N I

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-[illegible]

Edge Diffraction from Truncated Paraboloids and Hyperboloids

W. V. T. Rusch

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

June 1, 1967

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1113

Edge Diffraction from Truncated Paraboloids and Hyperboloids

W. V. T. Rusch

Approved by:

P. D. Potter

P. D. Potter, *Manager*
Communications Elements
Research Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

June 1, 1967

TECHNICAL REPORT 32-1113

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

Contents

Figures

Abstract

The geometrical theory of edge diffraction is applied to the problem of scattering from both paraboloidal and hyperboloidal reflectors. The results are compared with classical results from geometrical optics and vector diffraction theory. It is found that the edge-diffracted field provides an excellent approximation to the rigorously-determined field except in certain regions such as shadow-light boundaries.

Edge Diffraction from Truncated Paraboloids and Hyperboloids

I. Introduction

The useful theory developed by J. B. Keller (Ref. 1) to explain diffraction phenomena in terms of "diffracted rays" has recently been applied to the problem of scattering from certain shaped reflectors of interest in microwave antenna theory (Refs. 2, 3). The geometrical theory of diffraction enables important scattering behavior to be more easily understood and related to the parameters involved. The geometrical theory also provides an independent check on the classical Kirchhoff integral theory in regions where the validity of the Kirchhoff method is in question. Furthermore, the computational time required by the geometrical theory is considerably less than that required by the Kirchhoff theory.

The subject of this Report is the application of the geometrical theory of singly edge-diffracted rays to scattering from paraboloids and hyperboloids. While some aspects of the analysis have appeared previously in the literature, the present treatment presents: 1) a quantitative comparison of the geometrical and integral theories; 2) the application of axial caustic correction factors to the theory; and 3) some results which may indicate the usefulness of the integral theory in regions of previously doubtful validity.

II. Keller Theory of Edge-Diffracted Rays: Normal Incidence[1]

The Keller theory will be outlined briefly for the case of an electromagnetic wave incident upon a conducting, semi-infinite plane: $z = 0, x \leq 0$, Fig. 1. It will be assumed that the incident wave can be interpreted in terms of a ray or wave normal, and that this ray is normally incident upon the edge. In terms of the geometry of Fig. 1, the incident ray lies in a plane of constant y, and is incident from a direction α with respect to the z-axis. In accordance with the Keller theory, when such a ray strikes the edge, it generates "diffracted rays" which emerge radially in all directions but which remain in the same constant-y plane as the incident ray, Fig. 2.

The intensity of the edge-diffracted ray in an arbitrary direction θ is, for the case that the E-field of the incident ray is parallel to the edge:

$$H\text{-plane: } E_d = \frac{-E_{inc}\, e^{ikr}\, e^{i\pi/4}}{2\,(2\pi k)^{1/2}} \times [\sec \tfrac{1}{2}(\theta - \alpha) + \csc \tfrac{1}{2}(\theta + \alpha)]\,(r)^{-1/2} \qquad (1)$$

[1]The Keller theory is considerably more general. For this Report, however, only the normally incident case will be considered.

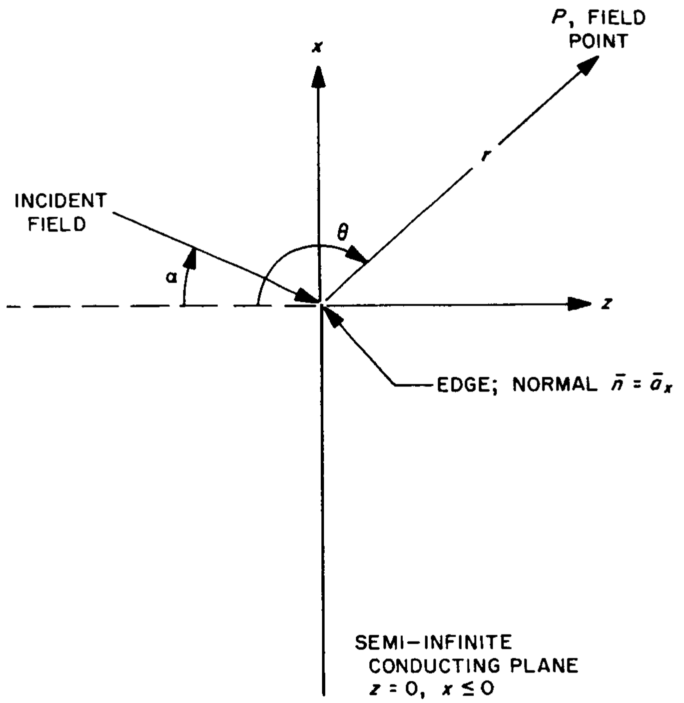


Fig. 1. Edge-diffracted geometry

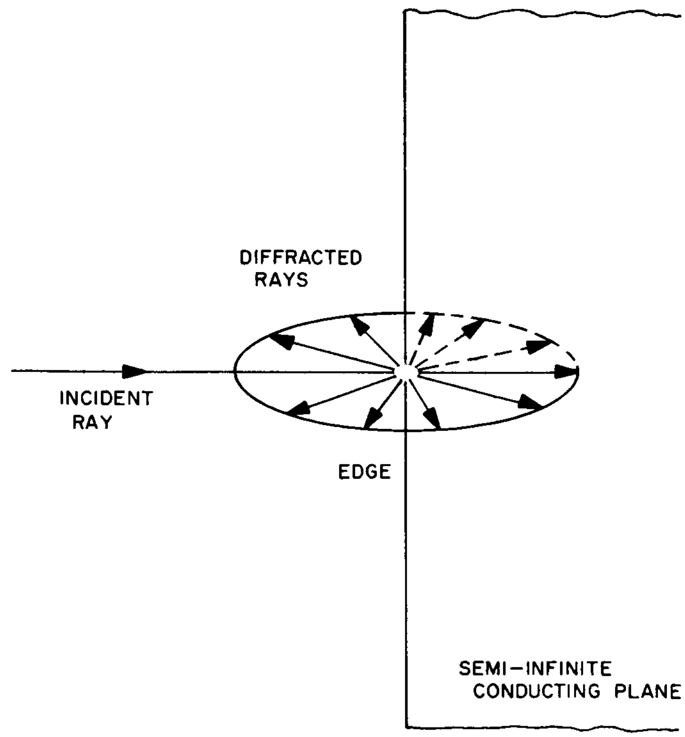


Fig. 2. Normally-incident edge-diffracted rays

and for the case that the H-field of the incident ray is parallel to the edge:

$$E\text{-plane: } H_d = \frac{-H_{inc}\, e^{ikr}\, e^{i\pi/4}}{2\,(2\pi k)^{1/2}} \left[\sec \tfrac{1}{2}(\theta - \alpha) - \csc \tfrac{1}{2}(\theta + \alpha)\right] (r)^{-1/2} \tag{2}$$

Equations (1) and (2) are based on asymptotic forms of the Sommerfeld solution of the half-plane problem (Ref. 4) and are not valid in the direction of geometrical reflection ($\theta = -\alpha$) or in the direction of the shadow-light boundary ($\theta = \alpha + \pi$). E_{inc} in Eq. (1) is the intensity of the E-field of the incident ray at the edge, and H_{inc} in Eq. (2) is the intensity of the H-field of the incident ray at the edge.

It is possible to extend Eqs. (1) and (2) to edge diffraction from a surface of revolution, illuminated by a spherical wave emerging from a point on the axis of symmetry; this surface is shown as a deep paraboloid in Fig. 3. Every incident ray impinges normally on the edge—or rim—of the reflector. Under the assumption that in the vicinity of every point on the rim diffraction takes place locally as a plane wave incident on a half-plane, the scattered field is again described by "diffracted rays" given by:

$$H\text{-plane:} \quad E_d = \frac{-E_{inc}\, e^{ikr}\, e^{i\pi/4}}{2\,(2\pi k)^{1/2}} \left[\sec \tfrac{1}{2}(\theta - \alpha) + \csc \tfrac{1}{2}(\theta + \alpha)\right] \left[r\left(1 + \frac{r}{\rho}\right)\right]^{-1/2} \tag{3}$$

$$E\text{-plane:} \quad H_d = \frac{-H_{inc}\, e^{ikr}\, e^{i\pi/4}}{2\,(2\pi k)^{1/2}} \left[\sec \tfrac{1}{2}(\theta - \alpha) - \csc \tfrac{1}{2}(\theta + \alpha)\right] \left[r\left(1 + \frac{r}{\rho}\right)\right]^{-1/2} \tag{4}$$

These expressions are identical to Eqs. (1) and (2) with the exception of the final factor. This factor accounts for the divergence of the tube of energy along the diffracted ray. For the initial case of a plane wave incident on a half-plane the energy of the diffracted rays diverged as a cylindrical wave in accordance with the factor $(r)^{-1/2}$ in

Eqs. (1) and (2). For the surface of rotation the edge-diffracted energy diverges as $[r(1 + r/\rho)]^{-1/2}$ where ρ is the distance along the ray from the edge to the axis. The point C where the ray, or its backward extension, intersects the axis is of singular interest, because here the cross-section of the tube of rays shrinks to zero. It should be noted that ρ is a signed distance, positive if the edge is between C and P; negative if C is between the edge and P.

Equations (3) and (4) are not valid for $\theta = -\alpha$ and $\theta = \alpha + \pi$. Also, Eqs. (3) and (4) become infinite for all points P on the axis, because the axis is a *caustic*, i.e., the locus of points where the rays intersect. An axial caustic correction factor must be applied to determine the edge-diffracted fields on the axis.

The above analysis represents a first-order theory, inasmuch as an edge-diffracted ray may be incident on the opposite edge, giving rise to doubly- or even multiply-diffracted edge rays. Other effects may occur: an edge diffracted ray may be reflected from the reflector in the same way that a geometrical-optics ray is reflected. Rays may "creep" along the surface or emerge tangentially from the surface. Although these effects are of considerable interest (Ref. 5), only the singly edge-diffracted ray will be included in the analysis following.

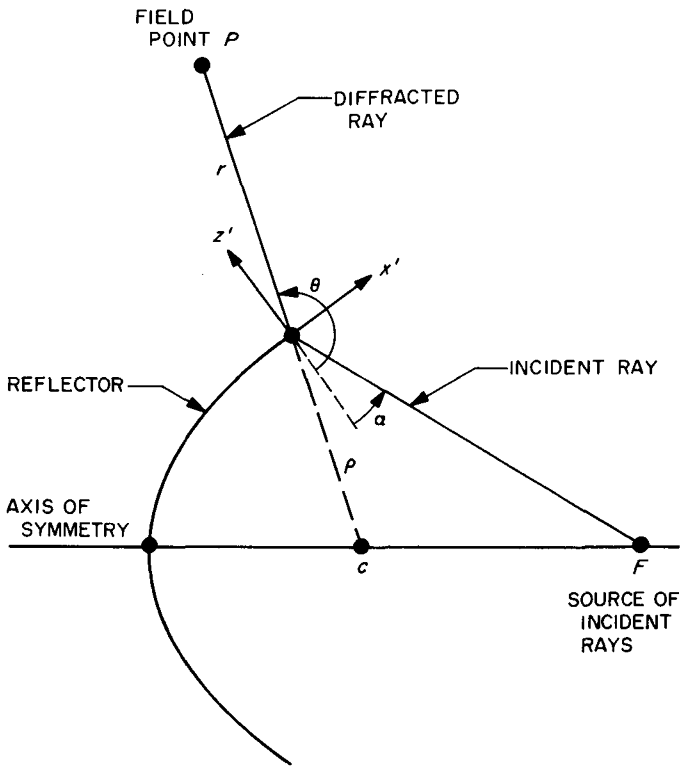


Fig. 3. Edge-diffracted geometry for surface of revolution

III. Edge Diffraction from a Paraboloid

The results of the previous section have been applied to compute the radiation field from a paraboloid fed by a point-source feed located at the focus, Fig. 4. With reference to Eqs. (3) and (4) the singly edge diffracted fields from Edge 1 are:

$$H\text{-plane:}\quad E_{d_1} = \frac{-E_{inc}\, e^{ikr_1} e^{i\pi/4}}{2(2\pi k)^{1/2}}\left[\sec \tfrac{1}{2}(\theta_1 - \alpha) + \csc \tfrac{1}{2}(\theta_1 + \alpha)\right]\left[r_1\left(1 + \frac{r_1}{\rho_1}\right)\right]^{-1/2} \tag{5}$$

$$E\text{-plane:}\quad H_{d_1} = \frac{-H_{inc}\, e^{ikr_1} e^{i\pi/4}}{2(2\pi k)^{1/2}}\left[\sec \tfrac{1}{2}(\theta_1 - \alpha) - \csc \tfrac{1}{2}(\theta_1 + \alpha)\right]\left[r_1\left(1 + \frac{r_1}{\rho_1}\right)\right]^{-1/2} \tag{6}$$

It may be easily shown that $1 + r_1/\rho_1 = r \sin \Psi/(D/2)$ and at great distances $r \cong r_1 + (D/2) \sin \Psi$ and $\theta_1 \cong \Psi - \alpha$. (For the paraboloid α is negative.) The result of these approximations is that, at great distances:

$$H\text{-plane:}\quad E_{d_1} = \frac{-E_{inc}\, e^{ikr} e^{-i[(kD/2)\sin\Psi - \pi/4]}}{2(2\pi)^{1/2}(kr)} \frac{(kD/2)^{1/2}}{(\sin\Psi)^{1/2}}\left[\frac{1}{\cos\left(\frac{\Psi}{2} - \alpha\right)} + \frac{1}{\sin\frac{\Psi}{2}}\right], \qquad 0 < \Psi < \pi \tag{7}$$

$$E\text{-plane:}\quad H_{d_1} = \frac{-H_{inc}\, e^{ikr} e^{-i[(kD/2)\sin\Psi - \pi/4]}}{2(2\pi)^{1/2}(kr)} \frac{(kD/2)^{1/2}}{(\sin\Psi)^{1/2}}\left[\frac{1}{\cos\left(\frac{\Psi}{2} - \alpha\right)} - \frac{1}{\sin\frac{\Psi}{2}}\right], \qquad 0 < \Psi < \pi \tag{8}$$

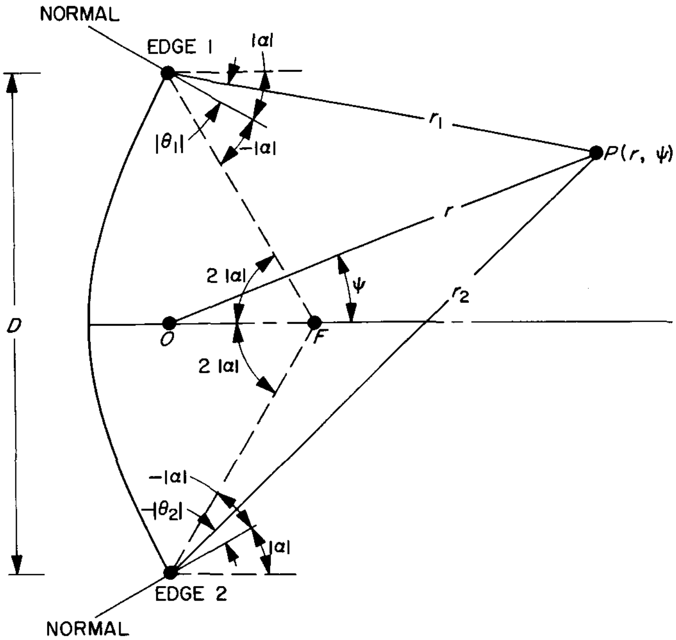


Fig. 4. Edge-diffracted geometry for paraboloid

Equations (7) and (8) are valid in the entire range of Ψ from 0 to π with the exception of $\Psi = 0$ and $\Psi = \pi$, which are the axial caustics, and $\Psi = \pi + 2\alpha$, which is the shadow-light boundary.

In a similar manner, the singly edge-diffracted ray from Edge 2 can be calculated, with the exception that this ray does not propagate directly into the region $\pi/2 < \Psi < \pi/2 - \alpha$ which is blocked by the paraboloid, Fig. 4. Consequently:

$$H\text{-plane:}\quad E_{d_2} = \frac{-E_{inc}\, e^{ikr}\, e^{-i\,[(kD/2)\sin\Psi - \pi/4]}}{2\,(2\pi)^{1/2}\,(kr)} \, \frac{(kD/2)^{1/2}}{(\sin\Psi)^{1/2}} \left[\frac{1}{\cos\left(\dfrac{\Psi}{2} - \alpha\right)} + \frac{1}{\sin\dfrac{\Psi}{2}} \right], \qquad 0 < \Psi < \pi/2 \tag{9}$$

$$E\text{-plane:}\quad H_{d_2} = \frac{-H_{inc}\, e^{ikr}\, e^{-i\,[(kD/2)\sin\Psi - \pi/4]}}{2\,(2\pi)^{1/2}\,(kr)} \, \frac{(kD/2)^{1/2}}{(\sin\Psi)^{1/2}} \left[\frac{1}{\cos\left(\dfrac{\Psi}{2} - \alpha\right)} - \frac{1}{\sin\dfrac{\Psi}{2}} \right], \qquad 0 < \Psi < \pi/2 \tag{10}$$

$$H\text{-plane:}\quad E_{d_2} = 0, \qquad \frac{\pi}{2} < \Psi < \frac{\pi}{2} - \alpha \tag{11}$$

$$E\text{-plane:}\quad H_{d_2} = 0, \qquad \frac{\pi}{2} < \Psi < \frac{\pi}{2} - \alpha \tag{12}$$

$$H\text{-plane:}\quad E_{d_2} = \frac{-E_{inc}\, e^{ikr}\, e^{i\,[(kD/2)\sin\Psi - \pi/4]}}{2\,(2\pi)^{1/2}\,(kr)} \, \frac{(kD/2)^{1/2}}{(\sin\Psi)^{1/2}} \left[-\frac{1}{\cos\left(\dfrac{\Psi}{2} + \alpha\right)} + \frac{1}{\sin\dfrac{\Psi}{2}} \right], \qquad \frac{\pi}{2} - \alpha < \Psi < \pi \tag{13}$$

$$E\text{-plane:}\quad H_{d_2} = \frac{-H_{inc}\, e^{ikr}\, e^{i\,[(kD/2)\sin\Psi - \pi/4]}}{2\,(2\pi)^{1/2}\,(kr)} \, \frac{(kD/2)^{1/2}}{(\sin\Psi)^{1/2}} \left[-\frac{1}{\cos\left(\dfrac{\Psi}{2} + \alpha\right)} - \frac{1}{\sin\dfrac{\Psi}{2}} \right], \qquad \frac{\pi}{2} - \alpha < \Psi < \pi \tag{14}$$

The total singly edge-diffracted ray in each of the principal planes is then the sum of the two individual rays, leading to the final result:

H-plane:

$$E_d = \frac{-E_{inc}\, e^{ikr}\,(kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left[\cos\tau\left(\frac{\cos\frac{\Psi}{2}\cos\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}\right)+i\sin\tau\left(\frac{\sin\frac{\Psi}{2}\sin\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}-\frac{1}{\sin\frac{\Psi}{2}}\right)\right], \qquad 0<\Psi<\frac{\pi}{2} \tag{15}$$

E-plane:

$$H_d = \frac{-H_{inc}\, e^{ikr}\,(kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left[\cos\tau\left(\frac{\cos\frac{\Psi}{2}\cos\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}\right)+i\sin\tau\left(\frac{\sin\frac{\Psi}{2}\sin\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}+\frac{1}{\sin\frac{\Psi}{2}}\right)\right], \qquad 0<\Psi<\frac{\pi}{2} \tag{16}$$

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ikr}\,(kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left\{\frac{e^{-i\tau}}{2}\left[\frac{1}{\cos\left(\frac{\Psi}{2}-\alpha\right)}+\frac{1}{\sin\frac{\Psi}{2}}\right]\right\}, \qquad \frac{\pi}{2}<\Psi<\frac{\pi}{2}-\alpha \tag{17}$$

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ikr}\,(kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left\{\frac{e^{-i\tau}}{2}\left[\frac{1}{\cos\left(\frac{\Psi}{2}-\alpha\right)}-\frac{1}{\sin\frac{\Psi}{2}}\right]\right\}, \qquad \frac{\pi}{2}<\Psi<\frac{\pi}{2}-\alpha \tag{18}$$

H-plane:

$$E_d = \frac{-E_{inc}\, e^{ikr}\,(kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left[\cos\tau\left(\frac{1}{\sin\frac{\Psi}{2}}-\frac{\sin\frac{\Psi}{2}\sin\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}\right)+i\sin\tau\left(\frac{-\cos\frac{\Psi}{2}\cos\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}\right)\right], \qquad \frac{\pi}{2}-\alpha<\Psi<\pi \tag{19}$$

E-plane:

$$H_d = \frac{-H_{inc}\, e^{ikr}\,(kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left[\cos\tau\left(\frac{-1}{\sin\frac{\Psi}{2}}-\frac{\sin\frac{\Psi}{2}\sin\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}\right)+i\sin\tau\left(\frac{-\cos\frac{\Psi}{2}\cos\alpha}{\cos^2\frac{\Psi}{2}-\sin^2\alpha}\right)\right], \qquad \frac{\pi}{2}-\alpha<\Psi<\pi \tag{20}$$

where $\tau = (kD/2)\sin\Psi - \pi/4$

The singularity caused by the axial caustic may be removed. For small values of Ψ it may be shown that the total singly edge-diffracted ray is:

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ik(D/2\cos\delta + x\sin\delta)}\,(D/2)^{1/2}}{(2\pi k\rho)^{1/2}\,[(D/2)^2 + x^2]^{1/4}} \left\{ \sec \tfrac{1}{2}\left[\delta - \left(\frac{\pi}{2} + 2\alpha\right)\right] + \csc\tfrac{1}{2}\left(\delta - \frac{\pi}{2}\right)\right\} \cos\left(k\rho\cos\delta - \frac{\pi}{4}\right) \tag{21}$$

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ik(D/2\cos\delta + x\sin\delta)}\,(D/2)^{1/2}}{(2\pi k\rho)^{1/2}\,[(D/2)^2 + x^2]^{1/4}} \left\{ \sec \tfrac{1}{2}\left[\delta - \left(\frac{\pi}{2} + 2\alpha\right)\right] - \csc\tfrac{1}{2}\left(\delta - \frac{\pi}{2}\right)\right\} \cos\left(k\rho\cos\delta - \frac{\pi}{4}\right) \tag{22}$$

where $x = r\cos\Psi$, $\rho = r\sin\Psi$, $\delta = \tan^{-1} x/(D/2)$. As x increases, $\delta \rightarrow \pi/2$ and the second term in brackets

$$\left[\csc\tfrac{1}{2}\left(\delta - \frac{\pi}{2}\right) \rightarrow \frac{-4x}{D}\right]$$

dominates Eq. (21) so that

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ik(D/2\cos\delta + x\sin\delta)}\,(D/2)^{1/2}}{(2\pi k\rho)^{1/2}\,[(D/2)^2 + x^2]^{1/4}} \left(\frac{-2x}{D/2}\right) \cos\left(k\rho\cos\delta - \frac{\pi}{4}\right) \tag{23}$$

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ik(D/2\cos\delta + x\sin\delta)}\,(D/2)^{1/2}}{(2\pi k\rho)^{1/2}\,[(D/2)^2 + x^2]^{1/4}} \left(\frac{2x}{D/2}\right) \cos\left(k\rho\cos\delta - \frac{\pi}{4}\right) \tag{24}$$

Equations (23) and (24) must be multiplied by the axial caustic correction factor

$$\tfrac{1}{2}\,(2\pi k\rho\cos\delta)^{1/2} \sec\left(k\rho\cos\delta - \frac{\pi}{4}\right) J_0\,(k\rho\cos\delta)$$

which has been obtained from solutions of the scalar wave equation in cylindrical coordinates.[2] This caustic correction, together with other simplifications reduces Eqs. (23) and (24) to:

$$H\text{-plane:}\quad E_d = E_{inc}\, e^{ik[x^2 + (D/2)^2]^{1/2}} J_0\,(k\rho\cos\delta) \rightarrow E_{inc}\, e^{ikx[1 + ik(D/2)^2/2x]} \text{ on axis} \tag{25}$$

$$E\text{-plane:}\quad H_d = -H_{inc}\, e^{ik[x^2 + (D/2)^2]^{1/2}} J_0\,(k\rho\cos\delta) \rightarrow -H_{inc}\, e^{ikx[1 + ik(D/2)^2/2x]} \text{ on axis} \tag{26}$$

If the point source feed is tapered to produce a uniform aperture distribution, then the geometrical optic ray reflected along the axis is

$$H\text{-plane:}\quad E_g = -E_{inc}\, e^{ikx} \tag{27}$$

$$E\text{-plane:}\quad H_g = H_{inc}\, e^{ikx} \tag{28}$$

The sum of the singly edge-diffracted ray and the geometrical ray on axis is then:

$$H\text{-plane:}\quad E_g = ik\left(\frac{D}{2}\right)^2 E_{inc}\frac{e^{ikx}}{2x} \tag{29}$$

$$E\text{-plane:}\quad H_g = -ik\left(\frac{D}{2}\right)^2 H_{inc}\frac{e^{ikx}}{2x} \tag{30}$$

Equations (29) and (30) for the main-lobe intensity of a paraboloid are in agreement with the results from integral diffraction theory (Ref. 6), a result which cannot be obtained from geometrical optics alone.

A similar correction procedure may be carried out to determine the axial field behind the paraboloid in the direction $\Psi = \pi$. The result is:

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ikr}}{kr}\,\frac{(kD/2)^{1/2}}{(2\pi)^{1/2}} \times \left[\frac{(2\pi)^{1/2}(kD/2)^{1/2}}{2}\left(\frac{1}{\sin\alpha} + 1\right)\right] \tag{31}$$

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ikr}}{kr}\,\frac{(kD/2)^{1/2}}{(2\pi)^{1/2}} \times \left[\frac{(2\pi)^{1/2}(kD/2)^{1/2}}{2}\left(\frac{1}{\sin\alpha} - 1\right)\right] \tag{32}$$

[2]See Appendix.

The principal-plane field intensities in the forward direction, as predicted by the geometrical theory, are equal to each other and are equal to the classical integral result. However, in the backward direction the principal-plane fields are not equal to each other[3], nor are they equal to the classical result (Ref. 7). However, manipulation of Eqs. (31) and (32) shows that the classical result for the back-lobe intensity is the geometric mean of the E- and H-plane back-lobe intensities as predicted by the geometrical theory. This result may tend to reinforce the usefulness of the integral theory in the back-lobe region.

[3]Resulting from the fact that the singly edge-diffracted fields alone do not strictly satisfy the field equations.

The H-plane results of a numerical example are presented in Fig. 5 along with the equivalent result computed from the integral diffraction theory. The comparison is excellent with the following three exceptions: 1) the theory of geometrical diffraction is not valid near the shadow-

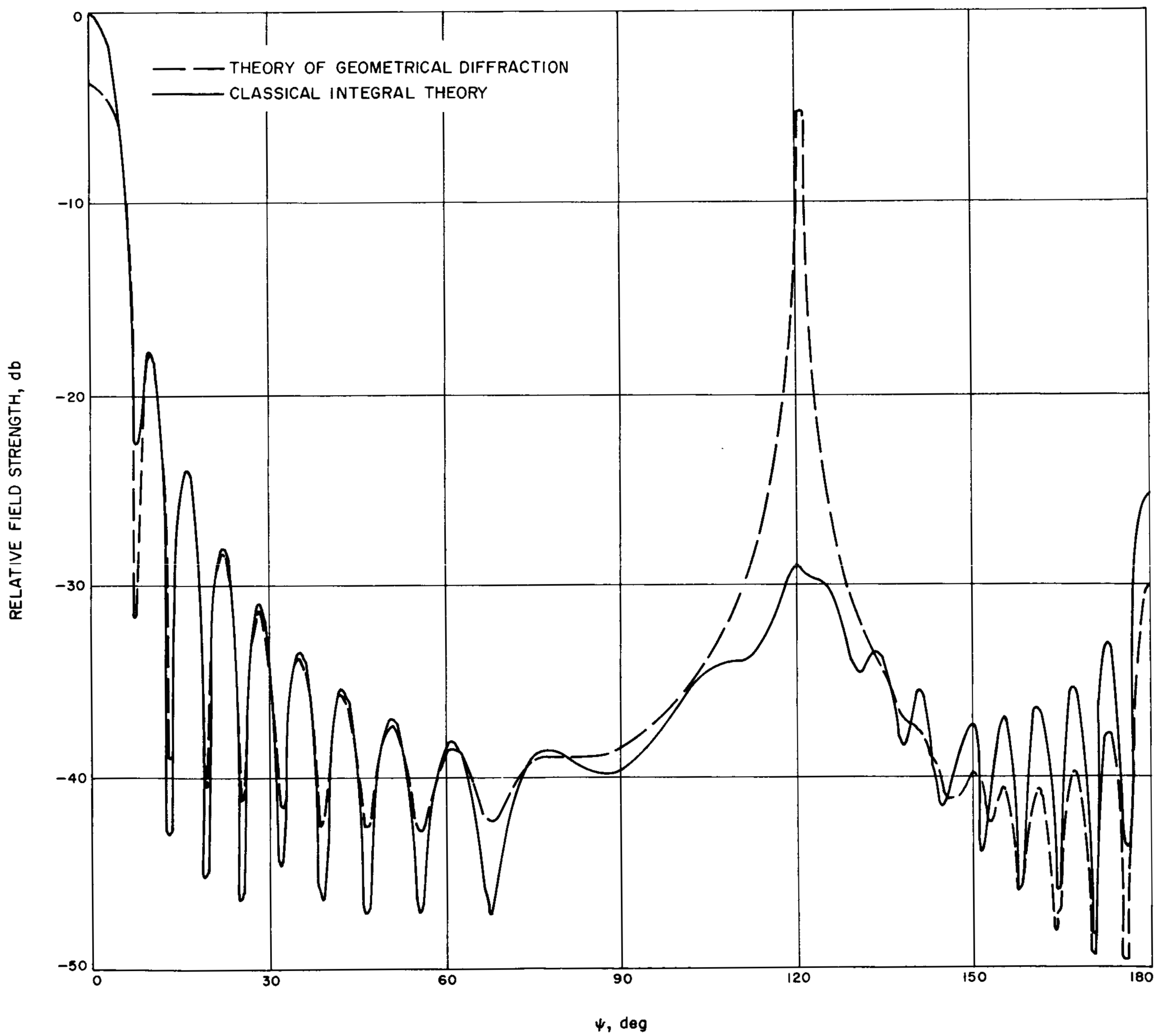


Fig. 5. H-plane field scattered from paraboloid; $F/D = 0.43$, $D = 10\lambda$

light boundary, where the field becomes singular (see below for a discussion of this); 2) the geometrical theory, while not valid in the forward direction, has been corrected to agree with the classical result; and 3) the geometrical result is considerably lower than the classical result in the backward direction. However, the E-plane back-lobe is correspondingly higher, as discussed above.

IV. Edge Diffraction from a Hyperboloid[1]

It is assumed that the hyperboloid is illuminated by a point-source feed located at F, one of the two hyperboloid foci, Fig. 6. The field emerging from F is:

$$\begin{aligned} H\text{-plane:}\quad E_F &= \frac{E_0\, e^{ik\rho}}{\rho}, \qquad 0 \leqq \theta' \leqq \theta'_e \\ &= 0, \qquad \theta' > \theta'_e \end{aligned} \tag{33}$$

$$\begin{aligned} E\text{-plane:}\quad H_F &= \frac{H_0\, e^{ik\rho}}{\rho}, \qquad 0 \leqq \theta' \leqq \theta'_e \\ &= 0, \qquad \theta' > \theta'_e \end{aligned} \tag{34}$$

Thus the field incident on the edges will be respectively

$$\frac{E_0\, e^{ik\rho_0}}{\rho_0} = E_{inc}$$

and

$$\frac{H_0\, e^{ik\rho_0}}{\rho_0} = H_{inc}.$$

These expressions will be used to determine the edge-diffracted fields. Using the methods of geometrical optics it can be shown that the geometrically scattered field is:

$$\begin{aligned} H\text{-plane:}\quad E_g &= -E_0 \left(\frac{e^2 - 1}{e^2 + 1 + 2e \cos \Psi} \right) \\ &\quad \times \frac{e^{ikr}}{r}\, e^{i\,(2ka + kx \cos \Psi)}, \qquad 0 \leqq \Psi \leqq \theta_e \end{aligned} \tag{35}$$

$$H\text{-plane:} = 0, \qquad \Psi > \theta_e$$

$$\begin{aligned} E\text{-plane:}\quad H_g &= H_0 \left(\frac{e^2 - 1}{e^2 + 1 + 2e \cos \Psi} \right) \\ &\quad \times \frac{e^{ikr}}{r}\, e^{i\,(2ka + kx \cos \Psi)}, \qquad 0 \leqq \Psi \leqq \theta_e \end{aligned} \tag{36}$$

$$E\text{-plane:} = 0, \qquad \Psi > \theta_e$$

where

$$\delta_t = \frac{\theta_e - \theta'_e}{2} \tag{37}$$

$$\alpha = -\left(\frac{\theta_e + \theta'_e}{2} \right) \tag{38}$$

$$x = D/2 \tan \theta_e \tag{39}$$

$$c = \frac{D}{4} \left(\frac{1}{\tan \theta_e} + \frac{1}{\tan \theta'_e} \right) \tag{40}$$

$$R_0 = \frac{D/2}{\sin \theta_e} \tag{41}$$

$$e = \frac{\dfrac{R_0}{c} + \left[\left(\dfrac{R_0}{c} \right)^2 + 4 \left(1 - \dfrac{R_0}{c} \cos \theta_e \right) \right]^{1/2}}{2 \left(1 - \dfrac{R_0}{c} \cos \theta_e \right)} \tag{42}$$

$$a = \frac{c}{e} \tag{43}$$

The total edge-diffracted rays can be determined in each of the principal planes by combining the rays from

[1]The hyperboloid is taken as a specific form of the general conic section of revolution. By choosing appropriate values of the eccentricity e, the formulas can be extended to discs, ellipsoids, etc.

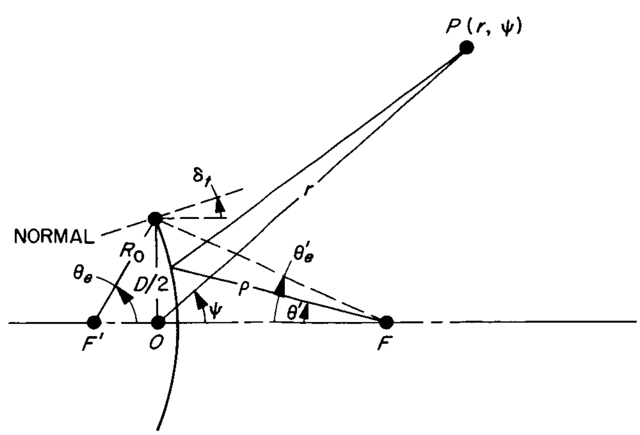


Fig. 6. Geometrical-optics scattering from hyperboloid

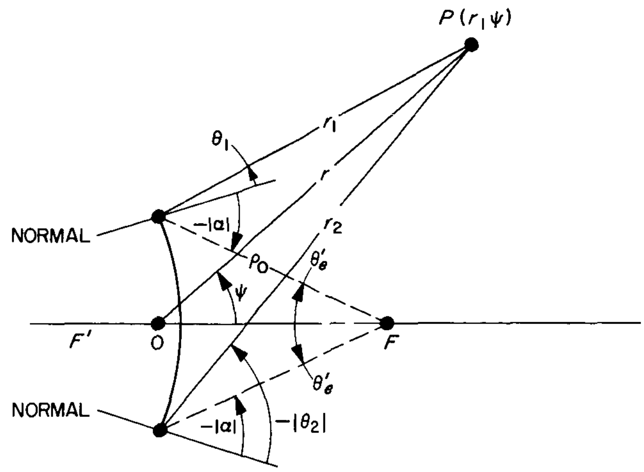


Fig. 7. Edge-diffracted geometry for hyperboloid

each edge as was done in the previous section. The results at great distances, Fig. 7, are:

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ikr}\, (kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)$$

$$\times\left[\cos\tau\left(\frac{\sin\frac{\gamma}{2} - \cos\frac{\gamma}{2}}{\sin\alpha - \sin\gamma} + \frac{\sin\frac{\beta}{2} + \cos\frac{\beta}{2}}{\sin\alpha + \sin\beta}\right) + i\sin\tau\left(\frac{\sin\frac{\gamma}{2} - \cos\frac{\gamma}{2}}{\sin\alpha - \sin\gamma} - \frac{\sin\frac{\beta}{2} + \cos\frac{\beta}{2}}{\sin\alpha + \sin\beta}\right)\right],\qquad 0 < \Psi < 90 - \delta_t \qquad (44)$$

where $\tau = (kD/2)\sin\Psi - \pi/4$; $\beta = \pi + \delta_t - \Psi$; $\gamma = \pi - \delta_t - \Psi$.

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ikr}\, (kD/2)^{1/2}}{(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$$

$$\times\left[\cos\tau\left(\frac{\sin\frac{\gamma}{2} + \cos\frac{\gamma}{2}}{\sin\alpha - \sin\gamma} + \frac{\sin\frac{\beta}{2} - \cos\frac{\beta}{2}}{\sin\alpha + \sin\beta}\right) + i\sin\tau\left(\frac{\sin\frac{\gamma}{2} + \cos\frac{\gamma}{2}}{\sin\alpha - \sin\gamma} - \frac{\sin\frac{\beta}{2} - \cos\frac{\beta}{2}}{\sin\alpha + \sin\beta}\right)\right],\qquad 0 < \Psi < 90 - \delta_t \qquad (45)$$

Equations (44) and (45) are valid everywhere in the range $0 < \Psi < 90 - \delta_t$ except at $\Psi = \theta_e$, the shadow-light boundary.

The singly edge-diffracted ray from Edge 2 is blocked by the hyperboloid in the range $90 - \delta_t < \Psi < 90$, with the result that:

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ikr}\, (kD/2)^{1/2}}{2\,(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left(e^{-i\tau}\right)\left(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\right)\left(\frac{\sin\frac{\beta}{2} + \cos\frac{\beta}{2}}{\sin\alpha + \sin\beta}\right),\qquad 90 - \delta_t < \Psi < 90 \qquad (46)$$

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ikr}\, (kD/2)^{1/2}}{2\,(2\pi)^{1/2}\,(kr)\,(\sin\Psi)^{1/2}}\left(e^{-i\tau}\right)\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)\left(\frac{\sin\frac{\beta}{2} - \cos\frac{\beta}{2}}{\sin\alpha + \sin\beta}\right),\qquad 90 - \delta_t < \Psi < 90 \qquad (47)$$

Results similar to Eqs. (44) and (45) may be obtained for the range $\Psi > 90$. However, these expressions are not of great physical interest and have not been included.

It is again necessary to apply a correction factor at $\Psi = 0$, the axial caustic. The results of this correction are:

$$H\text{-plane:}\quad E_d = \frac{-E_{inc}\, e^{ikr}\,(kD/2)}{2\,(kr)} \left(\sec\frac{\theta_e'}{2} - \csc\frac{\theta_e}{2} \right), \qquad \Psi = 0 \tag{48}$$

$$E\text{-plane:}\quad H_d = \frac{-H_{inc}\, e^{ikr}\,(kD/2)}{2\,(kr)} \left(\sec\frac{\theta_e'}{2} + \csc\frac{\theta_e}{2} \right), \qquad \Psi = 0 \tag{49}$$

The resulting H-plane scattered field is plotted in Fig. 8 for a specific numerical example. Plotted in the figure are curves computed from: Kirchhoff integral theory; geometrical optics; and geometrical diffraction theory, which contains the geometrical optic ray and the two singly edge-diffracted rays. In the geometrical "backscattering" region from 0 to 60 deg, the Kirchhoff result and the geometrical diffraction results virtually coincide, except near the shadow-light boundary at 60 deg. As discussed earlier the asymptotic expressions, upon which the geometrical theory is based, are not valid close to a shadow-light

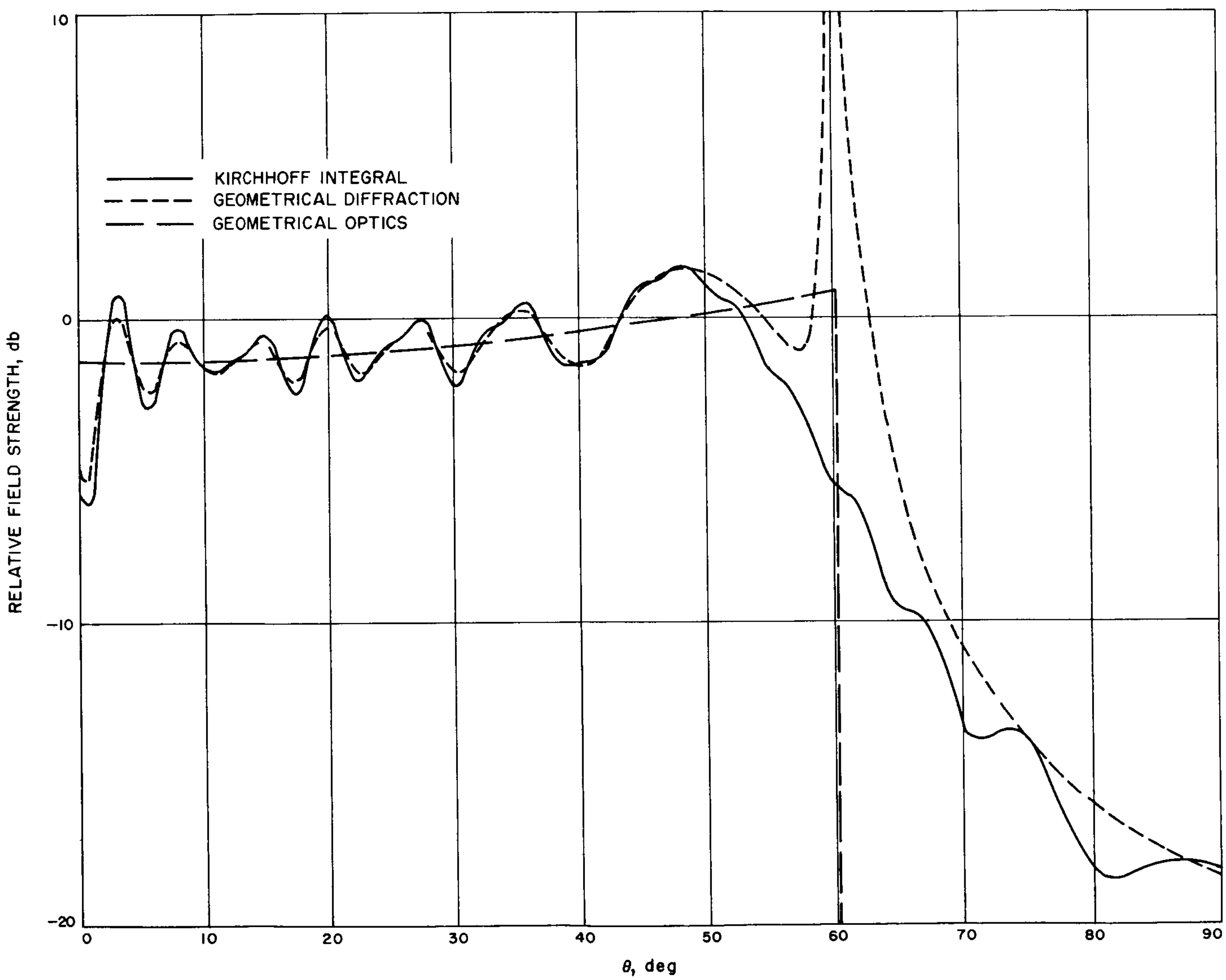


Fig. 8. *H*-plane field scattered from hyperboloid; $D = 24.6\lambda$, $e = 1.6707$

boundary. It can be shown, however, and verified experimentally, that *at* a shadow-light boundary the scattered field is ½ (i.e., −6 db) of the value predicted from geometrical optics. The theory must be modified to give accurate results on either side of the boundary. Such a modification has been outlined by Kay (Ref. 8) who employs Fresnel zones to calculate the field in this region.

V. Summary

The geometrical theory of singly edge-diffracted rays provides an excellent first order correction to the theory of geometrical optics in application to antenna theory. The theory compares favorably with the classical Kirchhoff integral theory, but the computational time is an order of magnitude smaller. The geometrical theory provides a clearer physical insight into diffraction phenomena. The greatest disadvantage of the geometrical theory lies in the singularities near the shadow-light boundary. Until more precise geometrical constructions are available for analysis near this singular region, it is necessary to apply the integral theory here. Consequently, the geometrical techniques and integral techniques complement each other in most reflector antenna problems.

Appendix

Derivation[5] of Axial-Caustic Correction Factor

A general solution of the scalar wave equation is:

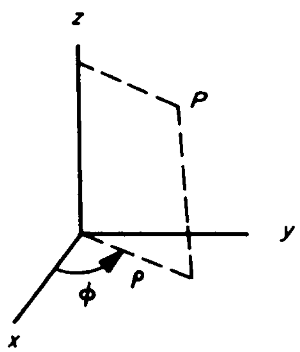
$$u = e^{ikz\sin\delta} J_n(k\rho\cos\delta)\cos n\phi \tag{A-1}$$

For large $k\rho >> 1$, the asymptotic value of the Bessel function can be used, yielding:

$$u = e^{ikz\sin\delta} \frac{(2)^{1/2}\cos n\phi}{(\pi k\rho\cos\delta)^{1/2}} \cos\left[k\rho\cos\delta - \frac{\pi}{2}(n + 1/2)\right] \tag{A-2}$$

Expanding the cosine factor:

$$u = \frac{\cos n\phi}{(2\pi k\rho\cos\delta)^{1/2}} \left[e^{ik(\rho\cos\delta + z\sin\delta) - i(n+1/2)\pi/2} + e^{-ik(\rho\cos\delta - z\sin\delta) + i(n+1/2)\pi/2}\right] \tag{A-3}$$



Hence two types of ray pass through each point P, one moving away from the z-axis making an angle $\pi/2 - \delta$ with the axis (the first term of Eq. A-3) and one moving toward the z-axis making an angle $\pi/2 - \delta$ with the axis. Since Eq. (A-2), and hence Eq. (A-3), blows up on the z-axis, it is necessary to multiply Eq. (A-2) by a correction factor which yields Eq. (A-1) for $k\rho << 1$ and the useful geometric result Eq. (A-3) for large $k\rho >> 1$. By inspection this correction factor is:

$$\frac{(\pi k\rho\cos\delta)^{1/2}}{(2)^{1/2}} J_n(k\rho\cos\delta)\sec\left[k\rho\cos\delta - \frac{\pi}{2}(n + 1/2)\right]$$

[5]This derivation is found in Appendix IV of Ref. 1.

References

1. Keller, J. B., "Diffraction by an Aperture," *Journal of Applied Physics,* Vol. 28, No. 4, pp. 426–444, April 1957.

2. Kinber, B. Y., "Lateral Radiation of Parabolic Antennas," *Radio Engineering and Electronic Physics,* pp. 481–492, January–June 1961.

3. Galindo, V., *Synthesis of Dual Reflector Antennas,* Technical Report 64-22, Electronics Research Laboratory, University of California, Berkeley, California.

4. Sommerfeld, A. J. W., *Optics,* p. 249, Academic Press, Inc., New York, 1954.

5. Konyoumajian, R. G., "Asymptotic High-Frequency Methods," *Proceedings IEEE,* pp. 864–876, August 1965.

6. Silver, S., *Microwave Antenna Theory,* p. 194, McGraw-Hill, New York.

7. Kritikos, H. N., "The Extended Aperture Method for the Determination of the Shadow Region Radiation of Parabolic Reflectors," *IEEE Transactions on Antenna and Propagation,* pp. 400–404, July 1963.

8. Kay, A. F., *The Application of Fresnel Zone Theory to Microwave Antenna Design,* TRG, Inc. Technical Report No. 3, November 15, 1962, East Boston, Massachusetts.