5-20) considering p as the independent variable is

$$
\begin{bmatrix}
\delta v^2(\varphi) \\ \delta\gamma(\varphi) \\ \delta\theta(\varphi)
\end{bmatrix}
=
\begin{bmatrix}
e^{-\frac{C_D(\varphi-\varphi_0)}{\sin\gamma_0}} & -v_0^2 C_D \frac{\cos\gamma_0}{\sin^2\gamma_0} \varphi e^{-\frac{C_D}{\sin\gamma_0}(\varphi-\varphi_0)} & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta v^2(\varphi_0) \\ \delta\gamma(\varphi_0) \\ \delta\theta(\varphi_0)
\end{bmatrix}
$$

(2.5-21)

The solution to the next order approximation,

$$
\begin{bmatrix} \dfrac{dv^{2(1)}}{dh} \\[2em] \dfrac{d\gamma^{(1)}}{dh} \\[2em] \dfrac{d\theta^{(1)}}{dh} \end{bmatrix} = \begin{bmatrix} -\dfrac{C_D \rho^{(0)}}{\sin\gamma^{(0)}} & +C_D \rho^{(0)} v^{2(0)} \csc\gamma^{(0)} \operatorname{ctn}\gamma^{(0)} & 0 \\[2em] 0 & 0 & 0 \\[2em] 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v^{2(0)} \\[2em] \gamma^{(1)} \\[2em] \theta^{(1)} \end{bmatrix}
$$

$$
+ \begin{bmatrix} -2 \\[2em] +\dfrac{C_L \rho}{\sin\gamma} + \left(1 - \dfrac{1}{v^{2(0)}}\right) \cot\gamma \\[2em] \cot\gamma \end{bmatrix} \tag{2.5-22}
$$

in terms of the transition matrix is

$$
\begin{bmatrix} v^{2(1)} \\ \gamma^{(1)} \\ \theta^{(1)} \end{bmatrix} = \int_{h_0}^{h} \begin{bmatrix} e^{-\frac{C_D(\varphi-\varphi_0)}{\sin\gamma_0}} & -v_0^2 C_D \dfrac{\cos\gamma_0}{\sin^2\gamma_0} e^{-\frac{C_D}{\sin\gamma_0}(\varphi^{(0)}-\varphi_0)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} -2 \\[2em] +\dfrac{C_L \rho^{(0)}}{\sin\gamma_0} + \left(1 - \dfrac{1}{v^{2(0)}}\right) \cot\gamma_0 \\[2em] \cot\gamma_0 \end{bmatrix} dh \tag{2.5-23}
$$

Now, at least, the last component of this equation may be integrated trivially. The other two components, as before, must be taulated.

In terms of this tabulation, the solution, correct to order ϵ, is

$$
\begin{aligned}
\mathcal{V}^2 &= \mathcal{V}^{2(0)} + \epsilon\, \mathcal{V}^{2(1)} \\
\gamma &= \gamma^{(0)} + \epsilon\, \gamma^{(1)} \\
\Theta &= \Theta^{(0)} + \epsilon\, \Theta^{(1)}
\end{aligned}
\tag{2.5-24}
$$

Improvements to the ballistic and skip solutions are of some interest.[10,103] It has been demonstrated that once these solutions have been produced within the context of an asymptotic expansion proceeding to a higher order the calculation of more accurate solutions is a straightforward, if algebraically complex, task. This is the major advantage of establishing these well known solutions within this systematic procedure.

A physiologically objectionable quality to flight in this regime is that the aerodynamic load factor, n, (total aerodynamic acceleration nondimensionalized with earth reference g, $g_{o\oplus}$) given by

$$
n = \frac{1}{2\epsilon}\left(C_L^2 + C_D^2\right)^{\frac{1}{2}} \rho \mathcal{V}^2 \, \frac{g_o}{g_{o\oplus}} \tag{2.5-25}
$$

is large (order $\frac{g_o}{\epsilon g_{o\oplus}}$). (See Table I) The maximum value of both the load factor and the aerodynamic heating may be easily calculated. Observe that both the aerodynamic heating and load factor may be expressed in the functional form as an aerodynamic effect, G, (see Appendix F)

$$
G = C_G \rho^i \mathcal{V}^{2j} \tag{2.5-26}
$$

and that the maximum of G, G^*, occurs when

$$
\frac{d \ln \mathcal{V}^2}{d \ln \rho} = -\frac{i}{j} \tag{2.5-27}
$$

But for both ballistic and skip trajectories

$$\frac{dN^2}{d\rho} = + \frac{C_D N^2}{\sin\gamma} \tag{2.5-28}$$

(Neglecting for the moment the variation of β.) Thus, for a ballistic trajectory, the maximum aerodynamic effect occurs when

$$\rho^* = \left(\frac{\rho'}{\frac{m\beta_0}{A}} \right)^* = -\frac{i}{j} \frac{\sin\gamma_0}{C_D} \tag{2.5-29}$$

which is at low altitudes for vehicles with large "ballistic coefficient," $\frac{m g_0}{C_D A}$, and large entry angles, γ_0. The maximum value of the aerodynamic effect is then

$$G^* = C_G \left(-\frac{i}{j} \frac{\sin\gamma_0}{C_D} \right)^i N_0^{2j} e^{-i/j} \tag{2.5-30}$$

It increases with increasing flight path angle and depends on the geometry of the vehicle only through the value of C_D. Further the maximum aerodynamic load factor, n^*, is

$$n^* = \frac{\sin\gamma_0}{\varepsilon} \frac{N_0^2}{e} \tag{2.5-31}$$

where the following constants have been substituted in Eq. (2.5-30)

$$C_G = \frac{C_D}{\varepsilon}$$

$$i = 1 \quad , \quad j = 1$$

It is completely determined by the initial values of flight path angle and velocity. These results were first reported by Allen and Eggers,[8, 9].

For the purposes here, it is important to observe that the surface of the planet may be encountered before the maximum aerodynamic effect occurs. Thus, a weapon entering at large entry angles may only have to tolerate a maximum aerodynamic effect of

$$G_1 = C_G \rho_1^{\,i} \left(v_0^{2j} e^{-j C_D \frac{p_1}{\sin \gamma_0}} \right) \qquad (2.5\text{-}32)$$

where ρ_1, p_1 and G_1 are the values of ρ, p and G at the surface of the planet. This value decreases as the ballistic coefficient, $mg_o/C_D A$, increases and as γ_o increases. The effect of negative lift is to further decrease the surface value of the aerodynamic effect. Thus, a large ballistic coefficient and flight path angle can possibly decrease the maximum heating and aerodynamic load encountered by a weapon prior to surface contact.

Notice that Eq. (2.5-32) is correct to lowest order in ϵ even for nonconstant β. A corresponding relation for maximum aerodynamic effect, when it does not occur at the planet surface, may be derived with some algebraic complexity. The density, ρ , is

$$\rho^* = -\frac{i}{j}\,\frac{\sin \gamma_0}{C_D}\left(1 + \frac{d \ln \beta}{dh}\right) \qquad (2.5\text{-}33)$$

and the maximum aerodynamic effect is

$$G^* = C_G \left(\frac{i}{j}\,\frac{\sin \gamma_0}{C_D}\left(1 + \frac{d \ln \beta}{dh}\right)\right)^i v_0^2\, e^{-\frac{i}{j}\left(1 + \frac{d \ln \beta}{dh}\right)} \qquad (2.5\text{-}34)$$

2.6 The Aero-Gravity Perturbed Regime

To this point, we have produced a Keplerian Regime and an Aero-dominated Regime. It is heuristically plausible to seek a regime where the aerodynamic and gravity forces enter to equal order. Such a regime must exist where the pressure is no longer of the same order as the wing loading.

The rescaling of the pressure, or equivalently the density, as

$$\rho' = \frac{\rho}{\varepsilon} \qquad (2.6\text{-}1)$$

and its substitution into Eq. (2.5-5) yields a new set of dynamical equations valid for $v^2 = 0(1)$, $\gamma = 0(1)$, $h = 0(\epsilon)$ and $\rho = 0(\epsilon)$. They are

$$\frac{dv^2}{dh} = -\epsilon c_D \frac{\rho v^2}{\sin\gamma} - \epsilon \frac{2}{(1+\epsilon h)^2}$$

$$\frac{d\cos\gamma}{dh} = -\frac{\epsilon}{2} c_L \rho - \epsilon \left(\frac{1}{1+\epsilon h} - \frac{1}{(1+\epsilon h)^2 v^2} \right) \cos\gamma$$

$$\frac{d\theta}{dh} = \epsilon \frac{\cot\gamma}{1+\epsilon h}$$

$$\frac{d\rho}{dh} = -\frac{\rho}{(1+\epsilon h)^2}$$

(2. 6-2)

To lowest order in ϵ , the perturbation equations are

$$\frac{dv^{2(0)}}{dh} = 0$$

$$\frac{d\cos\gamma^{(0)}}{dh} = 0$$

$$\frac{d\theta^{(0)}}{dh} = 0$$

$$\frac{d\rho^{(0)}}{dh} = -\beta\rho^{(0)}$$

(2. 6-3)

which integrate trivially to

$$
\begin{aligned}
v^{2(0)} &= v_0^{2(0)} \\
\cos\gamma^{(0)} &= \cos\gamma_0^{(0)} \\
\Theta^{(0)} &= \theta_0^{(0)} \\
\rho^{(0)} &= \rho_0^{(0)}\, e^{-\int\beta dh}
\end{aligned}
\qquad (2.6\text{-}4)
$$

To next order in ϵ the equations are

$$\frac{dv^{2(1)}}{dh} = -\frac{C_D\,\rho^{(0)}\,v^{2(0)}}{\sin\gamma^{(0)}} - z$$

$$\frac{d\cos\gamma^{(1)}}{dh} = -\frac{1}{2}C_L\,\rho^{(0)} - \left(1 - \frac{1}{v^{2(0)}}\right)\cos\gamma^{(0)}$$

$$\frac{d\Theta^{(1)}}{dh} = \cot\gamma^{(1)}$$

$$\frac{d\rho^{(1)}}{dh} = -\beta\,\rho^{(1)} \qquad (2.6\text{-}5)$$

These may be integrated by using the $p^{(0)} = p^{(0)}$ (h) dependence given in Eq. (2.6-4) to give

$$v^{2(1)} = \frac{C_D\,v^{2(0)}}{\sin\gamma^{(0)}}\left(\rho^{(0)} - \rho_0^{(1)}\right) - z\left(h - h_0\right)$$

$$\cos\gamma^{(1)} = \frac{C_L}{2}\left(\rho^{(0)} - \rho_0^{(0)}\right) + \cos\gamma^{(1)}\left(\frac{1}{v^{2(0)}} - 1\right)\left(h - h_1\right)$$

$$\Theta^{(1)} = \cos\gamma^{(0)}\left(h - h_0\right)$$

(2.6-6)

where

$$\rho^{(0)} = \rho_0^{(0)} e^{-\int \beta dh}$$

or in terms of the series correct to order ϵ^1

$$v^2 = v_0^2 + 2\frac{C_D v_0^2}{\sin\gamma_0}\left(\rho^{(0)} - \rho_0^{(0)}\right) - 2\varepsilon(h - h_0)$$

$$\cos\gamma = \cos\gamma_0 + \varepsilon\frac{C_L}{2}\left(\rho^{(0)} - \rho_0^{(0)}\right) - \varepsilon\cos\gamma_0\left(\frac{1}{v_0^2} - 1\right)(h - h_0)$$

$$\theta = \Theta_0 + \varepsilon\cot\gamma_0 (h - h_0)$$

(2.6-7)

Proceeding to next order is straightforward, though complex. This higher order solution is of less practical interest because rotating atmosphere effects must first be included for all planets except Venus. (See Chapter IV.) It is also simple to observe that, to next order, the effect of nonconstant g enters the problem. See Eqs. (2.6-2). Further, as variations in β are larger than $0(\epsilon)$, effects of the real atmosphere are more important than the next order correction to the solutions. It is interesting to observe that none of these effects were included in a solution for this regime reported by Shen[28] as correct to $0(\epsilon^2)$.

2.7 The Equilibrium Glide Regime

Until now, it has been tacitly assumed that the flight path angle γ is of order one. But it is commonly known that flight paths for entry are usually small. It is then appropriate to investigate the behavior of dynamical equations for $\gamma = 0(\epsilon)$, $v^2 = 0(1)$, $\rho = 0(\epsilon)$ and $h = 0(\epsilon)$. Letting $\gamma = \epsilon\gamma'$ in Eqs. (2.6-2) and as usual dropping the prime, one obtains

$$\frac{dv^2}{dh} = -\varepsilon C_D \frac{\rho v^2}{\sin(\varepsilon\gamma)} - \varepsilon\frac{2}{(1+\varepsilon h)^2}$$

$$\frac{d\cos(\varepsilon\gamma)}{dh} = -\varepsilon\frac{1}{2}C_L\rho - \varepsilon\left(\frac{1}{(1+\varepsilon h)} - \frac{1}{(1+\varepsilon h)^2 v^2}\right)\cos(\varepsilon\gamma)$$

$$\frac{d\theta}{dh} = \frac{\varepsilon\cot(\varepsilon\gamma)}{(1+\varepsilon h)}$$

$$\frac{d\rho}{dh} = -\frac{\rho}{(1+\varepsilon h)}$$

(2.7-1)

Substituting the appropriate series for $\sin\epsilon\gamma$, $\cos\epsilon\gamma$, and $\tan\epsilon\gamma$, and expanding the dependent variables, v^2, γ, p, in a power series in ϵ, viz.

$$
\begin{aligned}
v^2 &= \sum \epsilon^n v^{2(n)} \\
\gamma &= \sum \epsilon^n \gamma^{(n)} \\
\theta &= \sum \epsilon^n \theta^{(n)} \\
p &= \sum \epsilon^n p^{(n)}
\end{aligned}
\qquad (2.7\text{-}2)
$$

one obtains, to lowest order in ϵ

$$\frac{dv^2}{dh} = -C_D \frac{\rho v^2}{\gamma} \qquad (2.7\text{-}3)$$

$$0 = -\frac{1}{2} C_L \rho - \left(1 - \frac{1}{v^2}\right) \qquad (2.7\text{-}4)$$

$$\frac{d\theta}{dh} = \frac{1}{\gamma} \qquad (2.7\text{-}5)$$

$$\frac{dp}{dh} = -\rho \qquad (2.7\text{-}6)$$

Again, a singular set of equations have been produced. The derivative in Eq. (2.7-4) has disappeared. This reduces the equations to ordinary algebraic equations for v^2 and γ in terms of ρ. Notice that Eq. (2.7-4) specifies v^2 in terms of ρC_L and Eqs. (2.7-3) and (2.7-6) specify γ in terms of ρ, v^2 and C_D, so that

$$v^2 = \frac{1}{\frac{C_L}{2}\rho + 1} \qquad (2.7\text{-}9)$$

$$\gamma = -2\frac{C_D}{C_L}\frac{1}{v^2\left(-\beta + \frac{d\ln\beta}{dh} + \frac{d\ln C_L}{dh}\right)} \qquad (2.7\text{-}10)$$

where the possibility of β C_L and C_D varying with h is included. There are Sanger's equilibrium glide solutions. It is important to observe that arbitrary initial conditions cannot be met. Small deviations from this trajectory fall into another regime that will be described in Section 2.9. Some possibility of remaining on the trajectory is available through the modulation of C_L and C_D. This, in effect, requires a vehicle with variable C_L and C_D to fly such a trajectory.

The aerodynamic load factor given by

$$n = \frac{(C_L^2 + C_D^2)^{\frac{1}{2}}}{2}\rho v^2$$

$$= \frac{(C_L^2 + C_D^2)^{\frac{1}{2}}}{2}\frac{1}{\frac{C_L}{2} + \frac{1}{\rho}} \qquad (2.7\text{-}11)$$

$$\lim_{\rho \to \frac{1}{\epsilon}} n = \frac{(C_L^2 + C_D^2)^{\frac{1}{2}}}{C_L} \qquad (2.7\text{-}12)$$

increases monotonically to a value that is order one for $C_L > 0$.

The heating rate given by

$$\dot{q} = C_Q \rho^i v^{2j} \qquad (2.7\text{-}13)$$

does reach a maximum when

$$\frac{d\ln v^2}{d\ln\rho} = -\frac{i}{j} \qquad (2.7\text{-}14)$$

or using Eq. (2.7-9) when

$$\frac{1}{v^2}\frac{dv^2}{d\rho} = -\frac{1}{v^2}\left(\frac{1}{1+\frac{C_L}{2}\rho}\right)^2 \frac{C_L}{2} = \frac{i}{j}$$

$$\frac{C_L}{2}\rho = \frac{1}{j-i}$$

(2.7-15)

The maximum value of the heating rate is given by

$$\dot{q}^* = C_Q\left(\frac{2/C_L}{j-i}\right)^i\left(\frac{1}{1+\left(\frac{1}{j-1}\right)}\right)^j \tag{2.7-16}$$

so that increasing C_L decreases the maximum heating rate. The total heating is given by

$$\begin{aligned} q &= \int \frac{C_Q}{C_D}\rho^{i-1} v^{2(j-1)}\, dv \\ &= \int \frac{C_Q}{C_D}\left(\frac{2}{C_L}\left(\frac{1}{v^2}-1\right)\right)^{i-1} v^{2(j-1)}\, dv \end{aligned} \tag{2.7-17}$$

As $i < 1$ for convective heating, it is seen that high C_L increases the total heat input while high C_D decreases the total heat input. The converse is true for radiative heating.

The range equation may be integrated simply in terms of the velocity to give

$$\theta = \frac{1}{2}\frac{C_L}{C_D}\ln\left(\frac{1-v^2}{1-v_0^2}\right) \tag{2.7-18}$$

Notice, prior to leaving the equilibrium glide solution, that there is a nonuniformity of the solution at $v^2 = 1$. For $v^2 > 1$, Eq. (2.7-9) requires that $C_L < 0$ and Eq. (2.7-10) implies that $\gamma > 0$. For $v^2 < 1$ the two equations imply that $C_L > 0$ and $\gamma < 0$. At precisely $v^2 = 1$ Eq. (2.7-9) requires $C_L = 0$. Then if $\gamma = 0$ then

$\frac{dC_L}{d\rho} \to \infty$. To pursue this point in detail, define a new variable $v^{2'} = v^2 - 1$ and assume it is order ϵ. Writing the dynamical equations for $v^{2'} = 0(\epsilon)$, $\gamma^2 = 0(\epsilon)$, $C_L = 0(\epsilon)$, $h = 0(\epsilon)$, to lowest order in ϵ yields,

$$\frac{dv^{2'}}{dh} = -\frac{C_D\rho}{\gamma} - 2$$

$$\gamma\frac{d\gamma}{dh} = -\frac{C_L}{2}\rho - v^{2'} \qquad (2.7\text{-}19)$$

$$\frac{d\rho}{dh} = -\rho$$

For the first time, all terms in the dynamical equations enter to the same order. A general solution would indeed be difficult to obtain. But for the purposes here, it is sufficient to observe that the equation admits a $\frac{d\gamma}{dh} = 0$ solution, if

$$C_L = -\frac{v^{2'}}{\rho} \qquad (2.7\text{-}20)$$

which is the equilibrium glide solution near circular satellite velocity.

As this portion of the trajectory occurs over an altitude interval that is obviously small it is plausable to rescale the altitude. So for the original nondimensional variables $h = 0(\epsilon^2)$, $v^{2'} = v^2 - 1 = 0(\epsilon)$, $C_L = 0(1)$ and $\gamma = 0(\epsilon)$, the dynamical equations to lowest order in ϵ are

$$\frac{dv^{2(0)'}}{dh} = 0$$

$$-\gamma^{(0)}\frac{d\gamma^{(0)}}{dh} = \frac{C_L}{2}\rho^{(0)}$$

$$\frac{d\rho^{(0)}}{dh} = 0$$

$$(2.7\text{-}21)$$

These equations say that the velocity and the pressure (or equivalently the density) is constant to lowest order and that γ is controllable with variations in C_L of order one. We have produced an equation, presumably describing the short time behavior of the trajectory. This topic will be treated in a general context later. It is sufficient here to again observe that $\gamma^{(1)}$ = const is a solution if $C_L = 0(\epsilon)$.

The equations to next order in ϵ are, assuming $C_L = 0(\epsilon)$

$$\frac{dv^{2(1)'}}{dh} = \frac{C_D \rho^{(0)}}{\gamma^{(0)}} - 2$$

$$-\frac{1}{2}\frac{d\gamma^{(1)2}}{dh} = -\frac{C_L}{2}\rho^{(0)} - v^{2(0)'} \qquad (2.7\text{-}22)$$

$$\frac{dp^{(1)}}{dh} = -\rho^{(0)}$$

These may be integrated to give

$$v^{2(1)'} = -\frac{\rho^{(0)}}{\gamma^{(0)}}\int C_D dh - 2(h-h_0)$$

$$\gamma^{(1)2} = -\rho^{(0)}\int C_L dh - v_0^{2'}(h-h_0) \qquad (2.7\text{-}23)$$

The expansion for v^2 and γ^2 valid to order ϵ is

$$v^{2'} = v_0^{2'} + \epsilon\frac{\rho_0}{\gamma_0}\int C_D dh - 2\epsilon(h-h_0)$$

$$\gamma^2 = \gamma_0^2 - \epsilon\frac{\rho_0}{2}\int C_L dh - \epsilon v_0^{2'}(h-h_0) \qquad (2.7\text{-}24)$$

This solution goes smoothly through $\gamma = 0$ and $\epsilon v^{2'} = v^2 - 1 = 0$ for arbitrary C_D and C_L.

This rather lengthy investigation of the singularity in the equilibrium glide solution serves to illustrate a procedure for investigating the non-uniformities in any perturbation solution. Generally, a non-uniformity exists where one of the assumptions made in obtaining the solution fails. Here the assumption that failed was that the quantity $(v^2 - 1)$ was order one. The remedy for the problem is always to rescale the small quantity and perform another straight forward perturbation expansion.

2.8 The Orbital Decay Regime

Observing that the integrals for the equilibrium glide regime, Eqs. (2.7-9 - 10) are poorly behaved for $C_L = 0(\epsilon)$, it is natural to seek solutions valid for this case. Returning to Eqs. (2.7-1) and letting $C_L = \epsilon\, C_L'$ one obtains, to lowest order in ϵ,

$$\frac{dv^2}{dh} = c_D \frac{\rho v^2}{\gamma}$$

$$0 = 0 - \left(1 - \frac{1}{v^2}\right) \tag{2.8-1}$$

$$\frac{d\varphi}{dh} = -\rho$$

The equations may only be satisfied if ρ is also rescaled. This implies that there is no $\gamma = 0(\epsilon)$ flight regime when C_L and $\rho = 0(\epsilon)$. With this motivation, the behavior of the dynamical equations, Eqs. (2.7-1), will be investigated for $\gamma = 0(\epsilon)$, $v^2 = 0(1)$, $\rho = 0(\epsilon^2)$, $h = 0(\epsilon)$ and for the moment $C_L = 0(1)$. They are

$$\frac{dv^2}{dh} = -\frac{\epsilon^2 C_D \rho v^2}{\sin(\epsilon\gamma)} - \epsilon\frac{2}{(1+\epsilon h)^2}$$

$$\frac{d\cos(\epsilon\gamma)}{dh} = -\epsilon^2 \frac{1}{2} C_L \rho - \epsilon\left(\frac{1}{1+\epsilon h} - \frac{1}{(1+\epsilon h)^2 v^2}\right)\cos(\epsilon\gamma) \tag{2.8-2}$$

$$\frac{d\varphi}{dh} = -\frac{\rho}{(1+\epsilon h)^2}$$

To lowest order in ϵ, the expansion equations are

$$\frac{dv^{2(0)}}{dh} = 0$$

$$0 = 0 - \left(1 - \frac{1}{v^{2(0)}}\right) \tag{2.8-3}$$

$$\frac{d\varphi^{(0)}}{dh} = -\rho^{(0)}$$

or simply that the velocity to lowest order is circular satellite velocity. To order ϵ the expansion equations are

$$\frac{d v^{2(1)}}{dh} = - \frac{C_D \rho^{(0)} v^{2(0)}}{\gamma^{(0)}} - 2 \tag{2.8-4}$$

$$- \gamma^{(0)} \frac{d\gamma^{(0)}}{dh} = - \frac{1}{2} C_L \rho^{(0)} - \left(- h + \frac{2h}{v^{2(0)}} + \frac{v^{2(1)}}{v^{4(0)}} \right) \tag{2.8-5}$$

By the use of the last equation of Eqs. (2.8-3) and the result from Eqs. (2.8-3) that $v^{2(0)} = 1$, $v^{2(1)}$ may be eliminated from Eq. (2.8-5) to give

$$\frac{1}{2} \frac{d^2 \gamma^{2(0)}}{dh^2} + \frac{C_D \rho^{(0)}}{\gamma^{(0)}} = \frac{1}{2} C_L \frac{d\rho^{(0)}}{dh} - 1 \tag{2.8-6}$$

where

$$\rho^{(0)} = \beta \rho_0^{(0)} e^{-\int \beta dh} \tag{2.8-7}$$

This is clearly the equation for a nonlinear oscillation, valid for $C_L = 0(1)$ or $C_L = 0(\epsilon)$, and with equilibrium point given by

$$\gamma_0^{(0)} = - \frac{C_D \rho^{(0)}}{1 - \frac{1}{2} C_L \rho^{(0)} \left(\frac{d \ln \beta}{dh} - \beta \right)} \tag{2.8-8}$$

If this equilibrium flight path angle is substituted in Eq. (2.8-5) for $C_L = 0$, one obtains the interesting result

$$\frac{d v^{2(1)}}{dh} = \frac{C_D \rho^{(0)} v^{2(0)}}{C_D \rho^{(0)}} - 2 \tag{2.8-9}$$

$$= - 1$$

or

$$v^2 = v^{2(0)} + \varepsilon v^{2(1)} = 1 - \varepsilon h \tag{2.8-10}$$

so that the velocity to order ϵ is just the local satellite velocity. This agrees with the well known phenomenon that bodies undergoing orbital decay speed up to maintain local circular satellite velocity $\sqrt{rg}$. It is interesting to observe that we have shown that this behavior is only possible in a $\rho = 0(\epsilon^2)$ flight regime.

2.9 The Moderate Flight Path Angle Low-Density Regime

To this point, only scaling of the variables in integer powers of ϵ have been considered. As γ enters the dynamic equations to first and second powers for small γ it is natural to seek a distinguished form of the equation for $\gamma = 0(\epsilon^{\frac{1}{2}})$. The writing of the dynamic equations for $\gamma = 0(\epsilon^{\frac{1}{2}})$, $\rho = 0(\epsilon)$, $h = 0(\epsilon)$, $v^2 = 0(1)$ produces

$$\frac{dv^2}{dh} = -\varepsilon \frac{C_D \rho v^2}{\sin(\varepsilon^{\frac{1}{2}}\gamma)} - \varepsilon \frac{2}{(1+\varepsilon h)^2}$$

$$\frac{d\cos(\varepsilon^{\frac{1}{2}}\gamma)}{dh} = -\frac{\varepsilon}{2} C_L \rho - \varepsilon \left(\frac{1}{1+\varepsilon h} - \frac{1}{(1+\varepsilon h)^2 v^2} \right) \cos(\varepsilon^{\frac{1}{2}}\gamma)$$

$$\frac{d\rho}{dh} = -\frac{\rho}{(1+\varepsilon h)^2} \tag{2.9-1}$$

It is clear that one must seek an expansion in powers of $\epsilon^{\frac{1}{2}}$. To lowest order

$$\frac{dv^{2(0)}}{dh} = 0$$

$$\frac{1}{2}\frac{d(\gamma^{2(0)})}{dh} = \frac{1}{2} C_L \rho^{(0)} + \left(1 - \frac{1}{v^{2(0)}}\right) \tag{2.9-2}$$

$$\frac{d\rho^{(0)}}{dh} = -\rho^{(0)}$$

To next order for the velocity only

$$\frac{dv^{2(1/2)}}{dh} = -\frac{C_D \rho^{(0)} v^{2(0)}}{\gamma^{(0)}} \tag{2.9-3}$$

These equations can be integrated to give

$$v^{2(0)} = v_0^2$$

$$\frac{\gamma^{2(0)}}{2} - \frac{\gamma_0^2}{2} = \frac{1}{2} C_L (\rho - \rho_0) + \left(1 - \frac{1}{v_0^2}\right)(h - h_0)$$

$$v^{2(1/2)} = C_D v_0^2 \int_{\rho_0}^{\rho} \left(\frac{\gamma_0^2}{2} + \frac{1}{2} C_L (\rho - \rho_0) + \left(1 - \frac{1}{v_0^2}\right)(h - h_0)\right)^{-\frac{1}{2}} d\rho \qquad (2.9\text{-}4)$$

where

$$\rho^{(0)} = \rho_0 \, e^{-\int \beta dh}$$

The expansion correct to order ϵ^0 in γ and $\epsilon^{\frac{1}{2}}$ in v^2 is then

$$v^2 = v_0^2 + \epsilon^{\frac{1}{2}} C_D v_0^2 \int_{\rho_0}^{\rho} \left(\frac{\gamma_0^2}{2} + \frac{1}{2} C_L (\rho - \rho_0) + \left(1 - \frac{1}{v_0^2}\right)(h - h_0)\right)^{-\frac{1}{2}} d\rho$$

$$\frac{\gamma^2}{2} = \frac{\gamma_0^2}{2} + \frac{1}{2} C_L (\rho - \rho_0) + \left(1 - \frac{1}{v_0^2}\right)(h - h_0) \qquad (2.9\text{-}5)$$

where

$$\rho^{(0)} = \rho_0 \, e^{-\int \beta dh}$$

This solution describes an oscillatory type of trajectory that occurs when the initial conditions are not correct for an equilibrium glide. See Section 2.7. Recently, Hanin[65] has obtained interesting approximations to this type of trajectory by considering linear perturbations from equilibrium glide. It is expected that the above expression should give accurate description of the first nonlinear oscillation where Hanin has shown his results to be in poor agreement with numerical integration.

For flight at higher ρ and low C_L a slightly more complicated set of perturbation equations results. Consider the dynamic equations for $v^2 = 0(1)$, $\rho = 0(\epsilon^{\frac{1}{2}})$, $h = 0(\epsilon)$, $\gamma = 0(\epsilon^{\frac{1}{2}})$:

$$\frac{dv^2}{dh} = -\frac{\varepsilon^{\frac{1}{2}} C_D \rho v^2}{\sin(\varepsilon^{\frac{1}{2}}\gamma)} - \varepsilon \frac{2}{(1+\varepsilon h)^2}$$

$$\frac{d\cos(\varepsilon^{\frac{1}{2}}\gamma)}{dh} = -\varepsilon C_L \rho - \varepsilon \left(\frac{1}{1+\varepsilon h} - \frac{1}{(1+\varepsilon h)^2 v^2} \right) \cos(\varepsilon^{\frac{1}{2}}\gamma)$$

$$\frac{d\varphi}{dh} = -\frac{\rho}{(1+\varepsilon h)^2}$$

(2.9-9)

To lowest order in ϵ,

$$\frac{dv^2}{dh} = \frac{C_D \rho v^2}{\gamma}$$

$$-\frac{1}{2}\frac{d\gamma^2}{dh} = -C_L \rho - \left(1 - \frac{1}{v^2}\right)$$

$$\frac{d\varphi}{dh} = -\rho$$

These are the form of the dynamic equations assumed by both Eggers[46] and Chapman[13] in their analysis of entry dynamics. It is interesting to observe that a simpler form exists to lowest order for most other regimes of flight. This implies that these analyses are too accurate in these regimes. To next order, the flight path component of gravity enters in most regimes so that these analyses are not uniformly valid to order ϵ for all regimes of hypervelocity flight. With some manipulation the equations may be put in the following form:

$$\frac{d^2\rho}{d(\ln v^2)^2} = \frac{1}{C_D}\left(\frac{C_L}{2C_D} + \left(1 - \frac{1}{v^2}\right) \frac{1}{C_D \rho} \right)$$

(2.9-11)

where

$$\frac{d\rho}{d\ln v^2} = \frac{\gamma}{C_D}$$

and β, C_L and C_D were assumed constant. This is the Eggers' equation for constant C_L and C_D entry. Series solutions have been presented by Eggers[46], Citron and Meir[33], and Wang and Chu[11]. These solutions may be more adequate if one restricts their use to the $C_L = 0(\epsilon^{\frac{1}{2}})$, ρ 1 $0(\epsilon^{\frac{1}{2}})$, $\gamma = 0(\epsilon^{\frac{1}{2}})$ regime, a restriction not imposed by these authors.

Larrabee [37] has arranged the equations in the following form:

$$\frac{d^2 D}{d v^2} - \frac{3}{v}\frac{dD}{dv} + \frac{4}{v^2} D = \frac{1-v^2}{D} - \frac{C_L}{C_D}$$

where

$$\gamma = \frac{2D}{v^2} - \frac{1}{v}\frac{dD}{dv}$$

(2.9-12)

and

$$D = C_D \rho \frac{v^2}{2}$$

and shown its relation to the Chapman function

$$\bar{u}\ \bar{z} = D \qquad (2.9\text{-}13)$$

which is tabulated in Ref. (14).

One final form of the dynamic equation for moderate flight path angles occurs for velocities close to orbital velocity. Consider the dynamic equations scaled for $\gamma = 0(\epsilon^{\frac{1}{2}})$, $v^2 - 1 = v^{2'} = 0(\epsilon^{\frac{1}{2}})$, $\rho = 0(\epsilon)$, $C_L = 0(1)$ and $h = 0(\epsilon)$.

$$\epsilon^{\frac{1}{2}}\frac{dv^{2'}}{dh} = -\epsilon\,\frac{C_D\rho\,(\epsilon^{\frac{1}{2}} v^{2'} + 1)}{\sin(\epsilon^{\frac{1}{2}}\gamma)} - \epsilon\,\frac{2}{(1+\epsilon h)^2}$$

$$\frac{d\cos\epsilon^{\frac{1}{2}}\gamma}{dh} = -\epsilon\,\frac{1}{2}C_L\rho$$

(2.9-14)

$$-\epsilon\left(\frac{1}{1+\epsilon h} - \frac{1}{(1+\epsilon h)^2(\epsilon v'^2 - 1)}\right)\cos(\epsilon^{\frac{1}{2}}\gamma)$$

$$\frac{d\rho}{dh} = -\frac{\rho}{(1+\epsilon h)^2}$$

Then, to lowest order in ϵ, one obtains:

$$\frac{dv^{2\prime}}{dh} = -\frac{C_D \rho}{\gamma}$$

$$\gamma \frac{d\gamma}{dh} = \frac{C_L}{2} \rho \qquad (2.9\text{-}15)$$

$$\frac{d\varphi}{dh} = -\rho$$

which can be integrated to give

$$v^{2\prime} = v_0^{2\prime} - \frac{2 C_D}{C_L} (\gamma - \gamma_0)$$

$$\gamma^2 - \gamma_0^2 = -C_L (\varphi - \varphi_0) \qquad (2.9\text{-}16)$$

or for $C_L = (\epsilon^{\frac{1}{2}})$

$$v^{2\prime} = v_0^{2\prime} - \frac{C_D}{\gamma_0} (\varphi - \varphi_0)$$

$$\gamma = \gamma_0 \qquad (2.9\text{-}17)$$

Interestingly, these are the limiting forms of the "skip" and "ballistic" equation for small γ. See Section 2.5. Understandably, the same equations that apply when aerodynamic forces mask gravity and centrifugal accelerations are valid when these forces are small due to small flight path angle and near balance of gravity and centirfugal accelerations. These are the lowest order equation used by Lees et.al, in Ref. (7).

2.10 The Near Sonic Flight Regime

Until now, only trajectories with velocity the order of orbital velocity have been considered. The next distinguished form of the dynamic equations occurs when the velocity is order $\epsilon^{\frac{1}{2}}$ or when $v^2 = 0(\epsilon)$. This is the order of sonic velocity in all atmospheres. This is easily seen by expressing the Mach number, M, in terms of current parameters

$$M = \frac{v\sqrt{g_0' r_0'}}{\sqrt{\bar{\gamma} R T}} = \frac{v}{\sqrt{\epsilon\bar{\gamma}}} \tag{2.10-1}$$

Where $\bar{\Upsilon}$ is the ratio of specific heats and is near one for all gases. So if $v^2 = 0(\epsilon)$, $M = 0(1)$. The writing of the dynamical equations for $v^2 = 0(\epsilon)$, $h = 0(\epsilon)$, $\theta = 0(\epsilon)$, $\Upsilon = 0(1)$ and $\rho = 0(1)$ yields

$$\begin{aligned}
\frac{dv^2}{dh} &= -C_D\frac{\rho v^2}{\sin\gamma} - \frac{2}{(1+\epsilon h)^2} \\
\frac{d\cos\gamma}{dh} &= -\frac{C_L}{2}\rho - \left(\frac{\epsilon}{1+\epsilon h} - \frac{1}{(1+\epsilon h)^2 v^2}\right)\cos\gamma \\
\frac{d\theta}{dh} &= \frac{\epsilon\cot\gamma}{1+\epsilon h} \\
\frac{d\rho}{dh} &= -\frac{\rho}{(1+\epsilon h)^2}
\end{aligned} \tag{2.10-2}$$

To lowest order in ϵ one has

$$\begin{aligned}
\frac{dv^2}{dh} &= -C_D\frac{\rho v^2}{\sin\gamma} - 2 \\
\frac{d\cos\gamma}{dh} &= -\frac{C_L}{2}\rho + \frac{1}{v^2}\cos\gamma \\
\frac{d\theta}{dh} &= \cot\gamma \\
\frac{d\rho}{dh} &= -\rho
\end{aligned} \tag{2.10-3}$$

These are equations describing flight over a flat, constant gravity earth. It is interesting to observe that these are the assumptions commonly made in deriving the dynamic equations for flight at velocities the order of sonic velocity.

Solutions for these equations are difficult. One interesting set of solutions can be obtained by assuming flight is at small flight path angles. Thus for $\gamma = 0(\epsilon)$ and $h = 0(\epsilon^2)$ the equations carried to lowest order are

$$
\begin{aligned}
\frac{dv^2}{dh} &= -C_D \frac{\rho v^2}{\gamma} \\
0 &= -\frac{C_L}{2}\rho + \frac{1}{v^2} \\
\frac{d\theta}{dh} &= \frac{1}{\gamma} \\
\frac{dp}{dh} &= \rho
\end{aligned}
\qquad (2.10\text{-}4)
$$

These equations are singular, completely specifying the ρ, v^2, and γ necessary to fly along the trajectory. The range equation may be integrated to give

$$v^2 - v_0^2 = \int \frac{2C_D}{C_L}\, d\theta \qquad (2.10\text{-}5)$$

which is valid for any variation of $\frac{C_D}{C_L}$ with θ that is capable of maintaining $\gamma = 0(\epsilon)$. So, integrals for the sonic regime are available for values of $\frac{C_D}{C_L}$ which keep $\gamma = 0(\epsilon)$.

2.11 The Low Velocity Flight Regime

To this point, flight at velocities the order of orbital velocity and velocities the order of sonic velocity have been considered. Another distinguished form of the dynamic equation occurs when the velocity is of order ϵ. To see this, write the dynamic equations for $v^2 = 0(\epsilon^2)$, $h = 0(\epsilon)$, $\gamma = 0(1)$, and $\rho = 0(\frac{1}{\epsilon})$:

$$\epsilon \frac{d v^2}{dh} = - C_D \frac{\rho v^2}{\sin\gamma} - \frac{2}{(1+\epsilon h)^2}$$

$$\epsilon \frac{d\cos\gamma}{dh} = - \frac{C_L}{2} \rho - \left(\frac{\epsilon^2}{1+\epsilon h} - \frac{1}{(1+\epsilon h)^2 v^2} \right) \cos\gamma \qquad (2.11\text{-}1)$$

$$\frac{d\theta}{dh} = \frac{\cot\gamma}{1+\epsilon h}$$

$$\frac{d\varphi}{dh} = - \frac{\rho}{(1+\epsilon h)^2}$$

To lowest order in ϵ the equations become

$$0 = - \frac{C_D \rho v^2}{\sin\gamma} - 2$$

$$0 = - \frac{C_L}{2} \rho - \left(- \frac{1}{v^2} \right) \cos\gamma \qquad (2.11\text{-}2)$$

$$\frac{d\theta}{dh} = \cot\gamma$$

$$\frac{d\varphi}{dh} = - \rho$$

These simply say that the aerodynamic and gravity forces are in balance to lowest order

$$C_D \frac{\rho v^2}{2} = - \sin\gamma$$

$$C_L \frac{\rho v^2}{2} = \cos\gamma \qquad (2.11\text{-}3)$$

These equations have formed the back bone of "quasi-steady"[47] performance of aircraft. It is understandable that such analysis has proved inadequate for computing the performance of vehicles capable of near sonic velocity.

2.12 A Systematic Procedure for Identifying Regimes of Flight

The procedure for identifying the flight regime up to now has been ad hoc, based to some extent on intuition. This procedure can readily be systematized. Observe that to this point we have tried various possible scalings of the variables in the dynamic equations. When a meaningful lowest order balance occurred in the equations between acceleration, aerodynamic and gravity terms, this was identified as a regime of flight.

A systematic exhaustion of all such regimes is possible. It simply requires that an arbitrary scaling be applied to the variables in the dynamic equations and that all possible balances between terms be investigated. However, some difficulty will occur.

The dynamic equations for $v^2 = 0(\epsilon^{2n_v})$, $p = 0(\epsilon^{n_p})$, $h = 0(\epsilon^{n_h})$, $\gamma = 0(\epsilon^{n_\gamma})$, $C_L = 0(\epsilon^{n_L})$ are

$$\epsilon^{2n_v - n_h} \frac{dv^2}{dh} = -\epsilon^{n_p + 2n_v - 1} \frac{C_D \beta p v^2}{\sin(\epsilon^{n_\gamma}\gamma)} - \frac{2}{(1-\epsilon^{n_h} h)^2} \tag{2.12-1}$$

$$\epsilon^{-n_h} \frac{d\cos(\epsilon^{n_\gamma}\gamma)}{dh} = -\frac{1}{2}\epsilon^{n_p + n_L - 1} C_L \beta p - \left(\frac{1}{1+\epsilon^{n_h} h} - \frac{\epsilon^{-2n_v}}{(1+\epsilon^{n_h} h)^2 v^2}\right)\cos(\epsilon^{n_\gamma}\gamma) \tag{2.12-2}$$

$$\epsilon^{n_p - n_h} \frac{dp}{dh} = -\epsilon^{n_p - 1} \frac{\beta p}{(1+\epsilon^{n_h} h)^2}$$

where the range equation has not been included. Cursory analysis will reveal that a myriad of possibilities can exist. The situation may be made tractable by considering first $h = 0(\epsilon)$ and $\gamma \leq 0(1)$. For this value of scaling, h never explicitly enters the lowest order problem. Its implicit dependence may be eliminated by use of the hydrostatic equation, Eq. (2.12-3). Further, $\sin\gamma$ and $\cos\gamma$ will be conveniently expressed in terms of their appropriate series, retaining only the first term. With these simplifications, Eqs. (2.12-1 -3) become, to lowest order,

$$\epsilon^{2n_v} \frac{dv^2}{dp} = \epsilon^{2n_v + n_p - n_\gamma} \frac{C_D v^2}{\sin\gamma} + 2\epsilon^{1} \frac{1}{\beta p} \tag{2.12-4}$$

$$\epsilon^{2\gamma} \frac{d\cos\gamma}{d\rho} = \epsilon^{n_\rho + n_L} \frac{C_L}{2} + \left(\epsilon^{1} - \frac{\epsilon^{1-2n_v}}{v^2}\right) \frac{\cos\gamma}{\beta\rho} \qquad (2.12\text{-}5)$$

Considering first Eq. (2.12-4) the following possibilities exist:

(1) Flight path acceleration dominates if

$$2n_v < 2n_v + n_p - n_\gamma,\ 1$$

(2) Drag dominates if

$$2n_v + n_p - n_\gamma < 2n_v,\ 1$$

(3) The flight path component of gravity dominates if

$$1 < 2n_v,\ n_v + n_p - 2n_\gamma$$

These inequalities are easiest to interpret in terms of the regions of a Euclidean space where $2n_v$, n_p and n_γ are the coordinates (see Fig. 2.12-1). The inequalities then define regions of this space where the appropriate term in the differential equation dominates. As n_v, n_p and n_γ are the scaling in ϵ of v, p and γ, the plot may be interpreted as an inverse log plot of regions in state space.

In Fig. 2.12-1 it is seen that the state space is divided into three regions. Flight path acceleration dominates for low densities and high velocities. Drag dominates for high densities and small flight path angles, and the velocity component of gravity dominates for low velocities. Notice only when the velocities are order of sonic velocity ($v^2 = 0(\epsilon)$) and densities and flight path angles are of equal order do all terms enter to equal order in the differential equation.

Proceeding in an analogous manner with Eq. (2.12-4) one arrives at the following conditions:

(1) Normal acceleration dominates if

$$2n_\gamma < n_p + n_L,\ 1,\ 1 - 2n_v$$

(2) Lift dominates if

$$n_p + n_L < 2n_\gamma,\ 1,\ 1 - 2n_v$$

(3) Centrifugal acceleration dominates if

$$1 < 2n_\gamma,\ n_p + n_L,\ 1 - 2n_v$$

(4) The normal component of gravity dominates if

$$1 - 2n_v < 2n_\gamma,\ n_p + n_L,\ 1$$

These conditions may be similarly interpreted in terms of partitions in state space. See Fig. 2.12-1. The scaling of lift coefficient and density have been combined to allow a three-dimensional sketch. There are now four divisions of the

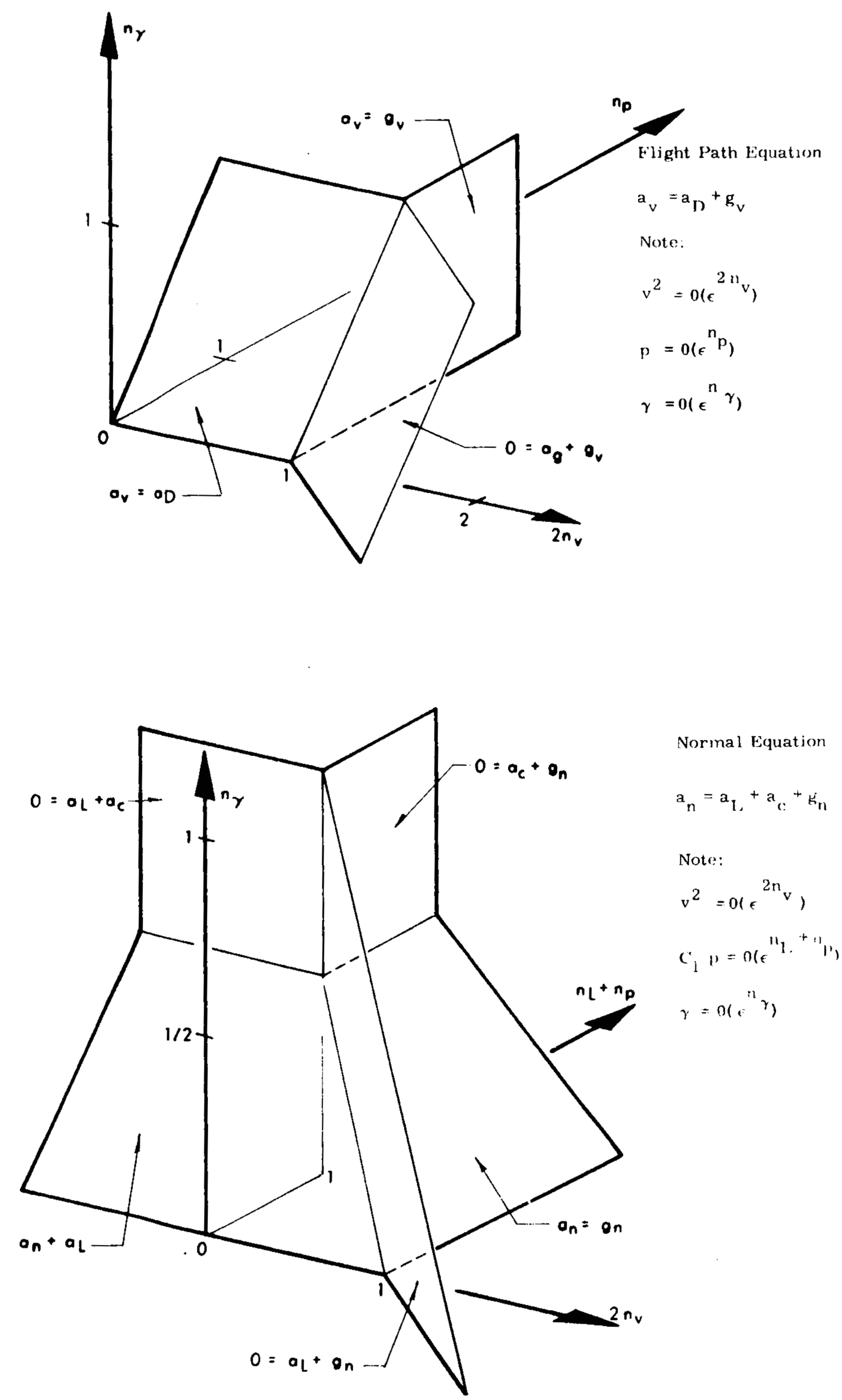


Fig. 2.12-1

Regions of Behavior of Dynamic Equations for $\gamma < 0(1)$, $h = 0(\epsilon)$

space, centrifugal acceleration dominates at large velocities, small densities (or p) and small flight path angles. The normal component of gravity dominates at small velocities, flight path angles and densities(or C_L). Normal acceleration dominates at large velocities and flight path angles. Lift dominates at large velocities and densities (or C_L).

It is interesting to observe that all terms in the differential equation are of the same order only when the velocity is the order of orbital velocity, flight path angle is order ϵ and the density C_L product is of order ϵ. As this region of state space does not correspond to the region where one obtains complete balance for the flight path equation, there is no region of state space where all terms in both dynamical equation enter to the same order.

To map the region of state space where the flight path angle is order $\pi/2$ requires that the dynamic equations be rewritten in terms of

$$\gamma' = \gamma - \frac{\pi}{2} \tag{2.12-6}$$

With this redefinition and for $\gamma' = 0(\epsilon^{n_{\gamma'}})$, $v^2 = 0(\epsilon^{2n_v})$, $p = 0(\epsilon^{n_p})$, $C_L = 0(\epsilon^{n_L})$, and $h = 0(\epsilon)$ the dynamic equations to lowest order in ϵ are:

$$\epsilon^{2n_v} \frac{dv^2}{dp} = \epsilon^{2n_v + n_p} \frac{C_D v^2}{\cos \gamma'} + 2\epsilon^1 \frac{1}{\beta p} \tag{2.12-7}$$

$$\frac{d \sin \gamma'}{dp} = \epsilon^{n_p + n_L - n_\gamma} \frac{C_L}{2} + \left(\epsilon^1 - \frac{\epsilon^{1-2n_v}}{v^2}\right) \frac{\sin \gamma'}{\beta p} \tag{2.12-8}$$

where only the first terms are to be retained in series expansions for $\sin \gamma'$ and $\cos \gamma'$. The regions of behavior of these equations are defined by the following conditions:

(1) For dominant flight path acceleration

$2n_v < 2n_v + n_p,\ 1$

(2) For dominant drag

$2n_v + n_p < 2n_v,\ 1$

(3) For dominant flight path component of gravity

$1 < 2n_v,\ 2n_v + n_p$

(4) For dominant normal acceleration

$$0 < n_p + n_L - n_{\gamma'}, \ 1, \ 1 - 2n_v$$

(5) For dominant lift

$$n_p + n_L - n_{\gamma'} < 0, \ 1 \quad 1 - 2n_v$$

(6) For dominant centrifugal acceleration

$$1 < 0, \ n_p + n_L - n_{\gamma'}, \ 1 - 2n_v$$

This condition obviously cannot be satisfied.

(7) For dominant normal component of gravity

$$1 - 2n_v < 0, \ n_p + n_L - n_{\gamma'}, \ 1$$

Again, these conditions are easist to interpret in terms of divisions of state space. See Fig. 2.12-2. There is a balance between drag and flight path accelerations at high velocities for densities order one. There is a balance between flight path acceleration and gravity component for velocities order $\epsilon^{\frac{1}{2}}$ and low densities. And finally, there is a balance between drag and the flight path component of gravity for high densities and low velocities.

Interestingly, a balance occurs only among all three terms in the flight path equation for densities of order one and velocities the order of sonic velocity. Since the flight path angles are large, this should be a short-lived portion of the trajectory.

In Fig. 2.12-2, it is seen that balance between normal gravity, lift and normal acceleration occur only for velocities of the order of sonic velocity, and where the order of γ' equals the order of the ρC_L product. As one normally has control over the value of C_L, this condition need never occur. Notice finally, that Figs. 2.12-1 and 2.12-2 join properly at $n_{v'} = 0$, $n_{\gamma} = 0$.

Prior to proceeding to the $h = 0(\epsilon^2)$ regime, a comment on its impliciation is in order. Notice that the independent variable h was introduced by use of the kinematic relation

$$\frac{dh}{dt} = v \sin \gamma$$

To produce a nonsingular set of equations, it is presumed that this equation is in balance. Specifically, that

$$o(h) = o(v)\, o(\sin\gamma)\, o(t)$$

so that specifying the order of h, v, and $\sin\gamma$ specifies the order of t. Thus, for example, a regime that has $h = 0(\epsilon^2)$, $v = 0(1)$, $\sin\gamma = 0(1)$, must occur on a time

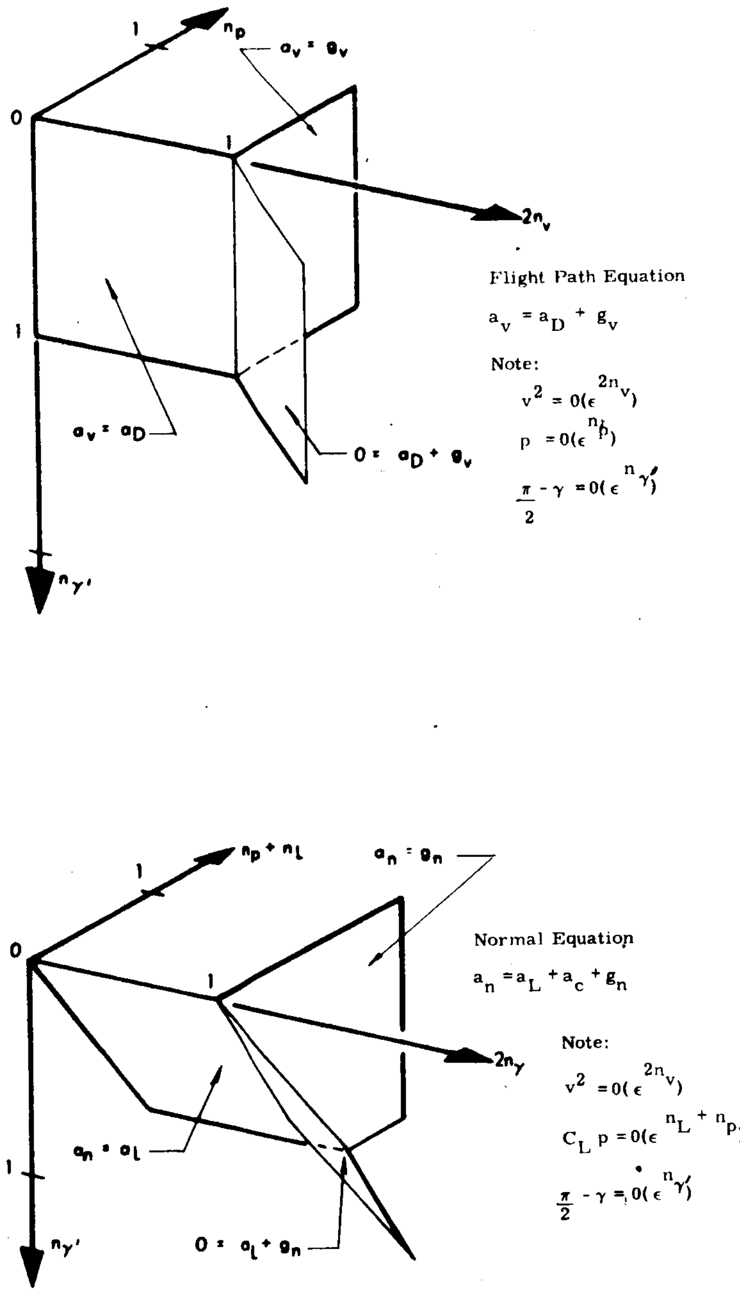


Fig. 2.12-2

Regions of Behavior of Dynamic Equations for $\gamma = \frac{\pi}{2} - \gamma < 0(1)$, $h = 0(\epsilon)$

scale of order ϵ^2 which is probably too short to be of interest in trajectory analysis but is exactly the time scale for stability analysis. Therefore, the implication of $h = 0(\epsilon^2)$ regime is normally flight at either small flight path angles, or n velocities or rapid time scales.

The equations, valid to lowest order in ϵ, for $h = 0(\epsilon^2)$, $v^2 = 0(\epsilon^{2n_v})$, $\rho = 0(\epsilon^{n_\rho})$, $C_L = 0(\epsilon^{n_L})$, $\gamma = 0(\epsilon^{n_\gamma}) \leq 0(1)$ are

$$\epsilon^{2n_v} \frac{dv^2}{dh} = -\epsilon^{2n_v + n_\rho - n_\gamma + 1} C_D \frac{\beta \rho v^2}{\sin\gamma} - 2\epsilon^2 \qquad (2.12\text{-}9)$$

$$\epsilon^{2n_\gamma} \frac{d\cos\gamma}{dh} = \epsilon^{n_\rho + n_L + 1} \frac{C_L}{2} \beta \rho - \left(\epsilon^2 - \frac{\epsilon^{2-2n_v}}{v^2}\right) \cos\gamma \qquad (2.12\text{-}10)$$

$$\frac{d\rho}{dh} = -\epsilon \beta \rho \qquad (2.12\text{-}11)$$

where appropriate series are assumed substituted for $\sin\gamma$ and $\cos\gamma$, as γ is presumed small. It is seen that p is a constant to lowest order. This is an assumption normally made in low velocity stability analysis. It is seen to be an equally valid assumption for velocities the order of sonic velocity if the altitude excursion of interest is $0(\epsilon^2)$.

The order of the variables required for each of the terms in the equations to be dominant are given by the following relations:

(1) For dominant flight path acceleration

$2n_v < 2n_v + n_p - n_\gamma + 1,\ 2$

(2) For dominant drag

$2n_v + n_p - n_\gamma + 1 < 2n_v,\ 2$

(3) For dominant flight path component of gravity

$2 < 2n_v + n_p - n_\gamma + 1,\ 2n_v$

(4) For dominant normal acceleration

$2n_\gamma < n_p + n_L + 1,\ 2,\ 2 - 2n_v$

(5) For dominant lift

$n_p + n_L + 1 < 2n_\gamma,\ 2,\ 2 - 2n_v$

(6) For dominant centrifugal acceleration

$$2 < 2n_\gamma;\ n_p + n_L + 1,\ 2 - 2n_v$$

(7) For dominant normal component of gravity

$$2 - 2n_v < 2n_\gamma,\ n_p + n_L + 1,\ 2$$

These conditions are illustrated in Fig. 2.12-3. By comparing these figures with Figs. 2.12-1, one sees that they are topologically similar. There is a stretching of the $2n_v$ and n_γ coordinates which reflects the interrelation of v, γ, and h that has already been discussed. The interesting feature of these figures is that motion occurs along constant n_p planes. Then, for example, the classical phugoid oscillation for a glider at $v^2 = 0(\epsilon^2)$, $\gamma = 0(1)$, $p\, C_L = 0(\frac{1}{\epsilon})$ can be interpreted as a motion where gravity and drag are in balance along the flight path. A transition is then made from lift and acceleration balance to a gravity and acceleration balance normal to the flight path. If the altitude excusion is allowed to be order ϵ then it is seen in Fig. 2.12-1 that the possibility for this type of motion exists at sonic velocities and at wing loadings, ($p\, C_L$) of order one. Such motion cannot exist without thrust, for velocities the order of orbital velocity, because drag always dominates the flight path component of gravity. This has led to some confusion in current literature on the subject.[43, 44, 45]

It has been shown in Section 2.9 that there are distinguished forms of the dynamic equations associated with velocities close to satellite velocity or small $v^2 - 1$. To investigate this behavior introduce the variable

$$v^{2'} = v^2 - 1 \qquad (2.12\text{-}12)$$

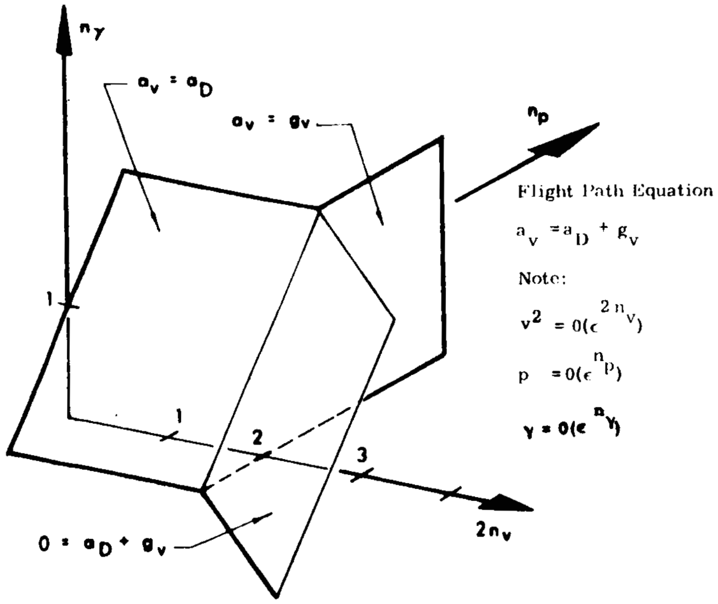
Then the dynamic equations, valid to lowest order in ϵ, for $h = 0(\epsilon)$ $v^{2'} = 0(\epsilon^{2n_v'}) < 0(1)$, $l_p = 0(\epsilon^{n_p})$, $C_L = 0(\epsilon^{n_L})$, $\gamma = 0(\epsilon^{n_\gamma}) \leq 0(1)$ are

$$\epsilon^{2n_v'} \frac{dv^{2'}}{dp} = \epsilon^{n_p - n_\gamma} \frac{C_D}{\sin\gamma} + 2\epsilon^1 \qquad (2.12\text{-}13)$$

$$\epsilon^{2n_\gamma} \frac{d\cos\gamma}{dp} = \epsilon^{n_L + n_p} \frac{C_L}{2} + \epsilon^{2n_v' + 1} v^{2'} \frac{\cos\gamma}{p} \qquad (2.12\text{-}14)$$

Where the appropriate series are assumed substituted for $\sin\gamma$ and $\cos\gamma$.

The orders in ϵ of the variables required for each of the terms in the



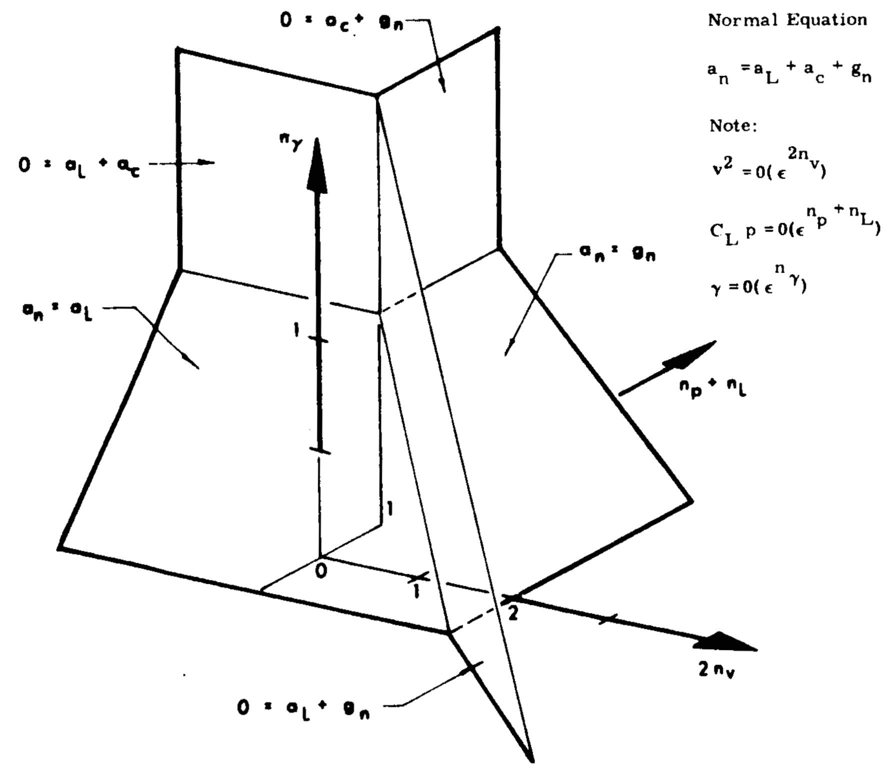


Fig. 2.12-3

Regions of Behavior of Dynamic Equations for $\gamma < 0(1)$, $h = 0(\epsilon^2)$

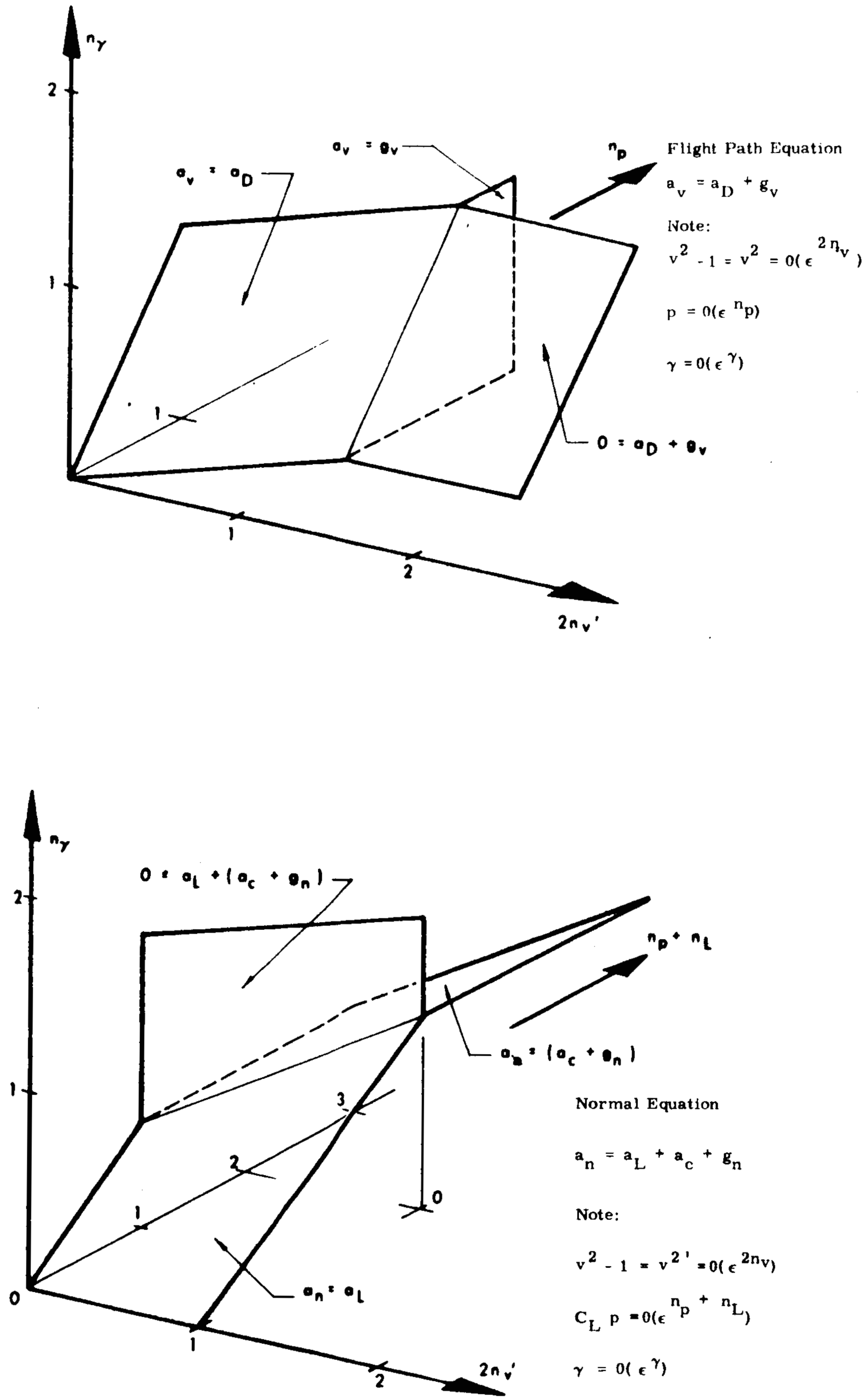


Fig. 2.12-4

Regimes of Behavior of Dynamic Equations for
$\gamma < 0(1)$, $h = 0(\epsilon)$, $v^{2'} = v^2 - 1 < 0(1)$

dynamic equations to be dominant, are given by the following relations:

(1) For dominant flight path acceleration

$$2n_{v'} < n_p - n_\gamma,\ 1$$

(2) For dominant drag

$$n_p - n_\gamma < 2n_{v'},\ 1$$

(3) For dominant flight path component of gravity

$$1 < 2n_{v'},\ n_p - n_\gamma$$

(4) For dominant-normal acceleratation

$$2n_\gamma < n_L + n_p,\ 2n_{v'} + 1$$

(5) For dominant lift

$$n_L + n_p < 2n_\gamma,\ 2n_{v'} + 1$$

(6) For dominant normal gravity centrifugal acceleration difference

$$2n_{v'} + 1 < 2n_\gamma,\ n_L + n_p$$

These conditions are illustrated in Fig. 2.12-4 . It is observed that there is a possible balance of all terms in the dynamic equations for $v'^2 = 0(\epsilon), C_L p = 0(\epsilon^2)$ $p = 0(\epsilon^2)$. This is the only condition for which such a balance is possible and it may be destroyed by choice of a C_L not of order one. The equilibrium glide condition $0 = a_L + a_c + g_n$ is seen to exist on a plane that requires γ and $p\,C_L$ to become arbitrarily small as v'^2 becomes small, a result that has already been observed. Orbital decay is seen to be characterized by the normal component of gravity and centrifugal acceleration in near balance, and drag nearly balancing the flight path component of gravity.

A particularly useful observation is that there is a relatively large region of the state space where only aerodynamic forces balance the normal and flight path accelerations. This is the near orbital velocity aerodynamically dominated regime.

2.13 A Multiple Regime Solution

After producing expansions for numerous regimes associated with flight at orbital velocities, it is natural to seek a single solution that to lowest order conforms to these expansions. Such a solution would preclude the practical problem of choosing the proper expansions associated with given initial conditions, if only lowest order results are desired.

It also holds the possibility of being valid for a complete trajectory that traverses a number of these regimes. This possibility may well not be realized for it will be shown in Chapter III that the matching principle sometimes requires a trajectory to be determined to order ϵ in one regime prior to establishing the initial conditions to lowest order in another regime. Certainly the next order correction to this lowest order solution would preclude this difficulty, but it will result that the solution is overly complex even to lowest order. A procedure that assures validity of a solution for the number of regimes of interest, and usually produces solutions of less complexity than this multiple regime solution, will be illustrated in Chapter III.

This multiple regime solution will now be developed. It will be shown to have a close but not exact similarity to Loh's "second order" solution. This, in effect, will supply an analytical justification of the numerical success experienced by Loh's empirically developed solution and also serve to define its region of validity. It will also correct Loh's solution and place it in a form that will allow systematic higher approximations.

Consider the following lowest order dynamic equations:

$$\frac{d\cos\gamma}{dh} = -\frac{1}{2}C_L\rho - \epsilon\left(1 - \frac{1}{v_0^2}\right)\cos\gamma_0 \qquad (2.13\text{-}1)$$

$$\frac{dv^2}{dh} = -\frac{C_D\rho v^2}{\sin\gamma} \qquad (2.13\text{-}2)$$

$$\frac{d\theta}{dh} = \frac{1}{\tan\gamma} \qquad (2.13\text{-}3)$$

$$\frac{dp}{dh} = -\rho \qquad (2.13\text{-}4)$$

It is observed that these equations neglect terms no larger than order ϵ for the following ranges of the dynamic variables:

(1) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(1)$, $p = 0(1)$ See Section 2.5.

(2) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(1)$, $p = 0(\epsilon)$ See Section 2.6.

(3) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(\epsilon^{\frac{1}{2}})$, $p = 0(\epsilon)$ See Section 2.9.

(4) $h = 0(\epsilon)$, $v^2 - 1 = 0(\epsilon^{\frac{1}{2}})$, $\gamma = 0(\epsilon^{\frac{1}{2}})$, $p = 0(\epsilon)$ See Section 2.9

A slight rearrangement of equations, Eqs. (2.13-1 - 2) and (2.13-4) yields:

$$d\cos\gamma = \frac{1}{2} c_L\, dp - \varepsilon\left(1 - \frac{1}{v_0^2}\right)\cos\gamma_0\, dh \tag{2.13-5}$$

$$d\ln v^2 = \frac{c_D\, dp}{\sin\gamma} \tag{2.13-6}$$

or

$$d\gamma = -\frac{c_L}{2c_D}\, d\ln v^2 + \varepsilon\left(1 - \frac{1}{v^2}\right)\cos\gamma_0 \frac{dh}{\sin\gamma} \tag{2.13-7}$$

which may be integrated to give

$$\cos\gamma - \cos\gamma_0 = \frac{1}{2} c_L\,(p - p_0) - \varepsilon\cos\gamma_0\left(1 - \frac{1}{v_0^2}\right)(h - h_0) \tag{2.13-8}$$

$$\ln\frac{v^2}{v_0^2} = c_D \int_{p_0}^{p} \left[\sin\cos^{-1}\left\{\cos\gamma_0 + \frac{1}{2} c_L (p - p_0) - \varepsilon\cos\gamma_0\left(1 - \frac{1}{v_0^2}\right)(h - h_0)\right\}\right]^{-1} dp \tag{2.13-9}$$

or

$$\gamma - \gamma_0 = \frac{c_L}{2c_D}\ln\frac{v^2}{v_0^2} + \varepsilon\left(1 - \frac{1}{v_0^2}\right)\int_{h_0}^{h}\left[\sin\cos^{-1}\left\{\cos\gamma_0 + \frac{1}{2} c_L (p - p_0) - \varepsilon\cos\gamma_0\left(1 - \frac{1}{v_0^2}\right)(h - h_0)\right\}\right]^{-1} dh \tag{2.13-10}$$

The range equation, Eq. (2.13-3), may be integrated to give

$$\theta - \theta_0 = \int_{h_0}^{h} \left[\tan\cos^{-1}\left\{ \cos\gamma_0 + \frac{1}{2} c_L (p - p_0) - \varepsilon \cos\gamma_0 \left(1 - \frac{1}{N_0}\right)(h - h_0) \right\} \right]^{-1} dh$$

(2.13-11)

where as usual, p and h are related by

$$p = p_0 e^{-\int \beta dh} \tag{2.13-12}$$

Loh's solution may be produced by making the following approximate integrations:

(1) The last term in Eq. (2.13-5):

$$\cos\gamma_0 \left(1 - \frac{1}{N_0^2}\right) \int dh = \cos\gamma_0 \left(1 - \frac{1}{N_0^2}\right) \int \frac{dp}{\rho} \cong \left(1 - \frac{1}{N_0^2}\right) \frac{p - p_0}{\rho}$$

(2.13-13)

(2) The last term in Eq. (2.12-7):

$$\cos\gamma_0 \left(1 - \frac{1}{N_0^2}\right) \int \frac{dh}{\sin\gamma} = -\cos\gamma_0 \left(1 - \frac{1}{N_0^2}\right) \int \frac{d \ln N^2}{\rho C_D} \cong -\cos\gamma_0 \left(1 - \frac{1}{N_0^2}\right) \frac{\ln \frac{N^2}{N_0^2}}{\rho C_D}$$

(2.13-14)

This transforms Eq. (2.13-8) and (2.13-9) into

$$\cos\gamma = \cos\gamma_0 + \frac{c_L}{2}(p - p_0) + \varepsilon \left(1 - \frac{1}{N_0^2}\right) \frac{\cos\gamma_0}{\rho} (p - p_0)$$

(2.13-15)

$$\gamma - \gamma_0 = \frac{C_L}{2C_D} \ln \frac{v^2}{v_0^2} \quad - \varepsilon \cos\gamma_0 \left(1 - \frac{1}{v_0^2}\right) \frac{\ln \frac{v^2}{v_0^2}}{\rho C_D}$$

(2.13-16)

These approximate integrations are not of lowest order importance if either $(1 - \frac{1}{v_o^2}) = 0(\epsilon^{\frac{1}{2}})$ or $\rho = 0(1)$. These are the trajectories that Loh predicts with the greatest accuracy. If one of these conditions is not satisfied, the solutions are not valid to lowest order. This has been numerically verified by Citron and Meir[(33)].

A final bit of tailoring is accomplished by observing that there is some degree of freedom left in the choice of the initial conditions in the term, $\cos \gamma_o (1 - \frac{1}{v_o^2})$. At the sacrifice of coupling the equation for γ and v as functions of p and ρ, or equivalently h, these initial conditions may be established instantaneously by use of the current values of γ and v^2. This may allow the solutions to traverse several of the regimes for which they are valid to lowest order without the bother of re-establishing the initial conditions on this term. Interestingly, it also makes the solution valid to lowest order for the equilibrium glide regime where $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(\epsilon)$, $p = 0(\epsilon)$. With this modification, Eqs. (2.13-15) and (2.13-16) become

$$\cos\gamma = \cos\gamma_0 + \frac{C_L}{2}(p - p_0) + \varepsilon\left(1 - \frac{1}{v^2}\right)\frac{\cos\gamma}{\rho}(p - p_0)$$

(2.13-17)

$$\gamma - \gamma_0 = -\frac{C_L}{2C_D} \ln \frac{v^2}{v_0^2} - \varepsilon \frac{\cos\gamma}{\rho C_D}\left(1 - \frac{1}{v^2}\right) \ln \frac{v^2}{v_0^2}$$

(2.13-18)

Where p and ρ are now assumed to have an exponential variation with h

$$\rho = \rho_0 e^{-\beta_0 (h - h_0)}$$

$$\rho = \rho_0 e^{-\beta_0 (h - h_0)} \qquad (2.13\text{-}19)$$

These are Loh's "second order" solutions for planetary entry. It is observed that they are only valid to lowest (or "first") order and not to "second order" as claimed. But they are valid to lowest order for a number of regimes. Specifically for:

(1) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(1)$, $\rho = 0(1)$

(2) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(1)$, $\rho = 0(\epsilon)$

(3) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(\epsilon^{\frac{1}{2}})$, $\rho = 0(\epsilon)$

(4) $h = 0(\epsilon)$, $v^2 - 1 = 0(\epsilon^{\frac{1}{2}})$, $\gamma = 0(\epsilon^{\frac{1}{2}})$, $\rho = 0(\epsilon)$

(5) $h = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(\epsilon)$, $\rho = 0(\epsilon)$

with the additional restriction that

$$\frac{\epsilon\left(1 - \frac{1}{v^2}\right)}{\rho} \leq 0(\epsilon^{\frac{1}{2}}) \qquad (2.13\text{-}20)$$

Eqs. (2.13-8 - 11) are valid in these same regimes without this final restriction if a similar ruse of instantaneously establishing the initial conditions on the quantity, $\cos\gamma_o \left(1 - \frac{1}{v_o^2}\right)$ is used. They are more complex. If a solution of this range of validity is not needed, an approach that may obtain simpler, more accurate solutions is to match the expansions for the range of interest. An example of this procedure will be given in Chapter III.

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CHAPTER III

PATCHING AND MATCHING THE ASYMPTOTIC EXPANSIONS

3.1 Introduction

To this point, only identification of the regimes of flight and solutions of the appropriate dynamic equation in asymptotic expansions have been considered. But if a trajectory that traverses several regimes of flight is to be analyzed, some method of combining the previously obtained solutions must be sought.

The simplest method of combining the solutions is to patch them together at their common boundary. As the two solutions were obtained by a different scaling in ϵ of one of the variables in the dynamic equations, ostensibly there is an intermediate scaling of that variable which can be used to define a boundary. The arbitrary constants in one solution can be picked so that the two solutions agree at this boundary. This is a method suggested by Battin[17] and Pontryagin, et al.[55]

It is clear that such a procedure creates a corner. A term in the dynamic equation that was of order one in one region may be of order ϵ in the other. One has arbitrarily picked a point to change the dynamic equations from one form to another. Though this is heuristically not pleasing, it may not create an appreciable error in the combined solution, especially if the two solutions were accurate to an order ϵ higher than was needed.

The procedure for combining solutions that will be used here is the "Matching Principle" of Kaplan and Lagerstrom[4, 5, 1] (see Appendix B). It is slightly more sophisticated and consequently more complicated. It is not apparent that the results always warrant the extra complexity, but at least conceptually it is more pleasing.

The principle is founded on the assumption that the two expansions share some common region of validity, an "overlap domain". If the two expansions are to match smoothly, they must be algebraically identical, to each order in ϵ, in this region. The arbitrary constants in the two expansions are chosen so that this matching occurs. A solution that smoothly transitions from one expansion to another is then formed by summing the two expansions and subtracting their common part. This solution is called a composite expansion for the two regimes that are matched.

Two solutions will be matched. The first will produce the well established procedure for patching Keplerian conics to ballistic trajectories within this more general context. The second will produce a solution not previously published, valid for an interesting class of lifting trajectories.

3.2 Matching a Keplerian Conic With a Ballistic Trajectory

The first use of the matching principle will be to match a Keplerian conic and an aerodominated ballistic trajectory. The result will be, in some sense, obvious, but will serve to illustrate some features of the principle on a simple example.

The ballistic trajectory solution, to lowest order in ϵ, is (see Section 2.5)

$$\begin{aligned} v_1^2 &= v_{01}^2 \, e^{\frac{c_D(\rho_1 - \rho_{01})}{\sin\gamma_{01}}} \\ \cos\gamma_1 &= \cos\gamma_{01} \\ \theta_1 &= \theta_{01} \\ \rho_1 &= \rho_0 \, e^{-\int \beta dh} \end{aligned} \tag{3.2-1}$$

The solution for a Keplerian conic is (see Section 2.4)

$$\begin{aligned} \frac{v_2^2}{2} - \frac{1}{1+\varepsilon h} &= \frac{v_{02}^2}{2} - \frac{1}{r_{02}} \\ (1+\varepsilon h)\, v_2 \cos\gamma_2 &= r_{02} v_{02} \cos\gamma_{02} \\ \frac{1-(1+\varepsilon h) v_2^2 \cos\gamma_2}{\cos\theta_2} &= \frac{1 - r_{02} v_{02}^2 \cos\gamma_{02}}{\cos\theta_{02}} \end{aligned} \tag{3.2-2}$$

where the ballistic trajectory variable (height), has been introduced by the relation,

$$r = 1 + \varepsilon h \tag{3.2-3}$$

The limiting form of the ballistic solution as $h \to \infty$ is

$$
\begin{aligned}
{}_2v_1^{\,2} &= v_{01}^{\,2}\, e^{-\frac{C_D \rho_{01}}{\sin \gamma_{01}}} \\
\cos {}_2\gamma_1 &= \cos \gamma_{01} \\
{}_2\theta_1 &= \theta_{01} \\
{}_2\rho_1 &= 0
\end{aligned}
\tag{3.2-4}
$$

This is the form of the ballistic solution valid in the "overlap domain". The limiting form of the Keplerian solution as $h \to 0$ to lowest order in ϵ is,

$$
\begin{aligned}
\frac{{}_1v_2^{\,2}}{2} - 1 &= \frac{v_{02}^{\,2}}{2} - \frac{1}{r_{02}} \\
{}_1v_2 \cos {}_1\gamma_2 &= r_{02}\, v_{02} \cos \gamma_{02} \\
\frac{1 - {}_1v_2^{\,2} \cos {}_1\gamma_2}{\cos {}_1\theta_2} &= \frac{1 - r_{02}\, v_{02}^{\,2} \cos \gamma_{02}}{\cos \theta_{02}}
\end{aligned}
\tag{3.2-5}
$$

This is the form of the Keplerian solution valid in the "overlap domain". The requirement that these two limiting forms match, i.e.,

$$
\begin{aligned}
{}_2v_1 &= {}_1v_2 \\
{}_2\gamma_1 &= {}_1\gamma_2 \\
{}_2\theta_1 &= {}_1\theta_2
\end{aligned}
\tag{3.2-6}
$$

determines the initial condition for the ballistic trajectory in terms of the initial condition for the Keplerian trajectory.

To reduce the algebraic complexity and to gain some insight into the meaning of these conditions, assume that the initial conditions for the Keplerian phase are given at the reference radius so that

$$
r_{02} = 1 \tag{3.2-7}
$$

Then Eqs. (3.2-5) reduce to

$$
\begin{aligned}
{}_1v_2^{\,2} &= v_{02}^2 \\
\cos {}_1\gamma_2 &= \cos \gamma_{02} \\
\cos {}_1\theta_2 &= \cos \theta_{02}
\end{aligned}
\qquad (3.2\text{-}8)
$$

Further, as p_{01} is arbitrary, to lowest order, assume that it is zero so that Eqs. (3.2-4) reduce to

$$
\begin{aligned}
{}_2v_1^{\,2} &= v_{01}^2 \\
\cos {}_2\gamma_1 &= \cos \gamma_{01} \\
{}_2\theta_1 &= \theta_{01}
\end{aligned}
\qquad (3.2\text{-}9)
$$

Then Eqs. (3.2-6) imply that the initial conditions for the Keplerian and ballistic trajectory are related as follows:

$$
\begin{aligned}
v_{01}^2 &= v_{02}^2 \\
\gamma_{01} &= \pm \gamma_{02} \\
\theta_{01} &= \pm \theta_{02}
\end{aligned}
\qquad (3.2\text{-}10)
$$

These are the relations commonly assumed in "patching" ballistic and Keplerian trajectories. So, to lowest order, the matching procedure conforms to intuition.

3.3 Matching the Aero-gravity Perturbed and the Aero-dominated Regimes.

A slightly more interesting application of the matching principle occurs when the aero-gravity perturbed phase is matched with an aero-dominated "skip." This will produce a composite expansion applicable for a currently interesting class of lifting weapon and manned vehicle trajectories.

The skip solution, correct to lowest or zeroth order in ϵ, for aero-dominated flight is

$$v_1^2 = v_{01}^2 \, e^{-\frac{2C_D}{C_L}(\gamma_1 - \gamma_{01})} \qquad (3.3\text{-}1)$$

$$\cos\gamma_1 = \cos\gamma_{01} + \frac{1}{2} C_L (p_1 - p_{01}) \qquad (3.3\text{-}2)$$

Recall that these equations are valid for $p = 0(1)$, $v^2 = 0(1)$, $\gamma = 0(1)$. See Section 2.5.

The aero-gravity perturbed solution, correct to first order in ϵ, is

$$v_2^2 = v_{02}^2 + \epsilon \frac{C_D v_{02}}{\sin\gamma_{02}} \left(\frac{p_1}{\epsilon} - p_{02} \right) - 2\epsilon (h - h_0) \qquad (3.3\text{-}3)$$

$$\cos\gamma_2 = \cos\gamma_{02} + \epsilon \frac{C_L}{2} \left(\frac{p_1}{\epsilon} - p_{02} \right) - \epsilon \cos\gamma_{02} \left(\frac{1}{v_{02}^2} - 1 \right)(h - h_0) \qquad (3.3\text{-}4)$$

Recall that his solution is valid for $p = 0(\epsilon)$, $v^2 = 0(1)$, $\gamma = 0(1)$. See Section 2.6. For both solutions,

$$p_1 = p_0 \, e^{-\int \beta \, dh} \qquad (3.3\text{-}5)$$

The limiting form of the latter solution as $p_1 \to \infty$ (the "overlap" domain form) to the lowest order in ϵ is

$${}_1v_2^2 = v_{02}^2 + \frac{C_D v_{02}^2}{\sin\gamma_{02}} p_1 \qquad (3.3\text{-}6)$$

$$\cos {}_1\gamma_2 = \cos\gamma_{02} + \frac{C_L}{2} p_1 \qquad (3.3\text{-}7)$$

Notice that this is equivalent to taking the first term in this expansion for small ϵ. See Appendix C. The limiting form of Eq. (3.3-2) as $p_1 \to 0$ (the "overlap" domain form) is seen to match Eq. (3.3-7) if

$$\begin{aligned} \cos \gamma_{01} &= \cos \gamma_{02} \\ \varphi_{01} &= 0 \end{aligned} \qquad (3.3\text{-}8)$$

This same result may be produced more formally by expressing Eq. (3.3-2) in the variable p_2

$$\cos \gamma_1 = \cos \gamma_{01} + \tfrac{1}{2} C_L (\epsilon \varphi_2 - \varphi_{01}) \qquad (3.3\text{-}9)$$

and truncating the series at the second term

$$\cos {}_2\gamma_1 = \cos \gamma_{01} - \tfrac{1}{2} C_L \varphi_{01} + \epsilon \tfrac{1}{2} C_L \varphi_2 \qquad (3.3\text{-}10)$$

Then requiring, from Eqs. (3.3-7) and (3.3-10), that ${}_2\gamma_1 = {}_1\gamma_2$ yields

$$\begin{aligned} \cos \gamma_{01} &= \cos \gamma_{02} \\ \varphi_{01} &= 0 \end{aligned} \qquad (3.3\text{-}11)$$

which is the result given in Eq. (3.3-8).

To obtain the limiting form of Eq. (3.3-1) as $p_1 \to 0$, notice that as $p_1 \to 0$, $\gamma_1 \to \gamma_{01}$. Then the limiting form of Eq. (3.3-1) may be expressed as

$${}_2N_1^{\,2} = N_{01}^{\,2} \left(1 - \frac{2 C_D}{C_L} ({}_2\gamma_1 - \gamma_{01})\right) \qquad (3.3\text{-}12)$$

Now to obtain a relation between an angle slightly different from γ_{01} and p_1, observe that for small ${}_2\gamma_1 - \gamma_{01}$,

$$\sin \gamma_{01} ({}_2\gamma_1 - \gamma_{01}) = \tfrac{1}{2} C_L ({}_2\varphi_1 - \varphi_{01}) \qquad (3.3\text{-}13)$$

from Eq (3.3-2)

so that

$$ {}_2\gamma_1 - \gamma_{01} = -\frac{1}{2}\,\frac{C_L\, p_1}{\sin\gamma_{01}} \tag{3.3-14} $$

Substitution of this into Eq. (3.3-12) yields

$$ {}_2 v_1^{\,2} = v_{01}^{\,2}\left(1 + \frac{C_D\, p_1}{\sin\gamma_{01}} \right) \tag{3.3-15} $$

This matches Eq. (3.3-6) if

$$ \begin{aligned} v_{01} &= v_{02} \\ \sin\gamma_{01} &= \sin\gamma_{02} \end{aligned} \tag{3.3-16} $$

which implies that to lowest order the initial conditions are not affected by a transition through this regime. This explains why this phase of the trajectory was neglected when matching a Keplerian conic to a ballistic trajectory.

Here the more formal matching is made algebraically complex by the form of the skip solution. Specifically, by expressing Eq. (3.3-1) in the variable p_2, one obtains

$$ v_1^{\,2} = v_{01}^{\,2}\; e^{-\frac{2C_D}{C_L}\left(\cos^{-1}\left(\cos\gamma_{01} + \frac{1}{2} C_L\left(\epsilon\, p_2 - p_{01}\right)\right) - \gamma_{01}\right)} \tag{3.3-17} $$

which for small ϵ may be expressed as

$$ v_1^{\,2} = v_{01}^{\,2}\left(1 + \epsilon\, \frac{C_D}{\sin\gamma_{01}}\, p_2 + O(\epsilon^2) \right) \tag{3.3-18} $$

where $p_{01} = 0$ from Eq. (3.3-11) was used. Truncating the series at the second term yields,

$$ {}_2v_1 = v_{01}^2 \left(1 + \varepsilon \frac{C_D\, p_2}{\sin\gamma_{01}} \right) \tag{3.3-19} $$

Requiring that ${}_1v_2 = {}_2v_1$ in Eqs. (3.3-6) and Eqs. (3.3-19) gives

$$ \begin{aligned} v_{01} &= v_{02} \\ \sin\gamma_{01} &= \sin\gamma_{02} \end{aligned} \tag{3.3-20} $$

which is the same result give in Eq. (3.3-16).

A description of the trajectory in terms of a "composite" expansion is convenient. This expansion is formed by summing the individual expansions and subtracting their common limit. (See Appendix C).

$$ \begin{aligned} v^2 &= v_1^2 + v_2^2 - {}_1v_2^2 \\ \cos\gamma &= \cos\gamma_1 + \cos\gamma_2 - \cos{}_1\gamma_2 \end{aligned} \tag{3.3-21} $$

or

$$ \begin{aligned} v^2 = {} & \frac{C_D v_0^2}{\sin\gamma_0} (-p_{02}) - 2\varepsilon (h - h_0 \\ & + v_0^2 e^{-\frac{2C_D}{C_L}(\gamma_1 - \gamma_0)} \end{aligned} $$

$$ \begin{aligned} \cos\gamma = {} & \cos\gamma_0 + \frac{C_L}{2}(p - p_{02}) \\ & - \varepsilon \cos\gamma_0 \left(\frac{1}{v_0^2} - 1 \right)(h - h_0) \end{aligned} \tag{3.3-22} $$

where

$$ \cos\gamma_1 = \cos\gamma_0 + \frac{C_L}{2} p $$

and

$$ p = p_0 e^{-\int \beta\, dh} $$

Notice that the matching principle has related all the arbitrary constants in the two solutions except p_{02} or equivalently h_{02}. This is of no consequence if the trajectory is coming down from the aero-gravity perturbed regime into the aero-dominated regime because its value is initially known and is not required for the lowest order skip problem. But, if the trajectory is going the other way, the value of P_{02} is needed to define the trajectory. To obtain this value, the skip trajectory must be calculated to next order. Thus, the aero-gravity perturbed problem has to be found to order ϵ to determine the arbitrary constants in the skip problem to order one. Now the skip problem has to be calculated to order ϵ to determine the arbitrary constants in the aero-gravity perturbed problem to order ϵ. This uneven balancing of orders of the problem to obtain matching is a common occurrence.

The composite expansion is closely related to Loh's "second-order" solution[2, 35, 33] given here for reference. (See Section 2.13).

$$\cos\gamma = \frac{\cos\gamma_0 + \frac{1}{2} C_L \rho \left(1 - \frac{\rho_0}{\rho}\right)}{1 + \varepsilon\left(\frac{1}{\nu^2} - 1\right)\left(1 - \frac{\rho_0}{\rho}\right)}$$

$$\ln\frac{\nu^2}{\nu_0^2} = \frac{\frac{2C_D}{C_L}(\gamma - \gamma_0)}{1 - \varepsilon C_L \frac{\cos\gamma}{\rho}\left(\frac{1}{\nu^2} - 1\right)} \qquad (3.3\text{-}23)$$

The composite solution has the advantage of: (1) conforming to the "second-order" solution in the $\rho = 0(1)$ regime to lowest order, (2) being valid in the $\rho = 0(\epsilon)$, $\gamma = 0(1)$ regime to order ϵ where Loh's solution is only valid to order one, (3) is numerically simpler to use.

Though it does not have the range of validity of Loh's solution, in its limited regimes it is more accurate. Its range of validity does include a number of currently interesting classes of trajectories. One is a super circular skip trajectory of an Apollo type which has the objective of losing velocity in excess of circular satellite velocity. The second trajectory is a negative lift trajectory characteristic of a lifting weapon.

Unfortunately, the solution in its present form is incapable of defining super circular orbit after a skip occurs, as the next order solution has not been included. But results up to this point are in excellent agreement with numerical solution. (See Fig. 3.3-1.) The lifting down trajectory starts in an aero-dominated regime near orbital velocity. It traverses the aero-gravity perturbed regime at large flight path angles and then goes back into an aero-dominated regime at low altitudes. This, also, is in agreement with the numerical solutions. (See Fig. 3.3-2.)

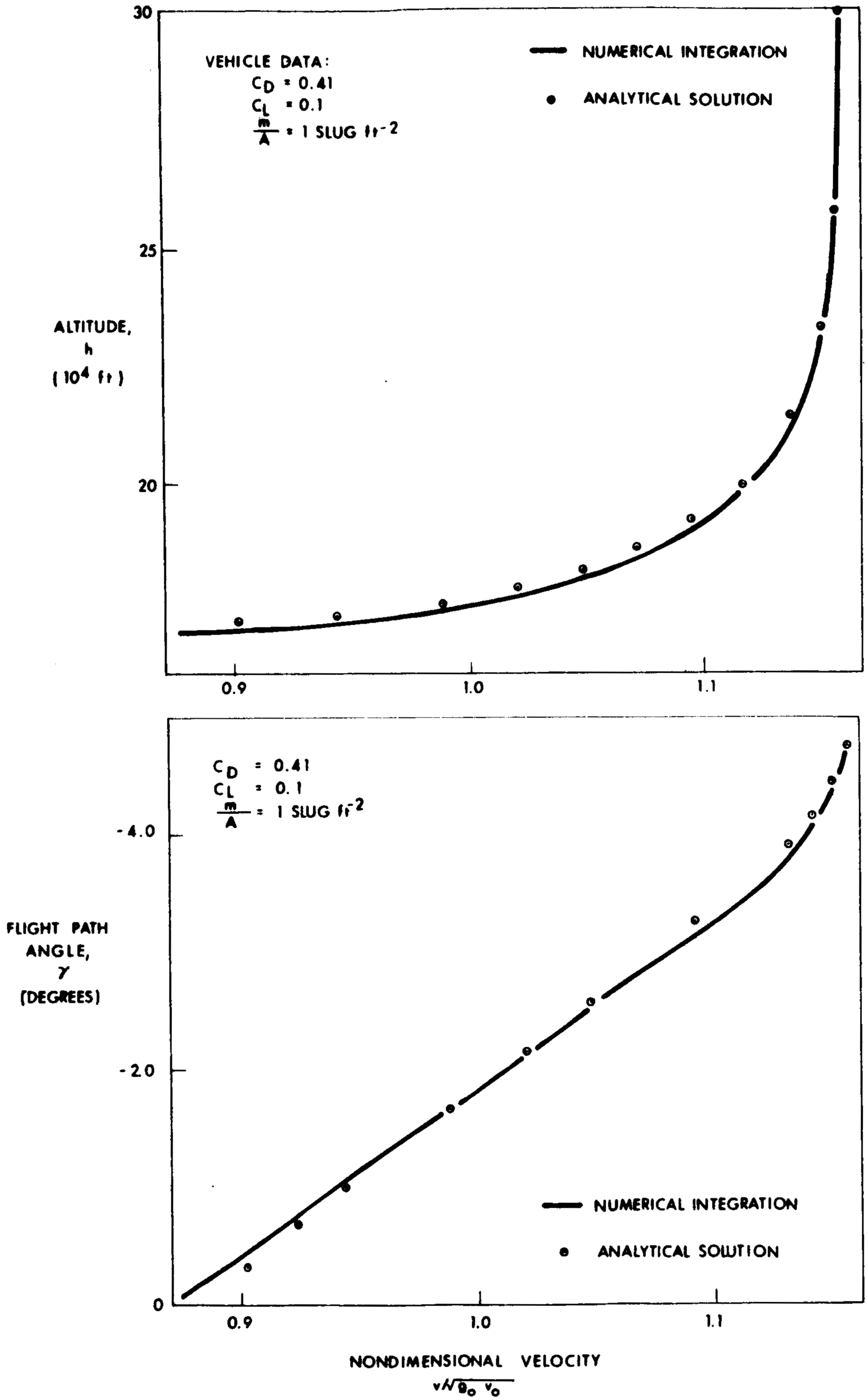

Fig. 3.3-1
A Numerical Verification of the Composite Solution for a Supercircular Skip Trajectory

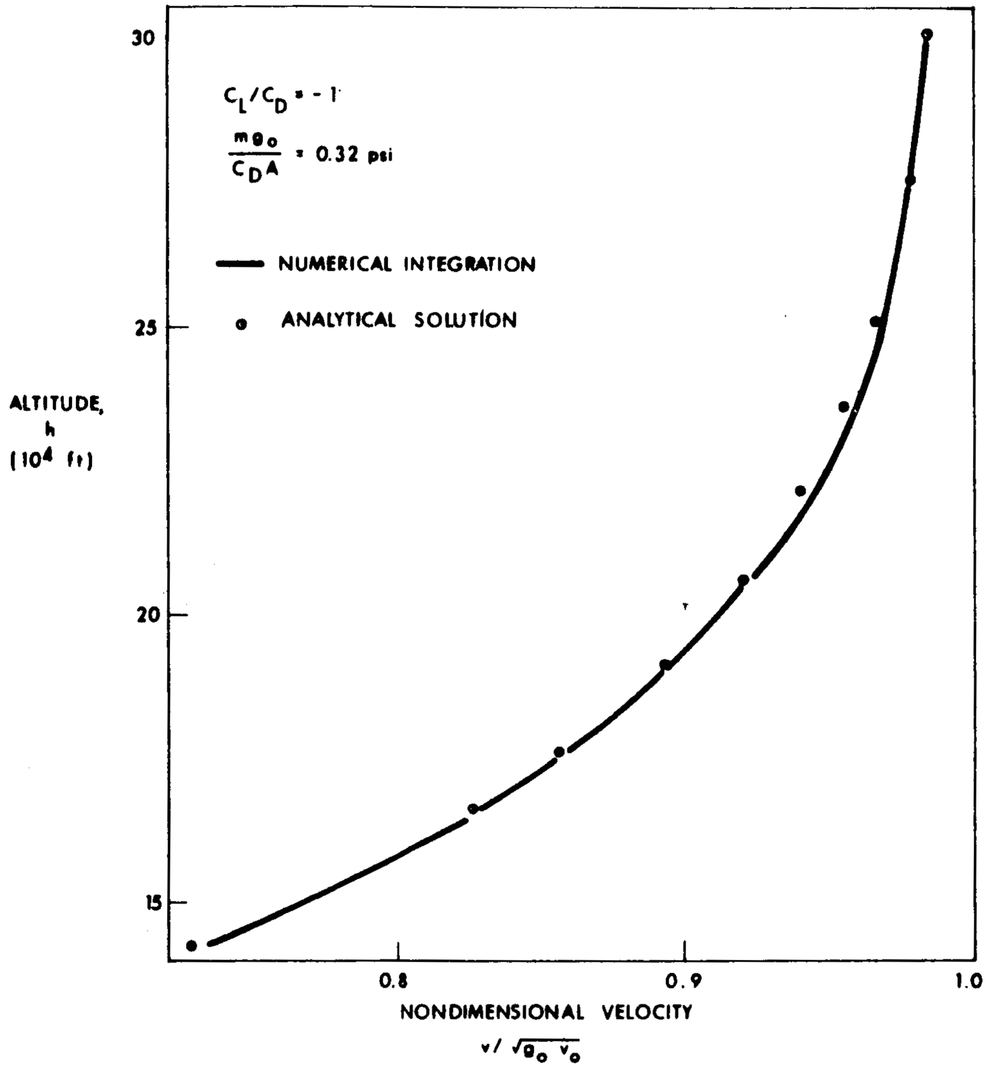

Fig. 3.3-2

A Numerical Verification of the Composite Solution for a Negative Lift Trajectory

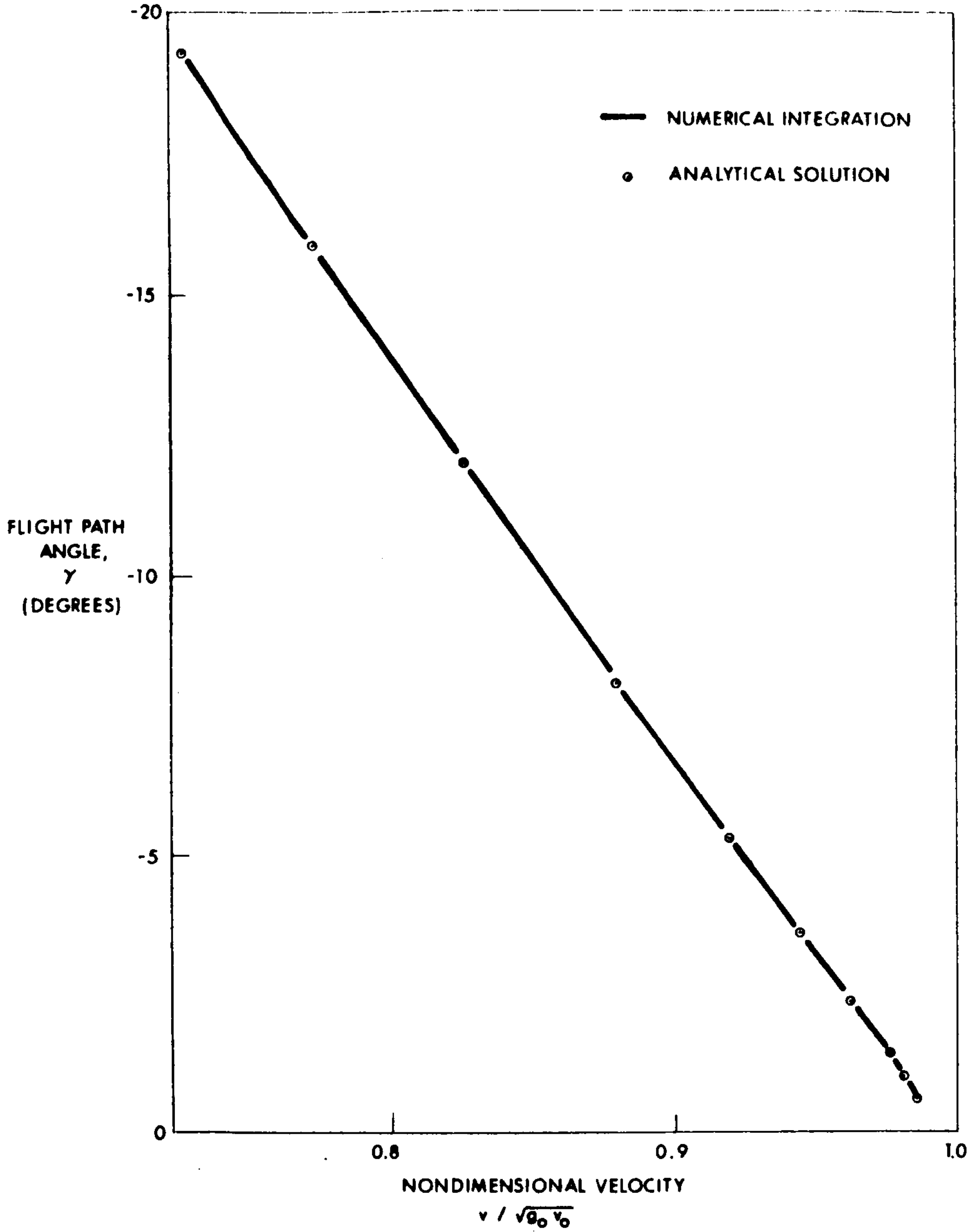
NUMERICAL INTEGRATION
ANALYTICAL SOLUTION
-20
-15
-10
-5
0
FLIGHT PATH ANGLE, γ (DEGREES)
0.8
0.9
1.0
NONDIMENSIONAL VELOCITY
$v / \sqrt{g_0 r_0}$

Fig. 3.3-2 (cont.)

CHAPTER IV

THE DYNAMICS OF THREE-DIMENSIONAL THRUSTING FLIGHT

4.1 Introduction

To this point, only two-dimensional nonthrusting flight about a spherically symmetric planet, with a nonrotating atmosphere, has been considered. The three-dimensional form of this problem is handled by noting that lift forces, out of the velocity-vector, planet-center plane, simply rotate this plane and leave the two-dimensional in plane problem unaltered. This, in effect, says that the dynamic equations for the in-plane out-of-plane problems are completely uncoupled and may be treated separately. The thrusting problem similarly uncouples if thrust is revolved into an in-plane and out-of-plane component.

The addition of planetary oblateness and a rotating atmosphere couples the in-plane and out-of-plane problem, but not necessarily to lowest order. In fact, it will be shown that these effects, though appreciable, always enter to higher order, so that the results of the two-dimensional nonrotating spherically symmetric analysis to lowest order will remain valid.

4.2 Nonthrusting Three-Dimensional Flight

Consider first, the problem of nonthrusting three-dimensional flight in a rotating atmosphere surrounding an oblate planet. The dynamic equations may be written in nondimensional* form as

$$\dot{\underline{r}} = \underline{v} \tag{4.2-1}$$

$$\dot{\underline{v}} = \underline{g} + \frac{\rho}{2}\left(v^2 - 2\epsilon_2\, \underline{v}\cdot\underline{v}_w + \epsilon_2^2\, v_w^2\right)\underline{c} \tag{4.2-2}$$

where $\underline{r}$ and $\underline{v}$ are the position and velocity vectors, ρ is the atmospheric density, $\underline{c}$ is a three-dimensional aero-dynamic coefficient, $\underline{v}_w$ is a wind velocity vector given as

$$\underline{v}_w = \underline{\Omega} \times \underline{r} \tag{4.2-3}$$

where $\underline{\Omega}$ is the angular velocity of the atmosphere (assumed to be the same as the angular velocity of the planet). The small parameter, ϵ_2, is the the ratio of the equatorial wind to the circular satellite velocity (or orbital period to rationalperiod).

*The same quantities are used to nondimensionalize mass, lengths and time as in Chapter II.

$$\epsilon_2 = \frac{\Omega r_0}{\sqrt{g_0 r_0}} = \Omega\sqrt{\frac{r_0}{g_0}} \tag{4.2-4}$$

The pressure is, as usual, assumed to be related to position by the hydrostatic equation.

$$\frac{\partial p}{\partial \underline{r}} = -\frac{\rho}{\epsilon_1}\left(\underline{g} - \underline{\Omega} \times (\underline{\Omega} \times \underline{r})\right) \tag{4.2-5}$$

where ϵ_1 is the ratio of atmospheric scale height to planetary radius and where a Coriolis correction has been included as the atmosphere is assumed to be rotating with the planet. Further pressure is related to density by the equation of state

$$p = \frac{\rho}{\beta} \tag{4.2-6}$$

The gravity acceleration vector, g, is the negative gradient of a potential, v, given as

$$\underline{g} = -\frac{\partial V}{\partial \underline{r}} \tag{4.2-7}$$

$$V = -\left[\frac{1}{r} - \left[\left(\frac{1}{r}\right)^3 \frac{J_2}{2}(3\sin^2 L - 1)\right]\right] \tag{4.2-8}$$

where J_2 is the second spherical harmonic coefficient and L is the planetary latitude. As might be expected J_2 is related to ϵ_2 and is approximately

$$J_2 \approx \frac{1}{3}\epsilon_2^2 \tag{4.2-9}$$

This simply states that the planetary rotation is the chief cause of the oblateness.

The additions of planetary rotation and oblateness have thus introduced only one new small parameter into the system, rather than two, as might have been originally surmised. A detailed analysis of the two parameter systems is overbearingly complex, if perfectly straightforward. Some enlightening observations are possible, though, without the benefit of the complete analysis.

The form of the hydrostatic equation still requires that heights above the planetary radius be rescaled to order ϵ_1 for a nonsingular solution. This will again introduce an aerodynamically dominated problem where gravitational accelerations do not enter to lowest order. The solution is therefore identical to the aero-dominated solution of Chapter II. It is interesting to observe that to next order, the effect of the rotating atmosphere should properly be accounted for, since $\epsilon_2^2 > \epsilon_1$ for all known planetary atmospheres except Venus.(see Table I)

For densities of order ϵ_1, aerodynamic forces do not enter to lowest order. This implies that the lowest order problem is a Keplerian conic. To next order the planetary oblatenss enters for all planets except Venus (as $\epsilon_2^2 > \epsilon_1$ for these planets). In fact, zero lift trajectories, perturbed by drag and oblateness, have been extensively treated in the literature.[85]

If a local rotating coordinate system is used to express the dynamic equations, as in Chapter II, there is no gravity contribution to lowest order. Further, if a local value of g_o and r_o are used to nondimensionalize the equations, the solutions developed in Chapter II are valid to order ϵ_1, including the effects of oblateness for motion in the planet vehicle-velocity-vector plane. The out-of-plane component of gravity can be made of little consequence by compensating for it with a lateral applications of lift.

When the velocities are the order of ϵ_2, the order of the planetary wind velocity, the problem is obviously better defined in terms of a coordinate system fixed to the planet. In such a coordinate system, the dynamic equations take the following form:

$$\dot{\underline{r}} = \underline{v} \tag{4.2-10}$$

$$\dot{\underline{v}} = \underline{g} - 2\epsilon_2 \underline{\Omega} \times \underline{v} + \frac{\rho}{2} v^2 \underline{c} \tag{4.2-11}$$

where $\underline{r}$ and $\underline{v}$ are position and velocity vectors in the rotating coordinate system and g is the negative gradient of a pseudo-potential associated with the rotating planet,

$$\underline{g} = -\frac{\partial V}{\partial \underline{r}} \tag{4.2-12}$$

where

$$V = \left[\frac{1}{r} - \left[\left(\frac{1}{r}\right)^3 \frac{J_2}{2} \left(3\sin^2 L - 1\right) + \Omega^2 r^2 \cos L \right]\right] \tag{4.2-13}$$

The hydrostatic equation in this coordinate system is simply

$$\frac{\partial p}{\partial \underline{r}} = -\frac{\rho}{\varepsilon_1} \underline{g} \tag{4.2-14}$$

Prior to considering the lower velocity flight, one interesting observation concerning the aerodyanamically dominated problem will be made. For altitudes of order, ϵ_1, and densities of order one, flight path angles of order one; or densities of order ϵ, flight path angles of order one, and velocities near circular velocities, the following lowest order problem applies:

$$\dot{\underline{v}} = \frac{\rho}{2} v^2 \underline{c} \tag{4.2-15}$$

$$\dot{p} = \underline{v} \cdot \left(\frac{\partial p}{\partial \underline{r}}\right) = -\rho \underline{g} \cdot \underline{v} \tag{4.2-16}$$

Notice that both density and time may be eliminated to give

$$\frac{d\underline{v}}{dp} = \frac{v^2 \underline{c}}{\underline{g} \cdot \underline{v}} \tag{4.2-17}$$

which indicates that flight in this regime follows a path along constant pressure surfaces. Observing that the range covered is of order ϵ_1 for the large flight path angle case, and of order $\epsilon_1^{\frac{1}{2}}$ for the small flight path angle case, then it is concluded that the direction of g to lowest order is constant. Defining the flight path angle as measured from the local g vector, one may integrate the equations to give the solutions in Chapter II. The lowest order oblateness term has thus been included. The next order term is of order, ϵ_2, and may simply be written in only geometric dependent form as

$$\underline{\dot{v}} = \frac{d\underline{v}}{dh}\frac{dh}{dt} = \frac{d\underline{v}}{dh}\left(\frac{\underline{g}}{g}\right)\cdot\underline{v} = \underline{\Omega}\times\underline{v} \qquad (4.2\text{-}18)$$

which may be integrated in its scalar form or its effect removed with the application of lift, as the acceleration is always perpendicular to the velocity vector. So a solution correct to order ϵ_2 is obtainable for the aerodominated regime about an oblate rotating planet with a real atmosphere in terms of the pressure and the geometry of the planet.

Finally, it is seen from Eq. (4.2-11) that there is no contribution of the rotating atmosphere, at low velocities, in this coordinate system. This implies that lowest order results previously obtained for two-dimensional flight at near sonic velocities are applicable in this coordinate system, if flight path angle is measured from the local $\underline{g}$ vector.

4.3 Thrusting Three-Dimensional Flight

The dynamic equation for thrusting three-dimensional flight in a rotating atmosphere surrounding an oblate planet may be written in nondimensional form as

$$\underline{\dot{r}} = \underline{v} \qquad (4.3\text{-}1)$$

$$\underline{\dot{v}} = \underline{g} + \underline{a}_A + \epsilon_3\,\underline{a}_T \qquad (4.3\text{-}2)$$

where $\underline{a}_A$ is the aerodynamic acceleration vector

$$\underline{a}_A = \frac{\rho}{2}\frac{v^2}{m}\underline{c} \qquad (4.3\text{-}3)$$

and $\underline{a}_T$ is the thrust acceleration

$$\underline{a}_T = \frac{\underline{T}}{m} \qquad (4.3\text{-}4)$$

Other variables are as defined in Eqs. (4.2-2). As might be expected, the addition of thrust to the problem adds a third parameter, ϵ_3, the ratio of the thrust acceleration to the reference value of gravitational acceleration (possibly large or small). A host of possibilities must now be considered, depending upon the relative size of the components of $\underline{v}$, $\underline{g}$, $\underline{a}_A$, and $\underline{a}_T$.

Some of the possibilities are discarded by recognizing that for thrust to be useful in aerodynamic flight, its velocity component must be of the same order as, or larger than, the aerodyanamic drag. Further, if the lift-to-drag ratio is greater than one, there is little advantage to thrusting in a direction other than along the velocity vector. Lateral forces are more efficiently developed with the use of lift. Hence, the thrust vector can be expected to be mainly along the velocity vector and of a magnitude greater than the drag for aerodynamic flight.

For orbital flight where the aerodynamic forces are negligible the thrust vector can be scaled only in reference to the local value of g. Cases where the value of the thrust acceleration is small in comparison to g have recently been treated extensively.[20, 72] The intermediate range where thrust accelerations and gravitational accelerations are of the same order is more difficult and has largely not yet been treated in the literature.

The classical case of impulsive thrust, where the thrust acceleration dominates over any gravity–and possibly aerodynamic–acceleration takes on an interesting interpretation in the light of the method presented here. For ϵ_3 large, the lowest order problem is

$$\begin{aligned} \dot{\underline{r}}^{(0)} &= \underline{v}^{(0)} \\ \dot{\underline{v}}^{(0)} &= \underline{a}_T \end{aligned} \qquad (4.3\text{-}5)$$

where the time scale is now of order $\frac{1}{\epsilon_3}$. These can be integrated to give the familiar result,

$$\begin{aligned} \underline{v}^{(0)}_{(t)} - \underline{v}^{(0)}_{(t_0)} &= \int_{t_0}^{t} \underline{a}_T \, dt \\ \underline{r}^{(0)}_{(t)} - \underline{r}^{(0)}_{(t_0)} &= \int_{t_0}^{t} \underline{v}^{(0)}_{(t)} \, dt + \underline{v}^{(0)}_{(t_0)} (t - t_0) \end{aligned} \qquad (4.3\text{-}6)$$

The effect of $\underline{g}$ and $\underline{a}_A$ can now be included in the higher order problem as

$$\begin{aligned} \dot{\underline{r}}^{(1)} &= \underline{v}^{(1)} \\ \dot{\underline{v}}^{(1)} &= \underline{g}(\underline{r}^{(0)}) + \underline{a}_A(\underline{r}^{(0)}, \underline{v}^{(0)}) \end{aligned} \qquad (4.3\text{-}7)$$

which are easily integrated to give,

$$\underline{v}^{(1)}(t) = \int_{t_0}^{t} \left(\underline{g}(\underline{r}^{(0)}) + \underline{a}_n(\underline{r}^{(0)}, \underline{v}^{(0)}) \right) dt$$

$$\underline{r}^{(1)}(t) = \int_{t_0}^{t} \underline{v}^{(1)}(t)\, dt \tag{4.3-8}$$

Consequently, the solution for "impusive" trajectories of relatively long duration may be written as an expansion

$$\underline{r}(t) = \underline{r}^{(0)}(t) + \frac{1}{\epsilon_3}\, \underline{r}^{(1)}(t)$$

$$\underline{v}(t) = \underline{v}^{(0)}(t) + \frac{1}{\epsilon_3}\, \underline{v}^{(1)}(t) \tag{4.3-9}$$

which correctly tends to the classical impulsive limit as $\epsilon_3 \rightarrow \infty$.

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CHAPTER V

OPTIMAL FLIGHT TRAJECTORIES

5.1 Introduction

A number of numerical methods are now available for the computation of complicated optimal flight trajectories. [54, 75, 76, 78, 81, 27] Analytical treatment of similar but often simpler problems must therefore be prefaced with a few justifying comments.

The structure of optimal flight trajectory problems is extremely complex. The dynamical system and the associated cost functions are highly nonlinear. There are numerous bounds on both the state space and the control. As a consequence, many optimal trajectory problems have local extremals, trajectories that minimize the cost with reference to all neighboring trajectories, but are not the absolute minima. A numerical optimization scheme will always locate one of these extremals, but can never tell if this is the absolute minimum, or even how many other extremals exist.

Analytical formulation of the problem can often identify the possibility of extremal solutions, even when the problem cannot be solved in detail. Numerical investigation can then be made to determine the optimal trajectory from the admitted extremals. When analytical solution for portions of the problem can be obtained, they are valuable for checking existing numerical solutions, interpreting numerical results and formulating more complex problems. But even short of formulating the complete problem, information concerning the general structure of the dynamical system can often be useful for the numerical investigator. For example, the knowledge that a maximum range problem will never enter the high load factor aerodynamically dominated regime, where range payoffs are small, would allow dispensing with a numerically cumbersome constraint on the trajectory load factor.

Analytical investigation of optimal flight trajectories is therefore worthy of pursuit. This has been generally recognized. [47, 60] The problem encountered was that the dynamical system was far too complicated for analytical treatment. Any ad hoc simplification of the dynamic system leads to a trajectory that violates the assumptions made in the simplification. Here considerable care has been taken in developing a systematic approximation procedure to avoid this difficulty.

An optimal trajectory associated with any lowest order problem will only be presumed valid for that particular regime of flight. When the trajectory leaves one regime and goes into another regime, another optimal trajectory problem must be solved. The two optimal trajectories then may be matched to give a trajectory valid for the two regimes. This has the distasteful aspect of requiring that a given optimal trajectory problem be solved in a large number of regimes. The resultant arcs must then be arranged in some combinatorial fashion to obtain the total trajectory. The precedure's redeeming feature is that it gives considerable information about an otherwise analytically untractable problem.

5.2 A Formulation of a General Flight Vehicle Optimization Problem

It seems advantageous to formulate a problem, substantially more complicated than one can presently solve, in order to illustrate the features that are common to all the problems that will be considered here.

The dynamic equations describing motion of a flight vehicle in a rotating planetary atmosphere are, (See Chapter IV)

$$\dot{\underline{r}} = \underline{v}$$
$$\dot{\underline{v}} = \underline{g} - 2\underline{\Omega} \times \underline{v} + \underline{a} \tag{5.2-1}$$

where $\underline{r}$ and $\underline{v}$ are position and velocity in the rotating coordinate system, $\underline{\Omega}$, is the angular velocity of the rotating atmosphere, $\underline{g} = \underline{g}(\underline{r})$ is the pseudo gravitational field in this rotating coordinate system, and $\underline{a}$ is the thrust and aero-dynamic acceleration. The two components of $\underline{a}$ are given as,

$$\underline{a} = \underline{a}_A + \underline{a}_T$$
$$\underline{a}_A = \frac{\rho}{2}\frac{v^2}{m}\underline{c} \qquad a_T = \frac{\underline{T}}{m} \tag{5.2-2}$$

where $\rho = \rho(\underline{r})$ is the atmospheric density, m is the vehicle mass, $\underline{c}$ is a three-dimensional aerodynamic coefficient supposedly less sensitive to, but not necessarily independent of, atmospheric properties and velocity magnitude, $\underline{T}$ is the thrust, either of aerodynamic origin and thus also dependent on atmospheric properties and velocity, or of rocket origin, and thus at least independent of the velocity.

For the purposes here it will be sufficient to consider $\underline{a}_T$ and $\underline{a}_A$ and to be the control over the vehicle where each belong to a set A_T and A_A whose bound is possibly a function of position and velocity, specifically

$$\underline{a}_T \in A_T(\underline{r}, \underline{v}, m)$$
$$\underline{a}_A \in A_A(\underline{r}, \underline{v}, m) \tag{5.2-3}$$

Further, the total acceleration, $\underline{a}$, is bounded in magnitude, and possibly direction, due to the physical limitations of the vehicle and its contents. Explicitly,

$$\underline{a} = \underline{a}_T + \underline{a}_A \leq A \tag{5.2-4}$$

For the optimal control problem, the dynamic equations must be supplemented with an equation describing the variation of vehicle mass due to possible fuel expenditure. Changes of mass due to ablation of vehicle heat shield is usually negligible. So,

$$\dot{m} = f_m(\underline{r}, \underline{v}, \underline{a}_T) \tag{5.2.5}$$

where the $\underline{r}$ and $\underline{v}$ dependence has been included to allows for possible aerodynamic propulsion.

A general cost component of the state may be introduced as

$$\dot{x}_o = f_o(\underline{r}, \underline{v}, a_T) \tag{5.2-6}$$

which can include minimum heating, maximum range, minimum fuel expended, minimum time, or specifically all the problems that will be considered here.

Then Eqs. (5.2-1, 5, 6) are a set of equations sufficient to define a state of the aerodynamic vehicle given suitable boundary conditions and specified control. The associated Hamiltonian is then (See Appendix B.)

$$\begin{aligned} H = {} & \underline{\lambda}_r^T \underline{v} + \underline{\lambda}_v^T (\underline{g} - 2\underline{\Omega} \times \underline{v} + \underline{a}_T + \underline{a}_A) \\ & + \lambda_m f_m(\underline{r}, \underline{v}, a_T) + \lambda_o f_o(\underline{r}, \underline{v}; a_T) \end{aligned} \tag{5.2-7}$$

where the operator $(\underline{\Omega}\, \mathrm{x})$ is taken as an antisymmetric matrix.

Now requiring that H be a minimum with respect to $\underline{a}_T$ and $\underline{a}_A$ yields a surprising result. First as $\underline{a}_T$ and $\underline{a}_A$ are related only by the constraint on the total acceleration $\underline{a}$, then for regions of state space when the system is not capable of reaching the acceleration boundary, the minimization of H may be done independently for $\underline{a}_T$ and $\underline{a}_A$. Further as $\underline{a}_T$ only enters f_m and f_o in terms of its magnitude a_T, then the direction of $\underline{a}_T$ is determined by the direction of $\underline{\lambda}_v$. In fact, $\underline{a}_T$ is anti-parallel to $\underline{\lambda}_v$. (See Fig. 5.2-1.) This is the Lawden[20] result extended to this more complex system.

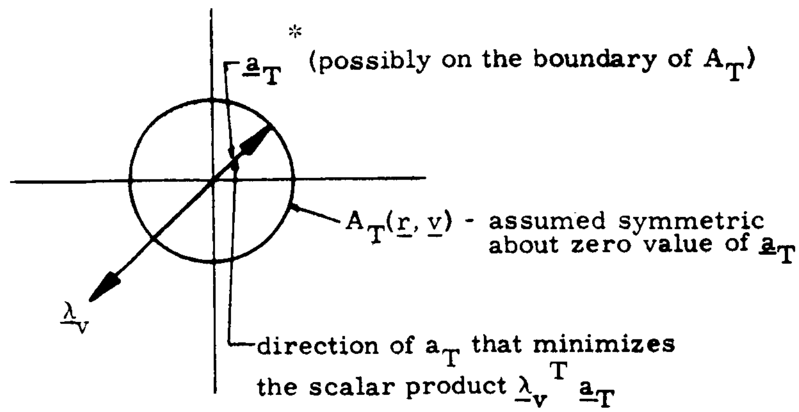


Fig 5.2-1 Optimal Direction for $\underline{a}_T$

Then the part of the Hamiltonian that depends on a_T is

$$H_{a_T} = -\lambda_v a_T + \lambda_m f_m(\underline{r}, \underline{v}, a_T) + \lambda_o f_o(\underline{r}, \underline{v}, a_T) \tag{5.2-8}$$

which is an equation from which the scalar value of a_T that minimizes H can be determined.

The conditions for minimum H with respect to $\underline{a}_A$ are somewhat simpler as $\underline{a}_A$ does not appear in f_o or f_m but is complicated by the unsymmetrical nature of A_A. (See Fig. 5.2-2.)

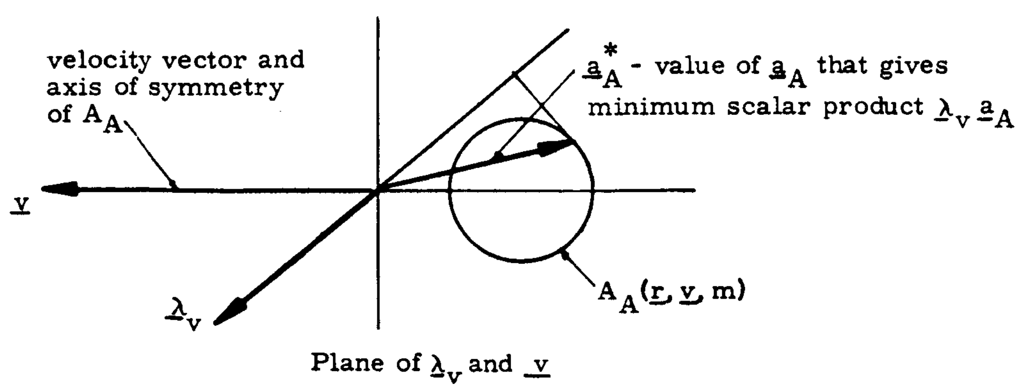


Fig. 5.2-2 Optimal Value of $\underline{a}_A$

A_A is symmetric with respect to an axis (the axis coincident with the velocity vector and the orientation of $\underline{a}_A$ about this axis is determined in the same manner that the direction of $\underline{a}_T$ was specified. Thus $\underline{a}_A^*$ always lies in the plane of $\underline{v}$ and $\underline{\lambda}_v$. This determines the required "angle of bank" of the vehicle. But the determination of the other two components of $\underline{c}$ associated with the vehicle's "angle of attack" is more difficult.

Notice that the optimal value of $\underline{a}_A$, $\underline{a}_A^*$ is specified by the condition that a plane perpendicular to $\underline{\lambda}_v$ is tangent to A_A at the point $\underline{a}_A^*$. Determining $\underline{a}_A^*$ in terms of a tangent to the surface A_A is a source of endless analytic grief in the literature[47, 60, 52, 11]. This is basically because the aerodynamic coefficients, $\underline{c}$, for hypersonic flight velocities, is specified in terms of a Newtonian drag polar whose trigonometric representative makes it near impossible to invert. (See Appendix F.).*

The method of circumventing this impasse, proposed here, is to consider the set, A_A, only composed of a number of discreet points so that $\underline{a}_A^*$ can only take onthesevalues. As this number may be made arbitrarily large, the loss of generality seems unimportant. The advantage accrued is that the troublesome tangency condition now becomes a switching condition. Specifically, for the two components of $\underline{\lambda}_v$, λv_1, and λv_2 along and normal to the axis of symmetry of A_A, a switching occurs when a possible change in $\underline{a}_A$ is related to $\underline{\lambda}_v$ by

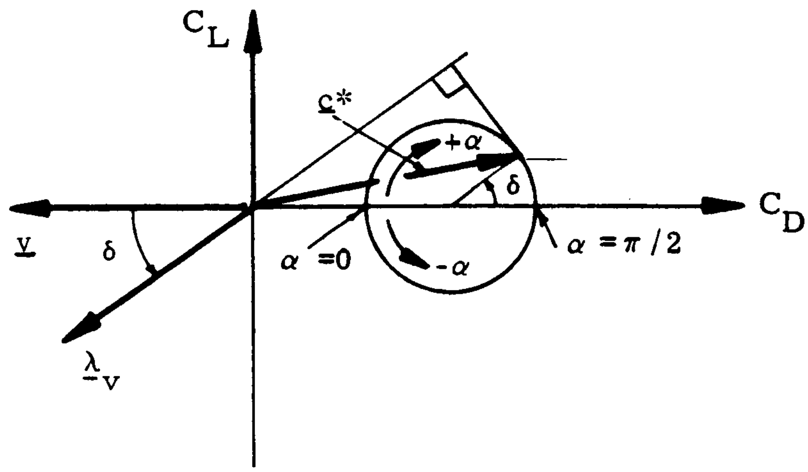
$$\frac{\lambda v_1}{\lambda v_2} = - \frac{\Delta a_{A_2}}{\Delta a_{A_1}} \tag{5.2-9}$$

(See Fig. 5.2-4). Thus, the piecewise constant value of the aerodynamic control, $\underline{a}_A$, that minimizes H, is determined as a function of $\underline{\lambda}_v$.

The other advantage of this approximation is that if the dynamic equations are integrable for constant aerodynamic coefficient $\underline{c}$, then relations for the $\underline{\lambda}_v$ associated with the piecewise constant control may be obtained. They are simply related to the perturbation variables around the solutions for constant $\underline{c}$. Specifically, if a solution for $\underline{r}$ and $\underline{v}$ exists and is expressible in terms of $\underline{v}$ and $\underline{r}$ at some later time,

---

* An interesting practical method of representing empirical drag polar data for optimal control purposes has been suggested by Prof. W.E. Vander Velde. It basically involves determining the relation between the angle of attack of the vehicle and the angle between the drag polar's normal and its axis of symmetry (see Fig. 5.2-3). As this angle is also the angle between the velocity vector and $\underline{\lambda}_v$ for an optimal value of $\underline{c}$, once $\underline{\lambda}_v$ and $\underline{v}$ are known $\underline{c}^*$ is easily determined.



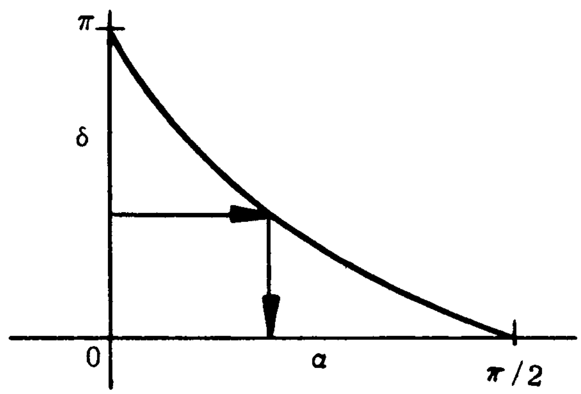


Fig 5.2-3 An Emperical Method of Determining $\underline{c}^*$ from $\underline{\lambda}_v$

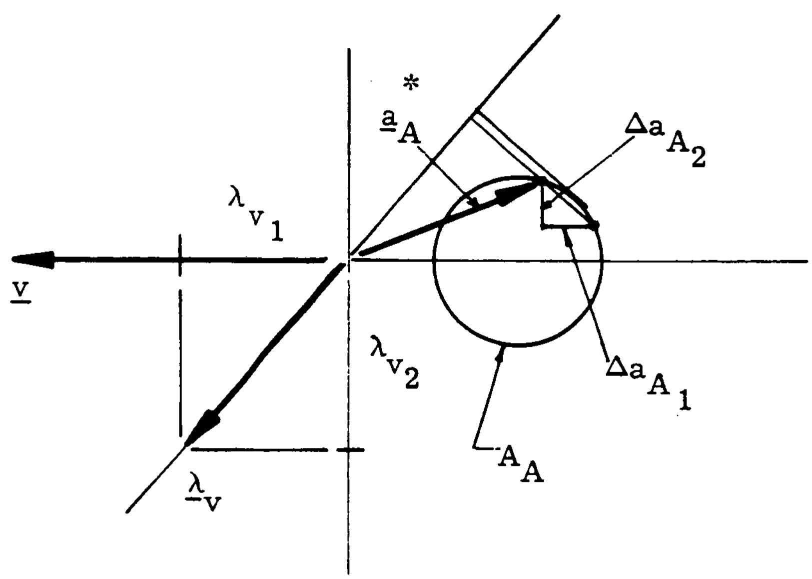


Fig 5. 2-4 Switching Condition for a Discrete Drag Polar

$$
\begin{aligned}
\underline{r}(t_1) &= \underline{r}_1(\underline{r}(t_2), \underline{v}(t_2), t_1, t_2) \\
\underline{v}(t_1) &= \underline{v}_1(\underline{r}(t_2), \underline{v}(t_2), t_1, t_2)
\end{aligned}
\tag{5.2-10}
$$

then

$$
\begin{bmatrix} \delta \underline{r}(t_1) \\ \delta \underline{v}(t_1) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \underline{r}_1}{\partial \underline{r}_2} & \dfrac{\partial \underline{r}_1}{\partial \underline{v}_2} \\[2ex] \dfrac{\partial \underline{v}_1}{\partial \underline{r}_2} & \dfrac{\partial \underline{v}_1}{\partial \underline{v}_2} \end{bmatrix} \begin{bmatrix} \delta \underline{r}(t_2) \\ \delta \underline{v}(t_2) \end{bmatrix}
\tag{5.2-11}
$$

and the relation for $\underline{\lambda}\, v_1(1)$ is (see Appendix A)

$$
\underline{\lambda}_v(t_1) = \left(\frac{\partial \underline{r}_2}{\partial \underline{v}_1}\right)^T \underline{\lambda}_r(t_2) + \left(\frac{\partial \underline{v}_2}{\partial \underline{v}_1}\right)^T \underline{\lambda}_v(t_2)
\tag{5.2-12}
$$

So, complete specification of the trajectory, the adjoint variables and the control in a piecewise fashion is possible for trajectories that have integrals for constant values of the aerodynamic coefficient. This approach is obviously of little advantage for numerical computation, but will be useful in the analytical investigation here, in determining the general character of optimal trajectory problems.

One other device that will be continually used is the elimination of time from the system equations. Some care must be made in choice of an alternate independent variable. It must be monotonically increasing for a well defined optimal control problem. An alternate choice of variable that obviously satisfies this condition is the cost coordinate for the particular problem. The preceding dynamical equations, written with x_o as the independent variable, are

$$
\begin{aligned}
\frac{dx_0'}{dx_0} &= 1 \\[1ex]
\frac{d\underline{r}}{dx_0} &= \frac{\underline{v}}{f_0(a_T, \underline{r}, \underline{v})} \\[1ex]
\frac{d\underline{v}}{dx_0} &= \frac{\underline{g} + \underline{\Omega} \times \underline{v} + \underline{a}}{f_0(a_T, \underline{r}, \underline{v})} \\[1ex]
\frac{dm}{dx_0} &= \frac{f_m(a_T, \underline{r}, \underline{v})}{f_0(a_T, \underline{r}, \underline{v})}
\end{aligned}
\tag{5.2-13}
$$

The advantage that has been obtained is not apparent here, (except the possible reduction of the number of variables) but will be in the work that will follow. It is interestingly observed that these equations define a general class of optimization problem known as "minimum time" problems, heavily studied in the field of optimal control.

5.3 Optimum Aerodynamic Plane Changes

5.3.1 Introduction

Currently, there is a large interest in aerodynamic plane change maneuvers[71, 79, 80] This is due to the sizeable advantages accrued in aerodynamic plane changes of orbits over normal impulsive techniques. A cursory analysis, indicating the reason for this advantage, will be given later.

5.3.2 The Aerodynamically Dominated Plane Change

Over a phase of the trajectory where the velocity is the order of orbital velocity and the pressure is the order of the wing loading, or, alternately where $(v^2 - 1) = 0(\epsilon^{\frac{1}{2}})$ and $p = 0(\epsilon)$ the following nondimensional dynamical equations are valid, to lowest order,

$$\begin{aligned} \frac{dv}{dt} &= -\frac{C_D}{2} \rho v^2 \\ v \frac{d\gamma}{dt} &= \frac{C_L}{2} \cos\phi \, \rho v^2 \\ v \cos\gamma \frac{d\psi}{dt} &= \frac{C_L}{2} \sin\phi \, \rho v^2 \end{aligned} \qquad (5.3.2\text{-}1)$$

where ϕ is the vehicle roll angle and ψ a heading angle.

A rearrangement convenient for the optimization to be done here is

$$\begin{aligned} \frac{d \ln v'}{d \ln v} &= 1 \\ \frac{d\gamma}{d \ln v} &= -\frac{C_L}{C_D} \cos\phi \\ \frac{d\psi}{d \ln v} &= -\frac{C_L}{C_D} \frac{\sin\phi}{\cos\gamma} \end{aligned} \qquad (5.3.2\text{-}2)$$

where v' has been introduced for analytical convenience. The Hamiltonian is

$$H = \lambda_{\ln v'} - \lambda_\gamma \frac{C_L}{C_D}\cos\phi - \lambda_\psi \frac{C_L}{C_D}\frac{\sin\phi}{\cos\gamma} \tag{5.3.2-3}$$

As the independent variable $\ell n\, v$ does not appear in H, H is a constant. The associated adjoint equations are

$$\frac{d\lambda_{\ln v'}}{d\ln v} = 0$$

$$\frac{d\lambda_\gamma}{d\ln v} = \lambda_\psi \frac{C_L}{C_D}\frac{\sin\phi}{\cos^2\gamma}\sin\gamma \tag{5.3.2-4}$$

$$\frac{d\lambda_\psi}{d\ln v} = 0$$

We will seek a minimum loss of velocity during the maneuver so*

$$\lambda_{\ln v'}(v_f) = -1, 0 \tag{5.3.2-5}$$

The final velocity is unspecified, so

$$H(v_f) = H = 0$$

The initial and final values of ψ are presumed given to make the problem wellposed. Therefore, $\lambda\psi = \lambda\psi_o \neq 0$. The final of γ is either given or

$$\lambda_\gamma(v_f) = 0$$

For a minimum of H with respect to ϕ, the vector $(C_L \cos\phi,\ C_L \sin\phi)$ should be colinear and opposite in direction to $\left(-\lambda_\gamma,\ -\frac{\lambda_{\psi_o}}{\cos\gamma}\right)$ (See Fig. 5.3.2-1). So

$$\tan\phi = \frac{\lambda_{\psi_o}}{\lambda_\gamma \cos\gamma} \tag{5.3.2-6}$$

*The possible zero value of the cost component of the adjoint vector will be included for correctness. See Appendix B.

$$H = \lambda_{\ln v}' - \frac{C_L}{C_D}\left(\lambda_\gamma^2 + \frac{\lambda\psi_o^2}{\sin^2\gamma}\right)^{\frac{1}{2}} = 0 \qquad (5.3.2\text{-}7)$$

For H to be a minimum with respect to $\frac{C_L}{C_D}$, $\frac{C_L}{C_D}$ is at its maximum value if

$$\left(\lambda_\gamma^2 + \frac{\lambda\psi_o^2}{\cos^2\gamma}\right) \neq 0 \qquad (5.3.2\text{-}8)$$

But, notice that $(\lambda_\gamma^2 + \frac{\lambda\psi_o^2}{\cos^2\gamma})$ cannot be zero as $\lambda\psi_o \neq 0$ for a well posed problem, so $\frac{C_L}{C_D}$ is at its maximum value for all v. To obtain an expression for the optimal value of ϕ, substitute Eq. (5.3.2-7) into Eq. (5.3.2-6), and eliminate λ_γ this yields

$$\sin\phi = \left.\frac{C_L}{C_D}\right|_{max} \frac{\lambda\psi_o}{\lambda_{\ln v_o}} \frac{1}{\cos\gamma} \qquad (5.3.2\text{-}9)$$

This expression seems to be satisfied by two angles, but the ambiguity is resolved by realizing only one angle satisfies the minimum principle (see Fig. 5.3.2-1). With the control specified, it can be substitute into the dynamic equations, Eqs. (5.3.2-2), and they integrated to determine the arbitrary constant $\frac{\lambda\psi_o}{\lambda_{\ln v_o}}$ and the final velocity interms of the initial and final values of ψ and γ. A rather simple observation will preclude this investigation.

Observe that the bank angle is symmetric with respect to $\gamma = 0$ and that as

$$\gamma \to \pm\frac{\pi}{2} \qquad \phi \to \frac{\pi}{2}$$

$$\gamma \to 0 \qquad \phi \to 0, \pi$$

If the initial and final values of γ are zero, the only bank angle that satisfies these conditions is $\frac{\pi}{2}$.

The relationship between ψ and v for this special case is obtained from Eq. (5.3.2-1) as

$$\psi - \psi_o = \left.\frac{C_L}{C_D}\right|_{max} \ln\frac{v}{v_o} \qquad (5.3.2\text{-}10)$$

But observe that the original dynamic equations, Eqs. (5. 3. 2-1) were insensitive to how the plane of the maneuver was defined, as gravity accelerations did not enter. So if the initial and final directions of the velocity vector are taken to define the plane of the maneuver, the optimal control is simply to apply maximum $\frac{C_L}{C_D}$ in this plane. The velocity change required to perform the maneuver is then simply

$$v_f = v_o \, e^{-\left.\frac{C_D}{C_L}\right|_{max} \Delta\psi'} \qquad (5.3.2\text{-}11)$$

where $\Delta\psi'$ is the angle between the initial and final velocity vectors, a now obvious result.

To make the advantages of aerodynamic turning over impulsive thrusting similarly apparent, observe that the plane change accomplished with a single impulsive thrust is given by

$$\sin \frac{\Delta\psi}{2} = \frac{\Delta v}{2 v_o} \qquad (5.3.2\text{-}12)$$

(see Fig. 5. 3. 2-1).

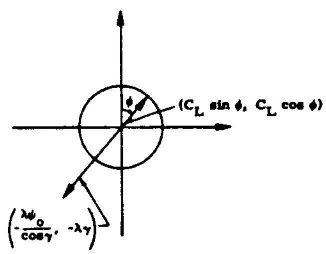


Fig. 5. 3. 2-1 Optimal Direction of the Lift Vector

For a small plane change, Eqs. (5. 3. 2-11) and (5. 3. 2. -12) reduce simply to

$$\begin{aligned} \Delta\psi_a &= \frac{C_L}{C_D} \frac{\Delta v}{v_o} \\ \Delta\psi_i &= \frac{\Delta v}{v_o} \end{aligned} \qquad (5.3.2\text{-}13)$$

so that a vehicle with a lift-to-drag ratio of greater than one has the possibility of performing a turn more efficiently by using lift to develop side force and propulsive force to cancel drag.

Notice when the Δv is the order of twice that necessary to escape the planetary gravity field, an impulsive maneuver can be accomplished that takes the vehicle to infinity, performs a plane change at no cost, and returns to the original orbit. Using this to define the interesting limit of aerodynamic plane change maneuvers gives

$$\Delta\psi = \frac{C_L}{C_D}\Big|_{max} \ln \frac{v_0}{v_f}$$

$$\frac{v_0 - v_f}{v_0} \leq \frac{2(\sqrt{2}\,v_0 - v_0)}{v_0} \tag{5.3.2-14}$$

$$\Delta\psi \leq .55 \frac{C_L}{C_D}\Big|_{max}$$

So, for example, a vehicle with a $\frac{C_L}{C_D}$ of 2 can out-perform an impulsive maneuver for plane changes of less than one radian. This is a much reported result, (71, 79, 80) unfortunately surrounded with an unnecessary amount of complexity.

5.4 The Minimum Velocity Lost Problem

5.4.1 Introduction

The objective is to control a lifting re-entry vehicle so that it looses minimum velocity in flight to a prescribed altitude, and possibly a prescribed flight path angle. This type of trajectory has been proposed (78) for a reconnaissance vehicle desiring a close planetary encounter with minimum loss of kinetic energy. It also has obvious application to a lifting weapon trajectory with an objective of reaching some near surface target with minimum loss of velocity. Numerical verfication for some of the analytical observations that follow may be found in Ref. (78).

5.4.2 The Minimum Velocity Lost Trajectories in the Aerodynamically Dominated Regime

The dynamical equations, correct to lowest order in ϵ, for nondimensional pressure, p, of order one, nondimensional velocity, v, of order on and flight path angle, γ, of any order, are (or $p = O(\epsilon)$, $v^2 - 1 = O(\epsilon)$, $\gamma = O(\epsilon^{\frac{1}{2}})$; (See Sections 2.5 and 2.9.)

$$\frac{dv^{2\prime}}{dv^2} = 1$$

$$\frac{d\gamma}{dv^2} = -\frac{1}{2}\frac{C_L}{C_D}\frac{1}{v^{2\prime}} \tag{5.4.2-1}$$

$$\frac{dp}{dv^2} = \frac{\sin\gamma}{C_D v^{2'}}$$

where $v^{2'}$ has been introduced for analytical convenience. The Hamiltonian is

$$H = \lambda_{v^{2'}} + \lambda_\gamma\left(-\frac{1}{2}\frac{C_L}{C_D}\frac{1}{v^{2'}}\right) + \lambda_p\left(\frac{\sin\gamma}{C_D v^{2'}}\right) = 0 \qquad (5.4.2\text{-}2)$$

It is both a constant and zero as the independent variable, v^2, does not appear on the right hand side of the system equations and the final value of v^2 is unspecified. The associated adjoint equations are

$$\frac{d\lambda_{v^{2'}}}{dv^2} = \left(-\frac{\lambda_\gamma}{2}\frac{C_L}{C_D} + \lambda_p\frac{\sin\gamma}{C_D}\right)\frac{1}{v'^4} = \frac{-\lambda_{v'^2}}{v'^2}$$

$$\frac{d\lambda_\gamma}{dv^2} = -\lambda_p\frac{\cos\gamma}{C_D v'^2} \qquad (5.4.2\text{-}3)$$

$$\frac{d\lambda_p}{dv^2} = 0$$

The boundary conditions for Eqs. (5.4.2-1, 3) are specified intial and final values of the state variables, v'^2, γ, and p or zero values for the associated adjoint variables with the possible exception of the final value of the adjoint variable, $\lambda_v'^2$ which may be -1. For minimum H with respect to $\frac{1}{C_D}$, the scalar product of $(\lambda p_o \sin\gamma, -\frac{1}{2}\lambda_\gamma)$ and $(\frac{1}{C_D}, \frac{C_L}{C_D})$ must be a minimum. This has a simple interpretation in terms of a a two dimensional Euclidian plot of the two vectors. It is seen from Fig. 5.4.2-1, that minimum H occurs when $(\frac{1}{C_D}, \frac{C_L}{C_C}$ has the largest projection on $(\lambda_p \sin\gamma_1 - \frac{1}{2}\lambda\,\theta)$. It is then easily seen that the following possibilities exist:

(1) When $\lambda_p \sin\gamma = 0$ and $\lambda\gamma \neq 0$ then

$\frac{C_L}{C_D}$ is maximum for $\lambda\gamma < 0$ and

$\frac{C_L}{C_D}$ is minimum for $\lambda\gamma > 0$.

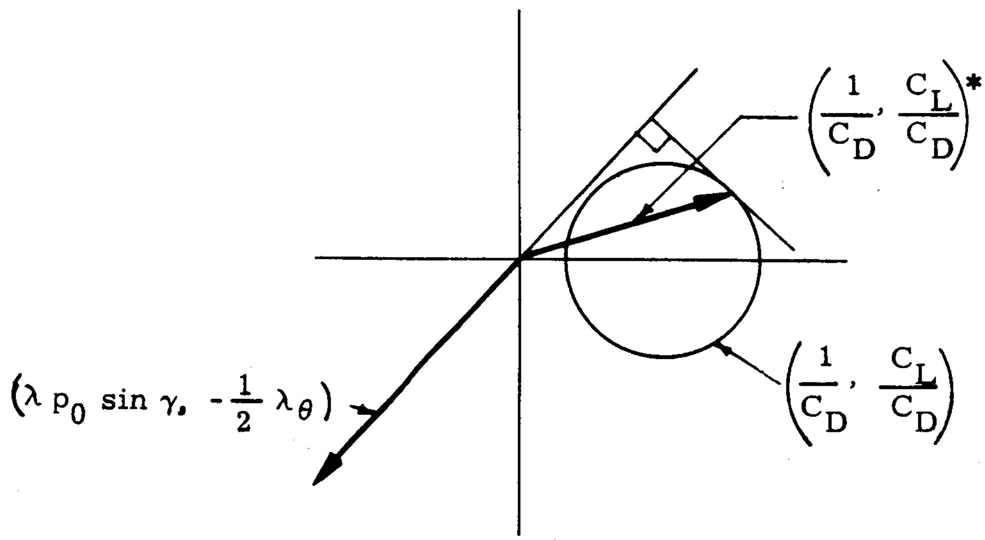


Fig. 5.4.2-1 Optimal Value of $\frac{1}{C_D}, \frac{C_L}{C_D}$

(2) When $\lambda_p \sin\gamma \neq 0$ and $\lambda_\gamma = 0$ then

$\frac{1}{C_D}$ is minimum for $\lambda_p \sin\gamma > 0$

$\frac{1}{C_D}$ is maximum for $\lambda_p \sin\gamma < 0$

(3) When $\lambda_p \sin\gamma = 0$ and $\lambda_\gamma = 0$ any value of $\frac{C_L}{C_D}$ and $\frac{1}{C_D}$ is admissible.

(4) For $\lambda_p \sin\gamma = 0$ and $\lambda_\gamma \neq 0$ over any interval of the trajectory,

$\lambda_p = 0$ and $\frac{d\lambda_p}{dv^2} = 0$ The final pressure is unspecified or,

$\sin\gamma = 0$ and $\frac{d\gamma}{dv^2} = -\frac{1}{2}\frac{C_L}{C_D}\frac{1}{v2'} = 0$. But, condition (1) above,

requires $\frac{C_L}{C_D} \neq 0$, so this exists only when final p is unspecified,

(5) For $\lambda_p \sin\gamma \neq 0$, $\lambda\gamma = 0$ over any interval of the trajectory,

$$\frac{d\lambda_\gamma}{dv^2} = -\lambda_p \frac{\cos\gamma}{C_D v'^2} = 0$$

$$\lambda_\gamma = 0$$

which is only possible if $\cos\gamma = 0$. (A vertical trajectory.)

(6) For $\lambda_p \sin\gamma = 0$ and $\lambda\gamma = 0$ over any interval of the trajectory,

$\lambda_p = 0$ (unspecified final p)

or

$$\sin\gamma = 0$$

$$\frac{d\gamma}{dv^2} = -\frac{1}{2}\frac{C_L}{C_D}\frac{1}{v'^2} = 0$$

and

$$\lambda_\gamma = 0$$

$$\frac{d\lambda_\gamma}{dv^2} = -\lambda_p \frac{\cos\gamma}{C_D v'^2} = 0$$

(unspecified final γ)

which is only possible if the final γ and p are unspecified, a poorly posed problem.

(7) When $\lambda_p \sin\gamma \neq 0$ and $\lambda_\gamma \neq 0$ it is convenient to make the assumption that the $(\frac{1}{C_D}, \frac{C_L}{C_D})$ vector can only taken discreet values. Then and integrable equation for $(\lambda_p \sin\gamma, \lambda_\gamma)$ may be obtained by combining Eq. (5. 4. 2 - 1) and (5. 4. 2-3) to give

$$\frac{d\lambda_\gamma}{d\gamma} = 2\lambda_{\varphi_0}\frac{\cos\gamma}{C_L}$$

or for piecewise const. values of C_L,

$$\lambda\gamma_{n+1} - \lambda\gamma_n = 2\frac{\lambda_{\varphi_0}}{C_{L_n}}(\sin\gamma_{n+1} - \sin\gamma_n)$$

where (5. 4. 2-4)

It is seen from these conditions that the vehicle:

(1) turns at max. $\frac{C_L}{C_D}$ to a specified γ if the final p is unspecified,

(2) turns at non-max. $\frac{C_L}{C_D}$ for final p specified, final γ unspecified, passing through max. $\frac{C_L}{C_D}$ when $\gamma = 0$.

(3) stops turning and descends or ascends at min. C_D when $\gamma = \pm\frac{\pi}{2}$ when final p specified and final γ is unspecified.

(4) conducts a lifting up-down or down-up maneuver specified by Eq. (5. 4. 2-4), passing through max. $\frac{C_L}{C_D}$ when $\gamma = 0$, to meet fixed final values of p and γ.

5. 4. 3 Minimum Velocity Lost Trajectories in the Aero-Gravity Perturbed Regime.

All optimal trajectory problems will have a particular sample structure in this regime. The optimal value of the control will be a constant value of C_L and C_D. This minimum velocity lost problem will serve to illustrate this result.

The first-order perturbation equations describing flight for $v^2 = 0(1)$, $\gamma = 0(1)$, $p = 0(\epsilon)$, $h = 0(\epsilon)$ are

$$\frac{dv^2}{dp} = C_D \frac{v_0^2}{\sin\gamma_0} + \frac{1}{p}$$
$$\frac{d\cos\gamma}{dp} = \frac{C_L}{2} - \frac{\cos\gamma_0}{p}\left(\frac{1}{v_0^2} - 1\right) \tag{5.4.3-1}$$

where p has been introduced as the independent variable. These equations are suitable for system equations for an optimal control problem if p is restricted to monotonic variation and the fictitious variable p' is introduced by the equation,

$$\frac{dp'}{dp} = 1 \tag{5.4.3-2}$$

The Hamiltonian associated with Eqs. (5.4.3-1 - 2) is then

$$H = \lambda_{v^2}\left(C_D \frac{v_0^2}{\sin\gamma_0} + \frac{1}{p'}\right) + \lambda_{\cos\gamma}\left(\frac{C_L}{2} - \frac{\cos\gamma_0}{p'}\left(\frac{1}{v_0^2} - 1\right)\right) + \lambda_{p'} \tag{5.4.3-3}$$

which is independent of p and thus constant. The associated adjoint equations are,

$$\frac{d\lambda_{v^2}}{dp} = 0 \quad , \quad \frac{d\lambda_{\cos\gamma}}{dp} = 0 \tag{5.4.3-4}$$

to that $\lambda_{p'}$ variation with p is specified by Eq. (5.4.3-3). The boundary conditions are specified initial and final values of p' and γ, or the associated $\lambda_{p'}$ and λ_y zero, with the additional condition that $H = 0$ if either the initial or final value ot p is not specified. As we are seeking a maximum final v^2, $\lambda v^2_f = -1$ (or possibly 0).

Requiring that H be a minimum with respect to the control gives the surprising result that C_L and C_D are constant, as both λ_{v^2} and $\lambda_{\cos\lambda}$ are constant. (See Fig. 5.4.3-1.)

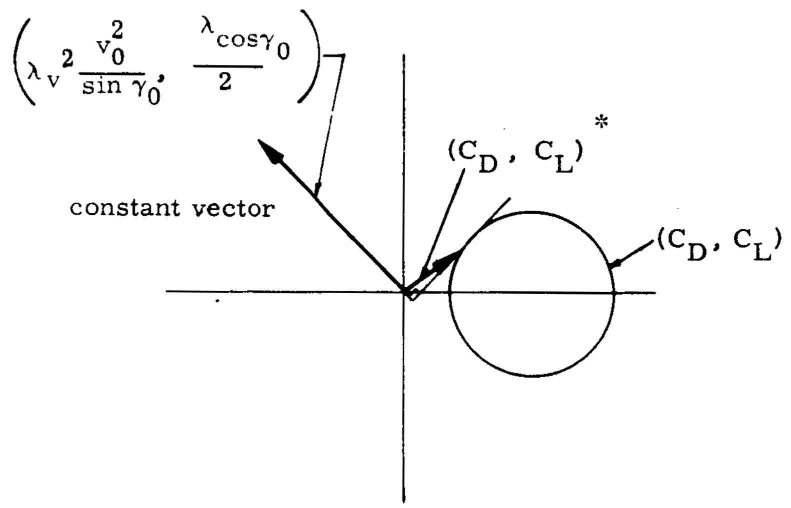


Fig 5.4.3-1 Optimal Control for Minimum Velocity Loss Aerogravity Perturbed Problem

If the final flight path angle, γ_f, is unspecified and occurs in this regime, then $\lambda \cos \gamma_o = 0$ and the optimal value of C_L is zero.

5.4.4 Minimum Velocity Lost Trajectories in the Small Flight Path Angle of Low Density Regime

The behavior of the dynamic equation in the low density regime is somewhat different if flight path angles are order, $\epsilon^{\frac{1}{2}}$. For $v^2 = 0(1)$, $\lambda = 0(\epsilon)$, $\gamma = 0(\epsilon^{\frac{1}{2}})$, $h = 0(\epsilon)$, the perturbation equations, correct to lowest order in γ and order $\epsilon^{\frac{1}{2}}$ in v^2 are (See Section 2.9.)

$$\frac{dv^2}{dp} = \frac{C_D v_0^2}{\gamma}$$

$$\frac{d\gamma^2}{dp} = -C_L - 2\left(1 - \frac{1}{v_0^2}\right)\frac{1}{p} \qquad (5.4.4\text{-}1)$$

As γ might well go through zero, making p a non-monotonic variable, the equations must be rearranged in the following form:

$$\frac{dv^{2'}}{dv^2} = 1$$

$$\frac{dp}{dv^2} = \frac{\gamma}{C_D v_o^2} \qquad (5.4.4\text{-}2)$$

$$\frac{d\gamma}{dv^2} = -\frac{1}{2}\frac{C_L}{C_D}\frac{1}{v_o^2} - \left(1 - \frac{1}{v_o^2}\right)\frac{1}{p\, C_D v_o^2}$$

These are suitable system equations for the optimal control problem. The Hamiltonian is then

$$H = \lambda_{v^{2'}} + \lambda_p \left(+\frac{\gamma}{C_D v_o^2}\right) + \lambda_\gamma \left(-\frac{1}{2}\frac{C_L}{C_D}\frac{1}{v_o^2} - \left(1-\frac{1}{v_o^2}\right)\frac{1}{p\, C_D v_o^2}\right) \qquad (5.4.4\text{-}3)$$

and is both constant and zero as the independent variable, v_o^2 is also the cost. The adjoint equations are

$$\frac{d\lambda_p}{dv^2} = -\lambda_\gamma\left(1 - \frac{1}{v_o^2}\right)\frac{1}{p^2 C_D v_o^2}$$

$$\frac{d\lambda_\gamma}{dv^2} = \frac{\lambda_p}{C_D v_o^2} \qquad (5.4.4\text{-}4)$$

$$\frac{d\lambda_{v^{2'}}}{dv^2} = 0$$

The boundary conditions fix the initial and final values of p and γ or the respective values of λ_p and λ_γ are zero. As a maximum value of the final velocity is sought, $\lambda_{v_o^{2'}} = -1$ (or possibly 0).

Requiring that H be a minimum with respect to $\frac{C_L}{C_D}$ and $\frac{1}{C_D}$ gives the condition illustrated in Fig. 5.4.4-1.

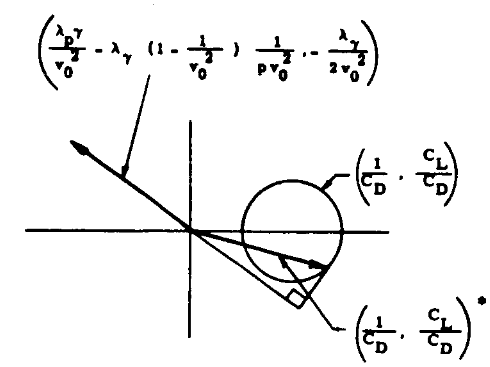


Fig. 5.4.4-1 Optimal Control for Minimum Velocity Lost in the Small γ Regime.

Notice that no arc where $\lambda_\gamma = 0$ exists because for λ_γ and $\frac{d\lambda_\gamma}{dp} = 0$ on any arc requires $\lambda_p = 0$. But for both λ_p and λ_γ to be zero would require $\lambda_v{}^2$ to be zero, if H = 0. As the complete adjoint vector cannot be zero, no such arc exists. Further, if the initial velocity is subcircular so that $(1 - \frac{1}{v_o{}^2})$ is negative, the condition $\lambda_\gamma = 0$ can never exist, because the adjoint equation for λ_γ is divergent. If $(1 - \frac{1}{vo^2})$ is positive, the value of λ_γ can oscillate, so multiple positive and negative values can result.

To gain some insight into the structure of the problem, consider the case when $(1 - \frac{1}{vo^2}) = 0$. λ_p is then a constant. By examining the Hamiltonian, Eq. (5.4.4-3), it is seen when $\gamma = 0$, $\frac{C_L}{C_D}$ is at is limiting value and is the same sign as λ_γ. The adjoint equation may be implicitly integrated as

$$\lambda_\gamma - \lambda_{\gamma_f} = \frac{\lambda_{p_0}}{v_0^2} \int_{v_f^2}^{v^2} \frac{dv^2}{C_D} \qquad (5.4.4\text{-}5)$$

which indicates that λ_γ can pass through zero (and thus $\frac{C_L}{C_D}$) at least once. If the final value of γ is unspecified then $\lambda_\gamma = 0$ is the final value and no switchings occur. These conditions may be interpreted to mean that a vehicle entering at circular satellite velocity will at most have its trajectory turned up and down once to meet some specified final p and γ with minimum loss of velocity.

5.4.5 Minimum Velocity Lost Trajectories in Equilibrium Glide Regime

The dynamic equations previously used are adequate for describing flight at small flight path angles where the flight path angle is changing rapidly. For small flight path angles that are varying slowly the following dynamic equations are valid, to lowest order in ϵ

$$\frac{dv^2}{dp} = \frac{C_D v^2}{\gamma} + \frac{2}{p} \tag{5.4.5-1}$$

$$0 = \frac{C_L}{2} - \left(\frac{1}{v^2} - 1\right)\frac{1}{p}$$

These are simply algebraic equations relating v^2, p, C_L and C_D.

$$v^2 = \frac{1}{\frac{C_L}{2} p + 1} \tag{5.4.5-2}$$

$$\gamma = -2\frac{C_D}{C_L}\left(\frac{C_L}{2} p + 1\right) \tag{5.4.5-3}$$

The first equation, Eq. (5.4.5-1) implies that for a maximum v^2 at a given p, C_L should be as large negative as possible. The second equation gives the small flight path angle necessary to maintain this condition.

These equations serve to define the upper limit boundary on all capture maneuvers, that is, the maximum velocity that a vehicle may have and still be kept in the close proximity of a planet with negative lift. Thus, any vehicle with super circular velocity desiring to be captured must dip into an atmosphere and fly up to the boundary from below.

5.5 The Maximum Range Problem

5.5.1 Introduction

An obvious objective in the control of a hypervelocity flight vehicle is to obtain maximum range. Some insight into proper regimes for the formulation of this problem may be obtained from the form of the dynamic equation. Observing that a nonthrusting vehicle has an amount of total energy that is expended in the pursuit of range, it is natural to express the rate of change of this energy with respect to range. For $v^2 = 0(1)$, $h = 0(\epsilon)$, $\gamma = 0(1)$, $\theta = 0(1)$, $\rho = 0(1)$ this equation is

$$\frac{d\left(v^2 - \frac{2}{1+\epsilon h}\right)}{d\theta} = -C_D \frac{\rho}{\epsilon}\frac{v^2}{\cos\gamma}(1+\epsilon h) \tag{5.5.1-1}$$

This equation must have θ or ρ rescaled, which implies that there are only range gains the order of the planetary radius when the density is order ϵ . Then for

$v^2 = 0(1)$, $h = 0(\epsilon)$, $\gamma = 0(1)$, $\theta = 0(1)$, $\rho = 0(\epsilon)$, to lowest order, the equation is

$$\frac{dv^2}{d\theta} = - \frac{C_D \rho v^2}{\cos\gamma} \qquad (5.5.1-2)$$

This implies that there are no lowest order range contributions due to the vehicle's potential energy.

Two of the maximum range problems will now be investigated all in the low density regime.

5.5.2 Maximum Range in the Equilibrium Glide Regime

A particularly simple maximum range problem is maximum range in equilibrium glide. The dynamic equation for $\gamma = 0(\epsilon)$, $v^2 = 0(1)$, $\rho = 0(\epsilon)$, $h = 0(\epsilon)$ with θ as the independent variable are (see Section 2.7)

$$\begin{aligned} \frac{dv^2}{d\theta} &= - C_D \rho v^2 \\ 0 &= - \frac{1}{2} C_L \rho - \left(1 - \frac{1}{v^2}\right) \end{aligned} \qquad (5.5.2-1)$$

Substituting the value of ρ given in the second equation gives

$$\frac{dv^2}{d\theta} = \frac{C_D}{C_L}(v^2 - 1) \qquad (5.5.2-2)$$

Now introduce a fictitious range variable,

$$\frac{d\theta'}{d\theta} = 1 \qquad (5.5.2-3)$$

The Hamiltonian is then,

$$H = \lambda_{\theta'} + \lambda_{v^2} \frac{C_D}{C_L}(v^2 - 1) = 0 \qquad (5.5.2-4)$$

The adjoint equations are

$$\frac{d\lambda_{\theta'}}{d\theta} = 0$$

$$\frac{d\lambda_{v^2}}{d\theta} = -\lambda_{v^2}\frac{C_D}{C_L} \qquad (5.5.2\text{-}5)$$

The boundary conditions are a fixed Δv^2, a free final θ which requires that $H = 0$, and a maximum θ' which requires that $\lambda_{\theta}' = -1$ (or possibly zero).

To show that $\frac{C_L}{C_D}$ at its maximum positive value, satisfies the necessary conditions, notice that v^2-1 is always negative. So that if λ_{v^2} is negative, $\frac{C_L}{C_D} = \frac{C_L}{C_D}\Big|_{max}$ minimizes H. If λ_{v^2} is initially negative it will remain negative (see Eq. 5.5.2-5) so that $\frac{C_L}{C_D} = \frac{C_L}{C_D}\Big|_{max}$ is a possible extremal. In fact, it is the only extremal in this regime because $\lambda_{v^2} = 0$ is not admissible, and λ_{v^2} positive requires $C_L = 0$ which takes the trajectory out of this regime. The range covered, as given in Section 2.7. is

$$\theta = \frac{1}{2}\frac{C_L}{C_D}\Big|_{max} \ln\left[\frac{1-v^2}{1-v_0^2}\right] \qquad (5.5.2\text{-}6)$$

5.5.3 Maximum Range in the Aerodominated Low-Density Regime

When flight is in the low density, near orbital velocity regime, the situation is slightly more complicated. For $\gamma = 0(\epsilon^{\frac{1}{2}})$, $v^2 - 1 = 0(\epsilon^{\frac{1}{2}})$, $\rho = 0(\epsilon)$, $h = 0(\epsilon)$ the dynamic equations are

$$\begin{aligned}
\frac{d\theta}{dv^2} &= -\frac{1}{C_D\rho} \\
\frac{d\gamma}{dv^2} &= -\frac{1}{2}\frac{C_L}{C_D} \\
\frac{d\rho}{dv^2} &= \frac{\gamma}{C_D} \\
\frac{dv^{2\prime}}{dv^2} &= 1
\end{aligned} \qquad (5.5.3\text{-}1)$$

The Hamiltonian is

$$\begin{aligned}
H = {} & \lambda_\theta\left(-\frac{1}{C_D\rho}\right) + \lambda_\gamma\left(-\frac{1}{2}\frac{C_L}{C_D}\right) \\
& + \lambda_\rho\left(\frac{\gamma}{C_D}\right) + \lambda_{v'^2} = 0
\end{aligned} \qquad (5.5.3\text{-}2)$$

The adjoint equations are

$$\frac{d\lambda_\theta}{dv^2} = 0$$

$$\frac{d\lambda_\gamma}{dv^2} = -\frac{\lambda_p}{C_D}$$

$$\frac{d\lambda_p}{dv^2} = -\lambda_\theta \frac{1}{C_D p^2} \qquad (5.5.3\text{-}3)$$

$$\frac{d\lambda_{v^{2'}}}{dv^2} = 0$$

Requiring that H be a minimum with respect to $\frac{C_L}{C_D}$, $\frac{1}{C_D}$ gives the usual condition, (see Fig. 5.5.3-1).

The control can be considered piecewise constant and the system and adjoint equations integrated. To gain some insight into the structure of the problem, consider C_D = constant and C_L variable between a positive and negative limit. This is valid for a low-lift drag (Apollo Type) vehicle.

Then for $\lambda_\gamma \neq 0$

$$C_L = C_{L\,max} \qquad \lambda_\gamma > 0$$
$$C_L = -C_{L\,max} \qquad \lambda_\gamma < 0 \qquad (5.5.3\text{-}4)$$

But λ_γ cannot be zero over any finite segment of the trajectory (see the adjoint equation), so the trajectory is of a skip variety. It clearly leaves this regime and enters a Keplerian regime.

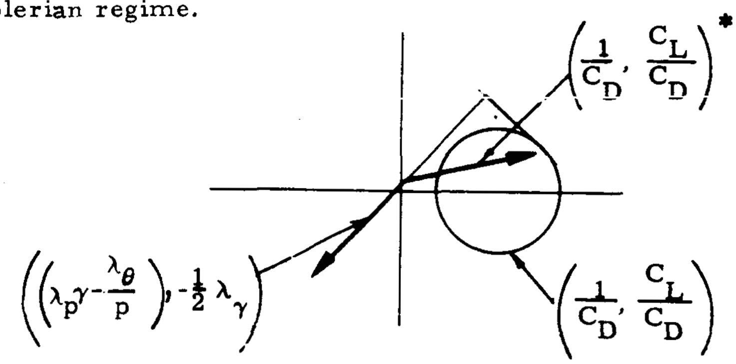


Fig 5.5.3-1 Optimal Control for Maximum Range - Low Density - Aerodominated Problem

Maximum range in the Keplerian regime is easily handled by observing that there is an optimum initial flight path angle γ_o^* associated with a given initial velocity, v_o, to obtain a maximum range, θ^*. Specifically,

$$\tan \gamma_o^* = \sqrt{1 - v_o^2} \tag{5.5.3-5}$$

where the maximum range is given by

$$\tan \theta^* = \frac{1}{2} \frac{v_o^2}{\sqrt{1 - v_o^2}} \tag{5.5.3-6}$$

See, for example, Ref. (102).

For $(1-v^2) = 0(\epsilon^{\frac{1}{2}})$ the optimum initial flight path angle $\gamma_o^* = 0(\epsilon^{\frac{1}{4}})$. The velocity lost in the skip is

$$\Delta v^2 = \frac{2C_D}{C_L} \Delta \gamma = O(\epsilon^{\frac{1}{4}}) \tag{5.5.3-7}$$

and the range gained is,

$$\begin{aligned} \Delta \Theta &\doteq \frac{1}{C_D \, p} \Delta v \\ &\doteq O(\epsilon^{\frac{1}{4}}) \end{aligned} \tag{5.5.3-8}$$

But the range gained in the Keplerian regime from Eq. (5.5.3-7) is

$$\begin{aligned} \tan \Theta^* &= O\left(\frac{1}{2} \frac{1}{\epsilon^{1/4}}\right) \\ \Theta^* &= O(1) \end{aligned} \tag{5.5.3, 9}$$

So the dominant range is gained in the Keplerian regime with minimum velocity lost skips conducted to reach the proper flight path angle. Minimum velocity lost in the aerodynamically dominated regime were treated in Section 5.4.2. It is sufficient for the purpose here to note that turning is done at maximum C_L / C_D to $\gamma = 0$ and p unspecified bottom of the trajectory and then at non-maximum C_L/C_D to meet the final specified value of γ_o^* for maximum range in the Keplerian regime.

Two widely different maximum range trajectories have been identified. One an equilibrium glide, the other a Keplerian free-fall connected with aerodominated skips. It is interesting to note both types of trajectories have been reported as the results of numerical computation. [75, 81]

5.5.4 Maximum Range in Near Sonic Flight

A problem of interest for a vehicle, in sonic flight, attempting to obtain a landing site, is the maximum range problem. The dynamic equations for small flight path angle sonic flight or for $\rho = 0(1), v^2 = 0(\epsilon), \gamma = 0(\epsilon)$ in nondimensional form, are (Section 2.10).

$$\frac{dv}{dt} = -\frac{C_D \rho v^2}{2}$$

$$0 = \frac{C_L \rho v^2}{2} - 1 \tag{5.5.4-1}$$

or eliminating ρv^2, they may be written simply as

$$\frac{dv}{dt} = -\frac{C_D}{C_L} \tag{5.5.4-2}$$

The range equation is

$$\frac{d\theta}{dt} = v \tag{5.5.4-3}$$

The optimal control problem could be formulated, but it is completely avoided by eliminating the independent variable in the usual manner, vis.

$$\frac{dv^2}{d\theta} = -\frac{2C_D}{C_L} \tag{5.5.4-4}$$

Thus for small flight path angle the maximum range for a given loss of velocity is obtained by maintaining the maximum C_L/C_D. The expression for the maximum range is then

$$\theta - \theta_0 = \frac{1}{2} \left.\frac{C_L}{C_D}\right|_{MAX} (v_0^2 - v^2) \tag{5.5.4-5}$$

As the vehicle will decelerate from velocities order $\epsilon^{\frac{1}{2}}$ to velocities order ϵ, the range covered in this regime is order ϵ. This is order ϵ smaller than the ranges traversed in the hypersonic regime.

5.6 The Minimum Heating Problem

5.6.1 Introduction

A vehicle attempting to establish a satellite orbit, or land, may enter a planetary atmosphere to lose part of its kinetic energy. A natural objective of this type of flight path is to minimize the portion of the kinetic energy that is transferred to the vehicle in the form of heat. Some general observations about the structure of the problem, not widely understood, can easily be made.

The heating rate may be expressed in the general form (see Appendix F)

$$\frac{dq}{dt} = C_Q \rho^i v^{2j} \tag{5.6.1-1}$$

The rate of change of the vehicle's energy with time is,

$$\frac{d\left(v^2 - \frac{2}{1+h}\right)}{dt} = - C_D \rho v^3 \tag{5.6.1-2}$$

The rate of change of vehicle energy with energy absorbed is then, for $h = 0(\epsilon)$,

$$\frac{d\left(v^2 - \frac{2}{1+\epsilon h}\right)}{dq} = - \frac{C_D \rho v^3}{2 C_Q \rho^i v^{2i}}$$

or to lowest order in ϵ

$$v^{(2j-3)} \frac{dv^2}{dq} = - \frac{1}{2} \frac{C_D}{C_Q} \rho^{1-i} \tag{5.6.1-3}$$

For a fixed change in kinetic energy, the energy absorbed by the vehicle is only a function of the density profile of the trajectory (if $i \neq 1$), the value of (C_D/C_Q) and the value of i. The value of i is determined by the type of heating being considered (see Appendix F). For convective heating $i \cong \frac{1}{2}$ and for radiative heating $i \cong \frac{3}{2}$*. By inspection of Eq. (5.6.1-3) it is seen that minimum convective heating trajectories are associated with large ρ and minimum radiation heating is associated with small ρ. As the values of j are $\frac{3}{2}$ and 10 respectively, for convective and radiative heating, the radiation heating occurs at high velocities, and the convective

*Some investigators [8, 96], have suggested a value of i = 1 for both total convective heating and radiation heating. If this is the case, the trajectory is insensitive to the density-velocity profile and the optimum value of C_D/C_Q is it's largest value for the complete trajectory. Even values of i suggested above, seem to offer so little density pay-off that typical minimum heating trajectories spend much of their time at near maximum C_D/C_Q. (See Ref. (76).

heating occurs at lower velocities. Thus, a minimum heating trajectory is characterized by a high-altitude-high-velocity minimum radiation heating phase followed by a lower velocity low altitude minimum convective heating phase.

In the low altitude phase aerodynamic load factors are quite high, (order $1/\epsilon$) so that load factor constraints must be considered. These factors make realistic minimum heating trajectories analytically the most difficult to treat.

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CHAPTER VI

GUIDANCE TECHNIQUES

6.1 Introduction

In the preceding chapters we have dealt extensively with a procedure for obtaining a uniformly valid analytic approximation to flight trajectories. As guidance applications represent one of the most stringent uses of such solutions, it is appropriate that methods of implementation be discussed.

It is significant to observe that the major advantage of using an asymptotic expansion for guidance applications is that one obtains an analytical solution whose accuracy is both uniform and estimatable. This, hopefully, precludes the extensive numerical investigation commonly made to estimate the accuracy of analytical solutions obtained in a less systematic fashion. It should also reduce the tailoring of the guidance system necessary to cope with regions of poor accuracy.

It is important to point out that the method of systematic approximation presented here does not always reduce the dynamic equations to a set tractable for analytical integration. But numerical integration of the reduced set is always possible with higher approximation and thus more accurate solutions expressed in terms of these numerical solutions. In this case, one has lost the advantage of analytical expressions for the lowest order solution but retained the option of obtaining a solution of a prescribed degree of accuracy, perhaps in a simpler form than a raw integration of the trajectory.

Independent of whether one chooses to handle these analytically difficult portions of the trajectory in this or more conventional methods (for example, a least squares fit), they may be matched or patched to the portions of a trajectory for which one obtains analytical asymptotic expansions. The combination should represent a relatively simple and accurate method of describing a complete flight trajectory.

6.2 Explicit Guidance Schemes[87]

The most obvious way to use an analytical solution in a guidance scheme is to explicitly compute the control necessary to take the vehicle to its desired objective. For example, if the state of a system is described by the differential equation

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}) + \epsilon \underline{g}(\underline{x}, \underline{u}) \qquad (6.2\text{-}1)$$

where a first approximation

$$\underline{\dot{x}}^{(0)} = \underline{f}(\underline{x}^{(0)}, \underline{u}) \qquad (6.2\text{-}2)$$

is valid. Then if a solution to Eq. (6.2-2) is given by

$$\underline{x}^{(0)} = \underline{x}^{(0)}(\underline{x}_f, t_f, t, \underline{u}(t)) \qquad (6.2\text{-}3)$$

where the control $\underline{u}(t)$ was chosen to satisfy some objective and $\underline{x}_f$ is final value of the state at time t_f, then Eq. (6.2-3) represents an expression for the control, given the current and final values of the state and time. It is clear that this control misses the final state with an error that is order ϵ, if the control is only computed once, as a function of the initial state. But the error is arbitarily small if the control is continually computed as a function of the current state. If this computation is complex, it may be avoided by computing instead the control necessary to "fly" the vehicle down the approximate solution. Specifically, if the approximate solution, Eq. (6.2-2) is substituted into the original differential equation, Eq.(6.2-1), the following equation results:

$$\frac{\partial \underline{x}^{(0)}}{\partial t} = \underline{f}(\underline{x}^{(0)}, \underline{u}) + \epsilon \underline{g}(\underline{x}^{(0)}, \underline{u}) \qquad (6.2\text{-}4)$$

This is an algebraic expression for the $\underline{u}(t)$ as a function of the current state. It is important to observe that the control computed in this manner is often not realizable. This simply states that the vehicle cannot fly the approximate trajectory with its real control. The previous method does not have the deficiency.

All of these methods perform somewhat poorly with the addition of uncertainty into the system. The explicit computation of the control based on the initial value of the state not only misses the final state by order ϵ, but, takes no account of initial uncertainties or possible "noise" driving the system away from the final state. A continuous computation of control based on current estimate of the state may spend large amounts of control chasing a randomly forced state vector. This is especially true near the final time when the control is invariably missing the final state due to random errors. A common method of correcting this erratic behavior is to only allow the control to be calculated for a finite number of sample intervals. This compensates the over control of the continuous control scheme with the sluggish behavior of the pre-computed control scheme. The character of the noise and the desired accuracy is accounted for by the choice of the sample times.

A strategy, probably closer to the optimal one,* is to choose the control to drive the system to an error ellipsoid rather than a point in state space. The relative size of the error ellipsoid should be chosen to represent the expected uncertainty in final state, conditioned on the current estimate of the state, and the character of the noise driving the system. The absolute size of the ellipsoid should reflect the final error tolerance. This type of system will react sluggishly when its knowledge of the final state is poor and the tolerances are lax but will react rapidly when the knowledge of its final state is good and tolerances are tight.

6.3 Nominal Guidance Schemes

Normally, analytical solutions are not considered suitable as nominal solutions for nominal guidance schemes. This is because errors introduced by the inaccuracies in the nominal solutions may be as large as the errors introduced by noise in the system. Though the system noise may be handled in a rational manner, there is no systematic method of treating inaccuracies in the nominal solution. Because a scheme for computing the solution to an arbitrary degree of accuracy has been presented, a method of theoretically circumventing this problem is available.

The method is basically predicated on analytically computing the nominal trajectory to an accuracy higher than is required for the system performance. The practical execution of such a computation may prove overly complex; but it offers the hope of being able to compute nominal trajectories in flight. This would remove one of severest restrictions on the use of this type of guidance scheme, their inflexible dependence on precomputed trajectories. A detailed example of this nominal guidance philosophy follows:

Nominal guidance schemes, as based on linear regulator theory, have a particularly elegant, if somewhat artificial statement.[78, 99] It is only necessary that this statement be placed in a context suitable for use here. Consider the nonlinear system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) + \epsilon \underline{g}(\underline{x}, \underline{u}) \tag{6.3-1}$$

with a valid first approximation,

$$\dot{\underline{x}}^{(0)} = \underline{f}(\underline{x}^{(0)}, \underline{u}^{(0)}) \tag{6.3-2}$$

and its solution,

*The problem of optimal control of a nonlinear system in the presence of noise has a rigorous formulation, but unfortunately few solutions. See Refs. (88, 89).

$$\underline{x}^{(0)} = \underline{x}^{(0)}(\underline{u}^{(0)}(t), \underline{x}_f, t, t_f) \tag{6.3-3}$$

written in terms of the desired final state, $\underline{x}_f$, at the final time, t_f.

The next approximation is

$$\dot{\underline{x}}^{(1)} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} \underline{x}^{(1)} + \frac{\partial \underline{f}}{\partial \underline{u}}\bigg|_{\underline{x}^{(0)}} \delta \underline{u} + \underline{g}(\underline{x}^{(0)}, \underline{u}^{(0)}) + \underline{w}(t) \tag{6.3-4}$$

where the control, $\underline{u}$, is assumed to deviate from $\underline{u}^{(0)}$ by an amount $\delta \underline{u}$, which is order ϵ. It is also assumed that the system is driven with order ϵ white noise, $\underline{w}(t)$. The mean and covariance of the noise are assumed to be

$$\begin{aligned} &\mathcal{E}[\underline{w}(t)] = 0 \\ &\mathcal{E}[\underline{w}(t)\underline{w}(t')^T] = Q\,\delta(t-t') \end{aligned} \tag{6.3-5}$$

The difficulty that occurs in the application of linear regulator theory is associated with the non-zero mean value of $\underline{g}(\underline{x}^{(0)}, \underline{u}^{(0)})$, and thus $\underline{x}^{(1)}$. A zero mean state variable may be obtained by simply realizing that the effect of $\underline{g}(\underline{x}^{(0)}, \underline{u}^{(0)})$ on the system is computable. In fact, it is the normal second approximation when noise and variation in the control are not included. It is given as the solution to the linear equation

$$\dot{\underline{x}}^{(1)'} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} \underline{x}^{(1)'} + \underline{g}(\underline{x}^{(0)}, \underline{u}^{(0)}) \tag{6.3-6}$$

Subject to the boundary condition, $\underline{x}^{(1)'}(t_f) = 0$, this solution can be written explicitly as

$$\underline{x}^{(1)'}(t) = \int_{t_f}^{t} \Phi(t,\tau)\, \underline{g}(\underline{x}^{(0)}, \underline{u}^{(0)})\, d\tau \,. \tag{6.3-7}$$

where

$$\Phi(t, t_f) = \frac{\partial \underline{x}^{(0)}}{\partial \underline{x}_f}(\underline{x}_f, t, t_f)$$

A zero mean state variable is then

$$\delta \underline{x} = \underline{x}^{(1)} - \underline{x}^{(1)'} \tag{6.3-8}$$

which clearly satisfies the linear differential equation,

$$\delta \dot{\underline{x}} = F(t)\, \delta \underline{x} + G(t)\, \delta \underline{u} + \underline{w} \tag{6.3-9}$$

To conform to a standard notation, the following matrices have been introduced:

$$F(t) = \left. \frac{\partial \underline{f}}{\partial \underline{x}} \right|_{\underline{x}^{(0)}(t)}$$

$$G(t) = \left. \frac{\partial \underline{f}}{\partial \underline{u}} \right|_{\underline{x}^{(0)}(t)} \tag{6.3-10}$$

The covariance of $\delta \underline{x}$ will be indicated by $P(t)$

$$\mathcal{E}\left[\delta \underline{x}(t)\, \delta \underline{x}(t)^T \right] = P \tag{6.3-11}$$

where its initial value P_o is presumed known. It is now only necessary to proceed with a straightforward statement of linear regulator theory.

A measurement of $\delta \underline{x}$, $\delta \underline{z}$, is assumed to be related to $\delta \underline{x}$ and corrupted with white noise, $\underline{v}$.

$$\delta \underline{z}(t) = H(t)\, \delta \underline{x}(t) + \underline{v}(t) \tag{6.3-12}$$

The noise, $\underline{v}(t)$, is assumed to have zero mean and covariance, $R(t)\, \delta(t - t')$ and to be correlated with $\underline{w}(t)$.

$$\mathcal{E}\left[\underline{v}(t)\, \underline{v}(t')^T \right] = R(t)\, \delta(t - t')$$

$$\mathcal{E}\left[\underline{v}(t)\, \underline{w}(t')^T \right] = M(t)\, \delta(t - t') \tag{6.3-13}$$

The control $\delta \underline{u}$ is picked to minimize the expected value of a quadratic cost function,

$$J = \frac{1}{2} \mathcal{E}\left[\delta \underline{x}^T(t_f) S_f \delta \underline{x}(t_f) + \int_{t_o}^{t_f} (\delta \underline{x}^T A_{(t)} \delta \underline{x} + 2 \delta \underline{x}^T N_{(t)} \delta \underline{u}^T + \delta \underline{u}^T B_{(t)} \delta \underline{u}) dt \right] \tag{6.3-14}$$

To determine this optimal control $\delta \underline{u}$ one must first form a minimum variance (maximum likelihood) estimate of the state, $\delta \underline{x}$, given by the Kalman-Busey,[90] Battin,[17] Gauss[91, 97], Filter

$$\delta \dot{\hat{\underline{x}}} = F_{(t)} \delta \hat{\underline{x}} + G_{(t)} \delta \underline{u} + K_{(t)} \left[\delta z - H \delta \hat{\underline{x}} \right] \tag{6.3-15}$$

where

$$K_{(t)} = (P_{(t)} H_{(t)}^T + M_{(t)}) R^{-1}_{(t)}$$

$$\dot{P}_{(t)} = F_{(t)} P_{(t)} + P_{(t)} F_{(t)}^T - (P_{(t)} H_{(t)}^T + M_{(t)}) H_{(t)} (M_{(t)}^T + H_{(t)} P_{(t)}) + Q_{(t)} \tag{6.3-16}$$

Then the optimal control, $\delta \underline{u}$, is linearly related to this estimate of the state, $\delta \underline{x}$, by a linear time varying gain,

$$\delta \underline{u}(t) = - C(t) \, \delta \hat{\underline{x}}(t) \tag{6.3-17}$$

where the gain matrix, C(t) is given by the solution of the deterministic state regulator problem

$$C(t) = B^{-1}(t) G^T(t) S(t)$$

$$\dot{S}(t) = - S(t) F(t) - F_{(t)}^T S(t) + (S(t) G(t) + N(t)) B^{-1}(t) (N_{(t)}^T + G_{(t)}^T S(t)) - A \tag{6.3-18}$$

In the flight dynamic problems considered here, the noise driving the system is usually correlated. This corresponds predominantly to uncertainties in atmospheric density. The extension of the preceding formulation to correlated noise only requires that the linear system be augmented by the addition of a shaping filter driven by white noise. This has been treated by Bryson[99] and Deyst.[92]

In some case, a quadratic cost function seems inadequate. Specifically, in hypervelocity flight problems velocity lost during turning is related to the absolute value of the turning angle. Potter and Deyst [94] and Wonham [88] have treated the linear regular problem with nonquadratic cost functions.

Thus, this theory with the included capability to handle correlated noise and nonquadratic cost function seems capable of handling a large class of problems encounted in the guidance of a hypervelocity flight vehicles about the nominal path. The small contribution made here is to show how uniformly valid analytical expansion can be rationally used with this theory. This offers the possibility of analytically stored nominal trajectories with the associated versatility usually ascribed only to explicit guidance techniques.

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CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary

This thesis has applied the method of matched asymptotic expansions to the problem of analytically describing flight trajectories. The dominant emphasis has been directed toward trajectories of the hypervelocity or atmospheric entry class. New and previously known analytical solutions for flight trajectories have been produced in one systematic procedurethat is capable of identifying their region of validity, proceeding to higher order more accurate solutions, and combining these solutions to obtain expressions valid over several regions of interest.

Specifically, the region of validity of all first approximations to the flight dynamic equations have been carefully identified. The analyses of Allen and Eggers[8, 9] Chapman[4], Shen[28], Lees [7] and Arthur[11] have all been shown to be rational approximations within this context. Their region of validity and accuracy has thus been established. The systematic procedure by which they may be extended to higher order has been demonstrated. Loh's "second order" solution has been shown to be a multiple regime "first order" approximation that could be corrected to a rational lowest order solution.

Two expansions have been matched to produce a composite expansion valid for a currently interesting class of lifting trajectories. Numerical results have been presented showing that this expansion is in excellent agreement with exact integration, for the types of trajectories for which it is presumed valid. A composite expansion is thereby illustrated to be relatively simple analytical solutions with predictable accuracy and range of validity; a result of considerable importance for guidance applications.

The three-dimensional problem of thrusting flight in a rotating atmosphere surrounding an oblate planet has been treated. The problem of two-dimensional flight in a non-rotating atmosphere surrounding a spherically symmetric planet has been shown to be systematically imbedded in this larger three dimensional problem.

Simple but valid models for optimal flight trajectory problems have been introduced. An optimal plane change, some maximum range and minimum velocity lost problems have been worked. Observations concerning the structure of these problems and the minimum heating problem have been made.

Methods of incorporating uniformly valid asymptotic expansions in guidance schemes have been suggested. Specifically, the advantages of uniformly valid solutions have been presented and methods of performing both explicit and linear nominal guidance with these solutions have been demonstrated.

7.2 Recommendation for Future Study

Only a small fruition of the potential application of the method to the flight dynamic problem have been made in this work. Higher order solutions, including the practical and important effects of rotation atmosphere and planetary oblateness, await computation in all flight regimes. Matching of both lower and these higher order solutions, for the numerous regimes of flight, awaits completion.

Analytic computation of optimal trajectories is difficult, due to the general non-integrability of the adjoint equation, except in a piecewise fashion. But interpretation of numerical results, in terms of models produced in this context, seem both fruitful and useful. A host of numerical optimal trajectories await such simple interpretation.

Application to guidance problems seem the most promising. This technique offers the possibility of formulating analytical guidance schemes for which accuracy and range of validity can be estimated and extended. Precise tailoring to specific application, though complex, is straightforward and holds the promise of excellent results.

A closely related topic is the utilization of lowest order analytical solutions in numerical integrating techniques of either a variation of parameters or Encke type. Such an implementation would certainly produce fast and accurate integration for numerically difficult hypervelocity flight trajectories.

Other perturbation techniques are available, specifically, the method of multiple scale. It is capable of handling a whole class of oscillatory trajectories for which matched asymptotic expansions seems particularly poorly suited.

7.3 Conclusions

In identifying the significant contribution of this thesis, it would be presumptious of the author to claim ultimate originality for much of its contents. Certainly, most of the observations about the flight dynamic problem made here, have occurred to many investigators before. But scarcely, if ever, have so many results been the output of a single investigation. This can in no sense be attributed to the skill of the investigator but rather to the efficiency of a systematic mathematical technique not previously applied in flight mechanics. The major contribution of this thesis is then the demonstration of the usefulness of the method of matched asymptotic expansions to problems in hypervelocity flight mechanics. It is the opinion of the author that this thesis only initiated investigation in an area where both significant amounts of useful and interesting results remain to be obtained, and an efficient technique for the production of these results is currently available.

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APPENDIX A

VECTOR MATRIX ANALYSIS

A.1 Vector Differential Equations

A set of n nonlinear equations may be conveniently represented in vector form as*

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{A.1-1}$$

with a solution

$$\underline{x} = \underline{x}(\underline{x}_o, t, t_o) \tag{A.1-2}$$

where $\underline{x}_o$ are initial conditions given at some time t_o. Note that a "variational equation" governing small perturbations about some solution to the differential equation may be written as

$$\delta\dot{\underline{x}} = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}} \delta\underline{x} \tag{A.1-3}$$

where $\left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}}$ is an n x n matrix of time varying coefficients evaluated along the given solution. The solution to Eq. (A.1-3) may be written in terms of a transition matrix, $\Phi(t, t_o)$ as

$$\delta\underline{x}(t) = \Phi(t, t_o)\,\delta\underline{x}(t_o) \tag{A.1-4}$$

where $\Phi(t, t_o)$ satisfies the equation

$$\frac{\partial \Phi(t, t_o)}{\partial t} = F(t)\,\Phi(t, t_o) \tag{A.1-5}$$

*Notice that time may always be artifically introduced into $\underline{f}(\underline{x})$ with the equation $\dot{x}_n = 1$. This is the approach that will be consistantly used here.

and

$$F(t) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}^{(0)}} \qquad \underline{\Phi}(t_0, t_0) = I \tag{A. 1-6}$$

If the solution to the original nonlinear equation is known in analytical form then the transition matrix may be obtained directly. If $\underline{x} = \underline{x}(\underline{x}_0, t, t_0)$ is the solution, then

$$\delta \underline{x} = \frac{\partial \underline{x}}{\partial \underline{x}_0} \delta \underline{x}_0 \tag{A. 1-7}$$

Comparing (A. 1-4) with (A. 1-7) reveals that

$$\underline{\Phi}(t, t_0) = \frac{\partial \underline{x}}{\partial \underline{x}_0}(\underline{x}_0, t, t_0) \tag{A. 1-8}$$

The adjoint set of equations for (A. 1-3) are

$$\dot{\underline{\lambda}} = -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T \underline{\lambda} \tag{A. 1-9}$$

which have the convenient property that

$$\frac{d}{dt}(\delta \underline{x}^T \underline{\lambda}) = \delta \dot{\underline{x}}^T \underline{\lambda} + \delta \underline{x}^T \dot{\underline{\lambda}} = 0 \tag{A. 1-10}$$

or

$$\delta \underline{x}^T \underline{\lambda} = \text{constant} \tag{A. 1-11}$$

Using this relation between t and t_o,

$$\delta \underline{x}^T(t) \underline{\lambda}(t) = \delta \underline{x}^T(t_0) \underline{\lambda}(t_0) \tag{A. 1-12}$$

and the propagation relation for $\delta \underline{x}$, Eq. (A. 1-4)

$$\delta \underline{x}(t) = \underline{\Phi}(t, t_0) \delta \underline{x}(t_0) \tag{A. 1-13}$$

gives the result,

$$\delta \underline{x}^T(t_0) \underline{\Phi}^T(t, t_0) \underline{\lambda}(t) = \delta \underline{x}^T(t_0) \underline{\lambda}(t_0) \tag{A. 1-14}$$

which must be valid for arbitrary, $\delta \underline{x}(t_o)$, so

$$\underline{\lambda}(t_o) = \Phi^T(t, t_o)\, \underline{\lambda}(t) \tag{A.1-15}$$

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APPENDIX B

OPTIMAL CONTROL THEORY*

B. 1 The Necessary Conditions of the Pontryagin Minimum Principle

Given a system of differential equations with control $\underline{u}$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \tag{B. 1-1}$$

that has a cost component x_o as the first component of the state vector $\underline{x}$. Then assume there exists some control $\underline{u}^*(t)$ over the interval of time t_o to t_f that minimizes the cost x_o at the final time, t_f. Any nonoptimum control, u(t), applied over a vanishingly small interval of time Δt must produce a perturbation away from trajectory given by

$$\begin{aligned} \delta \underline{x}(t) &= (\dot{\underline{x}} - \dot{\underline{x}}^*)\Delta t \\ &= (\underline{f}(\underline{x}^*, \underline{u}) - \underline{f}(\underline{x}^*, \underline{u}^*))\Delta t \end{aligned} \tag{B. 1-2}$$

The perturbation at the final time caused by any admissible control of or a perturbation of the final time must lie on or above a hyperplane passing through the final state (see Fig. B. 1). If this were not so, the controls producing the two perturbations lying on both sides of the hyperplane could be combined to produce a trajectory meeting the required boundary condition with lower cost. This condition may be analytically represented by requiring the normal to the hyperplane, $\underline{\lambda}(t_f)$, and the perturbation at t_f, $\delta \underline{x}'(t_f)$ have the relation,

$$\underline{\lambda}^T(t_f)\, \delta \underline{x}'(t_f) \geq 0 \tag{B. 1-3}$$

A positive or zero scalar product. Notice the λ_o has the possibility of being zero (if the hyperplane is "vertical"). Otherwise, it may be normalized to -1.

If $\underline{\lambda}(t)$ is chosen to satisfy the equation,

*A complete treatment of the ideas presented in this Appendix, together with more information about optimal control theory and optimal flight trajectories, may be found in the following Refs.: (55, 44, 56, 47, 19, 20, 21, 27, 54, 60, 60, 74, 76, 78, 81, 86).

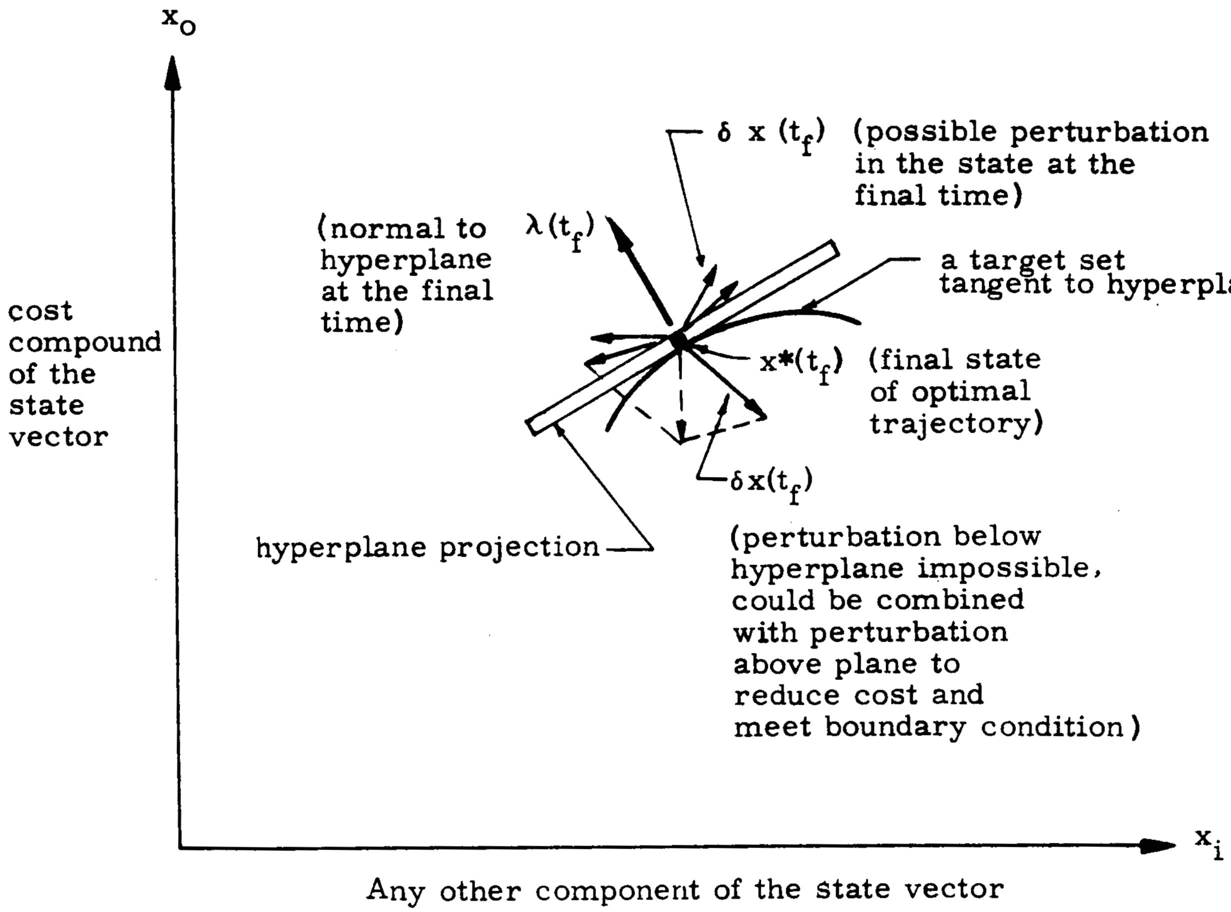

Fig B.1 Schematic Representation of the Minimum Principle

$$\dot{\underline{\lambda}} = -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T \underline{\lambda} \tag{B. 1-4}$$

then from Eqs. (A. 1-11), (B. 1-2) and (B. 1-3),

$$\begin{aligned} \underline{\lambda}^T(t_f)\,\delta\underline{x}'(t_f) &= \underline{\lambda}^T(t_f)\left(\delta\underline{x}(t_f) + \underline{f}(\underline{x}^*(t_f), \underline{u}^*)\,\Delta t_f\right) \\ &= \underline{\lambda}^T(t)\,\delta\underline{x}(t) + \underline{\lambda}^T(t_f)\,\underline{f}(\underline{x}^*(t_f), \underline{u}^*)\,\Delta t_f \geq 0 \end{aligned}$$

or (B. 1-5)

$$\underline{\lambda}^T(t)\,\underline{f}(\underline{x}^*, \underline{u})\,\Delta t + \underline{\lambda}^T(t_f)\,\underline{f}(\underline{x}^*(t_f), \underline{u}^*)\,\Delta t_f \geq \underline{\lambda}^T(t)\,\underline{f}(\underline{x}^*, \underline{u}^*)\,\Delta t$$

Then by defining a Hamiltonian, H, as

$$H(\underline{x}, \underline{u}, \underline{\lambda}) = \underline{\lambda}^T(t)\,\underline{f}(\underline{x}, \underline{u}) \tag{B. 1-6}$$

and by choosing, $H(x^*(t_f), \underline{u}^*(t_f), \underline{\lambda}^*(t_f)) = 0$ if $\Delta t_f \neq 0$, then one has the minimum principle

$$H(\underline{x}^*, \underline{u}, \underline{\lambda}) \geq H(\underline{x}^*, \underline{u}^*, \underline{\lambda}) \tag{B. 1-7}$$

Or that H evaluated along the optimal trajectory must be a minimum with respect to the control. It is noted here that $\underline{\lambda}(t_f)$ must be normal (transversal) to any target set,or a perturbation that satisfied the boundary conditions to first order would be below the hyperplane.

A particularly useful interpretation of this condition is in terms of a Euclidian norm in an n-dimensional space. Note $\underline{f}(\underline{x}, \underline{u})$ can be considered an n-dimensional mapping of the m-dimensional control $\underline{u}$. Then requiring that $H = \underline{\lambda}^T(t)\,\underline{f}(\underline{x}^*, \underline{u})$ be a minimum with respect to $\underline{u}$ is equivalent to requiring the control that gives the minimum $\underline{f}(\underline{x}^*, \underline{u})$ projection on $\underline{\lambda}(t)$. See Fig. B. 2.

To motivate the naming of the Hamiltonian note that the system equations and the adjoint equations, Eqs. (B. 1-1) and (B. 1-4) may be written as

$$\begin{aligned} \dot{\underline{x}} &= \frac{\partial H}{\partial \underline{\lambda}} \\ \dot{\underline{\lambda}} &= -\frac{\partial H}{\partial \underline{x}} \end{aligned} \tag{B. 1-8}$$

which are Hamilton equations of classical mechanics.

Further to show that H is a constant for constant bounded control sets, note

$$\dot{H} = \left(\frac{\partial H}{\partial \underline{x}}\right)^T \dot{\underline{x}} + \left(\frac{\partial H}{\partial \underline{\lambda}}\right)^T \dot{\lambda} + \lim_{\Delta t \to 0} \underline{\lambda}^T(t) \frac{\underline{f}(\underline{u}(t+\Delta t)) - \underline{f}(u(t))}{\Delta t} \qquad \text{(B. 1-9)}$$

The first two terms cancel each other by use of Eqs. (B. 1-8) and the last term is identically zero by application of the minimum principle unless the bounds on the control set are functions of time.

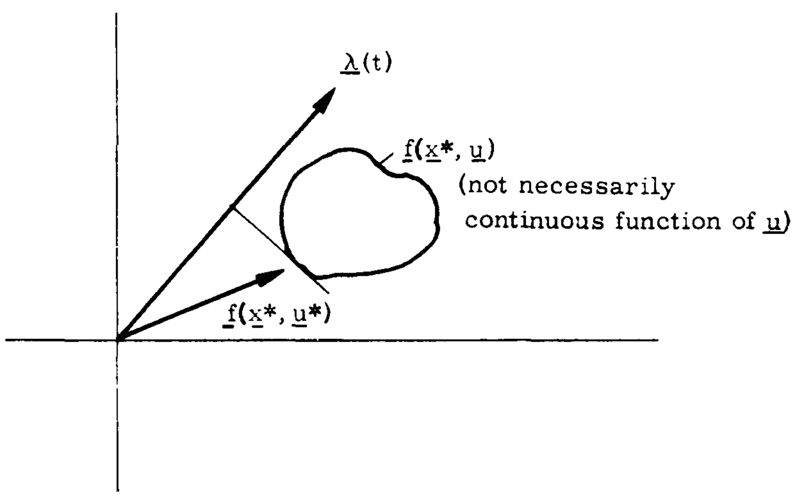

Fig B. 2 Geometric Interpretation of the Minimum Principle

APPENDIX C

THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

The method of matched asymptotic expansions is a powerful perturbation technique of applied mathematics. Its strength lies less in its rigorous formulation, which is in its infancy[3, 5], and more in the host of problems it has successfully handled[1, 4, 6, 12, 32, 57, 61, 64, 66, 67, 70]. A rather limited description of the technique will be presented that will suffice to describe the applications made of it in this thesis. For an elaborate and enlightening treatment of the method the reader is referred to Van Dyke[1].

C.1 <u>A Straight Forward Perturbation Expansion</u>

Given an ordinary differential equation in which a small parameter, ϵ, appears, for example:

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \epsilon \underline{g}(\underline{x}) \tag{C.1-1}$$

where both the independent variable, t, and the dependent variables, $\underline{x}$ are of order one. Then a straight forward perturbation expansion for the solution of this equation is produced by assuming that $\underline{x}$ can be represented as a power series in ϵ:

$$\begin{aligned} \underline{x}(t) &= \underline{x}^{(0)}(t) + \epsilon \underline{x}^{(1)}(t) + \epsilon^2 \underline{x}^{(2)}(t) + \cdots \\ &= \sum \epsilon^n \underline{x}^{(n)}(t) \end{aligned} \tag{C.1-2}$$

The new dependent variables, $\underline{x}^{(n)}_{(t)}$ are then assumed to satisfy a larger set of equations created by substituting the series, Eq. (C.1-2), into the original differential equation, Eq. (C.1-1), and equating terms multiplied by equal orders of ϵ, specifically:

$$\sum \epsilon^n \dot{\underline{x}}^{(n)} = \underline{f}\left(\sum \epsilon^n \underline{x}^{(n)}\right) + \epsilon \underline{g}\left(\sum \epsilon^n \underline{x}^{(n)}\right) \tag{C.1-3}$$

Expanding the right hand terms in a Taylor series gives,

$$\sum \epsilon^n \dot{\underline{x}}^{(n)} = \underline{f}(\underline{x}^{(0)}) + \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} (\epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \cdots)$$

$$+ \frac{1}{2} \frac{\partial^2 \underline{f}}{\partial \underline{x} \partial \underline{x}}\bigg|_{\underline{x}^{(0)}} (\epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \cdots)(\epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \cdots)^T + \cdots$$

$$+ \epsilon \underline{g}(\underline{x}^{(0)}) + \epsilon \frac{\partial \underline{g}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} (\epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \cdots)$$

$$+ \frac{1}{2} \frac{\partial^2 \underline{g}}{\partial \underline{x} \partial \underline{x}}\bigg|_{\underline{x}^{(0)}} (\epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \cdots)(\epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \cdots)^T + \cdots$$

(C. 1-4)

Equating equal orders of ϵ yields,

$$\epsilon^0: \quad \dot{\underline{x}}^{(0)} = \underline{f}(\underline{x}^{(0)})$$

$$\epsilon^1: \quad \dot{\underline{x}}^{(1)} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} \underline{x}^{(1)} + \underline{g}(\underline{x}^{(0)}(t))$$

$$\epsilon^2: \quad \dot{\underline{x}}^{(2)} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} \underline{x}^{(2)} + \frac{1}{2} \frac{\partial^2 \underline{f}}{\partial \underline{x} \partial \underline{x}}\bigg|_{\underline{x}^{(1)}} \underline{x}^{(1)} \underline{x}^{(1)T} + \frac{\partial \underline{g}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} \underline{x}^{(1)} \qquad \text{(C. 1-5)}$$

It is important to observe that only the lowest order equation is nonlinear. All higher order equations are linear with time varying coefficients and forcing functions. In fact their solution may be written explicity as

$$\underline{x}^{(n)}(t) = \Phi(t, t_0)\, \underline{x}^{(n)}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\, \underline{h}^{(n)}(\tau)\, d\tau \qquad \text{(C. 1-6)}$$

where $\Phi(t, t_0)$ is the transition matrix for perturbations about the lowest order solution and $\underline{h}^{(n)}_{(t)}$ represents the forcing terms that appear in the $n^{\underline{th}}$ order perturbation equation. It is seen $\underline{h}^{(n)}_{(t)}$ depends on the solution to the lower order equations. Explicity,

$$\underline{h}^{(n)}(t) = \underline{h}^{(n)}\left(\underline{x}^{(0)}(t), \underline{x}^{(1)}(t), \cdots, \underline{x}^{(n-1)}(t)\right) \tag{C. 1-7}$$

Also, it has been shown Eq. (A. 1-8), that the transition matrix may be written in terms of the solution to the lowest order differential equation as

$$\Phi(t,t_0) = \frac{\partial \underline{x}^{(0)}(t, t_0, \underline{x}_0)}{\partial \underline{x}_0} \tag{C. 1-8}$$

so if the lowest order nonlinear problem is explicity integrable, then Eq. (C. 1-6) is the explicit solution for all higher order perturbations. The solution to order ϵ^N is then given by

$$\overset{N}{\underline{x}}(t) = \sum_{n=0}^{N} \epsilon^n \underline{x}^{(n)}(t) \tag{C. 1-9}$$

It is important to observe that non-uniformities can occur in this straightforward perturbation expansion. A classical example of such a non-uniformity is when ϵ multiplies one of the deriviatives in the differential equations, Eq. (C. 1-1). The order of the differential equations is then reduced by one and arbitary initial conditions can no longer be met. Another expansion valid in the region where the initial conditions are to be imposed must be sought. Other non-uniformities do not manifest themselves in this simple fashion. They are usually associated with an unbounded term in the differential equations that occurs for specific values of the equation variables. Similar unbounded terms usually occur in the expansions for the solutions, though not necessarily in the lowest order term. When such a non-uniformity exists, the expansion is presumed invalid in the neighborhood of the non-uniformity and another expansion, valid in this region, must be sought. Generally, when no single expansion is valid through the field of interest, the problem is called a singular perturbation problem. The process of seeking another expansion valid in the region of a non-uniformity is generally accomplished by a rescaling of the variables to a variable more characteristic of the region of the non-uniformity. This process will now be described.

C. 2 Scaling of the Variable

It is important to observe that the form of a non-linear differential equation in which a small parameter appears is in no sense permanent.

Specifically, given a nonlinear differential equation with a small parameter

$$\frac{d\underline{x}_1}{dt_1} = \underline{f}_1(\underline{x}_1) + \varepsilon \underline{g}_1(\underline{x}_1) \tag{C. 2-1}$$

it is possible to transform this equation into another equation in terms of the new variables, $\underline{x}_2(t_2)$ and t_2

$$\frac{d\underline{x}_2}{dt_2} = \underline{f}_2(\underline{x}_2) + \varepsilon \underline{g}_2(\underline{x}_2) \tag{C. 2-2}$$

by simply rescaling the variables. Let x_{1j} and t_1 be related to x_{2j} and t_2 by

$$\begin{aligned} x_{1j} + c_j &= \varepsilon^{n_j} x_{2j} \\ t_1 + c &= \varepsilon^{n_t} t_2 \end{aligned} \tag{C. 2-3}$$

where ϵ^{n_j} and ϵ^{n_t} are arbitary powers of the small parameter ϵ, and c, c_j are arbitrary constants. Then if $\underline{x}_1$ and t_1 are assumed to be of order one, the new variables, x_{2j} and t_2 are presumed to describe length scales and time scales order ϵ^{n_j} and ϵ^{n_t} respectively. Eq. (C. 2-2) lowest order approximation

$$\frac{d\underline{x}_2^{(0)}}{dt_2} = \underline{f}_2^{(0)}(\underline{x}_2) \tag{C. 2-4}$$

is presumed to be valid first approximation to the system of differential equations, Eq. (C. 2-1) when $x_{1j} + c_j$ is order ϵ^{n_j} and t + c is order ϵ^{n_t}.

It is apparent that a particular nonlinear system may have many such first approximations. The approach taken here will be to systematically exhaust all possible first approximations. An expansion will then only bepresumed valid in a region in which the correct lowest order approximation has been used.

C. 3 The Matching Principle

Given two forms of the same differential equation, for example,

$$\begin{aligned} \frac{d\underline{x}_1}{dt_1} &= \underline{f}_1(\underline{x}_1) + \varepsilon \underline{g}_1(\underline{x}_1) \\ \frac{d\underline{x}_2}{dt_2} &= \underline{f}_2(\underline{x}_2) + \varepsilon \underline{g}_2(\underline{x}_2) \end{aligned} \tag{C. 3-1}$$

where the two equations are related by a scaling ϵ^{n_j} and ϵ^{n_t} of one or more of the variables, x_j and t

$$
\begin{aligned}
x_{1j} + c_j &= \varepsilon^{n_j} x_{2j} \\
t_1 + c &= \varepsilon^{n_t} t_2
\end{aligned}
\tag{C. 3-2}
$$

and two different straightforward perturbation expansions for the two equations

$$
\begin{aligned}
\underline{x}_1^M(t_1) &= \sum^M \varepsilon^m \underline{x}_1^{(m)}(t_1) \\
\underline{x}_2^N(t_2) &= \sum^N \varepsilon^n \underline{x}_2^{(n)}(t_2)
\end{aligned}
\tag{C. 3-3}
$$

complete to order ϵ^M and ϵ^N respectively, it is possible that the expansions have some common region of validity (see Ref. (3)), an "overlap domain". Say, for example, where

$$
\begin{aligned}
x_{1j} + c_j &= O(\varepsilon^{n_j/2}) \\
t_1 + c &= O(\varepsilon^{n_t/2})
\end{aligned}
\tag{C. 3-4}
$$

If such a domain exists, then the two expansions must have the same algebraic form in this domain. So, if the two expansions are written in the same variable $\underline{x}_1$ and t_1, then when

$$
\begin{aligned}
x_{1j} + c_j &= O(\varepsilon^{n_j/2}) \\
t_1 + c &= O(\varepsilon^{n_t/2})
\end{aligned}
\tag{C. 3-4}
$$

the expansions should agree to each power in ϵ. This is the form of the asymptotic matching principle of Kaplan and Lagerstrom[4] that will generally be used here. Van Dyke[1] has expressed this principle in the following less intuitive but more formal statement:

"The M-term first expansion of (the N-term second expansion)
= the N-term second expansion of (the M-term first expansion)"

Here, N and M are two integers either equal or different by one. By definition, the M-term first expansion of (the N-term second expansion) is found by rewriting the N-term second expansion in terms of the first expansion variables, expanding asymptotically for small ϵ, and truncating the result to M terms. The N-term second expansion of (the M-term first expansion) is similarly defined.

Symbolically, the matching principle will be indicated as:

$$ {}^{M}_{1}\left[x_2^{N} \right] = {}^{N}_{2}\left[x_1^{M} \right] \tag{C.3-5} $$

where for convenience the brackets and superscripts will normally be dropped when only doing matching to lowest few orders.

C. 4 A Composite Expansion

A composite expansion is defined as any series that reduces to the first expansion when expanded asymptotically for small ϵ in the first variables, and to the second expansion when expanded asymptotically for small ϵ in the second variables. The existence of such a composite expansion is seen to avoid the awkward practical question of when to switch from one expansion to another when traversing from one region to another. As the two expansions are assumed to have a common region of validity, one method of forming a composite expansion is to add the two expansions and subtract their common part. Specifically, if the two expansions, correct to ϵ^M and ϵ^N, are x_1^M and x_2^N respectively, and their common part is determined by inspection, or as the M-term first expansion of (the N-term second expansion), then a composite expansion valid to ϵ^M in region one and to ϵ^N in region two is,

$$ \underline{x}_c^{M,N} = \underline{x}_1^{M} + \underline{x}_2^{N} - {}^{M}_{1}\left[\underline{x}_2^{N} \right] \tag{C. 4-1} $$

or,

$$ \underline{x}_c^{M,N} = \underline{x}_1^{M} + \underline{x}_2^{N} - {}^{N}_{2}\left[\underline{x}_1^{M} \right] \tag{C. 4-2} $$

C. 5 An Expansion Procedure for Optimal Control Problems

The procedure described in the preceding sections for obtaining the solution to a nonlinear problem in which a small parameter appears applies equally well to the set of system and adjoint equations encountered in optimal control problems (See Appendix B). Consider, for example, the dynamic system described by the equation,

$$ \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) + \epsilon\, \underline{g}(\underline{x}, \underline{u}) \tag{C. 5-1} $$

The associated Hamiltonian is

$$H = \underline{\lambda}^T (\underline{f}(\underline{x},\underline{u}) + \varepsilon \underline{g}(\underline{x},\underline{u})) \tag{C.5-2}$$

and the adjoint equations are

$$\dot{\underline{\lambda}} = -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}}^{T} \underline{\lambda} + \varepsilon \frac{\partial \underline{g}}{\partial \underline{x}}\bigg|_{\underline{x}}^{T} \underline{\lambda} \right) \tag{C.5-3}$$

Expanding both $\underline{x}$ and $\underline{\lambda}$ in the following form

$$\underline{x}(t) = \sum \varepsilon^n \underline{x}^{(n)}(t) \qquad \underline{\lambda}(t) = \sum \varepsilon^n \underline{\lambda}^{(n)}(t) \tag{C.5-4}$$

Then substituting into Eq. (C.5-1) and Eq. (C.5-3) and equating equal coefficients of ϵ yields a familiar sequence for the system equations

$$\begin{aligned} &\varepsilon^0: \qquad \dot{\underline{x}}^{(0)} = \underline{f}(\underline{x}^{(0)}, u) \\ &\varepsilon^1: \qquad \dot{\underline{x}}^{(1)} = \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} \underline{x}^{(1)} + \underline{g}(\underline{x}^{(0)}, \underline{u}) \\ &\vdots \end{aligned} \tag{C.5-5}$$

But a slightly more complicated sequence for the adjoint equations

$$\begin{aligned} \dot{\underline{\lambda}}^{(0)} + \varepsilon \dot{\underline{\lambda}}^{(1)} + \cdots &= - \frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)} + \varepsilon \underline{x}^{(1)}}^{T} (\underline{\lambda}^{(0)} + \varepsilon \underline{\lambda}^{(1)} + \cdots) - \varepsilon \frac{\partial \underline{g}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)} + \varepsilon \underline{x}^{(1)} + \cdots}^{T} (\underline{\lambda}^{(0)} + \varepsilon \underline{\lambda}^{(1)} + \cdots) \\ &= -\left(\frac{\partial \underline{f}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} + \frac{\partial^2 \underline{f}}{\partial \underline{x} \partial \underline{x}}\bigg|_{\underline{x}^{(0)}} (\varepsilon \underline{x}^{(1)} + \cdots) + \cdots \right)^T (\underline{\lambda}^{(0)} + \varepsilon \underline{\lambda}^{(1)} + \cdots) \\ &\quad - \varepsilon \left(\frac{\partial \underline{g}}{\partial \underline{x}}\bigg|_{\underline{x}^{(0)}} + \frac{\partial^2 \underline{g}}{\partial \underline{x} \partial \underline{x}}\bigg|_{\underline{x}^{(0)}} (\varepsilon \underline{x}^{(1)} + \cdots) + \cdots \right)^T (\underline{\lambda}^{(0)} + \varepsilon \underline{\lambda}^{(1)} + \cdots) \end{aligned} \tag{C.5-6}$$

or

$$\epsilon^0: \quad \dot{\underline{\lambda}}^{(0)} = - \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|^T_{\underline{x}^{(0)}} \underline{\lambda}^{(0)}$$

$$\epsilon^1: \quad \dot{\underline{\lambda}}^{(1)} = - \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|^T_{\underline{x}^{(0)}} \underline{\lambda}^{(1)} - \left.\left(\frac{\partial^2 \underline{f}}{\partial \underline{x} \partial \underline{x}}\right)\right|_{\underline{x}^{(0)}} \underline{x}^{(1)} \underline{\lambda}^{(0)T} - \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|^T_{\underline{x}^{(0)}} \underline{\lambda}^{(0)} \qquad \text{(C. 5-7)}$$

$$\vdots$$

The equivalent expression for the Hamiltonian is

$$H = \left(\underline{\lambda}^{(0)} + \epsilon \underline{\lambda}^{(1)} + \cdots\right)^T \Big(\underline{f}(\underline{x}^{(0)}, \underline{u}) + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}^{(0)}} (\epsilon \underline{x}^{(1)} + \cdots) + \cdots$$

$$+ \epsilon \underline{g}(\underline{x}^{(0)}, \underline{u}) + \epsilon \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\underline{x}^{(0)}} (\epsilon \underline{x}^{(1)} + \cdots) + \cdots \Big) \qquad \text{(C. 5-8)}$$

$$= \underline{\lambda}^{(0)T} \underline{f}(\underline{x}^{(0)}, u) + \epsilon \left(\underline{\lambda}^{(1)T} \underline{f}(\underline{x}^{(0)}, \underline{u}) + \underline{\lambda}^{(0)T} \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}^{(0)}} \underline{x}^{(1)} + \epsilon \underline{g}^T(\underline{x}^{(0)}, \underline{u}) \underline{\lambda}^{(0)} \right)$$

$$+ \cdots$$

Notice that if the problem is to be calculated to order ϵ that the control that minimizes H to order ϵ may be determined in terms of $\underline{\lambda}^{(0)}$, $\underline{\lambda}^{(1)}$, $\underline{x}^{(0)}$ and $\underline{x}^{(1)}$. So, in principle the "ϵ optimal control" may be calculated. This is true to any order in ϵ. Unfortunately, the expressions even to order ϵ are extremely complex.

For problems where only lowest order results are desired, the necessary conditions may be easily produced by writing the Hamiltonian and adjoint equations associated with a lowest order model of the system. This model is simply the lowest order approximation to the dynamic equations. These equations may then be treated as though they were exact, with the restriction, of course, that the results will only be accurate to lowest order. This approach is consistently taken in Chapter V, to preclude overbearing algebraic complexity.

Finally, multiple expansions can and must be matched, as a single optimal trajectory normally traverses several regions of expansion validity. This has the distasteful aspect of solving the two point boundary value problem associated with the optimal trajectory with multiple matching conditions in between. Fortunately, if one is willing to settle for general information about the trajectory short of detailed numerical results, the simplified dynamic systems often admit so few optimal trajectories that a general description of the composite trajectory is possible, without the benefit of the detailed matching. The generation of these simple descriptions is the primary objective of Chapter V.

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APPENDIX D

PLANETARY GRAVITATIONAL FIELD

D. 1 The Gravitational Potential

The gravitational field of any planet may be expressed as the negative gradient of a potential, V, expressed in spherical harmonics as

$$\underline{g} = -\frac{\partial V}{\partial \underline{r}}$$

$$V = -\frac{GM}{r_0}\left[\frac{r_0}{r} - \left[\left(\frac{r_0}{r}\right)^3 J_2 \frac{1}{2}(3\sin^2 L - 1)\right] + \cdots \right] \qquad \text{(D. 1-1)}$$

where G is the universal gravitational constant, M is the planetary mass, r_o is the equatorial radius, J_2 is the second spherical harmonic coefficient, L is the latitude, and $\underline{r}$ is the radius vector in a planet centered spherical coordinate system. Only the first two terms in the series have been retained. These are the inverse r field associated with a spherically symmetric body and the second spherical harmonic, which basically accounts for the planet's oblate mass attraction.

If the $\underline{g}$ vector is desired in a rotating coordinate system, a slightly different potential must be used. This accounts for both the gravitational attraction and the centripetal acceleration of a point rotating with the planet-fixed coordinate system,

$$V' = -\frac{GM}{r_0}\left[\frac{r_0}{r} - \left[\left(\frac{r_0}{r}\right)^3 J_2 \frac{1}{2}(3\sin^2 L - 1)\right] + \frac{\Omega^2 r_0^3}{GM}\left(\frac{r}{r_0}\right)^2 (\cos 2L) \right] \qquad \text{(D. 1-2)}$$

where Ω is the planet's rotation rate.

These relations will suffice to define the gravitational field to the extent that will be needed here. For an explanation of these expressions and a more elaborate representation of the gravitational field, the reader is referred to Refs. (41, 16, and 85).

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APPENDIX E

ATMOSPHERIC PROPERTIES*

E. 1 Equation of State of the Gas

For relatively low density gases that compose most atmospheres, the equation of state may be written as

$$p = \rho \frac{\bar{R}}{\bar{M}} T \qquad \text{(E. 1-1)}$$

where p is the pressure, ρ the density, T the temperature, $\bar{R}$ the gas constant and $\bar{M}$ is the molecular weight of the well mixed gas.

E. 2 The Hydrostatic Equation

It one neglects the vertical component of the wind velocity, convervation of momemtum in the $\underline{g}$ direction will give

$$\frac{\partial p}{\partial h} = -\rho g \qquad \text{(E. 1-2)}$$

where h is presumed measured along $\underline{g}$, and g is the gravitational acceleration in the planets' rotating coordinate system. The density may be eliminated from the hydrostatic equation by using Eq. (E. 1-1).

$$\frac{dp}{dh} = -\frac{g\bar{M}}{\bar{R}T} p \qquad \text{(E. 1-3)}$$

Given a temperature distribution $T = T(\underline{r})$, this differential equation may be solved for $p = p(h)$ and Eq. (E. 1-1) will specify $\rho(h)$.

Notice that the quantity,

$$\beta = \frac{g\bar{M}}{\bar{R}T}$$

has units of inverse height. It is convenient to introduce it as an inverse atmospheric scale height. With this substitution, Eq. (E. 1-3) is

*For a more adequate treatment of planetary atmospheres see Refs. (15, 16 and 83).

$$\frac{dp}{dh} = -\beta(h)\, p$$

or (E. 1-4)

$$\ln \frac{p}{p_0} = \int_{h_0}^{h} \beta(h)\, dh$$

This serves to define p, and thus the ρ, variation along the $\underline{g}$ vector for a real atmosphere. It only remains to describe $\underline{g} = \underline{g}(r)$ to completely define the atmospheric properties

and (E. 1-5)

$$p = p(\underline{r})$$
$$\rho = \rho(\underline{r})$$

APPENDIX F

AERODYNAMIC FORCES AND HEATING*

F.1 Hypersonic Lift-Drag Data

In hypersonic flight the lift and drag coefficients are not strongly Mach num-dependent. The interrelation of C_L and C_D for high C_L/C_D vehicles be expressed in terms of a Newtonian drag polar of the form,

$$C_D = C_{D_o} + C_{DL}|\sin^3\alpha|$$
$$C_L = C_{L_o}\sin\alpha\cos\alpha|\sin\alpha| \qquad (F.1-1)$$

where α is the angle of attack and C_{D_o}, C_{DL} and C_{L_o} are constants. At small α, these relations simplify to

$$C_D = C_{D_o} + C_{DL}|\alpha^3|$$
$$C_L = C_{L_o}\alpha^2 \qquad (F.1-2)$$

or

$$C_D = C_{D_o} + C_{DL}\left(\frac{C_L}{C_{L_o}}\right)^{3/2}$$

For vehicles with low L/D, relations of the following forms are valid:

$$C_D = C_{D_o} + C_{DL}\cos k\alpha$$
$$C_L = C_{L_o}\sin k\alpha \qquad (F.1-3)$$

where k is a constant with value near one.

F.2 Free Stream Energy Flux

The free stream energy flux, $\dot{q}_o$, of the gas flowing by the vehicle is,

$$\dot{q}_o = \frac{1}{2}\rho V^3 \qquad (F.2-1)$$

*Justification for the information that follows together with an adequate treatment of hypersonic aerodynamics may be found in the following Refs. (22, 23, 31, 47, 49, 59, 62, 63, 95, 96).

Only in free molecular flow does energy of this order of magnitude reach the vehicle. In other flow regimes there are blocking effects that have been discussed. It is still expected that ρ and v are the relevant parameters for expressing heat transfer to the vehicle. In fact, impirical relation in terms of ρ and v to varying powers are available.

F.3 Generalized Aerodynamic Effect

Ambrosio[62] has suggested that both heating and aerodynamic loading can be expressed in the general form

$$G = C_G \rho^i v^{2j} \tag{F.3-1}$$

where G is a generalized aerodynamic effect and C_G is a dimensional constant. The value of C_G is a function of the vehicle geometry and the planetary atmosphere composition. The values of i and j are also functions of the atmospheric composition. For comparison, it will be convenient to express the known values for earth's atmosphere.

Generalized Effect:	C_G	i	j	$\frac{i}{j}$
Deceleration	$\frac{1}{2}(C_L^2+C_D^2)^{\frac{1}{2}}\frac{A}{m g_{o\oplus}}$	1.0	1.0	1.0
Stagnation Convective Heating Rate	$K_S R_N^{-\frac{1}{2}}$	.5	1.5	.333
Stagnation Radiation Heat Rate	$K_R R_N$	1.7	10.6	.160

where R_N is the nose radius of the vehicle, K_S and K_R are constants,

$$K_S = 1.5 \times 10^{-8}$$
$$K_R = .814 \times 10^{-84} \tag{F.3-2}$$

Notice that a large nose radius reduces the convective heating rate, but increases the radiation heating rate.

The maximum aerodynamic effect occurs when,

$$d \ln C_G + d \ln \rho^i + d \ln v^{2j} = 0$$

or for constant C_G when

$$\frac{d \ln v^2}{d \ln \rho} = -\frac{i}{j} \qquad \text{(F. 3-4)}$$

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TABLE I

Planetary Data (2, 15, 84, 16)

Planet	Equatorial Radius	Equatorial Gravitational Acceleration	Inverse Atmospheric Scale Height	Orbital Velocity		Orbital Velocity Mach Number		
	r_o (ft)	g_o (ft/sec^2)	β_o (ft^{-1})	$(r_o g_o)^{\frac{1}{2}}$(ft/sec)	$\frac{1}{\epsilon_1} = \beta_o r_o$	M_o	$g_o/g_{o_\oplus}$	$g_o/g_{o_\oplus}\ \epsilon_1$
Venus	2.03×10^7	28.3	4.9×10^{-5}	2.4×10^4	1006	26	.88	1000
Earth	2.09×10^7	32.2	4.3×10^{-5}	2.6×10^4	900	25	1.0	900
Mars	1.11×10^8	12.2	1.1×10^{-5}	1.17×10^4	132	9.5	.39	5
Jupiter	2.27×10^8	83.7	1.7×10^{-5}	13.6×10^4	3600	49	2.61	9100
Saturn	1.87×10^8	36.6	1.6×10^{-5}	8.26×10^4	3000	45	1.12	3360
Titan	6.90×10^6	7.08	1.0×10^{-5}	7×10^3	64	6.5	.22	14.1

TABLE I

Planetary Data (Cont.)

Planet	Oblateness f	Planet Rotational Velocity Ω (rad/sec)	Orbital Velocity / Equatorial Velocity $\frac{1}{\epsilon_2}$
Venus	0	$\sim 3.20 \times 10^{7}$	3690
Earth	$\frac{1}{297}$	7.31958×10^{-5}	17.5
Mars	$\frac{1}{192}$	7.08821×10^{-5}	15
Jupiter	$\frac{1}{15.4}$	1.77341×10^{-4}	2.95
Saturn	$\frac{1}{9.5}$	1.73953×10^{-4}	2.5
Titan			

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BIOGRAPHICAL SKETCH

Richard Errol Willes was born in Fort Pierce, Florida, on April 4, 1934, the son of Marjorie Tylander and Errol Shippen Willes. He attended public school and a freshman year at the University of Florida. After receiving an appointment to the United States Naval Academy in June 1953, he graduated in June 1957, with a Bachelor of Science Degree and a commission in the United States Air Force. He received single engine pilot training from September 1957, to January 1959, and served as a jet instructor pilot until June 1960.

As a result of an educational tour at the Massachusetts Institute of Technology from June 1960, to June 1962, he received a Master of Science Degree and Engineer's Degree in Aeronautics and Astronautics. He served as an instructor at the United States Air Force Academy in the Department of Aeronautics from August 1962, until July 1964, when he was granted academic leave to pursue a doctorate at the Massachusetts Institute of Technology.

He is a member of the scholastic honoraries: Sigma Gamma Tau, Tau Beta Pi and Sigma Xi. He is married to the former JoEllen Peacock of Jacksonville, Florida, and has three children, Robert Errol, Leslie Jennifer, and Tylan Richard Willes.