SOME STABILITY THEOREMS FOR ORDINARY DIFFERENCE EQUATIONS*

by

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*The author has received financial support from the National Aeronautics and Space Administration under their Traineeship Program and the preparation and reproduction of this paper was supported by NASA under Contract No. NAS8-11264.

NGR-40-002-015
no per ltr 4/20/67

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LaSalle [1,2,3] and others have developed a generalization of the "second method" of Liapunov which utilizes certain invariance properties of solutions of ordinary differential equations. Invariance properties of solutions of ordinary difference equations are utilized here to develop stability theorems similar to those in LaSalle [1]. As illustrations of the application of these theorems, a region of convergence is derived for the Newton-Raphson and Secant iteration methods. A modification of one of these theorems is given and applied to study the effect of round-off errors in the Newton-Raphson and Gauss-Seidel iteration methods.

I. INTRODUCTION. An *ordinary difference equation* is an equation of the type given in (1),

$$x(k+1) = f(k, x(k)) \tag{1}$$

where each x and $f(k,x)$ are elements of X, an n-dimensional vector space. Since the notation used in (1) can become very clumsy, the somewhat neater E notation is used. If E is defined as the operator where $Ex(k) = x(k+1)$, then equation (1) can be written as in (1*)

$$Ex = f(k,x) \tag{1*}$$

where the arguments of x and Ex are understood to be k.

A function $x(k;k_o,x_o)$ is called a <u>solution</u> of the difference equation (1) if it satisfies the following three conditions.

a) $x(k;k_o,x_o)$ is defined for $k_o \leqq k \leqq k_o + K$ for some integer $K > 0$.

b) $x(k_o;k_o,x_o) = x_o$, the initial vector.

c) $x(k+1;k_o,x_o) = f(k,x(k;k_o,x_o))$ for $k_o \leqq k \leqq k_o + K-1$.

Hereafter, it is assumed that a solution to (1) exists and is unique for all $k \geqq k_o$ and that this solution is continuous in the initial vector x_o. More specifically, if $\{x_n\}$ is a sequence of n-vectors with $x_n \to x_o$ as $n \to \infty$, then the solutions through x_n converge to the solution through x_o:

$$x(k;k_o,x_n) \to x(k;k_o,x_o) \quad \text{as} \quad n \to \infty$$

For all k on any compact interval, this convergence is assumed to be uniform.

For any n-vector x, let $|x|$ denote any vector norm of x. For any non-empty set of n-vectors A, denote the distance from

x to A by $d(x,A)$.

$$d(x,A) = \inf \{|x-y| : y \in A\}.$$

Introduce the vector ∞ to X and define $d(x,\infty) = |x|^{-1}$. Let $A^* = A \cup \{\infty\}$ and $d(x,A^*) = \min \{d(x,A),\ d(x,\infty)\}$.

A point $p \in X$ is a <u>positive limit point</u> of $x(k)$ if there is a sequence $\{k_n\}$ with $k_{n+1} > k_n \to \infty$, and $x(k_n) \to p$ as $n \to \infty$. The union of all the positive limit points of $x(k)$ is the <u>positive limit set</u> of $x(k)$.

II. <u>THE GENERAL STABILITY THEOREM</u>. Let G be any set in the vector space X. G may be unbounded. Let $V(k,x)$ and $W(x)$ be real valued functions defined for all $k \geq k_0$ and all x in G. If $V(k,x)$ and $W(x)$ are continuous in x, $V(k,x)$ is bounded below, and

$$\Delta V(k,x) = V(k+1,f(k,x))-V(k,x) \leq -W(x) \leq 0$$

for all $k \geq k_0$ and all x in G, then V is called a <u>Liapunov function</u> for (1) on G. Let $\bar{G}$ be the closure of G, including ∞ if G is unbounded, and define the set A by (2).

$$A = \{x \in \bar{G} : W(x) = 0\} \qquad (2)$$

The following result is the difference analog to Theorem 1 in LaSalle [1].

THEOREM 1. If there exists a Liapunov function V for (1) on G, then each solution of (1) which remains in G for all $k \geqq k_o$ approaches the set $A^* = A \cup \{\infty\}$ as $k \to \infty$.

PROOF: Let $x(k)$ be a solution to (1) which remains in G for all $k \geqq k_o$. Then, by assumption, $V(k,x(k))$ is a monotone non-increasing function which is bounded from below. Hence, $V(k,x(k))$ must approach a limit as $k \to \infty$, and $W(x(k))$ must approach zero as $k \to \infty$. From the definition of A^* and the continuity of $W(x)$, we get $d(x(k),A^*) \to 0$ as $k \to \infty$. Note that if G is unbounded and there exists a sequence $\{x_n\}$ such that $x_n \in G$, $|x_n| \to \infty$, and $W(x_n) \to 0$ as $n \to \infty$, then it is possible to have an unbounded solution under the conditions of the theorem. If G is bounded or if $W(x)$ is bounded away from zero for all sufficiently large x, then all solutions which remain in G are bounded and approach a closed, bounded set contained in A as $k \to \infty$.

This theorem can be used to easily prove all of the usual Liapunov stability theorems. See, for example, Hahn [1] and Kalman and Bertram [1]. For example, if G is the entire space X and $W(x)$ is positive definite, then $A = \{o\}$ and all solutions approach the origin as $k \to \infty$. However, as the following example shows, other considerations can be used to determine if a solution $x(k)$ will remain in G. The difference equation is given in equation (3).

$$Ex = x^{-2} \qquad \text{for} \quad x > 0 \tag{3}$$

Let the set G be the set of positive numbers. Then, if $x > 0$, we get $Ex > 0$ from equation (3) and all solutions which start in G remain in G. The function $V(k,x) = V(x)$

$$V(x) = \frac{x}{1+x^2}$$

is a Liapunov function for (3) on G since $V(x) \geqq 0$ and

$$\Delta V(x) = \frac{x^{-2}}{1+x^{-4}} - \frac{x}{1+x^2} = \frac{x(1-x)(x^3-1)}{(1+x^2)(1+x^4)} = -W(x) \leqq 0$$

We have $W(x) = 0$ when $x = 0$, $x = 1$, and $W(x) \to 0$ as $x \to \infty$. Thus, the set A^* is the set $\{0, 1, \infty\}$. Each solution with $x_0 > 0$ approaches A^* as $k \to \infty$. A look at the solutions to (3)

$$x(k) = x_0^{(-2)^k}$$

shows that this is exactly the case. If $x_0 = 1$, then $x(k) = 1$ for all k. If $x_0 < 1$, then $x(k) \to 0$ for even k and $x(k) \to \infty$ for odd k.

Quite often the set G can be constructed so that all solutions which start in some smaller set G_1 remain in G. One such case is covered in the following corollary.

COROLLARY 1. Let $u(x)$ and $v(x)$ be continuous real-valued functions. Let $V(k,x)$ be such that

$$u(x) \leqq V(k,x) \leqq v(x)$$

for all $k \geqq k_o$. For some η, define the sets $G = G(\eta)$ and $G_1 = G_1(\eta)$ as

$$G(\eta) = \{x : u(x) < \eta\}$$

$$G_1(\eta) = \{x : v(x) < \eta\}$$

If V is a Liapunov function for (1) on $G(\eta)$, then all solutions which start in $G_1(\eta)$ remain in $G(\eta)$ and approach A as $k \to \infty$.

PROOF: Let $x(k)$ be a solution of (1) with $x(k_o) \in G_1(\eta)$. Then

$$u(x(k)) \leqq V(k,x(k)) \leqq V(k_o,x(k_o)) \leqq v(x(k_o)) < \eta$$

for all $k \geqq k_o$, implying that $x(k) \in G(\eta)$ for all $k \geqq k_o$.

Theorem 1 and Corollary 1 give sufficient conditons for the positive limit set of a solution $x(k)$ to be contained in A.

There is an art to finding the best V, W, u, and v, i.e., the functions V, W, u, and v which give the largest G, the largest G_1, and the smallest A. Often more information about the behavior of the solutions can be obtained by considering several different Liapunov functions and combining the results from each.

The following example is taken from Vidal and Laurent [1]. The sampled control systems covered in this paper are described by the difference equation (4).

$$Ex = M(k,x)x \tag{4}$$

where $M(k,x)$ is a matrix. For any vector norm, $|x|$, define the norm of the matrix $M(k,x)$ by

$$|M(k,x)| = \min \{b : |M(k,x)y| \leqq b|y| \quad \text{for all} \quad y \neq 0\}$$

Then clearly, $|M(k,x)x| \leqq |M(k,x)||x|$. For the difference equation (4), try the Liapunov function $V(k,x) = |x|$. Then

$$\begin{aligned} \Delta V(k,x) &= |M(k,x)x| - |x| \\ &\leqq (|M(k,x)| - 1)|x| \ . \end{aligned}$$

Let $u(x) = v(x) = V(k,x) = |x|$, then $G_1(\eta) = G(\eta) = \{x : |x| < \eta\}$. For all x in $G(\eta)$ and all $k \geqq k_o$ let $|M(k,x)| \leqq a(x)$ and

$W(x) = +(1-a(x))|x|$. Then we have

$$\Delta V(k,x) \leqq -W(x) .$$

If $a(x) < 1$ for all x in $G(\eta)$, then $-W(x) \leqq 0$, the set A is the origin and possibly something on the boundary of $G(\eta)$. Since $V(k,x(k))$ is a non-increasing function of k and the boundary of $G(\eta)$ is a level surface of $V(k,x)$, the solutions cannot approach the boundary of $G(\eta)$. Hence, all solutions which start in $G(\eta)$ remain in $G(\eta)$ and approach the origin as $k \to \infty$. The set $G(\eta)$ is called a <u>domain of stability</u> for the system (4). The best $G(\eta)$ is chosen by picking η as large as possible without violating the inequality $a(x) < 1$ for all x in $G(\eta)$.

Various choices for the vector norm will result in various $a(x)$ and various domains of stability. Since each is sufficient, the union of all these domains of stability is also a domain of stability.

If $M(k,0)$ is a constant matrix, independent of k, and the special radius of $M(k,0)$ is less than one, then there is a vector norm such that $a(x)$ is continuous in x and $a(0) < 1$, indicating that there is a non-empty domain of stability (see the Appendix).

The following example illustrates that the results obtained in Theorem 1 and Corollary 1 are the best possible without further assumptions. The difference equation is (5).

$$\begin{aligned} Ex &= y \\ Ey &= a^2 x + p(k)y \end{aligned} \tag{5}$$

where $0 < a < 1$ and $0 < \delta \leqq p(t) < 1-a^2$. If $p(k) = p$, a constant, then the conditons for stability are satisfied and all solutions approach the origin as $k \to \infty$.

Try the Liapunov function

$$V(k,x,y) = a^2 x^2 + y^2$$

Then

$$\begin{aligned} \Delta V(k,x,y) &= -a^2 p(k)(x-y)^2 + a^2(p(k)-(1-a^2))x^2 \\ &\quad + (p(k)+1)(p(k)-(1-a^2))y^2 \\ &\leqq -a^2 p(k)(x-y)^2 \leqq -a^2 \delta (x-y)^2 = -W(x,y) \leqq 0 \ . \end{aligned}$$

From Corollary 1, we see that all solutions are bounded and $x(k)-y(k) \to 0$ as $k \to \infty$.

If

$$p(k) = \frac{1-a^2}{1+a^{k+1}}$$

for all $k \geqq 0$, then this $p(k)$ satisfies the conditions given above and one solution of the difference equation (5) is

$$x(k) = 1 + a^{k} \to 1 \quad \text{as} \quad k \to \infty$$

$$y(k) = 1 + a^{k+1} \to 1 \quad \text{as} \quad k \to \infty .$$

The results obtained are the best possible. Notice, however, that this $p(k)$ approaches $1-a^2$ as $k \to \infty$. If, instead of $p(k) < 1-a^2$, we knew that $p(k) \leqq 1-a^2 - \epsilon$ for some $\epsilon > 0$, then we get

$$\Delta V(k,x,y) \leqq -a^2\delta(x-y)^2 - a^2\epsilon x^2 - (1+\delta)\epsilon\, y^2 = -W_1(x,y) \leqq 0$$

and the only point where $W_1(x,y) = 0$ is $x = y = 0$. In this case, all solutions approach the origin as $k \to \infty$.

III. AUTONOMOUS DIFFERENCE EQUATIONS. If the function $f(k,x)$ in (1) is independent of k, then the difference equation is said to be autonomous, as in equation (6).

$$Ex = f(x) . \tag{6}$$

Just as is the case for autonomous differential equations, solutions to (6) are essentially independent of k_o so we assume $k_o = 0$ and write the solution as $x(k;x_o)$. A function $x^*(k)$ is said to be a solution for (6) on $(-\infty,\infty)$ if, for any k_o in $(-\infty,\infty)$, we have for all $k \geqq k_o$

$$x(k-k_o;x^*(k_o)) = x^*(k).$$

A set B is an invariant set of (6) if $x_o \in B$ implies that there is a solution $x^*(k)$ for (6) on $(-\infty,\infty)$ such that $x^*(k) \in B$ for all k and $x^*(0) = x_o$.

LEMMA 1. The positive limit set B of any bounded soltuion of (6) is a nonempty, compact, invariant set of (6).

PROOF: Let $x(k)$ be a bounded solution of (6) and B its positive limit set. For each $p \in B$, there is a monotone sequence of integers $\{k_n\}$ such that $k_n \to \infty$ and $x(k_n) \to p$ as $n \to \infty$. Then each function $y_n(k) = x(k+k_n)$ is a solution of (6) with $y_n(0) \to p$ as $n \to \infty$. From continuity in the initial conditions, these functions approach the solution $x(k;p)$ as $n \to \infty$. By extending each function $y_n(k)$ to $-k_n$, we can extend the solution $x(k;p)$ to $-\infty$. The simultaneous convergence to $x(k;p)$ and B implies that $x(k;p) \in B$ for all k, and so B is an invariant set. The fact that B is nonempty and compact is obtained from the definition of a positive limit set and the boundedness of $x(k)$.

For an autonomous equation, Theorem 1 can be strengthened as follows.

THEOREM 2. If there exists a Liapunov function $V(x)$ for (6) on some set G, then each solution $x(k)$ which remains in G is either unbounded or approaches some invariant set contained in A as $k \to \infty$.

PROOF. From Theorem 1, $x(k) \to A \cup \{\infty\}$ as $k \to \infty$. If $x(k)$ is unbounded, then Lemma 1 does not hold. If $x(k)$ is bounded, then its positive limit set is an invariant set.

If the set M is defined as the union of all the invariant sets contained in A, then $x(k) \to M$ as $k \to \infty$ whenever $x(k)$ remains in G and is bounded. The set M may be considerably smaller than the set A. Under the conditions of Theorem 2, an unbounded solution can exist only if G is unbounded and there is a sequence $\{x_n\}$, $x_n \in G$, $x_n \to \infty$ and $\Delta V(x_n) \to 0$ as $n \to \infty$.

Corollary 1 can be restated in a similar manner.

COROLLARY 2. If, in Theorem 2, the set G is of the form

$$G = G(\eta) = \{x : V(x) < \eta\}$$

for some $\eta > 0$, then all solutions which start in G remain in G and approach M as $k \to \infty$.

This corollary can be used to obtain regions of convergence for various iterative methods which can be described by an autonomous difference equation. A <u>region of convergence</u> is a set $G \subset X$ such that, if $x(0) \in G$, then $x(k) \in G$ for all $k \geqq 0$ and $x(k)$ approaches the desired vector as $k \to \infty$. The largest region of convergence is the union of all regions of convergence. The Secant and Newton-Raphson methods are treated as examples. For a derivation and discussion of these methods see, for example, Traub [1] or Ostrowski [1].

The Secant method for finding a root of $f(z) = 0$ ($f(z)$ and z are complex numbers) is given by assuming values for z_1 and z_2, then forming the sequence $\{z_k\}$ by repeated application of equation (7).

$$z_{k+2} = z_{k+1} - \frac{(z_{k+1}-z_k)f(z_{k+1})}{f(z_{k+1})-f(z_k)} \tag{7}$$

We assume that, for every k, $z_{k+1} \neq z_k$ and $f(z_{k+1}) \neq f(z_k)$, so this iteration formula is well defined for all k. Let α be the desired root of $f(z) = 0$ and let

$$f(\alpha + e) = f'(\alpha)e + g(\alpha, e)e^2 .$$

Then, letting $z_k = \alpha + e_k$ for each k, equation (7) becomes

$$e_{k+2} = M(\alpha, e_k, e_{k+1})e_k e_{k+1}$$

where

$$M(\alpha, e_k, e_{k+1}) = \frac{g(\alpha, e_{k+1})e_{k+1} - g(\alpha, e_k)e_k}{f(\alpha+e_{k+1}) - f(\alpha+e_k)} \quad .$$

With the assumption that α is a simple root of $f(z) = 0$ and $g(\alpha, e)$ is continuous and bounded in e, then $M(\alpha, e_k, e_{k+1})$ is continuous and bounded for e_k, e_{k+1} small enough.

The difference equation (8) is obtained by letting $x_1(k) = e_k$ and $x_2(k) = e_{k+1}$.

$$\begin{aligned} Ex_1 &= x_2 \\ Ex_2 &= M(\alpha, x_1, x_2)x_1x_2 \end{aligned} \tag{8}$$

Consider the Liapunov function $V_q(x_1, x_2) = |x_1|^q + |x_2|^q$ for some $q \geqq 1$. Then

$$\Delta V_q(x_1, x_2) = -(1 - |M(\alpha, x_1, x_2)x_2|^q)|x_1|^q$$

and $\Delta V_q(x_1, x_2) \leqq 0$ if $|M(\alpha, x_1, x_2)x_2| \leqq 1$. Let $G_q(\eta)$ be the set $G_q(\eta) = \{(x_1, x_2) : (|x_1|^q + |x_2|^q)^{1/q} < \eta\}$. Since $x_2 = 0$ implies $|M(\alpha, x_1, x_2)x_2| = 0 < 1$, there is some $\eta > 0$ such that $|M(\alpha, x_1, x_2)x_2| \leqq 1$ for all (x_1, x_2) in $G_q(\eta)$. From Corollary 2, this $G_q(\eta)$ is a region of convergence for the Secant method. If the initial guesses z_1 and z_2 are such that $(x_1, x_2) \in G_q(\eta)$ for some q, then (x_1, x_2) will remain in $G_q(\eta)$ for all k and approach an invariant set contained in the set

$A = \{(x_1,x_2) \in \overline{G}_q(\eta) : x_1 = 0\}$. The only invariant set of equation (8) with $x_1 = 0$ is the origin $x_1 = x_2 = 0$, so we get $(x_1,x_2) \to (0,0)$ as $k \to \infty$, and the method converges.

If, for $|e| \leqq \eta_o$, we get

$$|f'(\alpha+e)| \geqq F , \quad |g(\alpha,e)| \leqq G$$

then we get that $|M(\alpha,x_1,x_2)x_2| < 1$ if $|x_2| < F/G$. Thus, η can be taken as the smaller of η_o and F/G. For the particular equation $f(z) = z^2-\alpha^2$, we get $|f'(\alpha+e)| \geqq 2|\alpha|-2\eta_o$ and $|g(\alpha,e)| = 1$ for $|e| < \eta_o$. In this case, we can choose $\eta = \eta_o = \frac{2}{3}|\alpha|$.

It should be noted that the set $G_q(\eta)$, or even the union of these sets for all $q \geqq 1$, is not always the largest region of convergence. For the simple equation $f(z) = z^2-\alpha^2$, almost any choice of z_1, z_2, provided only that $z_1 \neq z_2$ and $f(z_1) \neq f(z_2)$, will lead to a sequence which will converge either to $+\alpha$ or to $-\alpha$. However, if z_1 and z_2 are in the region defined by $G_q(\eta)$, then not only will the sequence converge to α but this convergence will be uniform in the sense that $|z_k-\alpha|^q + |z_{k+1}-\alpha|^q$ will be a decreasing function of k.

Corollary 2 can also be used to find a region of convergence for the Newton-Raphson method. The Newton-Raphson method for finding a root of $f(z) = 0$ ($f(z)$ and z are n-vectors) is given by assuming a value for z_1, then forming the sequence $\{z_k\}$

by repeated application of equation (9).

$$z_{k+1} = z_k - [\frac{\partial f}{\partial z}(z_k)]^{-1} f(z_k) \tag{9}$$

where $\frac{\partial f}{\partial z}(z_k)$ is the matrix of partial derivatives of f. Here, we assume that $[\frac{\partial f}{\partial z}(z_k)]$ always has an inverse. If the desired root is a simple root, then this is the case. By letting α be the desired root, expanding $f(\alpha+e)$ as

$$f(\alpha+e) = [\frac{\partial f}{\partial z}(\alpha)]e + f_o(e)$$

and letting $z_k = \alpha+e_k$, then the difference equation becomes

$$e_{k+1} = +M_1(e_k)[M_2(e_k)e_k - f_o(e_k)]$$

where $M_1(e) = [\frac{\partial f}{\partial z}(\alpha+e)]^{-1}$ and $M_2(e) = [\frac{\partial f}{\partial z}(\alpha+e) - \frac{\partial f}{\partial z}(\alpha)]$. Let $|e|$ be some vector norm (see the Appendix). If α is a simple root of $f(z) = 0$ and f is twice continuously differentiable at $z = \alpha$, then, for each $\eta > 0$, there exists a positive constant $k(\eta)$ such that, for all e with $|e| < \eta$, we have $|M_1(e)(M_2(e)e-f_o(e))| \leq k(\eta)|e|^2$. Then, letting $V(e) = |e|$, we get

$$\Delta V(e) \leq -(1-k(\eta)|e|)|e|$$

and $\Delta V(e) \leqq 0$ if $k(\eta)|e| \leqq 1$. Using Corollary 2, we get a region of convergence $G(\eta_o) = \{z : |z-\alpha| < \eta_o\}$ where $\eta_o = \min(\eta, \frac{1}{k(\eta)})$. We can choose η so as to maximize η_o, thus obtaining the best region of convergence obtainable with this Liapunov function.

For the case where z and $f(z)$ are complex numbers, if there is some $F > 0$ such that $|f_o(e)| \leqq F|e|^2$ for all z where $|z-\alpha| \leqq \eta$, some $\eta \geqq \frac{2}{5}\frac{|f'(\alpha)|}{F}$, then we can get

$$k(\eta) = \frac{3}{2}\frac{F}{|f'(\alpha)| - F\eta}$$

and the best (with this $k(\eta)$) region of convergence is given by $G(\eta_o)$ where

$$\eta_o = \frac{2}{5}\frac{|f'(\alpha)|}{F}$$

For the simple case $f(z) = z^2-\alpha^2$, we get $\eta_o = \frac{2}{5}|\alpha|$. However, a sharper estimate may be used for $k(\eta)$ which results in $\eta_o = \frac{2}{3}|\alpha|$. This latter case is the best possible. Any disc centered at α with radius larger than $\frac{2}{3}|\alpha|$ will have points inside the disc which will map outside the disc on the next iteration and $\Delta V(x)$ is positive for some values of x.

It should be noted that the region of convergence $G(\eta_o)$ is not always the largest region of convergence. For the simple equation $f(z) = z^2-\alpha^2$, any initial guess $z_1 \neq 0$ will lead to a sequence $\{z_k\}$ which will converge either to $+\alpha$ or to $-\alpha$.

IV. PERIODIC DIFFERENCE EQUATIONS. If, in the difference equation (1), $f(k,x)$ is T-periodic for some integer $T \geqq 1$ and fixed x, i.e., $f(k+T,x) = f(k,x)$ for all k,x, then the difference equation is said to be a T-periodic difference equation. A function $x^*(k)$ is said to be a solution for (1) on $(-\infty,\infty)$ if, for any k_o in $(-\infty,\infty)$, we have for all $k \geqq k_o$

$$x(k;k_o,x^*(k_o)) = x^*(k)$$

A set B is an invariant set of (1) if $x_o \in B$ implies that there is a k_o and a solution $x^*(k)$ for (1) on $(-\infty,\infty)$ such that $x^*(k_o) = x_o$ and $x^*(k) \in B$ for all k.

LEMMA 2. Let $x(k)$ be a solution of (1) that is bounded for all $k \geqq k_o$. Then the positive limit set of $x(k)$ is an invariant set of (1).

PROOF: This lemma is proven in a manner very similar to that used in Lemma 1. The k_o used in the definition of an invariant set is obtained in the following manner. If $\{k_n\}$ is a monotone sequence such that $x(k_n) \to p \in B$, the positive limit set of $x(k)$, then there is a sequence of integers $\{M_n\}$ such that $k_n-M_nT \in [0,T)$ for all n. The set $[0,T)$

consists of a finite number of integers, so at least one of these integers, k_o, must satisfy $k_o = k_n - M_n T$ for an infinite number of n's. The solution $x(k;k_o,p)$ is the shown to be the limit of the functions $y_n(k) = x(k+k_n)$ and is in B for all k, thus demonstrating that B is an invariant set of (1).

Theorem 1 can now be restated for T-periodic difference equations.

THEOREM 3. Let $V(k,x)$ be a T-periodic, continuous function which is bounded below for all x in some set G. For $k \geqq k_o$ and x in G, let $\Delta V(k,x) \leqq 0$ and define the set A by $A = \{(k,x) : \Delta V(k,x) = 0, x \in \overline{G}\}$. Let M be the union of all solutions $x(k)$ of (1) such that $(k,x(k)) \in A$ for all k. Then each solution of (1) which remains bounded and in G for all $k \geqq k_o$ approaches some invariant set contained in M as $k \to \infty$.

PROOF. The function $V(k,x(k))$ is non-increasing and bounded below, hence $\Delta V(k,x(k)) \to 0$ as $k \to \infty$. The continuity of V and ΔV implies that $d((k,x(k)),A) \to 0$ as $k \to \infty$. Since $x(k)$ must approach an invariant set as $k \to \infty$, it must approach M as $k \to \infty$.

An unbounded solution is possible under the conditions of Theorem 3 only if G is unbounded and there exists a sequence $\{(k_n,x_n)\}$ with $|x_n| \to \infty$, and $\Delta V(k_n,x_n) \to 0$ as $n \to \infty$. If G is bounded or if $\Delta V(k,x)$ is bounded away from zero for all sufficiently large x, then all solutions of (1) which remain in G are bounded and approach M as $k \to \infty$.

V. ASYMPTOTICALLY ATUONOMOUS DIFFERENCE EQUATIONS. If the difference equation (1) can be written in the form of equation (10)

$$Ex = H(x) + F(k,x) \tag{10}$$

where $F(k,x) \to 0$ as $k \to \infty$ uniformly for all x in any compact set, the difference equation is said to be an asymptotically autonomous difference equation. With each asymptotically autonomous difference equation, (10), there is the associated autonomous difference equation (11).

$$Ex = H(x) \tag{11}$$

LEMMA 3. The positive limit set of any bounded solution of the asymptotically autonomous difference equation (10) is an invariant set of the autonomous difference equation (11).

This lemma is proven in the same manner as Lemma 1. Theorem 1 could now be restated in a manner similar to Theorem 2, but the following, more general statement has proven more useful in its applications.

THEOREM 4. If a solution $x(k)$ of the difference equation (1) approaches a closed, bounded set A as $k \to \infty$, and if $x(k)$ is also a solution of the asymptotically autonomous difference equation (10), then it approaches the largest invariant set of (11) contained in A as $k \to \infty$.

As an example of the application of Theorem 4, consider the difference equation (12),

$$\begin{aligned} Ex &= cx - s(1-p(k))y \\ Ey &= sx + c(1-p(k))y \end{aligned} \qquad (12)$$

where $c = \cos \omega$, $s = \sin \omega$, $0 < \omega < 2\pi$, $0 < \delta \leqq p(k) \leqq 2-\epsilon < 2$.

With the Liapunov function $V(x,y) = x^2+y^2$, we get

$$\Delta V(x,y) = -p(k)(2-p(k))y^2 \leqq -\delta\, \epsilon\, y^2 \leqq 0$$

Applying Corollary 1, we get that all solutions for (12) are bounded and $y(k) \to 0$ as $k \to \infty$.

Let $x_1(k)$, $y_1(k)$ be a solution for (12), then $y_1(k)$ is bounded and approaches 0 as $k \to \infty$. Also, $x_1(k)$, $y_1(k)$ is a solution of the difference equation (13).

$$\begin{aligned} Ex &= cx - sy + p(k)y_1(k) \\ Ey &= sx + cy - p(k)y_1(k) \end{aligned} \qquad (13)$$

This difference equation is asymptotically autonomous to the difference equation (14).

$$\begin{aligned} Ex &= cx - sy \\ Ey &= sx + cy \end{aligned} \tag{14}$$

The only invariant set of (14) with $y = 0$ is the origin $x=y=0$ since $0 < \omega < 2\pi$. By Theorem 4, all solutions of (12) approach this invariant set, the origin, as $k \to \infty$.

VI. PRACTICAL STABILITY. For many difference equations a solution is considered a stable solution if it enters and remains in a sufficiently small set. For example, under the proper conditions all solutions of the Newton-Raphson equation (9) approach the desired solution as $k \to \infty$. But, when the effects of round-off errors are considered this is no longer the case. However, if all the solutions become and remain close to the desired solution, then the method is judged to be satisfactory. This type of stability is called practical stability. The following theorem and corollaries are concerned with practical stability for the difference equation (15).

$$Ex = f(k, x) \tag{15}$$

THEOREM 5. Given a set $G \subset X$, possibly unbounded. Let $V(x)$ and $W(x)$ be continuous, real valued functions defined on G

and such that, for all k and all x in G,

$$\text{(i)} \quad V(x) \geqq 0$$

$$\text{(ii)} \quad \Delta V(k,x) = V(f(k,x)) - V(x) \leqq W(x) \leqq a$$

for some constant $a \geqq 0$. Let the set S be the set

$$S = \{x \in \overline{G} : W(x) \geqq 0\}$$

Let $b = \sup \{V(x) : x \in S\}$ and the set A be the set

$$A = \{x \in \overline{G} : V(x) \leqq b + a\}$$

Then any solution $x(k)$ which remains in G and enters A when $k = k_1$ remains in A for all $k \geqq k_1$.

The properties of S, A, and $V(x)$ are used to show that, if $x(k)$ is in A, then $x(k+1)$ is in A. The theorem follows by induction.

COROLLARY 3. If $\delta = \sup \{-W(x) : x \in G-A\} > 0$, then each solution $x(k)$ of (15) which remains in G enters A in a finite number of steps.

If $x(k)$ does not enter A in a finite number of steps, then

$$V(x(k)) = V(x(k_0)) + \sum_{n=k_0}^{k-1} \Delta V(k,x(n))$$
$$\leqq V(x(k_0)) - (k-k_0)\delta$$

and $V(x(k)) \to -\infty$ as $k \to \infty$, a contradiction since $V(x) \geq b+a$ for all x in $G-A$.

COROLLARY 4. If G is of the form $G = G(\eta) = \{x : V(x) < \eta\}$ and the conditons of Theorem 5 and Corollary 3 are satisfied, then all solutions which start in G remain in G and enter A in a finite number of steps.

Corollary 4 can be used to study the effects of round-off errors in the Newton-Raphson method. Without errors, the Newton-Raphson method is given by equation (9). With errors, this method is given by equation (16).

$$z_{k+1} = z_k - [\frac{\partial f}{\partial z}(z_k)]^{-1} f(z_k) + h(k,z_k) \qquad (16)$$

where all that is known about the error term $h(k,z_k)$ is its upper bound, say $|h(k,z_k)| \leq \epsilon$ for some vector norm and some $\epsilon > 0$. A value for ϵ can be obtained by assuming that z_k is known exactly and studying the steps of the computations in great detail to estimate the error in z_{k+1}. This error term includes the effects of errors in the functions $f(z)$ and $\frac{\partial f}{\partial z}(z)$, errors in evaluating $f(z)$ and $[\frac{\partial f}{\partial z}(z)]^{-1}$, and any other errors that may be encountered. Often it is not very difficult to find an estimate for ϵ, the problem is to determine the net effect of the term $h(k,z)$ on the positive limit set of a solution $z(k)$.

With the same assumptions on $f(z)$ and the same expansions used before, the difference equation (16) becomes

$$e_{k+1} = M_1(e_k)[M_2(e_k)e_k - f_o(e_k)] + h_1(k, e_k)$$

With $V(e) = |e|$, we get

$$\Delta V(e) \leq -(1-k(\eta)|e|)|e| + \epsilon = + W(e) \leq \epsilon$$

The set S becomes

$$S = \{e : W(e) \geq 0\} = \{e : |e| \leq b\}$$

where

$$b(\eta) = b = \frac{1 - \sqrt{1-4k(\eta)\epsilon}}{2k(\eta)} = \epsilon + 2k(\eta)\epsilon^2 + \ldots$$

provided that $4k(\eta)\epsilon < 1$. If $4k(\eta)\epsilon \geq 1$, then $W(e) \geq 0$ everywhere and the iterations may not converge. The set A is defined by

$$A = \{e : V(e) \leq b + \epsilon\} = \{e : |e| \leq b + \epsilon\}$$

We note that, for η small enough, we have

$$W(e) \leq -(b - k(\eta)(b+\epsilon)^2) = -\delta .$$

From Corollary 4, we have, if $\delta > 0$, then all solutions which start in $G(\eta_o)$ remain in $G(\eta_o)$, enter A in a finite number of interations, and remain in A thereafter. Here, η_o is not quite the same as before. We must choose η_o such that $k(\eta_o)\eta_o \leqq 1$, $4k(\eta_o)\epsilon < 1$, and $b(\eta_o)-k(\eta_o)(b(\eta_o)+\epsilon)^2 > 0$. If η_1 is the smallest positive solution of $\eta_1 k(\eta_1) = 1$, then choosing $\eta_o < \eta_1$ will satisfy both $k(\eta_o)\eta_o < 1$ and $b-k(\eta_o)(b+\epsilon)^2 > 0$. The condition $4k(\eta_o)\epsilon < 1$ becomes a condition on the precision or accuracy required in the computations.

Thus one effect of round-off errors is to reduce the region of convergence. Another effect of round-off errors is that the error of each z_k cannot generally be reduced much below the value $b+\epsilon = 2\epsilon + 2k(\eta)\epsilon^2 + \ldots$ no matter how many iterations are preformed. The value $b+\epsilon$ is called the ultimate accuracy obtainable with round-off errors. Notice that, for small ϵ, the ultimate accuracy is approximately 2ϵ, or about twice the round-off errors committed at each step.

If the ultimate accuracy is large, then the method is judged to be a poor since the effect of small round-off errors is a large error in the computed solution. If the ultimate accuracy is small, then the method is judged to be a good one since small round-off errors have a small effect on the computed solution. In this sense, the Newton-Raphson method is judged to be a good method.

For a nonsingular matrix A, many iteration methods for solving $Ax = b$ for the vector x are described by the difference equation (17)

$$x_{k+1} = Bx_k + c \tag{17}$$

where the matrix B and the vector c are determined in some fashion by A and b. For example, if $A = Q + R$, then $B = -Q^{-1}R$ and $c = Q^{-1}b$ would be a possibility. B and c must have the property that $x_o = Bx_o + c$ if and only if $Ax_o = b$. The iterations x_k will converge to the solution x_o if and only if $\rho(B)$, the spectral radius of B, is less than one. For a derivation of several of these methods, see, for example, Kunz [1] or Hildebrand [1]. Choose a vector norm $|x|$ such that $|B| = \lambda < 1$. Since $\rho(B) < 1$, this can always be done (see the Appendix).

Let x_o be the desired solution and let $x_k = x_o + e_k$. Then the e_k satisfy the difference equation

$$e_{k+1} = Be_k + h(k, e_k)$$

where the term $h(k, e_k)$ represents the round-off errors commited at step k. We assume that there exists positive constants η and ϵ such that $|h(k,e)| \leq \epsilon$ for all k and all e, $|e| < \eta$.

Try the Liapunov function $V(e) = |e|$. Then

$$\Delta V(e) \leqq -(1-\lambda)|e| + \epsilon = W(e) \leqq \epsilon$$

Then the set S is given by

$$S = \{e : W(e) \geqq 0\} = \{e : |e| \leqq \frac{\epsilon}{1-\lambda}\}$$

and $b = \frac{\epsilon}{1-\lambda}$. The set A is given by

$$A = \{e : V(e) \leqq b + \epsilon\} = \{e : |e| \leqq \frac{2-\lambda}{1-\lambda}\epsilon\}.$$

If $\eta > \frac{2-\lambda}{1-\lambda}\epsilon$, then we can choose

$$G = \{e : V(e) < \eta\} = \{e : |e| < \eta\}$$

and Corollary 4 holds. Thus, if e_1 is in the set G, then the solution will remain in G, will enter A after a finite number of iterations, and will remain in A for all following iterations.

By looking at the set A, we see that the ultimate accuracy is given by $b + \epsilon$.

$$b + \epsilon = \frac{2-\lambda}{1-\lambda}\epsilon$$

We note that, if λ is very nearly one, then this ultimate accuracy may be large even if ϵ is small. For example, if $\lambda = 1-\alpha$, then

$b + \epsilon = (\alpha^{-1} + 1)\epsilon \geqq \epsilon/\alpha$, and ϵ/α may be large. This indicates that these iteration methods will give acceptable results only if $\lambda = |B|$ is considerably less than one.

APPENDIX -- A THEOREM ON MATRIX NORMS. Let x be an n-vector and x* its complex-conjugate transpose. Given some positive definite matrix B, let the norm of x, $|x|$, be defined by

$$|x|^2 = x^*Bx \qquad (A1)$$

Other vector norms are possible, but vector norms of this type are all that are considered here.

Given a matrix A, the matrix norm of A, $|A|$, can be defined in terms of the vector norm by

$$|A| = \min \{b : |Ax| \leqq b|x| \quad \text{for all} \quad x \neq 0\} \qquad (A2)$$

In addition to the usual properties of a norm, this matrix norm satisfies the following.

a) $|Ax| \leqq |A|\ |x|$

b) $|\lambda| \leqq |A|$ for any eigenvalue λ of A.

c) $\rho(A) \leqq |A|$

where $\rho(A)$, the spectral radius of A, is the absolute value of the largest eigenvalue of A.

THEOREM: Let A_o be a matrix with spectral radius $a_o = \rho(A_o)$. For each $a > a_o$, there exists a vector norm such that

$$a_o \leqq |A_o| \leqq a \tag{A3}$$

PROOF: This theorem is proven by considering the equation

$$|A_o x|^2 - a^2 |x|^2 = x^*(A_o^* B A_o - a^2 B)x = -a^2 x^* C x \leqq 0$$

where C is some positive definite matrix and

$$A_o^* B A_o - a^2 B = -a^2 C \tag{A4}$$

For any positive definite matrix C, let B be the positive definite matrix defined by

$$B = \sum_{k=0}^{\infty} a^{-2k} A_o^{*k} C A_o^k \tag{A5}$$

Since $a > a_o = \rho(A_o)$, this sum converges absolutely and B is perfectly well defined. Furthermore, this B satisfies equation (A4) and can be used to define a vector norm as in (A1). With this norm, we get

$$|A_o x|^2 - a^2|x|^2 \leqq 0$$

or

$$|A_o x| \leqq a|x|$$

From the definition of the matrix norm given in (A2), we get $|A_o| \leqq a$. The other half of the inequality (A3) is a basic property of matrix norms.

The significance of this theorem is that a vector norm can be chosen so that the matrix norm of a matrix is made as close to the spectral radius of the matrix as desired. If $a_o < 1$, then letting $a = \frac{1}{2}(1+a_o) < 1$ leads immediately to the following corollary.

COROLLARY. A necessary and sufficient condition for the spectral radius a_o of a matrix A_o to be less than one is that there exist a vector norm such that the matrix norm of A_o satisfies $|A_o| < 1$.

It should be emphasized that the vector norm in the theorem and corollary depends quite heavily on the matrix under consideration. Given two different matrices A_1 and A_2 both with spectral radii less than one, there may not exist one vector norm so that both $|A_1| < 1$ and $|A_2| < 1$.

While the vector norm used satisfies all the requirements of a vector norm, it may be an "acceptable" norm. For example, the "unit sphere" $S = \{x : |x| = 1\}$ is an ellipsoid and the ratio of the longest axis to the shortest axis may be very high. Equation (A5) almost never can be used to compute the matrix B and resort must be made to solving (A4) directly for B. This may be a difficult task and it may be impossible to compute B to any desired degree of accuracy. This means that it may be very difficult to compute this norm of a vector.

This theorem and corollary are easily extended to cover continuous linear operators in a Hilbert space.

Acknowledgments

The author wishes to thank Professors J.K. Hale , J.P.LaSalle, and M.A.Feldstein of Brown University for their encouragement, advice, and criticism in the preparation of this article. Also, the author is grateful for the financial support rendered to him by the National Aeronautics and Space Administration under their Traineeship program.

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