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SUMMARY

The resonant modes of a dielectric sphere immersed in a warm collisional plasma are determined for various electron densities, electron collision frequencies, and electron temperatures. It is found that variations in the electron density, collision frequency, and temperature cause sufficiently large changes in the resonant frequencies and damping factors of these modes so that a dielectric sphere may be useful as a plasma diagnostic tool.

INTRODUCTION

In the many fields where plasmas are used, it has become increasingly important to determine accurately the densities, temperatures, and collision rates of the various species of particles that constitute the plasma. High-frequency electromagnetic waves, or microwaves, have been found to be particularly useful as diagnostic tools for determining the properties of the electrons.

Wait (refs. 1 and 2) has recently pointed out that a spherical region in a plasma that is devoid of charged particles may be useful as a plasma diagnostic tool. This spherical region acts as an electromagnetic resonator with its resonant frequencies and damping factors being functions of the plasma parameters. Such a region can be achieved in practice by immersing a dielectric sphere, either solid or hollow, in the plasma.

This report presents an analysis of the resonant modes in a solid dielectric sphere that is immersed in a warm collisional plasma. The resonant frequencies and damping factors of several modes are computed for various electron densities, electron temperatures, and electron collision frequencies to determine the feasibility and limitations of using this technique as a plasma diagnostic tool. The associated problem of exciting the electromagnetic fields in the dielectric sphere is not discussed herein.

BASIC EQUATIONS

The dynamics of the plasma will be described by using the linearized hydrodynamic and Maxwell equations for a single fluid model of a warm plasma medium (refs. 1 to 3). These equations are

$$i\omega p + m_e n_o u^2 \nabla \cdot \bar{v} = 0 \tag{1}$$

$$i\omega m_e n_o \bar{v} + \nu m_e n_o \bar{v} = n_o q \bar{e} - \nabla p \tag{2}$$

$$\nabla x \bar{h} = n_o q \bar{v} + i\omega \epsilon_o \bar{e} \tag{3}$$

$$\nabla x \bar{e} = -i\omega \mu_o \bar{h} \tag{4}$$

where

$\bar{e}$ electric field intensity

$\bar{h}$ magnetic field intensity

$\bar{v}$ electron fluid velocity

p perturbation electron pressure

m_e mass of electron

q electronic charge

u acoustic velocity in electron gas

n_o equilibrium electron number density

ϵ_o electric permittivity of free space

μ_o magnetic permeability of free space

ω angular frequency

ν effective electron collision frequency for momentum transfer

It has been assumed that all nonstationary quantities vary with time as $e^{i\omega t}$ and that ω is sufficiently large so that only the electron motion has to be considered.

Equation (1) is a combination of the continuity and energy conservation equations for the electron gas. For the case where the electron gas obeys the equation of state for an ideal gas, the acoustic velocity u is given by

$$u = \left(\frac{\gamma k T_e}{m_e}\right)^{1/2} \tag{5}$$

where

k Boltzmann constant

T_e electron temperature

γ ratio of specific heats with constant pressure to constant volume of electron gas

Equation (2) is a statement of the conservation of momentum for the electrons, and equations (3) and (4) are Maxwell equations where the electric current density has been set equal to the electron convection current $n_o q\bar{v}$.

The basic equations can be solved more easily if they are expressed in the following form:

$$\nabla \times \nabla \times \bar{h} - k_e^2 \bar{h} = 0 \tag{6}$$

$$\nabla^2 p + k_p^2 p = 0 \tag{7}$$

$$\bar{e} = \frac{1}{i\omega\epsilon_o\epsilon_p} \nabla \times \bar{h} - \frac{q}{\omega^2 \epsilon_o \epsilon_p m_e \left(1 - i\frac{\nu}{\omega}\right)} \nabla p \tag{8}$$

$$\bar{v} = -\frac{q}{\omega^2 \epsilon_o \epsilon_p m_e \left(1 - i\frac{\nu}{\omega}\right)} \nabla \times \bar{h} + \frac{i}{\omega n_o \epsilon_p m_e \left(1 - i\frac{\nu}{\omega}\right)} \nabla p \tag{9}$$

where

$$\epsilon_p = 1 - \frac{\omega_p^2}{\omega^2 \left(1 - i\frac{\nu}{\omega}\right)}$$

is the effective relative dielectric constant of the plasma

$$\omega_p^2 = \frac{n_o q^2}{\epsilon_o m_e}$$

$$k_e^2 = k_o^2 \epsilon_p$$

$$k_p^2 = \frac{\omega^2}{u^2}\left(1 - i\frac{\nu}{\omega}\right)\epsilon_p$$

$$k_o = \frac{\omega}{c}$$

where

c velocity of light

Equations (6) to (9) show that the fields in a warm plasma medium can be determined by first solving for the magnetic and pressure fields, $\bar{h}$ and p, from equations (6) and (7), and then computing the electric and velocity fields, $\bar{e}$ and $\bar{v}$, from equations (8) and (9). Equations (6) to (9), of course, apply in a dielectric medium if $\bar{v}$ and p are set equal to zero and if ϵ_p is set equal to the relative dielectric constant of the medium. Thus, in a dielectric

$$\nabla \times \nabla \times \bar{h} - Kk_o^2\bar{h} = 0 \tag{10}$$

$$\bar{e} = \frac{1}{i\omega\epsilon_o K} \nabla \times \bar{h} \tag{11}$$

where K is the relative dielectric constant of the medium.

MODEL AND BOUNDARY CONDITIONS

The model to be considered, shown in figure 1, consists of a solid dielectric sphere, of radius a and relative dielectric constant K, that is immersed in an infinite

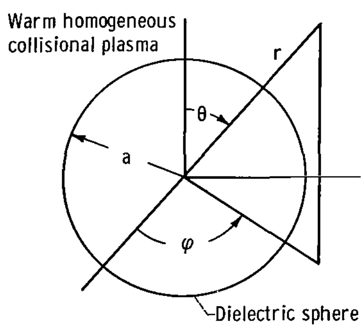


Figure 1. - Model of dielectric sphere.

homogeneous warm plasma. It will be assumed that the dielectric material has no electrical losses.

Before a solution for the field can proceed, boundary conditions must be specified for the field components at the plasma-dielectric interface $r = a$ and at infinity. At the plasma-dielectric interface, the components of the electric and magnetic fields that are tangent to the surface $r = a$ must be continuous. Thus,

$$\bar{e} \times \bar{a}_r \Big|_{r=a^-} = \bar{e} \times \bar{a}_r \Big|_{r=a^+} \tag{12}$$

$$\bar{h} \times \bar{a}_r \Big|_{r=a^-} = \bar{h} \times \bar{a}_r \Big|_{r=a^+} \tag{13}$$

where $\bar{a}_r$ is the unit vector in the radial direction. A boundary condition for the pressure or velocity fields must also be specified at $r = a$ to ensure that the fields will be unique (ref. 4). The boundary condition that will be used requires the normal component of the electron fluid velocity to vanish. Thus,

$$\bar{v} \cdot \bar{a}_r \Big|_{r=a^+} = 0 \tag{14}$$

This rigidity boundary condition is obviously an approximation. The validity of this condition has been the subject of much debate (refs. 5 to 7), since the basic equations (eqs. (1) to (4)) do not apply in the sheath region that exists at a plasma-dielectric interface and since, in many cases, this boundary condition overconstrains the solution in the limit of zero electron temperature. A more realistic approach is to approximate the sheath region with a thin layer of free space over the surface of the dielectric sphere. To simplify the analysis, however, this free-space layer is omitted. A discussion of its effect on the resonant frequencies is presented in the section NUMERICAL RESULTS. Since this analysis is concerned with resonance phenomena localized in the vicinity of the dielectric sphere, the proper boundary condition at infinity is to require the field to correspond to a divergent traveling or decaying wave as r approaches ∞.

It will be convenient in the following analysis to express all boundary conditions in terms of the magnetic and pressure fields. This can be done by combining equations (8), (9), and (11) with equations (12) to (14) to give

$$\frac{1}{K}\left[\frac{1}{\sin\theta}\frac{\partial h_r}{\partial\varphi} - \frac{\partial}{\partial r}(rh_\varphi)\right]\Bigg|_{r=a^-} = \frac{1}{\epsilon_p}\left[\frac{1}{\sin\theta}\frac{\partial h_r}{\partial\varphi} - \frac{\partial}{\partial r}(rh_\varphi)\right] - \frac{iq}{\omega\epsilon_p m_e\left(1 - i\frac{\nu}{\omega}\right)}\frac{\partial p}{\partial\theta}\Bigg|_{r=a^+} \tag{15}$$

$$\frac{1}{K}\left[\frac{\partial}{\partial r}(rh_\theta) - \frac{\partial h_r}{\partial \theta}\right]\Bigg|_{r=a^-} = \frac{1}{\epsilon_p}\left[\frac{\partial}{\partial r}(rh_\theta) - \frac{\partial h_r}{\partial \theta}\right] - \frac{iq}{\omega\epsilon_p m_e\left(1 - i\frac{\nu}{\omega}\right)}\frac{1}{\sin\theta}\frac{\partial p}{\partial \varphi}\Bigg|_{r=a^+} \tag{16}$$

$$h_\theta\Big|_{r=a^-} = h_\theta\Big|_{r=a^+} \tag{17}$$

$$h_\varphi\Big|_{r=a^-} = h_\varphi\Big|_{r=a^+} \tag{18}$$

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(h_\varphi \sin\theta) - \frac{\partial h_\theta}{\partial\varphi}\right] - \frac{i\omega\epsilon_o a}{n_o q}\frac{\partial p}{\partial r}\Bigg|_{r=a^+} = 0 \tag{19}$$

FORMAL SOLUTION FOR RESONANCE EQUATIONS

The general solution of equations (6) and (10) for the magnetic field can be shown (ref. 8) to consist of linear combinations of $\nabla \times (\bar{a}_r r\psi)$ and $\nabla \times \nabla \times (\bar{a}_r r\psi)$, where the generating function ψ satisfies the equations

$$\nabla^2\psi + Kk_o^2\psi = 0 \qquad r < a \tag{20}$$

$$\nabla^2\psi + k_e^2\psi = 0 \qquad r > a \tag{21}$$

Solutions of equations (20) and (21) that are appropriate for the problem under consideration are linear combinations of the eigenfunctions ψ_{nm} where

$$\psi_{nm} = e^{im\varphi}P_n^m(\cos\theta)j_n\left(\sqrt{K}\,k_o r\right) \qquad r < a \tag{22}$$

$$\psi_{nm} = e^{im\varphi}P_n^m(\cos\theta)k_n(ik_e r) \qquad r > a \tag{23}$$

The functions $P_n^m(\cos\theta)$ are the associated Legendre polynomials, and the $j_n\left(\sqrt{K}\,k_o r\right)$ and $k_n(ik_e r)$ are the spherical and modified spherical Bessel functions, respectively.

The spherical Bessel functions can be constructed by using the following recurrence equations:

$$j_o(x) = \frac{1}{x} \sin x$$

$$j_1(x) = \frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right)$$

$$\vdots \qquad \vdots$$

$$j_n(x) = \frac{2n - 1}{x} j_{n-1}(x) - j_{n-2}(x)$$

and

$$k_o(x) = \frac{\pi}{2x} e^{-x}$$

$$k_1(x) = \frac{\pi}{2x}\left(1 + \frac{1}{x}\right)e^{-x}$$

$$\vdots \qquad \vdots$$

$$k_n(x) = \frac{2n - 1}{x} k_{n-1}(x) + k_{n-2}(x)$$

The spherical Bessel functions of the first kind must be used in the solution for the region $r < a$ since the field must be bounded at the origin. The modified spherical Bessel functions must be used for $r > a$, where the branch of k_e, where $\mathrm{Re}\, k_e > 0$, must be selected to ensure that the field corresponds to a divergent wave, in order for the field to satisfy the proper boundary condition at infinity. If $\mathrm{Re}\, k_e = 0$, the branch where $\mathrm{Im} k_e < 0$ should be selected so that the field decays in amplitude as r approaches ∞.

The general solution for the magnetic field can now be constructed by forming linear combinations of the quantities $\nabla \times (\bar{a}_r r \psi_{nm})$ and $\nabla \times \nabla \times (\bar{a}_r r \psi_{nm})$ with the result

$$
\begin{aligned}
\bar{h} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Bigg(&\bar{a}_r \frac{B_{nm} n(n+1)}{\sqrt{K}\, k_o r} e^{im\varphi} P_n^m(\cos\theta) j_n\left(\sqrt{K}\, k_o r\right) \\
&+ \bar{a}_\theta \Bigg\{ A_{nm} im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} j_n\left(\sqrt{K}\, k_o r\right) \\
&+ \frac{B_{nm}}{\sqrt{K}\, k_o} e^{im\varphi} P_n'^m(\cos\theta) \left[j_n'\left(\sqrt{K}\, k_o r\right) + \frac{1}{r} j_n\left(\sqrt{K}\, k_o r\right) \right] \Bigg\} \\
&+ \bar{a}_\varphi \Bigg\{ -A_{nm} e^{im\varphi} P_n'^m(\cos\theta) j_n\left(\sqrt{K}\, k_o r\right) \\
&+ \frac{B_{nm}}{\sqrt{K}\, k_o} im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} \left[j_n'\left(\sqrt{K}\, k_o r\right) + \frac{1}{r} j_n\left(\sqrt{K}\, k_o r\right) \right] \Bigg\} \Bigg) \qquad r < a \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
\bar{h} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Bigg(&\bar{a}_r \frac{D_{nm}}{k_e r} n(n+1) e^{im\varphi} P_n^m(\cos\theta) k_n(ik_e r) \\
&+ \bar{a}_\theta \Bigg\{ C_{nm} im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} k_n(ik_e r) \\
&+ \frac{D_{nm}}{k_e} e^{im\varphi} P_n'^m(\cos\theta) \left[k_n'(ik_e r) + \frac{1}{r} k_n(ik_e r) \right] \Bigg\} \\
&+ \bar{a}_\varphi \Bigg\{ -C_{nm} e^{im\varphi} P_n'^m(\cos\theta) k_n(ik_e r) \\
&+ \frac{D_{nm}}{k_e} im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} \left[k_n'(ik_e r) + \frac{1}{r} k_n(ik_e r) \right] \Bigg\} \Bigg) \qquad r > a \qquad (25)
\end{aligned}
$$

where A_{nm}, B_{nm}, C_{nm}, and D_{nm} are constants to be determined, and the prime denotes differentiation with respect to the coordinate r or θ. The summations with respect to m in the previous equations need only be taken from -n to n since the associated Legendre polynomials $P_n^m(\cos\theta)$ are zero when $|m| > n$.

The general solution of equation (7) for the pressure is of the form

$$p = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_{nm} e^{im\varphi} P_n^m(\cos\theta) k_n(ik_p r) \qquad r > a \tag{26}$$

where the P_{nm} are constants to be determined. The branch of k_p where $\mathrm{Re}\, k_p > 0$, or $\mathrm{Im}\, k_p < 0$ if $\mathrm{Re}\, k_p = 0$, must be selected to ensure that the pressure satisfies the proper boundary condition at infinity. The pressure is, of course, zero in the region $r < a$.

Relations among the five sets of constants A_{nm}, B_{nm}, C_{nm}, D_{nm}, and P_{nm} can be determined by substituting the general solutions for the magnetic and pressure fields into the five boundary condition equations, equations (15) to (19). After a considerable amount of work, the five sets of equations can be expressed in the following forms:

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[(\mathscr{A}_{nm} - \mathscr{C}_{nm}) im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} + (\mathscr{B}'_{nm} - \mathscr{D}'_{nm}) e^{im\varphi} P_n'^m(\cos\theta) \right] = 0 \tag{27}$$

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[(\mathscr{B}'_{nm} - \mathscr{D}'_{nm}) im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} - (\mathscr{A}_{nm} - \mathscr{C}_{nm}) e^{im\varphi} P_n'^m(\cos\theta) \right] = 0 \tag{28}$$

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[(\mathscr{B}_{nm} - \mathscr{D}_{nm}) im e^{im\varphi} \frac{P_n^m(\cos\theta)}{\sin\theta} + (\mathscr{A}'_{nm} - \mathscr{C}'_{nm} + \mathscr{P}_{nm}) e^{im\varphi} P_n'^m(\cos\theta) \right] = 0 \tag{29}$$

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[(\mathscr{A}'_{nm} - \mathscr{C}'_{nm} + \mathscr{P}_{nm}) ime^{im\varphi}\frac{P_n^m(\cos\theta)}{\sin\theta} - (\mathscr{B}_{nm} - \mathscr{D}_{nm})e^{im\varphi}P_n'^m(\cos\theta)\right] = 0$$

(30)

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left\{\left[n(n+1)\mathscr{C}_{nm} - \mathscr{P}'_{nm}\right]e^{im\varphi}P_n^m(\cos\theta)\right\} = 0 \tag{31}$$

where

$$\mathscr{A}_{nm} = A_{nm}j_n\left(\sqrt{K}\,k_o a\right)$$

$$\mathscr{A}'_{nm} = A_{nm}\frac{1}{K}\left[j'_n\left(\sqrt{K}\,k_o r\right) + \frac{1}{a}\,j_n\left(\sqrt{K}\,k_o a\right)\right]\Bigg|_{r=a}$$

$$\mathscr{B}_{nm} = B_{nm}\frac{k_o}{\sqrt{K}}\,j_n\left(\sqrt{K}\,k_o a\right)$$

$$\mathscr{B}'_{nm} = B_{nm}\frac{1}{\sqrt{K}\,k_o}\left[j'_n\left(\sqrt{K}\,k_o r\right) + \frac{1}{a}\,j_n\left(\sqrt{K}\,k_o a\right)\right]\Bigg|_{r=a}$$

$$\mathscr{C}_{nm} = C_{nm}k_n(ik_e a)$$

$$\mathscr{C}'_{nm} = C_{nm}\frac{1}{\epsilon_p}\left[k'_n(ik_e r) + \frac{1}{a}\,k_n(ik_e a)\right]\Bigg|_{r=a}$$

$$\mathscr{D}_{nm} = D_{nm}\frac{k_e}{\epsilon_p}\,k_n(ik_e a)$$

$$\mathscr{D}'_{nm} = D_{nm}\frac{1}{k_e}\left[k'_n(ik_e r) + \frac{1}{a}\,k_n(ik_e a)\right]\Bigg|_{r=a}$$

$$\mathscr{P}_{nm} = \frac{iq}{\omega\epsilon_p m_e a\left(1 - i\frac{\nu}{\omega}\right)} P_{nm} k_n(ik_p a)$$

$$\mathscr{P}'_{nm} = \frac{i\omega\epsilon_o a}{n_o q} P_{nm} k'_n(ik_p r)\bigg|_{r=a}$$

Equations relating the quantities $\mathscr{A}_{nm}$, $\mathscr{A}'_{nm}$, $\mathscr{B}_{nm}$, etc. can be obtained by using the orthogonal properties of the exponential functions and the associated Legendre polynomials. Each term in equations (27) to (31) depends on the coordinate φ only through the factor $e^{im\varphi}$; therefore, these equations are also valid with the summation with respect to m eliminated because the function $e^{im\varphi}$ is orthogonal to $e^{il\varphi}$ if $l \neq m$.

Equations (27) and (28), with the summation over m and the factor $e^{im\varphi}$ removed, can now be combined. Multiplying each term in equation (27) by $imP_l^m(\cos\theta)/\sin\theta$ and each term in equation (28) by $P_l'^m(\cos\theta)$ and then adding yield

$$\sum_{n=0}^{\infty}\left((\mathscr{B}'_{nm} - \mathscr{D}'_{nm})\frac{im}{\sin\theta}\left[P_l^m(\cos\theta)P_n^m(\cos\theta)\right]' - (\mathscr{A}_{nm} - \mathscr{C}_{nm})\left\{P_n'^m(\cos\theta)P_l'^m(\cos\theta) + m^2\left[\frac{P_n^m(\cos\theta)P_l^m(\cos\theta)}{\sin^2\theta}\right]\right\}\right) = 0 \tag{32}$$

Next, each term in equation (32) is multiplied by $\sin\theta$ and integrated with respect to θ from 0 to π. Since

$$\int_0^{\pi}\left\{P_n'^m(\cos\theta)P_l'^m(\cos\theta) + m^2\left[\frac{P_n^m(\cos\theta)P_l^m(\cos\theta)}{\sin^2\theta}\right]\right\}\sin\theta\, d\theta$$

$$= \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!} \qquad \text{if } n = l$$

or

$$= 0 \qquad \text{if } n \neq l$$

and

$$\int_0^{\pi} m\left[P_n^m(\cos\theta)P_l^m(\cos\theta)\right]' d\theta = 0$$

the only way the integrated form of equation (32) can be satisfied is for $\mathscr{A}_{nm} = \mathscr{C}_{nm}$ for all values of n and m except for the special case where $n = m = 0$. If each term in equation (27) were multiplied by $P_l'^m(\cos\theta)$ and each term in equation (28) by $im\left[\frac{P_l^m(\cos\theta)}{\sin\theta}\right]$, the result would be $\mathscr{B}'_{nm} = \mathscr{D}'_{nm}$. Thus, equations (27) and (28) are satisfied only when $\mathscr{A}_{nm} = \mathscr{C}_{nm}$ and $\mathscr{B}'_{nm} = \mathscr{D}'_{nm}$. Since equations (29) and (30) are of the same form as equations (27) and (28), it follows that $\mathscr{A}'_{nm} = \mathscr{C}'_{nm} - \mathscr{P}_{nm}$ and $\mathscr{B}_{nm} = \mathscr{D}_{nm}$. Equation (31) simply requires that $\mathscr{P}'_{nm} = n(n+1)\mathscr{C}_{nm}$. Thus, the desired relations between the quantities $\mathscr{A}_{nm}$, $\mathscr{A}'_{nm}$, etc. are

$$\mathscr{A}_{nm} = \mathscr{C}_{nm} \tag{33}$$

$$\mathscr{B}'_{nm} = \mathscr{D}'_{nm} \tag{34}$$

$$\mathscr{A}'_{nm} = \mathscr{C}'_{nm} - \mathscr{P}_{nm} \tag{35}$$

$$\mathscr{B}_{nm} = \mathscr{D}_{nm} \tag{36}$$

$$\mathscr{P}'_{nm} = n(n+1)\mathscr{C}_{nm} \tag{37}$$

For the special case where $n = m = 0$, it can be shown that the quantities $\mathscr{A}_{nm}$, $\mathscr{A}'_{nm}$, etc., are completely arbitrary, except for $\mathscr{P}_{nm}$ and $\mathscr{P}'_{nm}$, which must be zero.

Comparing equations (33) to (37) with the definitions for the quantities $\mathscr{A}_{nm}$, $\mathscr{A}'_{nm}$, etc. reveals that the equations relating the constants A_{nm}, B_{nm}, C_{nm}, D_{nm}, and P_{nm} divide into two sets with one set involving only B_{nm} and D_{nm} and the second set involving only A_{nm}, C_{nm}, and P_{nm}. Combining equations (34) and (36) with the definitions for $\mathscr{B}_{nm}$, $\mathscr{B}'_{nm}$, $\mathscr{D}_{nm}$, and $\mathscr{D}'_{nm}$ reveals that B_{nm} and D_{nm} satisfy a set of homogeneous equations. Thus, if the solution for B_{nm} and D_{nm} is to be nontrivial, the determinant of the coefficients of these terms must vanish giving

$$\left.\frac{j_n'\left(\sqrt{K}\,k_o r\right) + \frac{1}{a}\,j_n\left(\sqrt{K}\,k_o a\right)}{j_n\left(\sqrt{K}\,k_o a\right)}\right|_{r=a} = \left.\frac{k_n'(ik_e r) + \frac{1}{a}\,k_n(ik_e a)}{k_n(ik_e a)}\right|_{r=a} \tag{38}$$

Similarly, using equations (33), (35), and (37) along with the definitions for $\mathscr{A}_{nm}$, $\mathscr{A}'_{nm}$, etc., reveals that A_{nm}, C_{nm}, and P_{nm} also satisfy a set of homogeneous equations. For a nontrivial solution, the determinant of the coefficients of these terms must also vanish giving

$$\left.\frac{k_n'(ik_e r) + \frac{1}{a}\,k_n(ik_e a)}{\epsilon_p k_n(ik_e a)}\right|_{r=a} = \left.\frac{j_n'\left(\sqrt{K}\,k_o r\right) + \frac{1}{a}\,j_n\left(\sqrt{K}\,k_o a\right)}{K j_n\left(\sqrt{K}\,k_o a\right)} + \frac{n(n+1)\omega_p^2}{\omega^2 \epsilon_p a^2\left(1 - i\,\frac{\nu}{\omega}\right)}\,\frac{k_n(ik_p a)}{k_n'(ik_p r)}\right|_{r=a} \tag{39}$$

Equations (38) and (39) may be appropriately called resonance equations since they uniquely determine the resonant frequencies or natural frequencies of the sphere after the plasma and sphere parameters (ω_p, u, ν, K, and a) have been specified. Before presenting the solutions of these equations, it is worthwhile to consider the possible spatial configurations of the electromagnetic field.

Solutions of equation (38) correspond to a field wherein $B_{nm} \neq 0$, $D_{nm} \neq 0$, and $A_{nm} = C_{nm} = P_{nm} = 0$. A study of equations (24) and (25) reveals that this field will have, in general, all three components of magnetic field (i.e., h_r, h_θ, and h_φ), but it can be shown by use of equations (8) and (11) that the electric field will have only θ and φ components. The pressure is zero since $P_{nm} = 0$. Thus, solutions of equation (38) generate a field that is transverse electric (TE) with respect to the the radial coordinate. Furthermore, it can be shown that solutions with $n = 0$ generate a null field and that solutions with $m = 0$ generate a field with only e_φ, h_r, and h_θ components. Finally, it should be noted that equation (38) does not contain a term that is dependent on the acoustic velocity, which in turn depends on the electron temperature. Thus, the resonant frequencies of the TE modes are independent of the electron temperature.

Solutions of equation (39) correspond to a field wherein $A_{nm} \neq 0$, $C_{nm} \neq 0$, $P_{nm} \neq 0$, and $B_{nm} = D_{nm} = 0$. It can be shown by use of equations (8), (11), (24), and (25) that these solutions correspond to a field that has, in general, all three components of the electric field (i.e., e_r, e_θ, and e_φ) but only the θ and φ components of the magnetic field. These solutions are transverse magnetic (TM) with respect to the radial coordinate. The solutions for $n = 0$ generate a null field, as in the TE case; and solutions with $m = 0$ generate a field with only e_r, e_θ, and h_φ components. The resonant frequencies of the TM modes are dependent on the electron temperature through the term k_p, which appears only in the last term in equation (39). As the electron temperature approaches zero, k_p approaches infinity, which makes the last term in equation (39) approach zero. Thus, the resonance equation for the TM modes in a cold plasma consists of the first two terms in equation (39).

NUMERICAL RESULTS

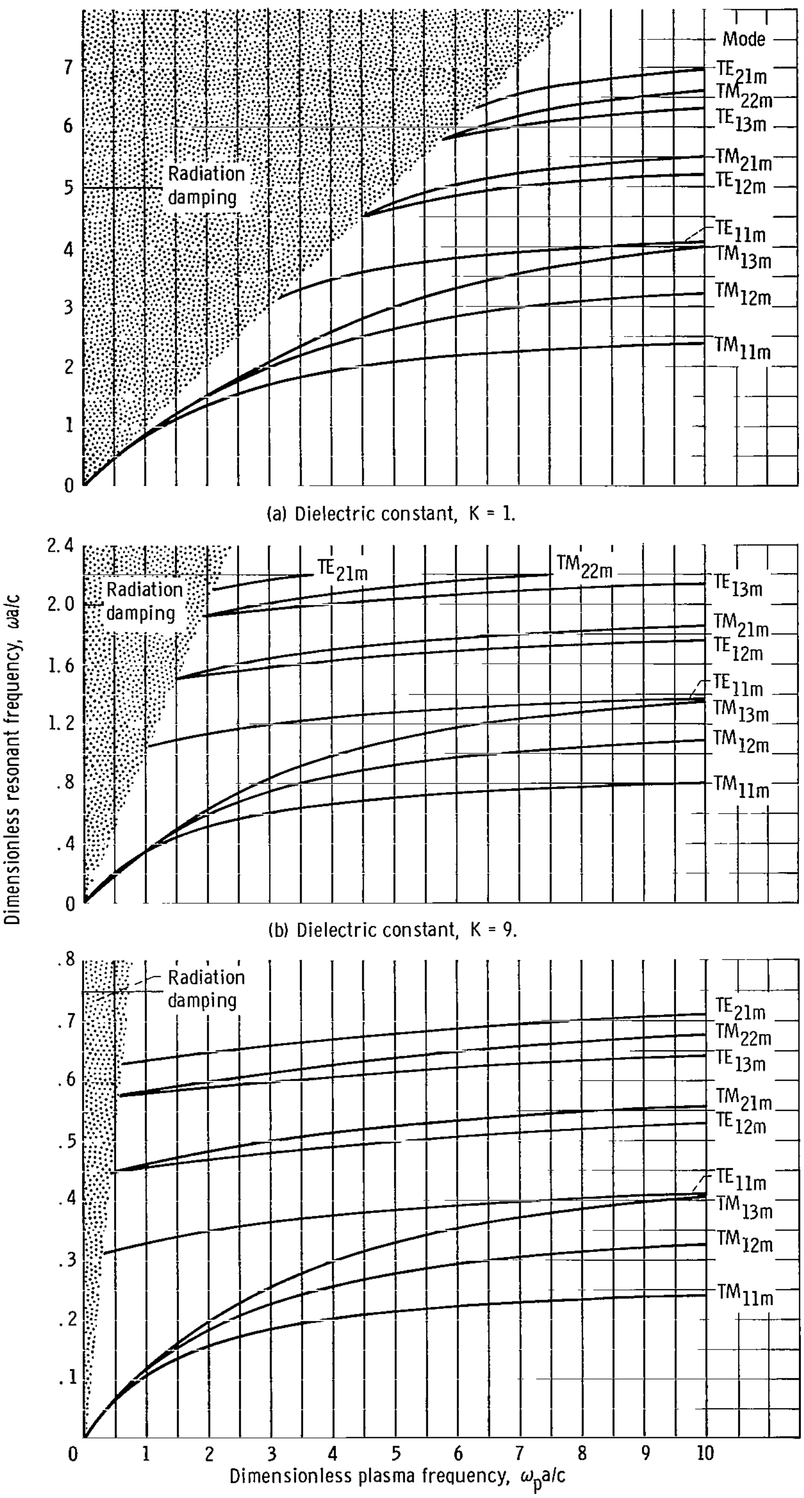
The resonant modes that correspond to the solutions of equations (38) and (39) will be denoted as TE_{lnm} and TM_{lnm}, respectively. The subscript l denotes the l^{th} solution of the resonance equation, with $l = 1$ being the solution with the lowest resonant frequency. The subscript n denotes the order of the spherical Bessel functions, and m corresponds to the number of cyclic variations in the field as φ varies from 0 to 2π. Since the resonance equations are independent of m, and m can take values from -n to n, the resonant modes will be (2n + 1)-fold degenerate. For example, the $TE_{1,1,-1}$, $TE_{1,1,0}$, and $TE_{1,1,1}$ modes will have the same resonant frequency.

Numerical solutions of the resonance equations were obtained by using an IBM 7090 computer. The first case considered was the resonant frequencies of a dielectric sphere in a cold, collisionless plasma; that is, both the electron collision frequency ν and the acoustic velocity u were set equal to zero. The results for this case are shown in figure 2 in the form of curves of dimensionless resonant frequency $\omega a/c$ against the dimensionless plasma frequency $\omega_p a/c$. The calculations were carried out with the relative dielectric constant of the sphere equal to 1, 9, and 100.

The results show that, as $\omega_p a/c$ approaches infinity, which corresponds to the electron density going to infinity, the resonant frequency of each mode approaches a limiting value. This result is expected since a cold, collisionless plasma of infinite electron density acts as a conductor with infinite conductivity. Thus, the resonant frequencies of a dielectric sphere immersed in a plasma of infinite electron density are equal to those of a dielectric sphere coated with a perfectly conducting surface. These limiting resonant frequencies can be shown to be only functions of K and a. Some numerical values for these frequencies are given in table I.

At moderate electron densities, the electromagnetic field penetrates a slight distance into the plasma. The field, however, is still totally reflected by the plasma. Since the effective point of reflection is now located outside the surface $r = a$, the effect on the field is equivalent to an increase in the radius of the cavity; thus, the resonant frequencies are less than their values for infinite electron density.

There are two cases to consider as the electron density becomes small. In the first case, the resonant frequency always remains less than the plasma frequency; thus,



(a) Dielectric constant, K = 1.

(b) Dielectric constant, K = 9.

(c) Dielectric constant, K = 100.

Figure 2. - Dimensionless resonant frequency plotted against dimensionless plasma frequency.

TABLE I. - RESONANT FREQUENCIES FOR INFINITE ELECTRON DENSITY

Mode	Dimensionless resonant frequency, $\sqrt{K}\,(\omega a/c)$
TM_{11m}	2.74
TM_{12m}	3.87
TE_{11m}	4.49
TM_{13m}	4.97
TE_{12m}	5.76

TABLE II. - RESONANT FREQUENCIES WHERE PLASMA FREQUENCY EQUALS RESONANT FREQUENCY

Mode	Dimensionless resonant frequency, $\sqrt{K}\,(\omega a/c)$
TM_{1nm}	0
TE_{11m}	3.14
TE_{12m} and TM_{21m}	4.49
TE_{13m} and TM_{22m}	5.76
TE_{21m}	6.28

solutions exist down to zero electron density. This condition only occurs for the TM_{1nm} modes. Only the TM_{11m}, TM_{12m}, and TM_{13m} modes of this class are shown in figure 2. It can be shown by using the asymptotic forms of the spherical Bessel functions for small argument along with equation (39) that the resonance curves for the TM_{1nm} modes are given by

$$\frac{\omega a}{c} = \frac{\omega_p a}{c}\left[\frac{n+1}{n(K+1)+1}\right]^{1/2} \tag{40}$$

when $\omega a/c \ll 1$. This result, with $K = 1$, is in agreement with the quasi-static analysis of Budden (ref. 9).

In the second case, the resonant frequency is equal to the plasma frequency for some electron density and exceeds the plasma frequency as the electron density is decreased from that value. This condition occurs for all TE modes and for the TM_{lnm} modes where l is greater than 1. It can be shown that the resonant frequencies where the resonant frequency is equal to the plasma frequency are only functions of K and a. Some numerical values for these frequencies are given in table II.

When the resonant frequency is above the plasma frequency, energy can be radiated from the dielectric sphere so that a shock-excited field in the sphere will decay with time. The calculation of the resonant frequencies and damping factors for this case will be deferred until the effect of electron collisions is introduced.

Tables I and II show that the total percentage change in the resonant frequency as ω_p varies from $\omega_p = \omega$ to $\omega_p = \infty$ is independent of K for each mode except for the TM_{1nm} modes. The TM_{1nm} modes offer the possibility of measuring very low electron densities (see fig. 2, p. 15) since the resonance curves extend to $\omega_p = 0$. However, since the resonance curves of all TM_{1nm} modes pass through $\omega_p = 0$, it may be difficult, in practice, to distinguish between these various modes for small values of ω_p unless a technique can be devised to excite only one of these modes. It should also be noted that the model used for the plasma is not valid when ω becomes too small. If any mode other than a TM_{1nm} mode is used, the lower limit of the range of usable electron densities is nonzero. For example, if the TE_{11m} mode is used, the range of electron densities is restricted to where $\sqrt{K}\ \omega_p a/c > 3.14$ (see table II). The lower limit of this range of densities can be decreased by increasing either K or a. The maximum value for K that can presently be attained, consistent with low electrical losses, is approximately 100. Increasing a increases the size of the sphere and thereby reduces the spatial resolution of the measurement.

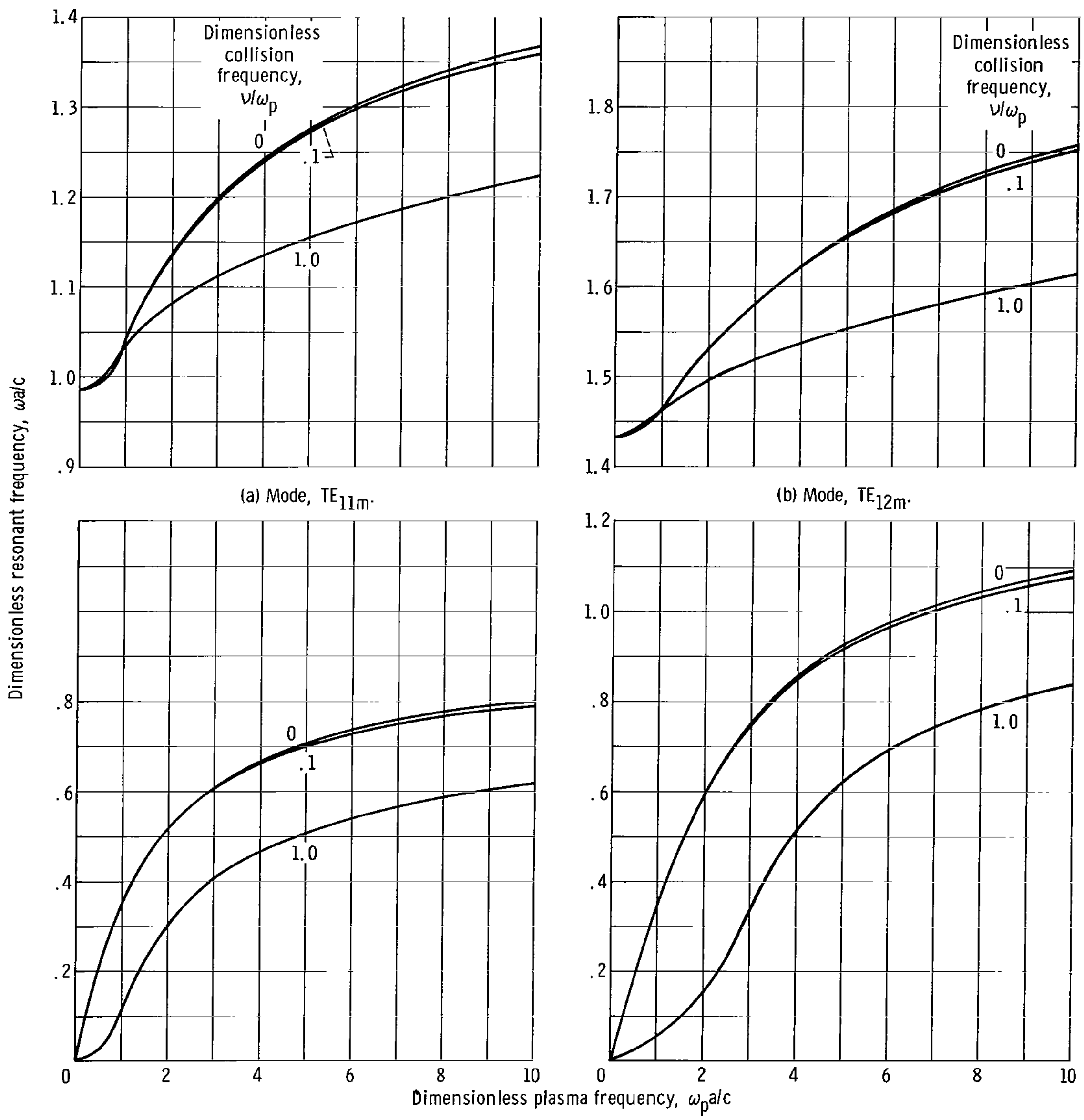
A study of figure 2 reveals that the resonant frequency changes most rapidly with plasma frequency when the resonant and plasma frequencies are equal. Since $\omega a/c$ is proportional to $\omega_p a/c$ for the TM_{1nm} modes when $\omega a/c << 1$ (eq. (40)), it can easily be shown that a 1-percent change in electron density corresponds to a 0.5-percent change in resonant frequency. This result is independent of K. For the case where K = 1, the initial slopes of the resonance curves for modes other than the TM_{1nm} modes correspond approximately to a 0.25-percent change in resonant frequency for a 1-percent change in electron density. These sensitivities are reduced for higher values of relative dielectric constant. The high sensitivity of the sphere with K = 1 is offset by the fact that the resonance curves for low K approach their limiting values for infinite electron density more rapidly than the curves for higher K and thereby reduce the range of electron densities that can be measured.

It was pointed out in the section MODEL AND BOUNDARY CONDITIONS that the model under consideration does not take into account the sheath that forms over the surface of the sphere. This sheath can be approximated by assuming that there is a thin free-space layer between the dielectric and the plasma. The effect of this layer can easily be taken into account for the case where K = 1 since the sphere and the free-space layer then have the same relative dielectric constants. The net effect in this case is to increase the sphere radius by the thickness of the free-space layer. Since typical sheath thicknesses are of the order 10^{-5} meter and typical sphere radii can be of the order 10^{-2} meter, the sheath effectively increases the sphere radius by approximately 0.1 percent. This increase in sphere radius produces changes in the resonant frequencies of less than 0.1 percent. The effect of the sheath when the relative dielectric constant of the sphere is other than 1 cannot be treated as readily. However, it is anticipated that the effect will be of similar magnitude.

It should be noted that the solutions for the case of zero electron temperature and collision frequency give rise to real values for $k_o a$ and imaginary values for $k_e a$ and $k_p a$. Thus, the field has an oscillatory behavior in the radial direction within the sphere and decays exponentially with radius outside the sphere.

Collisions between the electrons and other particles in the plasma are a source of energy loss to the field. A shock-excited field in the dielectric sphere varies with time as $e^{i\omega t-\alpha t}$, where the damping factor α accounts for the decay in the amplitude of the field with time. The resonant frequencies and damping factors when collisions are present were determined by replacing ω in the resonance equations by $\omega + i\alpha$ and then finding simultaneous values for ω and α that satisfied the equations.

Numerical results for the case of a dielectric sphere immersed in a cold collisional plasma are shown for several modes in figures 3 and 4 in the form of curves of $\omega a/c$



(c) Mode, TM_{11m}.

(d) Mode, TM_{12m}.

Figure 3. - Dimensionless resonant frequency plotted against dimensionless plasma frequency. Dielectric constant, K = 9.

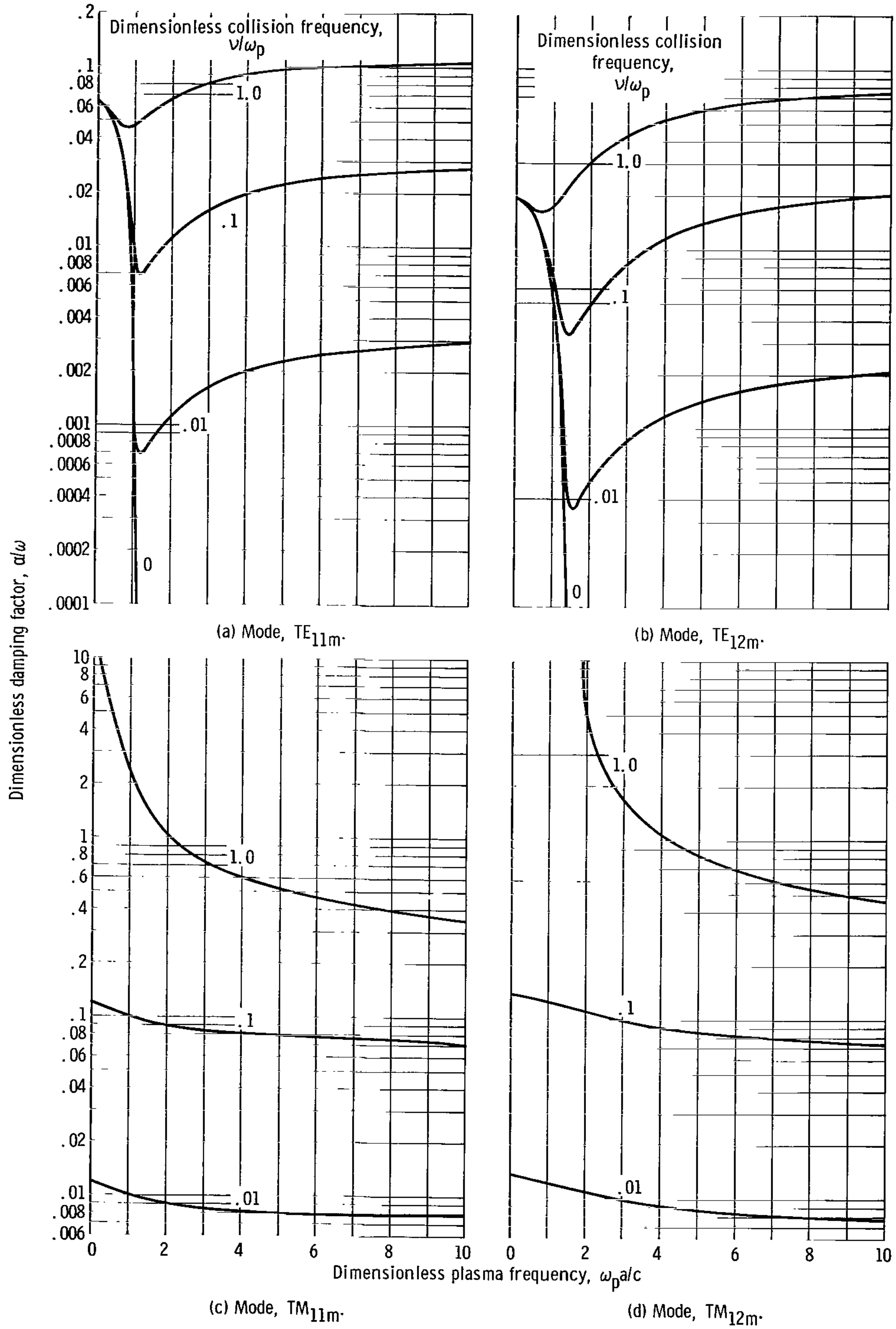


Figure 4. - Dimensionless damping factor plotted against dimensionless plasma frequency. Dielectric constant, K = 9.

and the dimensionless damping factor α/ω against $\omega_p a/c$ for various dimensionless collision frequencies ν/ω_p. The relative dielectric constant of the sphere was set equal to 9. The results show that the resonant frequency $\omega a/c$ is not greatly perturbed by the effect of collision frequencies ν/ω_p in the range 0 to 0.1. The damping factor α/ω is, however, very sensitive to the value of ν/ω_p. Thus, for low collision rates, the electron density can be determined by measuring $\omega a/c$ as in the collisionless case, and then the collision frequency can be determined by measuring α/ω. In practice, the Q of the cavity would normally be measured and then the damping factor would be computed from the equation

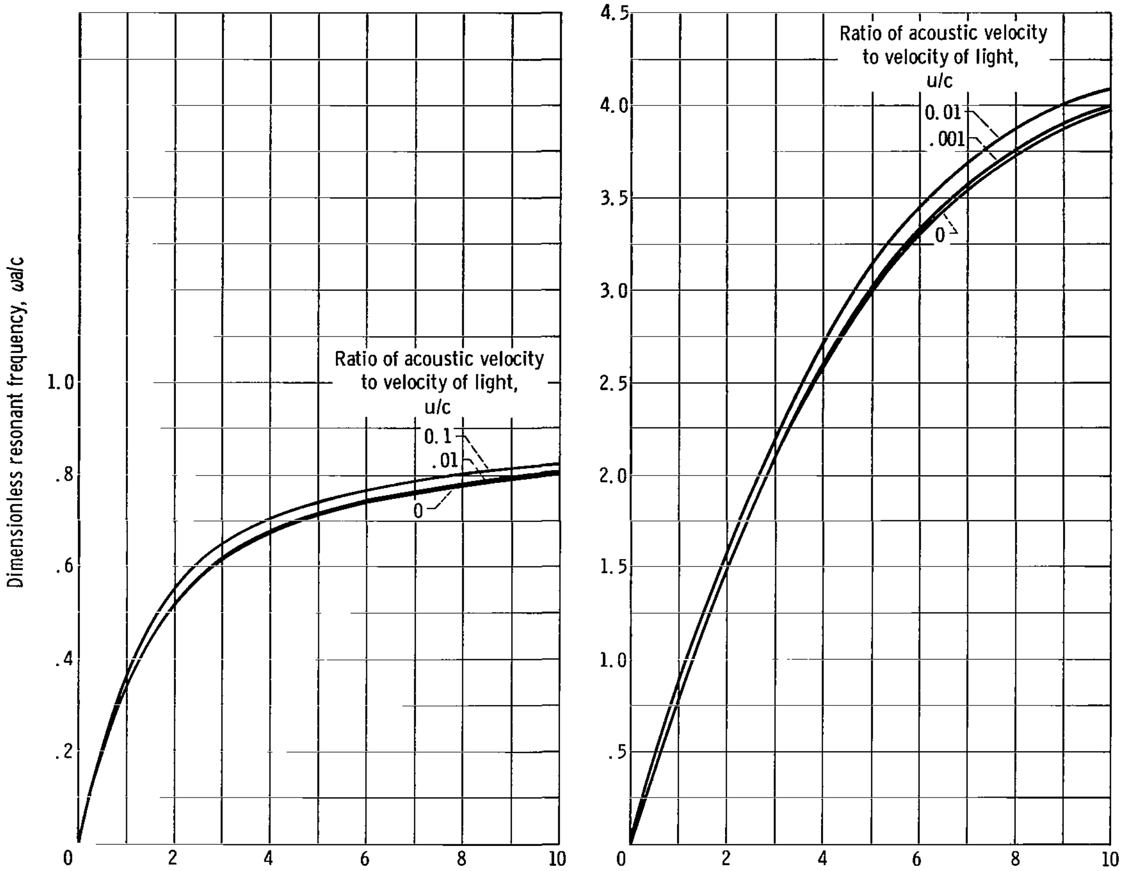
$$Q = \frac{\omega}{2\alpha}$$

When the collision frequency ν/ω_p is in the range 0.1 to 1, the resonant frequency $\omega a/c$ is reduced considerably from its value when $\nu/\omega_p = 0$. Thus, for moderate collision frequencies, an iterative procedure must be used to find values for $\omega_p a/c$ and ν/ω_p that are in agreement with measured values of $\omega a/c$ and α/ω. If ν/ω_p exceeds 1, the Q of the cavity becomes too small for this diagnostic technique to be useful.

For the case of the TE_{11m} and TE_{12m} modes (see figs. 2(a) and (b)), the damping factor, which is a strong function of ν/ω_p at resonant frequencies much below the plasma frequency, becomes very large at frequencies above the plasma frequency and approaches a value which is independent of ν/ω_p. This result is understandable since collisions represent the only energy loss mechanism at frequencies below the plasma frequency. At frequencies above the plasma frequency, however, energy can be lost from the sphere through both collisions and radiation. It is evident from figures 4(a) and (b) that, at frequencies above the plasma frequency, radiation is the dominant loss mechanism since the value of the damping factor is almost independent of collision frequency. Radiation losses are not important in the case of the TM_{1nm} modes since their resonant frequencies are always below the plasma frequency.

Solutions for the case of a nonzero electron collision frequency give rise to complex values for $k_o a = \omega a/c + i(\alpha a/c)$, $k_e a$, and $k_p a$. Thus, the field has both oscillatory and exponential behavior with radius within the sphere and, in most cases, is a traveling wave that decays extremely rapidly with radius external to the sphere.

The final case to be considered is the effect of the electron temperature on the resonant frequencies and damping factors of the various modes. As discussed previously the resonant frequencies and damping factors of all TE modes are independent of the electron temperature, within the limits imposed by the model; thus, no further consideration of the TE modes is necessary. Figure 5(a) shows a plot of $\omega a/c$ against $\omega_p a/c$ for the TM_{11m} mode for values of u/c of 0, 0.01, and 0.1. The collision frequency



(a) Mode, TM_{11m}; dielectric constant, K = 9.

(b) Mode, TM_{13m}; dielectric constant, K = 1.

Figure 5. - Dimensionless resonant frequency against dimensionless plasma frequency.

ν/ω_p was set equal to zero. As shown, the change in $\omega a/c$ due to varying u/c is not great. The value of $\omega a/c$ only changes approximately 0.25 percent in going from u/c = 0 to u/c = 0.01, which is equivalent to changing the electron temperature from 0^o to $3.7\times10^{5^o}$ K.

A study of the resonance equation for the TM modes (eq. (39)) reveals that the last term, which is the only temperature-dependent term, contains the factor n(n + 1). Thus, it may be anticipated that the resonant frequencies of the modes with large n will have a more pronounced temperature dependence. This effect was verified from extensive numerical solutions of the resonance equation. It was also found that the temperature effects could be increased by decreasing K. A typical result is shown in figure 5(b) for the TM_{13m} mode with K = 1. The shift in resonant frequency is now approximately 2.7 percent in going from u/c = 0 to u/c = 0.01.

As discussed in the section MODEL AND BOUNDARY CONDITIONS, the validity of the computed temperature effects is dependent on the validity of the rigidity boundary condition given by equation (14). An experimental study of the resonant frequencies of a dielectric sphere in a warm plasma should give a good understanding of the accuracy of this boundary condition.

CONCLUDING REMARKS

The resonant frequencies and damping factors of a dielectric sphere immersed in a warm collisional plasma were computed for various electron densities, electron collision frequencies, and electron temperatures. The resonant frequencies and damping factors are sufficiently strong functions of the electron density and collision frequency so that a dielectric sphere can be useful as a plasma diagnostic tool. The variations of the resonant frequencies with electron temperature are also sufficiently large to be of diagnostic importance. However, the validity of these computed temperature effects is dependent on the accuracy of the rigidity boundary condition.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 14, 1967,
129-01-07-05-22.

APPENDIX - SYMBOLS

a	radius of sphere
$\bar{a}_r, \bar{a}_\theta, \bar{a}_\varphi$	unit vectors
c	velocity of light
$\bar{e}$	electric field intensity
e_r, e_θ, e_φ	r, θ, and φ components of electric field intensity
$\bar{h}$	magnetic field intensity
h_r, h_θ, h_φ	r, θ, and φ components of magnetic field intensity
Im	imaginary part
i	$\sqrt{-1}$
K	relative dielectric constant of sphere
k	Boltzmann constant
k_e	wave number for magnetic field in plasma
k_o	free-space wave number, ω/c
k_p	wave number for pressure field in plasma
m_e	mass of electron
n_o	equilibrium electron number density
p	perturbation electron pressure
q	electronic charge
Re	real part
r	radial coordinate (see fig. 1)
T_e	electron temperature
t	time
u	acoustic velocity
$\bar{v}$	electron fluid velocity
α	damping factor
γ	ratio of specific heats with constant pressure to constant volume for electron gas
ϵ_o	electric permittivity of free space
ϵ_p	relative dielectric constant of plasma
θ	azimuthal coordinate (see fig. 1)
μ_o	magnetic permeability of free space
ν	effective electron collision frequency for momentum transfer
φ	polar coordinate (see fig. 1)
ψ	magnetic field generating function
ω	angular frequency
ω_p	plasma frequency

REFERENCES

1. Wait, James R.: Radiation From Sources Immersed in Compressible Plasma Media. Can. J. Phys., vol. 42, no. 9, Sept. 1964, pp. 1760-1780.

2. Wait, James R.: Resonances of a Spherical Void in a Compressible Isotropic Plasma. J. Geophys. Res., vol. 69, no. 19, Oct. 1, 1964, pp. 4189-4191.

3. Oster, L.: Linearized Theory of Plasma Oscillations. Rev. Mod. Phys., vol. 32, no. 1, Jan. 1960, pp. 141-168.

4. Sancer, Maurice I.: Boundary Conditions for a Unique Solution to the Linearized Warm-Plasma Equations. Radio Science, vol. 1, no. 9, Sept. 1966, pp. 1067-1072.

5. Lee, S. W.; Liang, C.; and Lo, Y. T.: Inconsistency of Boundary Conditions of Plasma Models in a Bounded Region. Electronics Letters, vol. 1, no. 5, July 1965, pp. 128-129.

6. Wait, J. R.: Modes of Propagation in a Bounded Compressible Plasma. Electronics Letters, vol. 1, no. 7, Sept. 1965, pp. 193-194.

7. Lee, S. W.; Liang, C.; and Lo, Y. T.: Further Remarks on Boundary Conditions of Plasma Models in a Bounded Region. Electronics Letters, vol. 1, no. 7, Sept. 1965, p. 215.

8. Morse, Philip M.; and Feshbach, Herman: Methods of Theoretical Physics. Vol. 2, McGraw-Hill Book Co., Inc., 1953, pp. 1759-1767.

9. Budden, K. G.: The Resonant Frequencies of Cavities in a Magnetoionic Medium. Can. J. Phys., vol. 42, no. 1, Jan. 1964, pp. 90-102.

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