NASA TECHNICAL NOTE

NASA TN D-4001

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NASA TN D-4001

APPROXIMATION FOR RADIANT ENERGY TRANSPORT IN NONGRAY, NONSCATTERING GASES

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1967

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SUMMARY

Radiant heat transfer is the most important mode of energy transfer in many high-temperature gases and plasmas; thus, it is necessary to be able to estimate its magnitude with confidence. In cases where the temperature varies appreciably over a photon mean free path, a transport analysis is generally required. When a plasma has a complex spectrum, such an analysis can be unduly time consuming even with a high-speed digital computer unless approximations are made.

An approximation for integrated intensity in nongray, nonscattering gases is presented that uses a new mean absorption coefficient in the gray-gas transport equation. This absorption coefficient is called an "effective absorption coefficient." The approximation was tested by calculating the integrated intensities by exact and by approximate methods for 39 cases. The integrated intensities calculated by the effective absorption coefficient approximation had errors of only -25 to +28 percent, which demonstrated the superiority of this approximation over a similar approximation using the Planck mean absorption coefficient. Despite certain restrictions on the form of the spectral absorption coefficient, the effective absorption coefficient approximation is expected to be useful for most problems of nonscattering plasmas in local thermodynamic equilibrium (LTE) even where large temperature and pressure differences and inhomogeneities are present.

INTRODUCTION

In very-high-temperature propulsion devices such as gaseous nuclear rockets, the dominant mechanism of heat transfer is radiant energy exchange between volumes of plasma and between plasma and the wall. There are two general methods of analyzing such heat transfer: transport analysis and the diffusion approximation. An introduction to these two methods is contained in reference 1.

The term "transport analysis" generally refers to a method of calculating radiant flux density by considering conditions at other points in space besides the point at which the radiant flux density is desired. In this report, transport analysis refers to single or multiple use of the transport equation (eq. (131) of ref. 1 or some modification of it) to calculate integrated intensity in some particular direction in space from the point where the radiant flux density is desired. This calculation is repeated for other directions, and an integration over solid angle is performed to obtain the radiant flux density. If no approximations are made, transport analysis gives the correct value of radiant flux density.

The usual diffusion approximation is really an approximate transport analysis in which the radiant flux density is inversely proportional to the Rosseland mean absorption coefficient and directly proportional to the temperature gradient at the point where the radiant flux density is desired (ref. 1). This approximation cannot be equaled for simplicity, but it is not applicable when temperature varies appreciably in a photon mean free path as it may far from the center of the core of a gaseous nuclear rocket.

The gray-gas transport equation is merely the monochromatic transport equation (eq. (131) of ref. 1) integrated over all wave numbers with the assumption that the spectral (or monochromatic) absorption coefficient is independent of wave number. The gray-gas transport equation is, in general, rigorously applicable only to a gray gas or to a gas that is optically thin at all important wave numbers (important wave numbers are those within about one order of magnitude of the wave number of the maximum specific intensity of the Planck function).

For other cases, the monochromatic transport equation is customarily applied at a sufficiently large number of wave numbers to enable calculation of integrated intensity by numerical quadrature. The radiant flux density is then obtained by integrating the integrated intensity over all solid angles. If the absorption coefficient has an intricate dependence on wave number and the problem does not involve slab geometry, calculation of radiant flux density by this method may be too time consuming even with a high-speed digital computer. For a uranium plasma, this difficulty can be appreciated as more than 9000 spectral lines have been observed in a uranium arc. For this and other reasons, radiant heat transfer far from the center of the core of a gaseous nuclear rocket has not been analyzed with any confidence.

Attempts to find satisfactory approximations for calculating radiant energy transport in nongray gases of any optical thickness have been made by a number of investigators. Stewart (ref. 2) has obtained a solution for slab-geometry steady-state problems where the spectral absorption coefficient can be factored into a part that is dependent on temperature and density and a part that is dependent on wave number. Grant (ref. 3) has obtained an approximation by dividing the spectrum into wave-number (or frequency) groups and by dividing the spectral absorption coefficient into absorption coefficient groups. Unfortunately, the latter approximation is only valid when the spectral absorption coefficient is independent of position. Sampson (ref. 4) has obtained an approximation for slab-geometry problems that gives radiant flux densities and their divergences that are "nearly always within a factor of 2 of the correct values." This approximation gives exact values of these quantities for three classes of problems: (1) an optically thick gas where the diffusion approximation is applicable, (2) a gas that is optically thin at all important wave numbers, and (3) a gray gas. All three methods lack either generality or accuracy. Viskanta (ref. 5) and Campbell and Nelson (ref. 6) used flux-weighted mean absorption coefficients that require calculation of spectral fluxes in order to evaluate them.

In this report, a new approximation is presented in which the integrated intensity is obtained from the gray-gas transport equation by using therein an effective absorption coefficient averaged over the spectrum once and for all. The use of the effective absorption coefficient in the gray-gas transport equation is expected to give a satisfactory approximation to integrated intensity for most cases involving a nonscattering gas in local thermodynamic equilibrium and possessing spectral lines, continua, or both. The effective absorption coefficient gives exact values of integrated intensity for five classes of problems. The effective absorption coefficient approximation was successfully applied to calculate integrated intensity for 39 cases, 33 of which did not belong to the five classes.

SYMBOLS

A	area, cm^2
Δa	perturbation in spectral linear absorption coefficient, cm^{-1}
a_e	effective linear absorption coefficient, cm^{-1}
a_g	gray linear absorption coefficient, cm^{-1}
a_{Pl}	Planck mean linear absorption coefficient, cm^{-1}
a_{Ro}	Rosseland mean linear absorption coefficient, cm^{-1}
a_V	spectral (or monochromatic) linear absorption coefficient at photon energy V, cm^{-1}
$a_{\tilde{\nu}}$	spectral (or monochromatic) linear absorption coefficient at wave number $\tilde{\nu}$, cm^{-1}
B	integrated Planck function (called integrated blackbody intensity in ref. 1), W/(cm^2)(steradian)
$B_{\tilde{\nu}}$	Planck (or blackbody) function, W/(cm)(steradian)

b_c	collision half-width of Lorentz line (line with a dispersion contour), cm^{-1}
$\bar{b}_c$	average collision half-width of Lorentz lines, cm^{-1}
c	velocity of light in a vacuum, cm/sec
$E_{\tilde{\nu}}$	power radiated by spontaneous emission, $\mathrm{W/(cm^2)(steradian)}$
$\vec{F}$	radiant flux density, $\mathrm{W/cm^2}$
F_n	component of $\vec{F}$ in the direction of $\vec{n}$, $\mathrm{W/cm^2}$
f	cumulative distribution
f^{-1}	inverse function of f
h	Planck constant, (J)(sec)
I	integrated intensity, $\mathrm{W/(cm^2)(steradian)}$
$I_{\tilde{\nu}}$	specific intensity, W/(cm)(steradian)
J	line strength parameter
k	Boltzmann constant, $\mathrm{J/^{o}K}$
L	distance from point P to wall (fig. 1), cm
M	parameter in series expansion of Q
$\vec{n}$	unit vector normal to dA (fig. 1)
P	probability parameter, $\mathrm{cm^2}$
p	pressure, atm
Q	fraction of Planck function between $\tilde{\nu}$ and infinite $\tilde{\nu}$
q	integral in denominator of effective absorption coefficient for narrow wave-number group, cm^{-1}
R	random number
S	distance along ray (fig. 1), cm
$\tilde{S}$	integrated linear absorption coefficient of spectral line, cm^{-2}
$\bar{S}$	average integrated linear absorption coefficient of spectral lines, cm^{-2}
T	temperature, $^{o}\mathrm{K}$
t	distance- or temperature-dependent factor of spectral linear absorption coefficient
V	dimensionless photon energy
W	square of V

w	wave-number-dependent factor of spectral linear absorption coefficient, cm^{-1}
Δw	perturbation in w, cm^{-1}
w_g	constant term in wave-number-dependent factor of spectral linear absorption coefficient, cm^{-1}
X	mole fraction
x	component of S in direction $\vec{n}$ (fig. 1), cm
y	dummy variable
z	integral in numerator of effective absorption coefficient for narrow wave-number group, cm^{-2}
θ	angle between ray and direction $\vec{n}$ (fig. 1), rad
$\tilde{\nu}$	wave number, cm^{-1}
$\Delta\tilde{\nu}$	difference of two wave numbers, cm^{-1}
σ	Stefan-Boltzmann constant, $W/(cm^2)(^{o}K)^4$
τ_e	effective optical depth
τ_g	gray optical depth
$\tau_{\tilde{\nu}}$	spectral (or monochromatic) optical depth
ψ	local conditions in gas (temperature and pressure or, if more than one gas, partial pressures of the different gases)
Ω	solid angle (fig. 1), steradians

Subscripts:

a	gas a
approx	approximate
b	gas b
d	gas d
e	gas e
i	isothermal, homogeneous, and constant pressure
j	alternate subscript for origin group or gas
k	alternate subscript for origin group or gas
l	alternate subscript for origin group or gas
line	spectral line

m	wave-number group, minimum wave number in wave-number group m, or minimum photon energy in wave-number group m
m+1	maximum wave number in wave-number group m or maximum photon energy in wave-number group m
max	maximum value
min	minimum value
n	component in direction $\vec{n}$ (fig. 1)
O	$s = 0$
o	center of spectral line
r	origin group, species, or gas
u	number of origin groups or gases
v	alternate subscript for origin group or gas
w	wall
α	number of wave-number groups

ANALYSIS

The derivation of the effective absorption coefficient for a single gas or plasma, together with refinements for large temperature ratios and for inhomogeneous mixtures, is contained in this section. The effective absorption coefficient is derived initially for use in the gray-gas transport equation in order to express the integrated intensity for a homogeneous, isothermal, isobaric, nongray gas. A hypothesis is then made that allows the use of this equation as an approximation to the integrated intensity for a gas with any kind of gradients.

This hypothesis has two principal defects, one of which causes appreciable errors for large temperature ratios and the other causes errors for inhomogeneous mixtures. The first defect is largely remedied by dividing the spectrum into wave-number groups, with a modified effective absorption coefficient and modified gray-gas transport equation for each group. The second defect is largely remedied by dividing the spectral absorption coefficient into origin groups according to the origin of the spectra and by assuming no correlation between the spectra of different origin groups. This division requires the assignment of an effective absorption coefficient to each origin group.

The foregoing approximations and hypothesis are extended to take into account a black wall at finite temperature.

Assumptions and Restrictions

The assumptions utilized in the following derivations are listed herein for convenience.

(1) The term "gas" includes "plasma."

(2) No scattering of radiation occurs.

(3) The gas is in local thermodynamic equilibrium, that is, the composition, degree of molecular excitation, and velocity distribution at any point correspond to thermodynamic equilibrium at some temperature, but the specific intensities do not have to correspond to the Planck function at that temperature.

(4) The gas is nonrelativistic.

(5) Changes in the state of the gas are slow enough that the time of flight of photons may be neglected.

(6) Doppler shift in the spectrum may be neglected. When narrow spectral lines are present, this is a more stringent requirement than (4) or (5).

(7) Any walls are black, that is, they do not reflect or transmit light.

(8) The ordinary index of refraction is 1.

(9) There is no interaction between photons.

(10) Only one photon interacts with a particle at any given time.

(11) The molecules of the gas have random orientation in space.

The analysis is neither limited to gases with only spectral lines or only continua, nor is it restricted as to optical thickness.

Basic Relations

Some fundamental relations are needed in subsequent sections and are given here. Let $a_{\tilde{\nu}}$ be the spectral linear absorption coefficient of a gas, including the induced emission factor (ref. 7). If S is the distance along a ray from the point P (fig. 1), then the spectral optical depth of S relative to P is

$$\tau_{\tilde{\nu}} \equiv \int_0^S a_{\tilde{\nu}}(S')dS' \tag{1}$$

The Planck function is given by

$$B_{\tilde{\nu}} = \frac{2hc^2\tilde{\nu}^3}{e^{hc\tilde{\nu}/kT} - 1} \tag{2}$$

Here h is the Planck constant, c is the velocity of light, $\tilde{\nu}$ is the wave number, k is the Boltzmann constant, and T is the temperature.

The value of the specific intensity $I_{\tilde{\nu}}$ at point P in the direction from the wall at L toward P is obtained by integration of equation (116) of Chandrasekhar (ref. 1) with appropriate boundary conditions.

$$I_{\tilde{\nu}} = \int_0^L B_{\tilde{\nu}}(S)e^{-\tau_{\tilde{\nu}}(S)}a_{\tilde{\nu}}(S)dS + B_{\tilde{\nu},w}e^{-\tau_{\tilde{\nu}}(L)} \tag{3}$$

This is the monochromatic transport equation. The quantity $B_{\tilde{\nu}}$ is the Planck function for the gas temperature, whereas $B_{\tilde{\nu},w}$ is the Planck function for the wall temperature.

The integrated intensity I at point P is found by integrating equation (3) over all wave numbers

$$I = \int_0^\infty \int_0^L B_{\tilde{\nu}}(S)e^{-\tau_{\tilde{\nu}}(S)}a_{\tilde{\nu}}(S)dS\, d\tilde{\nu} + \int_0^\infty B_{\tilde{\nu},w}e^{-\tau_{\tilde{\nu}}(L)}d\tilde{\nu} \tag{4}$$

According to equation (4), the contribution to I at point P due to the gas along an element of ray dS is

$$dI = \int_0^\infty B_{\tilde{\nu}}(S)e^{-\tau_{\tilde{\nu}}(S)}a_{\tilde{\nu}}(S)d\tilde{\nu}\, dS \tag{5}$$

For a gray gas the spectral linear absorption coefficient including the induced emission factor (ref. 7) is independent of wave number and is designated a_g. The corresponding gray optical depth at S is

$$\tau_g \equiv \int_0^S a_g(S')dS' \tag{6}$$

Hence, for a gray gas, equation (4) becomes

$$I = \int_0^L B(S)e^{-\tau_g(S)}a_g(S)dS + B_w e^{-\tau_g(L)} \tag{7}$$

where B is the integrated Planck function for the gas temperature, and B_w is the integrated Planck function for the wall temperature. According to equation (7), the contribution to I at point P due to the gas along an element of ray dS is

$$dI = B(S)e^{-\tau_g(S)}a_g(S)dS = B(S)e^{-\int_0^S a_g(S')dS'}a_g(S)dS \tag{8}$$

Effective Absorption Coefficient for Single Gas

In this section, the effective absorption coefficient is derived for an isothermal, isobaric, single gas or homogeneous mixture, and its mathematical behavior is discussed. A hypothesis is made that permits its application to gases with any kinds of gradients.

Homogeneous, isothermal, isobaric, nongray gases. - In order to use equations of the form of equations (7) and (8) for a homogeneous, isothermal, isobaric, nongray gas, an effective absorption coefficient a_e must be found to use in place of a_g. Specifically,

$$dI = Be^{-\int_0^{S_i} a_e(S_i')dS_i'} a_e(S_i)dS_i \tag{9}$$

where the subscript i has been added to S to indicate that the gas is homogeneous, isothermal, and isobaric. The effective absorption coefficient a_e must take whatever form is necessary so that equation (9) is valid for any homogeneous, isothermal, isobaric gas. For a given wave number, $a_{\tilde{\nu}}$ and $B_{\tilde{\nu}}$ are independent of S_i; thus equating equations (5) and (9) results in

$$\int_0^\infty B_{\tilde{\nu}} a_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S_i} d\tilde{\nu} = Be^{-\int_0^{S_i} a_e(S_i')dS_i'} a_e(S_i) \tag{10}$$

Integrating equation (10) with respect to S_i from 0 to S_i'' gives

$$-\int_0^\infty B_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S_i''} d\tilde{\nu} + B = -Be^{-\int_0^{S_i''} a_e(S_i')dS_i'} + B \tag{11}$$

Cancelling the two B's, dropping the double primes in equation (11), and dividing equation (10) by equation (11) yield

$$a_e(S_i) = \frac{\int_0^\infty B_{\tilde{\nu}} a_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S_i} d\tilde{\nu}}{\int_0^\infty B_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S_i} d\tilde{\nu}} \tag{12}$$

For a homogeneous, isothermal, isobaric, gray or nongray gas and a cold wall at L, equation (9) can be integrated to give

$$I = \int_0^L Be^{-\int_0^{S_i} a_e(S_i')dS_i'} a_e(S_i)dS_i = \int_0^{\tau_e(L)} Be^{-\tau_e} d\tau_e \tag{13}$$

where the effective optical depth τ_e is defined by

$$\tau_e(S_i) \equiv \int_0^{S_i} a_e(S_i')dS_i' \tag{14}$$

Substituting equation (12) into equation (14) and noting that the numerator in equation (12) is just the negative of the derivative of the denominator with respect to S_i give

$$\tau_e(S_i) = \ln \frac{B}{\int_0^\infty B_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S_i} d\tilde{\nu}} \tag{15}$$

The effective optical depth τ_e is always measured from the point P where the integrated intensity and radiant flux density are desired.

Behavior of a_e for a homogeneous, isothermal, isobaric gas. - The variation of a_e as S_i varies is relatively simple. As S_i approaches 0, it is clear from equation (12) that a_e approaches the Planck mean absorption coefficient a_{Pl} given by

$$a_{Pl} \equiv \frac{1}{B} \int_0^\infty B_{\tilde{\nu}} a_{\tilde{\nu}} \, d\tilde{\nu} \tag{16}$$

As S_i approaches infinity, a_e approaches the smallest value of $a_{\tilde{\nu}}$ in the spectrum. For intermediate values of S_i, the function $a_{\tilde{\nu}}$ which results in a maximum value of a_e with a_{Pl} held constant can be shown to be the constant value a_{Pl}. Hence $a_e \leq a_{Pl}$ always.

The variation of a_e with S_i has a physical significance for a nongray gas. When S_i is very small, virtually all the radiation emitted by gas along dS_i will reach point P (fig. 1). The appropriate mean absorption coefficient for use in the gray-gas transport equation for such optically thin cases is known to be a_{Pl} (ref. 4) because there is no measurable self-absorption, that is, the gas does not absorb its own radiation to any extent.

If the gas is gray and if τ_g is large, then radiation emitted by gas along dS_i will be strongly self-absorbed before it reaches point P; thus, only a small fraction of the radiation emitted along dS_i gets to P. All photons emitted along dS_i, regardless of wave number, will have the same probability of getting to P, namely, $\exp(-\tau_g)$. Equation (7), which contains $\exp(-\tau_g)$ in its first term, then gives the correct integrated intensity at P.

If, on the other hand, the gas is nongray, the dependence of the emission of the gas on $a_{\tilde{\nu}}$ must be considered. If equation (119) in reference 1 is used, the Kirchhoff law may be expressed as $E_{\tilde{\nu}} = a_{\tilde{\nu}}B_{\tilde{\nu}}$ where $E_{\tilde{\nu}}$ is the power per unit volume per unit solid angle per unit wave number radiated by spontaneous emission. In addition to the dependence of $E_{\tilde{\nu}}$ on $B_{\tilde{\nu}}$, $E_{\tilde{\nu}}$ is large at those wave numbers where $a_{\tilde{\nu}}$ is large. The absorption of radiation from a nongray gas by the same gas can therefore be termed "preferential self-absorption" because for a given path length the gas will absorb a larger fraction of its own radiation than of radiation from a blackbody at its temperature.

For an optically thick, nongray gas, the contribution to the integrated intensity at P by gas along dS_i far from P is smaller than would be calculated if a_{Pl} were used in equation (7) or (13). For a homogeneous, isothermal, isobaric, nongray gas, this effect due to preferential self-absorption is allowed for properly by using a_e in equation (13) instead of a_{Pl} because a_e is weighted properly to account for preferential self-absorption.

Nongray gases with gradients. - In cases where $a_{\tilde{\nu}}$ can be approximately or exactly factored into a product $w(\tilde{\nu})t(S)$, the concept of preferential self-absorption is valid. The degree of preferential self-absorption (or, equivalently, how much less a_e is than a_{Pl}) would be expected to depend on the effective optical depth τ_e rather than S.

The hypothesis is made here that τ_e can be used as an approximate index of the degree of preferential self-absorption in any gas of a single species whether or not $a_{\tilde{\nu}}$ can be exactly factored into a product $w(\tilde{\nu})t(S)$ and whether or not temperature and pressure are constant. The basis for this hypothesis is that the spectra of an actual gas at moderately different temperatures and pressures are not markedly different from one another except in magnitude; thus, in a practical problem $a_{\tilde{\nu}}$ does approximately factor into a product $w(\tilde{\nu})t(S)$. This hypothesis is used below in finding an expression for a_e, whose value relative to a_{Pl} is merely a measure of the degree of preferential self-absorption in the gas.

The value of a_e may be computed if the value of a_e at S in a gas with gradients is assumed to be the same as the value of a_e for an isothermal, isobaric gas having the same effective optical depth (same degree of preferential self-absorption) and same pressure and temperature as at S in the gas with gradients. This assumption may be called the equal-effective-optical-depth hypothesis. Before expressing this hypothesis mathematically, the quantity S_i is redefined as the distance along a ray in an isothermal, isobaric gas having temperature and pressure the same as at S and having such a value that the effective optical depths τ_e in the gas with gradients and in the isothermal, isobaric gas are the same. Then a_e for a gas with gradients is given by the expression for the associated isothermal, isobaric gas.

$$a_e\left[\psi(S), S_i\right] = \frac{\int_0^\infty B_{\tilde{\nu}}(\psi) a_{\tilde{\nu}}(\psi) e^{-a_{\tilde{\nu}}(\psi) S_i} \, d\tilde{\nu}}{\int_0^\infty B_{\tilde{\nu}}(\psi) e^{-a_{\tilde{\nu}}(\psi) S_i} \, d\tilde{\nu}} \tag{17}$$

where ψ stands for the temperature and pressure at S in the gas with gradients and, of course, everywhere in the associated isothermal, isobaric gas.

The effective optical depth for a gas with gradients is defined by

$$\tau_e(S) \equiv \int_0^S a_e\left[\psi(S'), S_i(S')\right] dS' \tag{18}$$

which is analogous to equation (14). The effective optical depth for an isothermal, isobaric gas of depth S_i with the same pressure and temperature as at S in a gas with gradients is, according to equation (15),

$$\tau_e = \ln \frac{B(S)}{\int_0^\infty B_{\tilde{\nu}}(S) e^{-a_{\tilde{\nu}}(S) S_i} \, d\tilde{\nu}} \tag{19}$$

The definition of S_i given previous to equation (17) may then be expressed mathematically by equating equations (18) and (19).

$$\int_0^S a_e\left[\psi(S'), S_i(S')\right] dS' \equiv \tau_e = \ln \frac{B(S)}{\int_0^\infty B_{\tilde{\nu}}(S) e^{-a_{\tilde{\nu}}(S) S_i(S)} \, d\tilde{\nu}} \tag{20}$$

In general S_i is not equal to S. Solving equation (20) for $S_i(S)$ and then equation (17) for $a_e(S)$ is a time consuming procedure by any method. This procedure may be avoided by noting that, for a given value of S, S_i and τ_e are uniquely related as shown by the right-hand side of equation (20). Hence a_e may be regarded as a function of $\psi(S)$ and τ_e; thus, differentiating the left-hand side of equation (20) gives

$$\frac{d\tau_e}{dS} = a_e\left[\psi(S), \tau_e\right] \tag{21}$$

and the quantity S_i no longer enters explicitly into the calculation of τ_e. Equation (21) may be integrated readily by numerical methods to get $\tau_e(S)$ if $a_e\left[\psi(S), \tau_e\right]$ is available in a form such as a table. This integration is started by using the boundary condition $\tau_e = 0$ at $S = 0$.

For a given gas, values of $a_e\left[\psi(S), \tau_e\right]$ may be tabulated once and for all in a three-dimensional table with arguments of pressure p, T, and τ_e because a_e depends only on p, T, and τ_e. Such a table may be constructed by selecting p, T, and a number of values of S_i. The quantities a_e and τ_e are calculated from equations (17) and (19), respectively, for each value of S_i, and the results are interpolated to find a_e at the desired values of τ_e for the table. This procedure is then repeated for other values of p and T.

The previous considerations make it possible to generalize equation (13) to give the effective transport equation for a gas with or without temperature and pressure gradients and with a cold wall at L, namely,

$$I = \int_0^L B(S)e^{-\int_0^S a_e\left[\psi(S'), \tau_e'\right]dS'} a_e\left[\psi(S), \tau_e\right]dS = \int_0^{\tau_e(L)} Be^{-\tau_e}\, d\tau_e \tag{22}$$

When I must be evaluated a sizable number of times for the same gas and when that gas possesses a complex spectrum, it is faster to tabulate $a_e\left[\psi(S), \tau_e\right]$ once and for all and then to solve equations (21) and (22) once for each I than to evaluate the double integral in the exact equation (4) once for each I. This advantage becomes clearer if it is considered that for a plasma with, say, 9000 spectral lines the integral with respect to S in equation (4) would probably have to be evaluated by numerical methods at 40 000 or more different wave numbers so that the integral with respect to wave number could be evaluated by numerical methods with reasonable accuracy. This procedure would have to be followed for each I if the exact equation (4) were used.

Exact classes of problems. - Equation (22) has already been shown to be exact for (1) isothermal, isobaric, single gases and homogeneous mixtures and is shown in the next two paragraphs to be exact for (2) gray or (3) optically thin gases with or without temperature or pressure gradients and with a cold, black wall behind them.

In the case of a gray gas, equation (17) gives $a_e = a_g$. Substituting this equality into equation (22) and utilizing equation (6) gives equation (7) except for the last term, which is appreciable only if the wall is not cold. Since equation (7) is exact, it follows that equation (22) is exact for gray gases with a cold, black wall behind them.

For a gas that is optically thin at all wave numbers, that is, $a_{\tilde{\nu}}S << 1$, $a_e = a_{Pl}$. If this substitution is made and the exponential is expanded in a power series, equation (22) becomes

$$I = \int_0^L B(S)\left[1 - \int_0^S a_{Pl}(S')dS' + \ldots\right]a_{Pl}(S)dS \tag{23}$$

Because $a_{\tilde{\nu}}S << 1$ at all wave numbers,

$$\int_0^S a_{Pl}(S')dS' << 1 \tag{24}$$

so all but the first term of the power series may be neglected, giving

$$I = \int_0^L B(S)a_{Pl}(S)dS \tag{25}$$

Some manipulation of exact equation (4) is necessary to show that it is identical to equation (25) for an optically thin gas. If the wall is cold, the last term of equation (4) may be dropped. Setting the exponential equal to 1 in equation (4), reversing the order of integration, and utilizing equation (16) gives equation (25). Hence, equation (22) is exact for a gas that is optically thin at all wave numbers and has a cold, black wall behind it.

Refinement by Using Wave-Number Groups

Error due to Planck function shift. - In problems involving large maximum-to-minimum temperature ratios, nongray gas, and spectral optical depths of order of magnitude 1 or more, an appreciable error is frequently present in equation (22) because of the shift of the curve of $B_{\tilde{\nu}}$ against $\tilde{\nu}$ to larger values of $\tilde{\nu}$ as temperature increases. This error is not evident from a cursory inspection of equation (22) because the dependence on the Planck function $B_{\tilde{\nu}}$ is hidden in the evaluation of a_e (eq. (17)). The error can be demonstrated, but the derivation is long. The error is principally due to the integrand of the exponential of equation (22) being calculated with values of a_e determined with Planck functions corresponding to temperatures at S' rather than at S.

Introduction of wave-number groups. - The error may be minimized by writing several equations analogous to equation (22) but with each applicable to a particular wave-number range. Thus, every important wave number is included in one of the wave-number ranges, which will hereafter be called "wave-number groups." The value of I may be obtained by simply adding the several equations. There is a precedent (ref. 3) for such a procedure. From the derivation that follows, it will become evident that when this procedure is carried to the extreme of having an infinite number of equations for an infinite number of very narrow wave-number groups, the value of I obtained is always exact.

The group intensity I_m of the m^{th} wave-number group is defined as all intensity with wave numbers between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$, and with the help of equation (3) I_m is given exactly by

$$I_m \equiv \int_m I_{\tilde{\nu}}\,d\tilde{\nu} = \int_m \int_0^{\tau_{\tilde{\nu}}(L)} B_{\tilde{\nu}}(\tau_{\tilde{\nu}})e^{-\tau_{\tilde{\nu}}}\,d\tau_{\tilde{\nu}}\,d\tilde{\nu} + \int_m B_{\tilde{\nu},w}e^{-\tau_{\tilde{\nu}}(L)}\,d\tilde{\nu} \qquad (26)$$

where m under the integral signs means to integrate from $\tilde{\nu}_m$ to $\tilde{\nu}_{m+1}$.

If the same derivation is followed as for equation (17), except that I_m is used wherever I was used, the group effective absorption coefficient is found to be

$$a_{e,m}\left[\psi(S), S_{i,m}\right] = \frac{\int_m B_{\tilde{\nu}}(\psi)a_{\tilde{\nu}}(\psi)e^{-a_{\tilde{\nu}}(\psi)S_{i,m}}\,d\tilde{\nu}}{\int_m B_{\tilde{\nu}}(\psi)e^{-a_{\tilde{\nu}}(\psi)S_{i,m}}\,d\tilde{\nu}} \qquad (27)$$

where an equal-effective-optical-depth hypothesis is required for each group. Thus, for the same value of S, values of $S_{i,m}$ are generally different for different groups.

Several other group quantities result from such a derivation. The group Planck function is defined by

$$B_m \equiv \int_m B_{\tilde{\nu}}\,d\tilde{\nu} \qquad (28)$$

The group effective optical depth $\tau_{e,m}$ is given by

$$\int_0^S a_{e,m}\left[\psi(S'), S_{i,m}(S')\right]dS' \equiv \tau_{e,m} = \ln \frac{B_m(S)}{\int_m B_{\tilde{\nu}}(S)e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)}\,d\tilde{\nu}} \qquad (29)$$

The group effective transport equation for a gas with or without gradients and with a cold wall at L is

$$I_m = \int_0^L B_m(S) e^{-\int_0^S a_{e,m}\left[\psi(S'), \tau'_{e,m}\right] dS'} a_{e,m}\left[\psi(S), \tau_{e,m}\right] dS$$

$$= \int_0^{\tau_{e,m}(L)} B_m e^{-\tau_{e,m}} d\tau_{e,m} \tag{30}$$

The integrated intensity may be obtained by summation of equation (30) over all α groups.

$$I = \sum_{m=1}^{\alpha} I_m \tag{31}$$

Exact classes of problem. - Equation (30) is exact for the following two additional classes of problem besides the three classes mentioned subsequent to equation (22). The first additional class of problem is the class where the elementary diffusion approximation (ref. 1) is applicable and where the wave-number group is narrow enough that the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ may be neglected. This exactness is proven in appendix A where it is also shown that the same radiant flux density is obtained either by integrating equation (30) or from the diffusion approximation. In the second additional class of problems where equation (30) is exact, $a_{\tilde{\nu}}$ must be of the form

$$a_{\tilde{\nu}}(S) = w_m(\tilde{\nu}) t_m(S) \tag{32}$$

for $\tilde{\nu}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$, and the wave-number group must again be narrow enough that the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ may be neglected. When these two conditions are satisfied, the exactness of equation (30) is proven in appendix B (eq. (B11)) for cases with or without temperature or pressure gradients.

Refinement by Using Origin Groups

Error due to $a_{\tilde{\nu}}$ not factoring. - The underlying assumption in the application of equation (30) to general problems involving temperature and pressure gradients is that $a_{\tilde{\nu}}$ may be factored approximately into the form given in equation (32). This assumption may break down in two types of problems:

(1) Problems involving inhomogeneous mixtures

(2) Problems involving large ratios of maximum to minimum temperature together with a gas possessing spectra of widely different excitation potentials

Introduction of origin groups. - Two spectral absorption coefficients $a_{\tilde{\nu},j}$ and $a_{\tilde{\nu},k}$ are said in this report to be uncorrelated within wave-number group m if

$$\int_m B_{\tilde{\nu}}(a_{\tilde{\nu},j} - a_{Pl,m,j})(a_{\tilde{\nu},k} - a_{Pl,m,k})d\tilde{\nu} = 0 \tag{33}$$

The quantities $a_{Pl,m,j}$ and $a_{Pl,m,k}$ are group Planck mean absorption coefficients for $a_{\tilde{\nu},j}$ and $a_{\tilde{\nu},k}$, respectively, and are defined by

$$a_{Pl,m,v} \equiv \frac{1}{B_m} \int_m B_{\tilde{\nu}} a_{\tilde{\nu},v}\, d\tilde{\nu} \tag{34}$$

where v can be j or k.

If spectra are sufficiently complex, then little or no correlation generally exists between $a_{\tilde{\nu}}$'s for different gases of a mixture or, for that matter, between the low-excitation-potential and high-excitation-potential contributions to $a_{\tilde{\nu}}$ of a single gas. This lack of correlation suggests the approach for refining equation (30). In many cases, a spectrum for which equation (32) is not even approximate may be so divided into u origin groups $r = 1, 2, 3, \ldots u$ that there is little or no correlation between the spectral absorption coefficients of any two origin groups, but the spectral absorption coefficient $a_{\tilde{\nu},r}$ of each origin group approximately factors the way $a_{\tilde{\nu}}$ does in equation (32). When origin groups are used, the spectral absorption coefficient $a_{\tilde{\nu}}$ of the gas or mixture is the sum of all the $a_{\tilde{\nu},r}$. For an example of the use of origin groups, if an inhomogeneous mixture contains three gases a, b, and c, and only gas c has spectra of widely different excitation potentials, then origin group 1 might correspond to all the spectrum of gas a, origin group 2 to all the spectrum of gas b, origin group 3 to the low-excitation-potential spectrum of gas c, and origin group 4 to the high-excitation-potential spectrum of gas c.

If no correlation is assumed to exist between the $a_{\tilde{\nu},r}$ of different origin groups, then there should be little or no preferential absorption of, say, radiation of origin group 1 by origin group 2. Consequently, $a_{e,m}$ should be replaced by the sum of the $a_{e,m,r}$ of the various origin groups, and the value of each $a_{e,m,r}$ should depend on the group effective optical depth $\tau_{e,m,r}$ for that origin group and no other. This dependence is obtained by defining the effective absorption coefficient of the r^{th} origin group in the m^{th} wave-number group by

$$a_{e,m,r}\left[\psi(S), S_{i,m,r}\right] \equiv \frac{\int_m B_{\tilde{\nu}}(\psi) a_{\tilde{\nu},r}(\psi) e^{-a_{\tilde{\nu},r}(\psi) S_{i,m,r}} \, d\tilde{\nu}}{\int_m B_{\tilde{\nu}}(\psi) e^{-a_{\tilde{\nu},r}(\psi) S_{i,m,r}} \, d\tilde{\nu}} \tag{35}$$

Here ψ now stands for temperature and partial pressures of all chemical species at S (which, in general, all affect the width of spectral lines of origin group r). The quantity $S_{i,m,r}$ is defined as the distance along a ray in an isothermal, isobaric gas having only the absorption coefficient $a_{\tilde{\nu},r}$ and having temperature and pressure the same as at S, such that the effective optical depths $\tau_{e,m,r}$ of the r^th^ origin group in the m^th^ wavenumber group in the isothermal, isobaric gas and in the gas with gradients are the same. In this definition both gases are assumed to have the same line widths.

The group effective optical depth $\tau_{e,m,r}$ is given by

$$\int_0^S a_{e,m,r}\left[\psi(S'), S_{i,m,r}(S')\right] dS' \equiv \tau_{e,m,r} = \ln \frac{B_m(S)}{\int_m B_{\tilde{\nu}}(S) e^{-a_{\tilde{\nu},r}(S) S_{i,m,r}(S)} \, d\tilde{\nu}} \tag{36}$$

The total group effective optical depth $\tau_{e,m}$ is

$$\tau_{e,m} = \sum_{r=1}^{u} \tau_{e,m,r} = \int_0^S \sum_{r=1}^{u} a_{e,m,r}\left[\psi(S'), \tau'_{e,m,r}\right] dS' \tag{37}$$

The group effective transport equation for a gas with or without gradients and with a cold wall at L is no different from equation (30) except for the evaluation of $\tau_{e,m}$.

$$I_m = \int_0^{\tau_{e,m}(L)} B_m(\tau_{e,m}) e^{-\tau_{e,m}} \, d\tau_{e,m} \tag{38}$$

It can be shown that the use of origin groups as outlined is equivalent to the assumption that radiation belonging to one origin group is absorbed by another origin group in the same proportion as radiation from a blackbody would be.

Exact classes of problems. - Equations (37) and (38) are either exact or an excellent approximation for six classes of problems.

The first class is a homogeneous mixture of any number of slightly nongray (slightly colored), isobaric gases with or without temperature gradients and with a cold, black wall behind it. The spectral absorption coefficients of the various gases are assumed to be uncorrelated and independent of position. For such a mixture, $a_{\tilde{\nu},r}$ of each gas r may be represented by

$$a_{\tilde{\nu},r} = a_{g,m,r} + \Delta a_r \tag{39}$$

where $a_{g,m,r}$ is the average value of $a_{\tilde{\nu},r}$ in the wave-number group, and Δa_r is any small wave-number-dependent perturbation. If the wave-number group is narrow enough that the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ may be neglected, then both the exact group transport equation obtained by integrating equation (3) assuming a cold wall

$$I_m = \int_m \int_0^{\tau_{\tilde{\nu}}(L)} B_{\tilde{\nu}}(\tau_{\tilde{\nu}}) e^{-\tau_{\tilde{\nu}}} \, d\tau_{\tilde{\nu}} \, d\tilde{\nu} \tag{40}$$

and the effective group transport equation (eq. (38)) may be expanded in series involving integrals of powers of $\Delta a_r/a_{g,m}$ where $a_{g,m}$ is the sum of all $a_{g,m,r}$ with the same m value. It is shown in appendix C that the two series expansions so obtained are identical through third order in $\Delta a_r/a_{g,m}$, which proves that equations (37) and (38) are an excellent approximation for this class of problem.

The second class is an inhomogeneous, isothermal mixture of any number of slightly nongray gases that may possess pressure gradients and have a cold, black wall behind them. The spectral absorption coefficients of the various gases are assumed to be uncorrelated. For such a mixture, $a_{\tilde{\nu},r}$ of each gas may be represented by

$$a_{\tilde{\nu},r} = (w_{g,m,r} + \Delta w_r) t_{m,r}(S) \tag{41}$$

where $w_{g,m,r}$ is a constant for the particular gas and wave number group, Δw_r is any small perturbation depending only on wave number and having an average value of 0 in the wave-number group, and $t_{m,r}$ is a factor depending only on S and due to concentration variation. Assuming the wave-number group may be any width, both the exact and the effective group transport equations may be expanded in a series involving integrals of powers of $\Delta w_r/w_{g,m,r}$. It is shown in appendix D that the two series expansions thus obtained are identical through third order in $\Delta w_r/w_{g,m,r}$, which shows that equations (37) and (38) are an excellent approximation for this class of problem.

The integrated or group intensity for class 3, consisting of a homogeneous, isothermal, isobaric mixture of any number u of gases with lines possessing Lorentz contours and with no continua, may be calculated by the Mayer-Goody statistical model (refs. 7 to 9) if the wave numbers of the lines have a random distribution, if the spectra of the u gases are uncorrelated, and if the integrated linear absorption coefficients $\tilde{S}_r$ of all lines of each gas meet two requirements. The integrated linear absorption coefficient of a line of gas r is defined by

$$\tilde{S}_r \equiv \int_0^\infty a_{\tilde{\nu},\text{line}}\, d\tilde{\nu} \tag{42}$$

where $a_{\tilde{\nu},\text{line}}$ is the linear absorption coefficient due to only the line being considered. The first requirement on $\tilde{S}_r$ is that the probability $P_r\, d\tilde{S}_r$ of $\tilde{S}_r$ of a line of gas r being between $\tilde{S}_r$ and $\tilde{S}_r + d\tilde{S}_r$ is

$$P_r\, d\tilde{S}_r = \frac{1}{\bar{S}_r} e^{-\tilde{S}_r/\bar{S}_r}\, d\tilde{S}_r \tag{43}$$

where $\bar{S}_r$ is the average integrated linear absorption coefficient (in units of, say, cm^{-2}) for all lines of gas r. The second requirement is that there is no correlation between the values of $\tilde{S}_r$ and the wave numbers of the respective lines. The gases may be present in any proportions. If the half-widths of the lines and the mean line spacings of all the gases are the same, then the Mayer-Goody statistical model and equations (37) and (38) give the same integrated or group intensities for all depths L of the gases, assuming the gases are bounded by a cold, black wall (fig. 1). Hence equations (37) and (38) are as good an approximation as the Mayer-Goody statistical model for this class.

The fourth class of problems may be called "independent, strong line" problems. If two or more gases have very narrow spectral lines and no continua and if the depth L of the gases is neither too small nor too large, the centers of the lines may be optically thick, but the wings of the lines may be optically thin and cover most of the spectrum. Because the coincidence of optically thick line centers of different gases is quite rare for gases with uncorrelated spectra, the integrated intensity of the mixture should be the sum of the integrated intensities due to each gas, or, from equation (30),

$$I_m = \sum_{r=1}^{u} \int_0^{\tau_{e,m,r}(L)} B_m(\tau_{e,m,r}) e^{-\tau_{e,m,r}}\, d\tau_{e,m,r} \tag{44}$$

where the mixture may possess temperature and pressure gradients. However, if the wings of the lines are optically thin and cover most of the spectrum, then $\tau_{e,m,r} \ll 1$, and equation (44) becomes

$$I_m = \sum_{r=1}^{u} \int_0^{\tau_{e,m,r}(L)} B_m\, d\tau_{e,m,r} \tag{45}$$

Equation (45) may be compared to equations (37) and (38) if it is assumed that $\tau_{e,m} \ll 1$; thus, equations (37) and (38) give

$$I_m = \int_0^L B_m \sum_{r=1}^{u} a_{e,m,r}\, dS \tag{46}$$

This equation is equivalent to equation (45). Thus, equations (37) and (38) give correct values of I_m for independent, strong line problems.

The last two of the six classes of problems are (5) mixtures of gray gases and (6) mixtures of gases such that the mixture is optically thin at all important wave numbers. For both classes, the mixtures may be inhomogeneous and have temperature and pressure gradients. For these classes, equations (37) and (38) give exact values of I_m.

Allowance for a Black Wall

Extension of approximations. - Extension of the approximations given in the preceding three subsections to problems where the wall is black but not cold is not difficult. If a wave-number group is narrow enough that the variations of the Planck functions of both the gas and the wall can be neglected between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ and if $a_{\tilde{\nu}}$ is given by equation (32), then it is shown in appendix B (eq. (B10)) that the group intensity is given exactly by

$$I_m = \int_0^{\tau_{e,m}(L)} B_m(\tau_{e,m}) e^{-\tau_{e,m}}\, d\tau_{e,m} + B_{m,w} e^{-\tau_{e,m}(L)} \tag{47}$$

even in the presence of temperature and pressure gradients. Here $B_{m,w}$ is the group Planck function for the wall temperature. When the spectrum is not divided into origin groups, equation (47) may also be written

$$I_m = \int_0^L B_m(S) e^{-\int_0^S a_{e,m}\left[\psi(S'),\tau'_{e,m}\right] dS'} a_{e,m}\left[\psi(S),\tau_{e,m}\right] dS + B_{m,w} e^{-\int_0^L a_{e,m}\left[\psi(S'),\tau'_{e,m}\right] dS'} \tag{48}$$

Substitution of equation (37) into the second term of equation (47) reveals that the use of origin groups is equivalent to assuming the multiplication property of transmissions (ref. 10). This property has been observed experimentally for molecular band spectra of different gases (ref. 10).

Exact classes of problems. - Equation (47) is exact for four classes of problems: (1) the class previously discussed for which $a_{\tilde{\nu}}$ is given by equation (32), (2) gray gases or mixtures of gray gases, no matter how wide the wave-number group is made, (3) gases or mixtures that are optically thin at all important wave numbers, irrespective of how wide the wave-number group is, and (4) the trivial case of an isothermal, isobaric gas with a black wall at the same temperature as the gas and with any width of wave-number group.

Hypothesis for other problems. - The hypothesis is made here that equation (47) may be applied as an approximation to cases not falling within the four classes of problems previously given. Support for this hypothesis is provided by specific cases in the following section.

RESULTS AND DISCUSSION

Behavior of the Effective Absorption Coefficient

The general behavior of the effective absorption coefficient has been discussed in the section Behavior of a_e for a homogeneous, isothermal, isobaric gas (p. 10). Here a specific example, profile A, is considered. Profile A is a spectral absorption coefficient consisting of the step function shown in figure 2 and tabulated in table I. The behavior of a_e for profile A is shown for an isothermal, isobaric gas in figure 3 together with the behaviors of the Planck mean absorption coefficient a_{Pl} and the Rosseland mean absorption coefficient a_{Ro} (ref. 1) defined by

$$a_{Ro} \equiv \frac{\int_0^\infty \frac{dB_{\tilde{\nu}}}{dT} d\tilde{\nu}}{\int_0^\infty \frac{1}{a_{\tilde{\nu}}} \frac{dB_{\tilde{\nu}}}{dT} d\tilde{\nu}} \tag{49}$$

All three mean absorption coefficients always lie between the minimum and the maximum spectral absorption coefficients $a_{\tilde{\nu},min}$ and $a_{\tilde{\nu},max}$, respectively. Where the spectral absorption coefficient and the pressure and temperature do not vary with S, as in the present case, a_{Pl} and a_{Ro} are independent of S, but a_e is dependent on S. The effective absorption coefficient a_e decreased from a_{Pl} to $a_{\tilde{\nu},min}$ as S varied from zero to infinity.

The effect of temperature on a_e is shown in figure 4 for an isothermal, isobaric gas. The dependence of a_e on temperature was caused by the shift of the Planck function as shown in figure 2.

The effective optical depth for an isothermal, isobaric gas with profile A is also shown in figure 4. The effective optical depth did not depend linearly on S but increased more slowly at high values of S due to preferential self-absorption.

Methods of Evaluating Integrated Intensity

Integrated intensity was calculated for specific cases by several approximate methods.

(1) The "a_{Pl}" method was evaluation of the gray-gas group transport equation using the group Planck mean absorption coefficient $a_{Pl,m}$ in place of $a_{e,m}\left[\psi(S), \tau_{e,m}\right]$ and $a_{e,m}\left[\psi(S'), \tau'_{e,m}\right]$ in equation (48). The equation for $a_{Pl,m}$ is equation (34) without the subscripts v. This procedure may be considered a first approximation to equation (48) because it gives exact values of I_m for gases that are optically thin at all important wave numbers.

(2) The "one-origin-group a_e" method was the evaluation of equation (48) and thus is equivalent to doing a problem without introducing origin groups.

(3) The "two-origin-group a_e" method was the evaluation of equation (47) using $\tau_{e,m}$ from equation (37) with $u = 2$.

Variations of these methods involved the use of different numbers of wave-number groups. For comparison, integrated intensity was also calculated exactly by equation (4). All calculations were done on the Lewis IBM 7094 computer.

Specific Cases with One Gas

Thirty-six specific isobaric cases involving eight different $a_{\tilde{\nu}}$ profiles were selected to test the accuracy of the one-origin-group a_e method. Thirty of these cases did not belong to any of the five classes of problems for which the one-origin-group a_e method is exact. The eight $a_{\tilde{\nu}}$ profiles are given in figures 2 and 5 to 8 and are tabulated in table I. Profiles B and C (fig. 5) are constant half-width Lorentz lines, profile D (fig. 6) is a step function, and profiles E and F (fig. 6) are linear variations. Profile G (fig. 7) is a step function selected to cause a very large error in I when only one wave-number group is used. Profile H (fig. 8) is a Lorentz line whose width varies as $T^{-1/2}$ as would be expected in many practical, isobaric problems. The arbitrarily assumed temperature distributions are designated 1 to 8 in figure 9. A black wall was located at $L = 50$ centimeters in all cases.

The integrated intensities calculated by the exact, one-wave-number-group a_{Pl}, and one-wave-number-group, one-origin-group a_e methods are given in tables II and III for profiles A to F. The ratios of approximate to exact integrated intensities are plotted in figure 10 for profile B. For profiles A to F the a_{Pl} method gave errors in I of -43 to 455 percent, whereas the one-origin-group a_e method demonstrated its superiority, at least for Lorentz lines, by giving errors in I of only -25 to 24 percent.

The calculated values of I for profile G are given in table IV. Both the a_{Pl} and the one-origin-group a_e methods with one wave-number group resulted in very large errors in I. The difficulty was remedied by using two wave-number groups. Because the gas was gray in both groups, the errors vanished for both approximations as shown in table IV.

The values of I for profile H are also given in table IV. Because of the large ratio of maximum to minimum temperature, one wave-number group for the one-origin-group a_e method gave an error of 35 percent. When three wave-number groups were used, this error was reduced to 27 percent. The a_{Pl} method was totally unsatisfactory with either one or three wave-number groups.

Specific Cases with Gas Mixtures

Three specific, isobaric cases with mixtures involving five hypothetical gases were selected to test the accuracy of the two-origin-group a_e method. None of the cases belonged to any of the six classes of problems for which the two-origin-group a_e method is exact or can be shown analytically to be an excellent approximation.

The five gases included one completely transparent gas c and four gases a, b, d, and e with Lorentz lines. Each of the latter gases was assumed to have 20 lines with various integrated absorption coefficients and random wave numbers of the line centers, as discussed in appendix E. Consequently, the spectra of gases a, b, d, and e were essentially uncorrelated. In mixtures, the contribution of gas r (a, b, d, or e) to $a_{\tilde{\nu}}$ was taken equal to the mole fraction X_r times the value of $a_{\tilde{\nu},r}$ for pure gas r.

The case referred to as binary mixture A involved gases a and b distributed as shown at the right of figure 11. A cold, black wall was located at 50 centimeters, and temperature distribution 8 (fig. 9) was chosen. The $a_{\tilde{\nu}}$ of the mixture is shown in figure 11. The calculated values of I are given in table V.

The case referred to as binary mixture B involved gases d and e in place of gases a and b, respectively, so that the $a_{\tilde{\nu}}$ of the mixture (not shown) was different. Otherwise binary mixture B was the same as binary mixture A. The calculated values of I are given in table V.

The case named ternary mixture was obtained by inserting a completely transparent gas c between gases a and b of binary mixture A as shown at the right of figure 12. The curves in figures 11 and 12 were plotted with the Lewis CACOMP digital computer program and a CALCOMP plotter. Insertion of the transparent gas increased the depth of the gas to 150 centimeters, to which point the cold, black wall was relocated. Temperature distribution 9 was chosen because it had the same maximum and minimum temperatures as for the binary mixtures. The $a_{\tilde{\nu}}$ of the mixture is shown in figure 12. The calculated values of I are given in table V. Despite first impressions conveyed by figures 11 and 12, binary mixture A and the ternary mixture had appreciable self-absorption ($\tau_{\tilde{\nu}}$ not negligible compared to 1) at most wave numbers.

A review of the accuracy of the various approximate methods for calculating I (table V) for the three mixtures shows that the a_{Pl} method with one or three wave-number groups was in some cases unsatisfactory. Calculations by the one-origin-group a_e and the three-origin-group a_e methods were performed with three wave-number groups. In these calculations, one origin group gave errors between -49 and 19 percent, while two origin groups gave errors between -17 and 28 percent, demonstrating the superiority of the two-origin-group a_e method for inhomogeneous mixtures containing two semi-opaque gases.

RECOMMENDATIONS FOR PRACTICAL APPLICATIONS

The approximations developed in this report are well suited to calculations with a high-speed digital computer but not well suited to hand calculations. Because the effective absorption coefficient approximation introduces errors of the order of ±28 percent in the calculation of integrated intensity and hence radiant flux density, the numerical methods used in practical calculations can be crude without introducing serious additional error. The proposed procedure for practical calculations of radiant flux density is outlined in the following sections.

Selection of Wave-Number Groups

Once the gases and their spectra involved in a given problem or series of problems have been determined, the next step is to select the wave-number groups. Because equation (47) is exact for gray gases, errors in I will be minimized if the spectrum is so divided that each wave-number group is as nearly gray as possible. A very conservative approach would be to divide the spectrum so that no Planck function corresponding to any temperature of the problem varied appreciably in any wave-number group. When a large number of closely related cases are to be calculated, a better approach would be to calculate a few cases exactly by applying equation (4) to the entire spectrum and then to determine by trial and error how many wave-number groups would be needed if equations (31) and (47) are used as an approximation.

Selection of Origin Groups

For problems involving chemical reactions, at least one origin group generally will be required for each chemical species possessing a spectrum in a given wave-number group.

Calculation of $a_{e,m,r}$ and $\tau_{e,m,r}$

The effective absorption coefficients $a_{e,m,r}$ can be calculated once and for all for a series of problems involving the same chemical species. The computer can then store them as multivariant polynomial fits or in multiargument tables. If one origin group is assumed per species, it will be necessary to have at most $u + 2$ arguments (T, $\tau_{e,m,r}$, and partial pressures of all u chemical species). However, in many cases fewer arguments will suffice because the various particles that cause significant collision broadening of spectral lines often produce the same line shape and because $a_{e,m,r}/p_r$ may be stored instead of $a_{e,m,r}$ itself (p_r is the partial pressure of species r). Thus, $a_{e,m,r}/p_r$ often may be stored with the three arguments T, $\tau_{e,m,r}$, and $\bar{b}_c$ where $\bar{b}_c$ is an average collision half-width.

It was mentioned in the section Nongray Gases with Gradients that in constructing a table of a_e values, a number of values of S_i are substituted into equations (17) and (19) to get a_e and τ_e. Similarly, in constructing a table of $a_{e,m,r}$ values, a number of values of $S_{i,m,r}$ may be substituted into equation (35) and the right-hand side of equation (36). A better procedure is to nondimensionalize equations (35) and (36) by defining dimensionless photon energies by

$$\left.\begin{aligned} V &\equiv \frac{hc\tilde{\nu}}{kT} \\ V_m &\equiv \frac{hc\tilde{\nu}_m}{kT} \\ V_{m+1} &\equiv \frac{hc\tilde{\nu}_{m+1}}{kT} \end{aligned}\right\} \tag{50}$$

From equations (2), (28), (36), and (50),

$$\tau_{e,m,r} = \ln \frac{\displaystyle\int_m \frac{V^3}{e^V - 1}\, dV}{\displaystyle\int_m \frac{V^3 e^{-a_{V,r} S_{i,m,r}}}{e^V - 1}\, dV} \tag{51}$$

where $a_{\tilde{\nu},r}$ has been rewritten $a_{V,r}$ for clarity, and m under the integral sign means to integrate from V_m to V_{m+1}. Evaluation of the integral in the numerator can be rapidly carried out using the expressions for Q(V) given in appendix F. From equations (2), (35), and (50),

$$a_{e,m,r} = \frac{\displaystyle\int_m \frac{a_{V,r} V^3 e^{-a_{V,r} S_{i,m,r}}}{e^V - 1}\, dV}{\displaystyle\int_m \frac{V^3 e^{-a_{V,r} S_{i,m,r}}}{e^V - 1}\, dV} \tag{52}$$

For spectra involving many lines, use of statistical models (refs. 7 to 9) may be preferable to use of equations (51) and (52).

Evaluation of Integrated Intensity

If one origin group is assumed per species, integrated intensity I_m may be evaluated at point P for a ray in a given direction (fig. 1) in an inhomogeneous mixture of u gases by integrating by numerical methods u first-order differential equations and a single integral. The u first-order differential equations are given by

$$\frac{d\tau_{e,m,r}}{dS} = a_{e,m,r}\left[\psi(S), \tau_{e,m,r}\right] \qquad (r = 1, 2, 3, \ldots, u) \tag{53}$$

which is equivalent to the left-hand side of equation (36). The single integral is given in equation (47), in which B_m may be evaluated by the method in appendix F.

Evaluation of Radiant Heat-Flux Density

The component F_n of the heat-flux density at the point P in the direction $\vec{n}$ normal to the infinitesimal area dA (fig. 1) may be calculated from the relation

$$F_n = \int_{4\pi} \sum_{m=1}^{\alpha} I_m \cos\theta \, d\Omega \tag{54}$$

where $d\Omega$ is an element of solid angle with the ray at its center, θ is the angle between the ray and the direction $\vec{n}$, and the integration is performed over 4π steradians.

CONCLUSIONS

1. The effective absorption coefficient approximation simplifies radiant-heat transport calculations for cases with complicated spectra that are optically thick at some important wave numbers. Integration over all important wave numbers is performed once and for all for all gas mixtures containing the same chemical species.

2. To obtain accurate integrated intensities where widely different temperatures occur in a problem, it may be necessary to divide the spectrum into wave-number groups, with an effective absorption coefficient and an effective transport equation for each group.

3. The use of origin groups in the calculation of the effective absorption coefficient is generally necessary to obtain accurate integrated intensities for inhomogeneous mixtures and may be desirable where spectra of the same gas have widely different excitation potentials. For valid results, spectra of the different origin groups must have no appreciable correlation with each other, and the spectral absorption coefficient of each origin group must factor approximately into wave-number-dependent and position-dependent functions.

4. When one to three wave-number groups were used as required and one or two origin groups were used as appropriate, the effective absorption coefficient approximation gave integrated intensities with errors of -25 to 28 percent for 39 cases. In contrast, use of Planck mean absorption coefficients in the gray-gas transport equations for the same number of wave-number groups gave integrated intensities with errors of -43 to 455 percent.

5. If hot, gray or black radiation falls on a cold gas with narrow, isolated spectral lines, the fraction of the radiation transmitted may be underestimated seriously by using the Planck mean absorption coefficient. The error can be reduced substantially by using the effective absorption coefficient approximation.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, February 16, 1967,
122-28-02-17-22.

APPENDIX A

DERIVATION OF GROUP DIFFUSION EQUATION FROM EFFECTIVE GROUP TRANSPORT EQUATION

It is shown in this appendix that if the wave-number group is narrow enough and if the usual assumptions of the elementary diffusion approximation (ref. 1) are made, then the effective group transport equation gives exact values of I_m. The group diffusion equation is derived from the equation for I_m and is found to be the same as derived from the literature.

EXACTNESS OF EFFECTIVE GROUP TRANSPORT EQUATION

Consider a gas where the Planck function does not vary appreciably in a few photon mean free paths at any wave number. Then the elementary diffusion approximation (ref. 1) should, in general, be applicable. Although it is not obvious from the usual derivation (ref. 1), this approximation entails assuming that $a_{\tilde{\nu}}$ is independent of temperature and position. This assumption is reasonable because when the approximation is applicable all significant contributions to the radiant flux density come from gas within a few photon mean free paths of the point in question, and the Planck function (and therefore temperature) must not vary appreciably in this region. Hence, in the following derivation, $a_{\tilde{\nu}}$ is assumed to be independent of temperature and position.

If a wave-number group is chosen narrow enough that the variation of $B_{\tilde{\nu}}$ with $\tilde{\nu}$ can be neglected, then according to equation (27), $a_{e,m}$ is given by

$$a_{e,m} = \frac{\int_m a_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S_{i,m}} \, d\tilde{\nu}}{\int_m e^{-a_{\tilde{\nu}} S_{i,m}} \, d\tilde{\nu}} \tag{A1}$$

Substitution of equation (A1) into equation (29) and differentiation of the result with respect to S give $dS_{i,m}/dS = 1$. By using the boundary condition $S_{i,m} = 0$ at $S = 0$, it is found that $S_{i,m} = S$ for all values of S, and equation (A1) becomes

$$a_{e,m} = - \frac{\frac{d}{dS} \int_m e^{-a_{\tilde{\nu}} S} \, d\tilde{\nu}}{\int_m e^{-a_{\tilde{\nu}} S} \, d\tilde{\nu}} \tag{A2}$$

Let

$$q_m \equiv \int_m e^{-a_{\tilde{\nu}} S} \, d\tilde{\nu} \tag{A3}$$

Thus,

$$a_{e,m} = - \frac{d \ln q_m}{dS} \tag{A4}$$

If the gas is assumed to be infinite, equation (30) becomes

$$I_m = \int_0^{\infty} B_m e^{-\int_0^S a_{e,m} dS'} a_{e,m} \, dS$$

or

$$I_m = - \int_{q_m(0)}^{q_m(\infty)} \frac{B_m e^{\ln q_m - \ln \Delta\tilde{\nu}_m}}{q_m} \, dq_m = - \int_{q_m(0)}^{q_m(\infty)} \frac{B_m}{\Delta\tilde{\nu}_m} \, dq_m \tag{A5}$$

where $q_m(0)$ is the value of q_m for $S = 0$, and $\Delta\tilde{\nu}_m$ is $\tilde{\nu}_{m+1} - \tilde{\nu}_m$. From equation (A3),

$$dq_m = - \int_m a_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S} \, d\tilde{\nu} \, dS \tag{A6}$$

Combining equations (A5) and (A6) gives

$$I_m = \frac{1}{\Delta\tilde{\nu}_m} \int_0^{\infty} B_m \int_m a_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S} \, d\tilde{\nu} \, dS \tag{A7}$$

which is exact because it may also be obtained from equation (3) by setting $L = \infty$, dropping the last term, integrating over the wave-number group, and neglecting the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$.

DERIVATION OF GROUP DIFFUSION EQUATION

The group diffusion equation may be obtained from equation (A7) if B_m is expanded in a Taylor series about $S = 0$.

$$B_m = B_m(O) + S\left(\frac{dB_m}{dS}\right)_O + \ldots \tag{A8}$$

and only the first two terms are retained. Combining equations (A7) and (A8) and changing the order of integration give

$$I_m = \int_m \int_0^\infty \left[B_m(O) + S\left(\frac{dB_m}{dS}\right)_O\right] \frac{e^{-\tau_{\tilde{\nu}}}}{\Delta\tilde{\nu}_m} d\tau_{\tilde{\nu}}\ d\tilde{\nu} \tag{A9}$$

This equation simplifies to

$$I_m = B_m(O) + \left(\frac{dB_m}{dS}\right)_O \int_m \frac{1}{a_{\tilde{\nu}}\ \Delta\tilde{\nu}_m} d\tilde{\nu} \tag{A10}$$

From Stewart and Pyatt (ref. 11), the local Rosseland mean absorption coefficient $a_{Ro,m}$ for the wave-number interval $\Delta\tilde{\nu}_m$ is defined by

$$\frac{1}{a_{Ro,m}} \equiv \int_m \frac{1}{a_{\tilde{\nu}}\,\Delta\tilde{\nu}_m} d\tilde{\nu} \tag{A11}$$

where the variation of $dB_{\tilde{\nu}}/dT$ in this wave-number interval has been neglected. Combining equations (A10) and (A11) gives

$$I_m = B_m(O) + \frac{1}{a_{Ro,m}}\left(\frac{dB_m}{dS}\right)_O \tag{A12}$$

The gradient of T is assumed to be in the direction $\vec{n}$ or $-\vec{n}$. The group heat flux density is (fig. 1)

$$\vec{F}_m = -\int_{4\pi} I_m \cos\theta\ d\Omega\,\vec{n} \tag{A13}$$

where, from equation (A12),

$$I_m = B_m(O) + \frac{\cos \theta}{a_{Ro,m}} \left(\frac{dB_m}{dx} \right)_O \tag{A14}$$

Here x is the component of S in the direction $\vec{n}$. Combining equations (A13) and (A14) gives

$$\vec{F}_m = - \frac{4\pi}{3a_{Ro,m}} \left(\frac{dB_m}{dx} \right)_O \vec{n} \tag{A15}$$

This is simply the group diffusion equation, which can also be derived from the spectral diffusion equation (ref. 1) by integrating it over the wave-number interval $\Delta\tilde{\nu}_m$ with the assumption that $\left(dB_{\tilde{\nu}}/dx\right)_O$ is constant in this interval.

APPENDIX B

DERIVATION OF EFFECTIVE GROUP TRANSPORT EQUATION FROM EXACT GROUP TRANSPORT EQUATION FOR A CLASS OF PROBLEMS

In this appendix the effective group transport equation is derived from the exact group transport equation for a gas whose $a_{\tilde{\nu}}$ can be separated into wave-number-dependent and position-dependent factors (eq. (32)). Temperature and pressure gradients may be present. The gas is bounded by a black wall at $S = L$, which may be at any temperature. The wave-number group must be narrow enough that the variation of the Planck functions of the gas and of the wall for wave numbers between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ may be neglected.

Equation (26) is the exact group transport equation. If the variation of $B_{\tilde{\nu}}$ and $B_{\tilde{\nu},w}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ is neglected and the order of integration is changed, equation (26) becomes

$$I_m = \frac{1}{\Delta\tilde{\nu}_m}\int_0^L B_m(S)\int_m e^{-\tau_{\tilde{\nu}}(S)} a_{\tilde{\nu}}(S)\,d\tilde{\nu}\,dS + \frac{B_{m,w}}{\Delta\tilde{\nu}_m}\int_m e^{-\tau_{\tilde{\nu}}(L)}\,d\tilde{\nu} \tag{B1}$$

The next step is to find a simple expression for the product $a_{\tilde{\nu}}(S)S_{i,m}(S)$. The quantity $S_{i,m}(S)$ is defined by equation (29).

If the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ is neglected, equation (29) becomes

$$\int_0^S a_{e,m}\left[S',S_{i,m}(S')\right]dS' = \ln\frac{\Delta\tilde{\nu}_m}{\displaystyle\int_m e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)}\,d\tilde{\nu}} \tag{B2}$$

Equation (27) defines $a_{e,m}$. If the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ is neglected, equation (27) becomes

$$a_{e,m}\left[S,S_{i,m}(S)\right] = \frac{\displaystyle\int_m a_{\tilde{\nu}}(S)e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)}\,d\tilde{\nu}}{\displaystyle\int_m e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)}\,d\tilde{\nu}} \tag{B3}$$

Equation (B3) is then substituted into equation (B2), equation (32) is substituted into the result, and the equation thus obtained is differentiated with respect to S.

$$\frac{dS_{i,m}(S)}{dS} + \frac{S_{i,m}(S)}{t_m(S)} \frac{dt_m(S)}{dS} = 1 \tag{B4}$$

The solution of the differential equation (B4) with the boundary condition $S_{i,m} = 0$ at $S = 0$ is

$$S_{i,m}(S) = \frac{\int_0^S t_m(S')dS'}{t_m(S)} \tag{B5}$$

Combining equations (32) and (B5) gives

$$a_{\tilde{\nu}}(S)S_{i,m}(S) = \int_0^S a_{\tilde{\nu}}(S')dS' = \tau_{\tilde{\nu}}(S) \tag{B6}$$

This equation gives the desired simple expression for $a_{\tilde{\nu}}(S)S_{i,m}(S)$.

Substituting equation (B6) into equation (B1) gives

$$I_m = \frac{1}{\Delta\tilde{\nu}_m} \int_0^L B_m(S) \int_m e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)} a_{\tilde{\nu}}(S)d\tilde{\nu}\; dS + \frac{B_{m,w}}{\Delta\tilde{\nu}_m} \int_m e^{-a_{\tilde{\nu}}(L)S_{i,m}(L)} d\tilde{\nu} \tag{B7}$$

If the variation of $B_{\tilde{\nu}}$ between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ is neglected, the right-hand side of equation (29) gives

$$e^{-\tau_{e,m}(S)} = \frac{1}{\Delta\tilde{\nu}_m} \int_m e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)} d\tilde{\nu} \tag{B8}$$

If equation (B8) is multiplied by equation (B3), the result is

$$e^{-\tau_{e,m}(S)} a_{e,m}(S) = \frac{1}{\Delta\tilde{\nu}_m} \int_m a_{\tilde{\nu}}(S)e^{-a_{\tilde{\nu}}(S)S_{i,m}(S)} d\tilde{\nu} \tag{B9}$$

Equations (B8) and (B9) are then substituted into equation (B7) with the result

$$I_m = \int_0^L B_m(S) e^{-\tau_{e,m}(S)} a_{e,m}(S) dS + B_{m,w} e^{-\tau_{e,m}(L)} \tag{B10}$$

The derivation of equation (B10) has shown that it is an exact equation for the class of problem considered. If the wall is cold ($B_{m,w}$ negligible), equation (B10) becomes

$$I_m = \int_0^L B_m(S) e^{-\tau_{e,m}(S)} a_{e,m}(S) dS \tag{B11}$$

Equations (B10) and (B11) are effective group transport equations.

APPENDIX C

SERIES EXPANSIONS OF TWO GROUP TRANSPORT EQUATIONS FOR HOMOGENEOUS MIXTURE OF NONGRAY GASES WITH OR WITHOUT TEMPERATURE GRADIENTS

In this appendix, series expansions of the exact group transport equation and of the effective group transport equation (eq. (38) with $\tau_{e,m}$ given by eq. (37)) are carried out and compared in order to help establish the validity of the effective group transport equation with more than one origin group.

The class of problem considered is restricted to a homogeneous mixture of u slightly nongray (slightly colored) isobaric gases in any proportions and with or without temperature gradients. The gases are assumed to have uncorrelated spectra and are numbered 1, 2, . . ., u. The two expansions give the group intensity at S = O (point P in fig. 1). The distance L along the ray to a cold, black wall can be any value including infinity. The optical depth of the gas along the full length of the ray is arbitrary. The wave-number group considered is assumed to be narrow enough that the variation of the Planck function $B_{\tilde{\nu}}$ with wave number can be neglected for the range of wave numbers included in the group.

The spectral absorption coefficient of each gas of the mth wave-number group is assumed to be independent of position along the ray and may be expressed as

$$a_{\tilde{\nu},r} = a_{g,m,r} + \Delta a_r \qquad r = 1, 2, \ldots, u \tag{C1}$$

where r designates the particular gas. The gray absorption coefficient $a_{g,m,r}$ is a constant for each wave-number group for each gas, so the perturbation Δa_r is some function of wave number. Integrating equation (C1) over the range of wave numbers included in the wave-number group m gives

$$\int_m a_{\tilde{\nu},r}\, d\tilde{\nu} = \int_m a_{g,m,r}\, d\tilde{\nu} + \int_m \Delta a_r\, d\tilde{\nu}$$

or

$$\int_m a_{\tilde{\nu},r}\, d\tilde{\nu} = a_{g,m,r}\, \Delta\tilde{\nu}_m + \int_m \Delta a_r\, d\tilde{\nu} \tag{C2}$$

If $a_{\tilde{\nu},r}$ is given, $a_{g,m,r}$ may be so selected that

$$\int_m a_{\tilde{\nu},r}\, d\tilde{\nu} = a_{g,m,r}\, \Delta\tilde{\nu}_m \qquad r = 1, 2, \ldots, u \tag{C3}$$

Subtracting equation (C3) from equation (C2) gives

$$\int_m \Delta a_r\, d\tilde{\nu} = 0 \qquad r = 1, 2, \ldots, u \tag{C4}$$

The requirement that each gas be slightly nongray may be specified by

$$\left| \frac{\Delta a_r}{a_{g,m,r}} \right| \ll 1 \tag{C5}$$

for all wave numbers in the wave-number group. The total spectral absorption coefficient is given by

$$a_{\tilde{\nu}} = \sum_{r=1}^{u} a_{\tilde{\nu},r} \tag{C6}$$

Equations (C1) to (C6) apply in both the series expansions which follow.

EXPANSION OF THE EXACT GROUP TRANSPORT EQUATION

From equations (1) and (26), the exact group transport equation with a cold, black wall is

$$I_m = \int_m \int_0^L B_{\tilde{\nu}} e^{-a_{\tilde{\nu}} S} a_{\tilde{\nu}}\, dS\, d\tilde{\nu} \tag{C7}$$

Because it was assumed that the variation of $B_{\tilde{\nu}}$ with wave number could be neglected, equation (C7) may be rewritten

$$I_m = \int_0^L \frac{B_m}{\Delta\tilde{\nu}_m} \int_m e^{-a_{\tilde{\nu}} S} a_{\tilde{\nu}}\, d\tilde{\nu}\, dS \tag{C8}$$

If equations (C1) and (C6) are used, equation (C8) becomes

$$I_m = \int_0^L \frac{B_m}{\Delta\tilde{\nu}_m} \int_m e^{-S\sum_{l=1}^{u}(a_{g,m,l}+\Delta a_l)} \sum_{r=1}^{u} (a_{g,m,r} + \Delta a_r)\, d\tilde{\nu}\, dS \tag{C9}$$

Define a total gray absorption coefficient for wave-number group m by

$$a_{g,m} \equiv \sum_{r=1}^{u} a_{g,m,r} \tag{C10}$$

Also, define a gray optical depth by

$$\tau_{g,m} \equiv a_{g,m} S \tag{C11}$$

Combining equations (C9), (C10), and (C11) and expanding the Δa_l terms of the exponential in powers of $\Delta a_l / a_{g,m}$ give

$$\begin{aligned} I_m = \int_0^{\tau_{g,m}(L)} \frac{B_m}{\Delta\tilde{\nu}_m} e^{-\tau_{g,m}} \int_m \prod_{l=1}^{u} \Bigg[1 - \tau_{g,m} \frac{\Delta a_l}{a_{g,m}} + \frac{\tau_{g,m}^2}{2} \left(\frac{\Delta a_l}{a_{g,m}}\right)^2 \\ - \frac{\tau_{g,m}^3}{6} \left(\frac{\Delta a_l}{a_{g,m}}\right)^3 + \ldots \Bigg] \left(1 + \sum_{r=1}^{u} \frac{\Delta a_r}{a_{g,m}}\right) d\tilde{\nu}\, d\tau_{g,m} \end{aligned} \tag{C12}$$

where $\tau_{g,m}(L)$ is the value of $\tau_{g,m}$ at $S = L$, and $\prod$ indicates a product.

Appropriate expressions will now be found to express the assumption that the spectra of the u gases are uncorrelated. The perturbation Δa_r at a given wave number may be associated with a hypothetical photon of the same wave number, the photon being a member of a hypothetical population r of photons with a constant frequency function between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ and a zero frequency function outside these limits. In other words, the probability of a photon having a wave number in an infinitesimal range $d\tilde{\nu}$ of fixed width is independent of wave number between $\tilde{\nu}_m$ and $\tilde{\nu}_{m+1}$ and is zero outside this range. If there are u statistically independent populations of photons with this type of frequency function, then the following averages exist (ref. 12).

$$\left.\begin{array}{ll}\displaystyle\int_m \Delta a_j \, \Delta a_k \, d\tilde{\nu} = 0 & j \neq k \\[2ex] \displaystyle\int_m \Delta a_j \, \Delta a_k \, \Delta a_l \, d\tilde{\nu} = 0 & \begin{array}{l} j \neq k \\ j \neq l \\ k \neq l \end{array} \\[2ex] \displaystyle\int_m \Delta a_j \left(\Delta a_k\right)^2 d\tilde{\nu} = 0 & j \neq k \end{array}\right\} \tag{C13}$$

Equation (C13) defines "u gases with uncorrelated spectra." It is a good assumption for atomic and ionic spectra consisting of many lines by themselves or superimposed on continua which do not have strong wave-number dependence.

Equations (C4) and (C13) reveal that many of the terms that occur in evaluating products in (C12) are zero, with the result that

$$I_m = \int_0^{\tau_{g,m}(L)} B_m e^{-\tau_{g,m}} \left[1 + \left(\frac{\tau_{g,m}^2}{2} - \tau_{g,m} \right) \frac{1}{\Delta\tilde{\nu}_m} \sum_{l=1}^{u} \int_m \left(\frac{\Delta a_l}{a_{g,m}} \right)^2 d\tilde{\nu} \right.$$

$$\left. + \left(\frac{\tau_{g,m}^2}{2} - \frac{\tau_{g,m}^3}{6} \right) \frac{1}{\Delta\tilde{\nu}_m} \sum_{r=1}^{u} \int_m \left(\frac{\Delta a_r}{a_{g,m}} \right)^3 d\tilde{\nu} \right] d\tau_{g,m} \tag{C14}$$

where terms higher than third order in $\Delta a_r / a_{g,m}$ have been dropped as insignificant. Equation (C14) is the truncated series expansion for the group intensity at point P based on the exact group transport equation.

EXPANSION OF THE EFFECTIVE GROUP TRANSPORT EQUATION

From equations (37) and (38), the effective group transport equation may be written

$$I_m = \int_0^{\tau_{e,m}(L)} B_m \prod_{r=1}^{u} e^{-\tau_{e,m,r}} \, d\tau_{e,m} \tag{C15}$$

Consequently, evaluation of equation (C15) requires two preliminary steps: the evaluation of $d\tau_{e,m,r}$ (from which $d\tau_{e,m}$ is obtained) and the evaluation of $\exp(-\tau_{e,m,r})$.

The value of $a_{e,m,r}$ is needed to evaluate $d\tau_{e,m,r}$. If the variation of $B_{\tilde{\nu}}$ with wave number within the wave-number group is neglected, equation (35) becomes

$$a_{e,m,r} = \frac{\int_m a_{\tilde{\nu},r} e^{-a_{\tilde{\nu},r} S_{i,m,r}} \, d\tilde{\nu}}{\int_m e^{-a_{\tilde{\nu},r} S_{i,m,r}} \, d\tilde{\nu}} \tag{C16}$$

Substitution of equation (C16) into equation (36) and differentiation of the result with respect to S gives $dS_{i,m,r}/dS = 1$. If the boundary condition $S_{i,m,r} = 0$ at $S = 0$ is used, it is found that $S_{i,m,r} = S$ for all values of S. From equations (C16) and (36),

$$d\tau_{e,m,r} = \frac{\int_m e^{-a_{\tilde{\nu},r} S} a_{\tilde{\nu},r} \, d\tilde{\nu} \, dS}{\int_m e^{-a_{\tilde{\nu},r} S} \, d\tilde{\nu}} \tag{C17}$$

Let the denominator of equation (C17) be designated by

$$q_{m,r} \equiv \int_m e^{-a_{\tilde{\nu},r} S} \, d\tilde{\nu} \tag{C18}$$

Substituting equation (C1) into equation (C18) gives

$$q_{m,r} = \int_m e^{-a_{g,m,r} S} e^{-\Delta a_r S} \, d\tilde{\nu} \tag{C19}$$

A gray optical depth for gas r may be defined by

$$\tau_{g,m,r} = a_{g,m,r} S \tag{C20}$$

Substituting equation (C20) into equation (C19) and expanding the second exponential in a power series give

$$q_{m,r} = e^{-\tau_{g,m,r}} \left[\Delta\tilde{\nu}_m + \frac{\tau^2_{g,m,r}}{2} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}} \right)^2 d\tilde{\nu} - \frac{\tau^3_{g,m,r}}{6} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}} \right)^3 d\tilde{\nu} + \dots \right] \tag{C21}$$

where the second term in the series has been omitted because it is zero according to equation (C4).

Similarly, the integral in the numerator of equation (C17) is designated by

$$z_{m,r} \equiv \int_m e^{-a_{\tilde{\nu},r} S} a_{\tilde{\nu},r}\, d\tilde{\nu} \tag{C22}$$

Substituting equation (C1) into equation (C22) gives

$$z_{m,r} = \int_m e^{-a_{g,m,r} S} e^{-\Delta a_r S} (a_{g,m,r} + \Delta a_r) d\tilde{\nu} \tag{C23}$$

Expanding the second exponential in a power series and evaluating the result give

$$z_{m,r} = e^{-\tau_{g,m,r}} \left[\Delta\tilde{\nu}_m + \left(\frac{\tau^2_{g,m,r}}{2} - \tau_{g,m,r} \right) \int_m \left(\frac{\Delta a_r}{a_{g,m,r}} \right)^2 d\tilde{\nu} + \left(\frac{\tau^2_{g,m,r}}{2} - \frac{\tau^3_{g,m,r}}{6} \right) \int_m \left(\frac{\Delta a_r}{a_{g,m,r}} \right)^3 d\tilde{\nu} + \dots \right] a_{g,m,r} \tag{C24}$$

where the second term in the series has been omitted because it is zero according to equation (C4).

Combining equations (C17), (C21), and (C24) gives

$$d\tau_{e,m,r} = \left[1 - \frac{\tau_{g,m,r}}{\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}} \right)^2 d\tilde{\nu} + \frac{\tau^2_{g,m,r}}{2\,\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}} \right)^3 d\tilde{\nu} \right] a_{g,m,r}\, dS \tag{C25}$$

where terms of higher than third order in $\Delta a_r/a_{g,m,r}$ have been neglected.

A different form of $d\tau_{e,m,r}$ is useful to evaluate $\exp(-\tau_{e,m,r})$. From equation (C17) and the right-hand side of equation (36),

$$d\tau_{e,m,r} = \frac{\int_m e^{-a_{\tilde{\nu},r}S} a_{\tilde{\nu},r} \, d\tilde{\nu} \, dS}{\Delta\tilde{\nu}_m e^{-\tau_{e,m,r}}} \tag{C26}$$

This equation can also be written

$$e^{-\tau_{e,m,r}} d\tau_{e,m,r} = \frac{z_{m,r} \, dS}{\Delta\tilde{\nu}_m} \tag{C27}$$

Combining equations (C24) and (C27) and truncating the series at third order in $\Delta a_r/a_{g,m,r}$ give

$$e^{-\tau_{e,m,r}} d\tau_{e,m,r} = e^{-\tau_{g,m,r}} \left[1 + \left(\frac{\tau_{g,m,r}^2}{2} - \tau_{g,m,r}\right) \frac{1}{\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}}\right)^2 d\tilde{\nu} \right.$$

$$\left. + \left(\frac{\tau_{g,m,r}^2}{2} - \frac{\tau_{g,m,r}^3}{6}\right) \frac{1}{\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}}\right)^3 d\tilde{\nu}\right] d\tau_{g,m,r} \tag{C28}$$

Integrating both sides from the origin of the ray out to S and simplifying the resulting expression give

$$e^{-\tau_{e,m,r}} = e^{-\tau_{g,m,r}} \left[1 + \frac{\tau_{g,m,r}^2}{2\,\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}}\right)^2 d\tilde{\nu} - \frac{\tau_{g,m,r}^3}{6\,\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}}\right)^3 d\tilde{\nu}\right] \tag{C29}$$

It is now possible to evaluate equation (C15). First equation (C25) is substituted into the differential form of equation (37) with the result

$$d\tau_{e,m} = \sum_{r=1}^{u} \left\{ \left[1 - \frac{\tau_{g,m,r}}{\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}}\right)^2 d\tilde{\nu} + \frac{\tau^2_{g,m,r}}{2\,\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m,r}}\right)^3 d\tilde{\nu} \right] a_{g,m,r} \right\} dS \tag{C30}$$

Substituting equations (C29) and (C30) into equation (C15) and using equations (C10) and (C11) give

$$\begin{aligned} I_m = \int_0^{\tau_{g,m}(L)} B_m e^{-\tau_{g,m}} \prod_{r=1}^{u} \left[1 + \frac{\tau^2_{g,m}}{2\,\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m}}\right)^2 d\tilde{\nu} - \frac{\tau^3_{g,m}}{6\,\Delta\tilde{\nu}_m} \int_m \left(\frac{\Delta a_r}{a_{g,m}}\right)^3 d\tilde{\nu} \right] \\ \times \left[1 - \frac{\tau_{g,m}}{\Delta\tilde{\nu}_m} \sum_{l=1}^{u} \int_m \left(\frac{\Delta a_l}{a_{g,m}}\right)^2 d\tilde{\nu} + \frac{\tau^2_{g,m}}{2\,\Delta\tilde{\nu}_m} \sum_{k=1}^{u} \int_m \left(\frac{\Delta a_k}{a_{g,m}}\right)^3 d\tilde{\nu} \right] d\tau_{g,m} \end{aligned} \tag{C31}$$

Evaluating equation (C31) and retaining only third- or lower-order terms in $\Delta a_r/a_{g,m}$ gives equation (C14). Hence, through third order in $\Delta a_r/a_{g,m}$, both the exact group transport equation and the effective group transport equation give the same value for I_m.

APPENDIX D

SERIES EXPANSIONS OF TWO GROUP TRANSPORT EQUATIONS FOR INHOMOGENEOUS, ISOTHERMAL MIXTURE OF NONGRAY GASES

In this appendix, just as in appendix C, series expansions of the exact group transport equation and of the effective group transport equation (eq. (38) with $\tau_{e,m}$ given by eq. (37)) are carried out and compared in order to help establish the validity of the effective group transport equation with more than one origin group.

The class of problems considered is different than in appendix C. It is restricted to an isothermal mixture of u slightly nongray gases in any proportions and mixed in a homogeneous or inhomogeneous manner. Pressure gradients may be present. The gases are assumed to have uncorrelated spectra and are numbered 1, 2, . . ., u. The two expansions give the group intensity at S = O (point P in fig. 1). The distance L along the ray to a cold, black wall can be any value. The optical depth of the gas along the full length of the ray is arbitrary. The wave-number group considered may be any width.

It is assumed that the absorption coefficient of each gas in wave-number group m may be expressed as

$$a_{\nu,r} = w_{m,r}(\nu)t_{m,r}(S) \qquad r = 1, 2, \ldots, u \tag{D1}$$

where $t_{m,r}$ is a factor dependent only on S and due to concentration variation. In equation (D1),

$$w_{m,r} = w_{g,m,r} + \Delta w_r \tag{D2}$$

where $w_{g,m,r}$ is a constant, and Δw_r is a small perturbation, dependent only on wave number. Multiplying equation (D2) by B_ν and integrating over the wave-number group give

$$\int_m B_\nu w_{m,r}\, d\nu = B_m w_{g,m,r} + \int_m B_\nu \Delta w_r\, d\nu \tag{D3}$$

If $w_{m,r}$ is given, $w_{g,m,r}$ may be so selected that

$$\int_m B_\nu w_{m,r}\, d\nu = B_m w_{g,m,r} \qquad r = 1, 2, \ldots, u \tag{D4}$$

Subtracting equation (D4) from equation (D3) gives

$$\int_m B_{\tilde{\nu}} \Delta w_r \, d\tilde{\nu} = 0 \qquad r = 1, 2, \ldots, u \qquad \text{(D5)}$$

The condition that each gas be slightly nongray is specified by

$$\left| \frac{\Delta w_r}{w_{g,m,r}} \right| << 1 \qquad \text{(D6)}$$

for all wave numbers in wave-number group m. Equations (D1) to (D6) apply in both the series expansions which follow.

EXPANSION OF THE EXACT GROUP TRANSPORT EQUATION

From equation (26), the exact group transport equation for an isothermal gas with a cold, black wall is

$$I_m = \int_m B_{\tilde{\nu}} \left[1 - e^{-\tau_{\tilde{\nu}}(L)}\right] d\tilde{\nu} \qquad \text{(D7)}$$

From equations (1), (D1), and (D2),

$$e^{-\tau_{\tilde{\nu}}(L)} = e^{-\sum_{r=1}^{u} w_{g,m,r} \int_0^L t_{m,r}(S)dS} \; e^{-\sum_{r=1}^{u} \Delta w_r \int_0^L t_{m,r}(S)dS} \qquad \text{(D8)}$$

Define a gray optical depth $\tau_{g,m}$ for $S = L$ by

$$\tau_{g,m}(L) \equiv \sum_{r=1}^{u} w_{g,m,r} \int_0^L t_{m,r}(S)dS \qquad \text{(D9)}$$

Substituting equation (D9) into equation (D8) and expanding the second exponential give

$$e^{-\tau_{\tilde{\nu}}(L)} = e^{-\tau_{g,m}(L)} \prod_{r=1}^{u} \left\{1 - \Delta w_r \int_0^L t_{m,r}(S)dS + \frac{1}{2}\left[\Delta w_r \int_0^L t_{m,r}(S)dS\right]^2 - \frac{1}{6}\left[\Delta w_r \int_0^L t_{m,r}(S)dS\right]^3 + \dots\right\} \qquad \text{(D10)}$$

Substituting equation (D10) into equation (D7) gives

$$I_m = \int_m B_{\tilde{\nu}} \left(1 - e^{-\tau_{g,m}(L)} \prod_{r=1}^{u} \left\{1 - \Delta w_r \int_0^L t_{m,r}(S)dS + \frac{1}{2}\left[\Delta w_r \int_0^L t_{m,r}(S)dS\right]^2 - \frac{1}{6}\left[\Delta w_r \int_0^L t_{m,r}(S)dS\right]^3 + \dots\right\}\right) d\tilde{\nu} \qquad \text{(D11)}$$

In a manner similar to that for equation (C13), the lack of correlation of the spectra of the u gases is specified by

$$\left.\begin{array}{ll} \int_m B_{\tilde{\nu}} \Delta w_j \Delta w_k \, d\tilde{\nu} = 0 & j \neq k \\ \int_m B_{\tilde{\nu}} \Delta w_j \Delta w_k \Delta w_l \, d\tilde{\nu} = 0 & j \neq k \\ & j \neq l \\ & k \neq l \\ \int_m B_{\tilde{\nu}} \Delta w_j (\Delta w_k)^2 \, d\tilde{\nu} = 0 & j \neq k \end{array}\right\} \qquad \text{(D12)}$$

Define a gray optical depth for gas r for $S = L$ by

$$\tau_{g,m,r}(L) \equiv w_{g,m,r} \int_0^L t_{m,r}(S)dS \qquad \text{(D13)}$$

With the help of equations (D5), (D12), and (D13), equation (D11) becomes

$$I_m = B_m\left\{1 - e^{-\tau_{g,m}(L)}\left[1 + \sum_{r=1}^{u} \frac{\tau^2_{g,m,r}(L)}{2} \int_m \frac{B_{\tilde{\nu}}}{B_m}\left(\frac{\Delta w_r}{w_{g,m,r}}\right)^2 d\tilde{\nu}\right.\right.$$

$$\left.\left. - \sum_{r=1}^{u} \frac{\tau^3_{g,m,r}(L)}{6} \int_m \frac{B_{\tilde{\nu}}}{B_m}\left(\frac{\Delta w_r}{w_{g,m,r}}\right)^3 d\tilde{\nu} + \ldots\right]\right\} \tag{D14}$$

This equation is the desired expansion of the exact group transport equation.

EXPANSION OF THE EFFECTIVE GROUP TRANSPORT EQUATION

From equations (38) and (37), the effective group transport equation for an isothermal gas is

$$I_m = B_m\left[1 - e^{-\tau_{e,m}(L)}\right] = B_m\left[1 - \prod_{r=1}^{u} e^{-\tau_{e,m,r}(L)}\right] \tag{D15}$$

From equations (35) and (36),

$$d\tau_{e,m,r} = \frac{\int_m B_{\tilde{\nu}} e^{-a_{\tilde{\nu},r} S_{i,m,r}} a_{\tilde{\nu},r}\, d\tilde{\nu}\, dS}{B_m e^{-\tau_{e,m,r}}} \tag{D16}$$

By a derivation analogous to the derivation of equation (B5),

$$t_{m,r}(S) S_{i,m,r} = \int_0^S t_{m,r}(S')dS' \tag{D17}$$

for all values of S.

From equations (D1), (D16), and (D17),

$$e^{-\tau_{e,m,r}}\, d\tau_{e,m,r} = \frac{\displaystyle\int_m B_{\tilde{\nu}} w_{m,r} t_{m,r} e^{-w_{m,r}\int_0^S t_{m,r}(S')dS'}\, d\tilde{\nu}\, dS}{B_m} \qquad \text{(D18)}$$

Integrating equation (D18) from $S = O$ to L gives

$$e^{-\tau_{e,m,r}(L)} = \int_m \frac{B_{\tilde{\nu}}}{B_m} e^{-w_{m,r}\int_0^L t_{m,r}(S)dS}\, d\tilde{\nu} \qquad \text{(D19)}$$

Substituting this equation into equation (D15), making use of equation (D2), expanding into a power series, and making use of equations (D5), (D9), and (D13) give equation (D14) through third order. Hence, through third order in $\Delta w_r/w_{g,m,r}$, both the exact group transport equation and the effective group transport equation give the same value for I_m.

APPENDIX E

SPECTRAL ABSORPTION COEFFICIENTS FOR GASES IN GAS MIXTURES

The method of obtaining the spectral absorption coefficients for hypothetical gases a, b, d, and e in the gas mixtures is given herein as well as the spectral absorption coefficients obtained. Each gas was assumed to have 20 spectral lines whose wave numbers and integrated intensities were given by the Mayer-Goody statistical model (refs. 7 to 9).

WAVE NUMBERS OF LINE CENTERS

For each gas the 20 lines were restricted to wave numbers between 0 and 400 000 reciprocal centimeters because the Planck function for the maximum assumed temperature (40 000° K) was negligible at larger wave numbers. The wave numbers ν_0 of the line centers were assumed to have random values as required by the Mayer-Goody statistical model. These values were selected by taking six-digit random numbers from the table in reference 13 and discarding numbers higher than 400 000. Rearranged in order of increasing wave number, these values are given in tables VI and VII for gases a, b, d, and e.

INTEGRATED ABSORPTION COEFFICIENTS OF LINES

The probability $P_r \, d\tilde{S}_r$ of the integrated linear absorption coefficient $\tilde{S}_r$ of a line of pure gas r being between $\tilde{S}_r$ and $\tilde{S}_r + d\tilde{S}_r$ was assumed to be given by equation (43). Let $f(\tilde{S}_r)$ be the cumulative distribution of $\tilde{S}_r$ (ref. 14).

$$f(\tilde{S}_r) \equiv \int_0^{\tilde{S}_r} P_r(\tilde{S}_r')d\tilde{S}_r' = 1 - e^{-\tilde{S}_r/\bar{S}_r} \tag{E1}$$

The inverse function f^{-1} is defined by

$$f^{-1}\left[f(\tilde{S}_r)\right] \equiv \tilde{S}_r \tag{E2}$$

Hence,

$$f^{-1}(y) = -\bar{S}_r \ln(1 - y) \tag{E3}$$

where y is a dummy variable. Let R be a member of a set of random numbers with rectangular distribution between 0 and 1 and no numbers outside this range. Then values of $\tilde{S}_r$ with the probability distribution given by equation (43) may be found from the relation (ref. 14)

$$\tilde{S}_r = f^{-1}(R) = -\bar{S}_r \ln(1 - R) \tag{E4}$$

Let $J \equiv \tilde{S}_r/\bar{S}_r$. Then, equation (E4) becomes

$$J = -\ln(1 - R) \tag{E5}$$

Because of the rectangular distribution of R between 0 and 1, 1 - R can be replaced with R; thus equation (E5) becomes

$$J = -\ln R \tag{E6}$$

Values of J for the 80 lines were calculated from equation (E6) by taking 80 six-digit random numbers for R from the random-number table of reference 13 and are listed in tables VI and VII.

Two other quantities were needed to specify the Lorentz lines in a pure gas. The collision half-width (ref. 7) was chosen as 750 reciprocal centimeters for all lines of all gases. The integrated linear absorption coefficient $\bar{S}_r$, which is roughly an average value of $\tilde{S}_r$ for all lines of gas r, was chosen as 838 reciprocal centimeters squared for each pure gas.

SPECTRAL ABSORPTION COEFFICIENTS FOR PURE GASES

The spectral absorption coefficients for pure gases a, b, d, and e were calculated using the results from the preceding two sections. The spectral absorption coefficient $a_{\tilde{\nu},a}$ for pure gas a is the curve at S = O in figure 11, $a_{\tilde{\nu},b}$ is the curve at S = 50, while $a_{\tilde{\nu},d}$ and $a_{\tilde{\nu},e}$ are not shown but have the same general appearance as $a_{\tilde{\nu},a}$ and $a_{\tilde{\nu},b}$.

Because of the manner in which they were obtained, $a_{\tilde{\nu},a}$, $a_{\tilde{\nu},b}$, $a_{\tilde{\nu},d}$, and $a_{\tilde{\nu},e}$ are essentially uncorrelated.

APPENDIX F

GROUP PLANCK-FUNCTION COMPUTER SUBPROGRAM

Near the end of this appendix is a listing of the group Planck-function subprogram written in FORTRAN IV for the IBM 7094 digital computer. This subprogram was used in several of the programs used to calculate integrated intensity by different approximations and should be useful to other investigators for numerous applications.

The group Planck function B_m was defined by equation (28). Let

$$Q(\tilde{\nu}, T) \equiv \frac{\int_{\tilde{\nu}}^{\infty} B_{\tilde{\nu}}\, d\tilde{\nu}}{B} \qquad \text{(F1)}$$

Then,

$$B_m(T) = \frac{\sigma T^4}{\pi}\left[Q(\tilde{\nu}_m, T) - Q(\tilde{\nu}_{m+1}, T)\right] \qquad \text{(F2)}$$

where σ is the Stefan-Boltzmann constant. It is readily shown that

$$Q(V) = \frac{15}{\pi^4}\int_V^{\infty} \frac{V'^3}{e^{V'} - 1}\, dV' \qquad \text{(F3)}$$

where V is given by equation (50). Hence,

$$B_m(T) = 1.805\times10^{-12}\, T^4\left[Q(V_m) - Q(V_{m+1})\right] \qquad \text{(F4)}$$

Expressions for Q(V) have been given by Wiebelt (ref. 15) and are repeated. Let $W = V^2$.

$$Q(V) = 1 - VW\left[0.051330 - V\left(0.019249 - V\left\{0.0025665 - W\left[3.0554\times10^{-5} - W(5.6581\times10^{-7} - 1.1573\times10^{-8}\, W)\right]\right\}\right)\right] \qquad V < 2 \qquad \text{(F5)}$$

$$Q(V) = 0.15399 \sum_{M=1}^{5} \frac{e^{-MV}}{M^4} \left\{\left[(MV + 3)MV + 6\right]MV + 6\right\} \qquad V \geq 2 \tag{F6}$$

The subprogram for the group Planck function, based on equations (F4) to (F6), with the FORTRAN variable BLACK representing the group Planck function B_m follows:

```
      FUNCTION BLACK(T)
      COMMON /B3/OMEGA
      DIMENSION OMEGA(2),QUAD(2)
      FACTOR=1.4388/T
      DO 7 I=1,2
      V=FACTOR*OMEGA(I)
      IF(V-2.)1,10,10
   10 IF(V-60.)2,11,11
   11 QUAD(I)=0.
      GO TO 7
    1 W=V**2
      QUAD(I)=1.-V*W*(.051330-V*(.019249-V*(.0025665-W*(3.0554E-5-W*
     1 (5.6581E-7-1.1573E-8*W)))))
      GO TO 7
    2 ZEXP=EXP(-V)
      SUM=0.
      WEXP=1.
      DO 4 M=1,5
      IF(V*FLOAT(M)-60.)5,4,4
    5 WEXP=WEXP*ZEXP
      PROD=FLOAT(M)*V
      SUM=SUM+WEXP*(((PROD+3.)*PROD+6.)*PROD+6.)/FLOAT(M)**4
    4 CONTINUE
      QUAD(I)=.15399*SUM
    7 CONTINUE
      BLACK = 1.805E-12*T**4*(QUAD(1)-QUAD(2))
      RETURN
      END
```

The arguments are actually T, OMEGA (1), and OMEGA (2), although the last two are in COMMON. OMEGA (1) is the wave number at the lower limit of the group, and OMEGA (2) is the wave number at the upper limit.

The program is accurate to within 0.00002 B.

REFERENCES

1. Chandrasekhar, S.: An Introduction to the Study of Stellar Structure. Dover Publications, Inc., 1957, pp. 183-215.
2. Stewart, John C.: Non-Grey Radiative Transfer. J. Quant. Spectrosc. Radiat. Transfer, vol. 4, no. 5, Sept.-Oct. 1964, pp. 723-729.
3. Grant, Ian P.: On the Representation of Frequency Dependence in Non-Grey Radiative Transfer. J. Quant. Spectrosc. Radiat. Transfer, vol. 5, no. 1, Jan.-Feb. 1965, pp. 227-243.
4. Sampson, Douglas H.: Choice of an Appropriate Mean Absorption Coefficient for Use in the General Grey Gas Equations. J. Quant. Spectrosc. Radiat. Transfer, vol. 5, no. 1, Jan.-Feb. 1965, pp. 211-225.
5. Viskanta, R.: Concerning the Definitions of the Mean Absorption Coefficient. Int. J. Heat Mass Transfer, vol. 7, no. 9, Sept. 1964, pp. 1047-1049.
6. Campbell, Philip M.; and Nelson, Robert G.: Numerical Methods for Nonlinear Radiation Transport Calculations. Rep. No. UCRL-7838, University of California Lawrence Radiation Lab., Sept. 29, 1964.
7. Penner, S. S.: Quantitative Molecular Spectroscopy and Gas Emissivities. Addison-Wesley Publ. Co., Inc., 1959, pp. 10-318.
8. Mayer, Harris: Methods of Opacity Calculations. Rep. No. LA-647 (AECD-1870), University of California, Los Alamos Scientific Lab., Mar. 1948.
9. Goody, R. M.: A Statistical Model for Water-Vapour Absorption. Quart. J. Roy. Meteorol. Soc., vol. 78, no. 336, Apr. 1952, pp. 165-169.
10. Goody, Richard M.: Theoretical Basis. Vol. 1 of Atmospheric Radiation. Oxford University Press, 1964, pp. 122-243.
11. Stewart, John C.; and Pyatt, Kedar D., Jr.: Theoretical Study of Optical Properties. Photon Absorption Coefficients, Opacities, and Equations of State of Light Elements; Including the Effect of Lines. Rep. No. GA-2528, vol. I (AFSWC-TR-61-71, vol. I), General Atomic Div., General Dynamics Corp., Sept. 1961.
12. Cramér, Harold: Random Variables and Probability Distributions. Second ed., Cambridge University Press, 1962, p. 23.
13. Dixon, Wilfrid J.; and Massey, Frank J., Jr.: Introduction to Statistical Analysis. Second ed., McGraw-Hill Book Co., Inc., 1957, pp. 366-370.
14. Hammersley, J. M.; and Handscomb, D. C.: Monte Carlo Methods. John Wiley and Sons, Inc., 1964.
15. Wiebelt, John A.: Engineering Radiation Heat Transfer. Holt, Rinehart, and Winston Publishing Co., 1966, p. 134.

TABLE I. - EQUATIONS OF SPECTRAL ABSORPTION COEFFICIENTS DESIGNATED A TO H

Profile[a]	Spectral absorption coefficient, $a_{\tilde{\nu}}$, cm^{-1}	Range of wave number $\tilde{\nu}$ for which expression is valid, cm^{-1}
A	0.10 .02	$0 \leq \tilde{\nu} \leq 60\,830$ $60\,830 < \tilde{\nu} \leq \infty$
B	$\dfrac{3\,767\,000}{(\tilde{\nu} - 60\,830)^2 + 18\,840\,000}$	$0 \leq \tilde{\nu} \leq \infty$
C	$\dfrac{376\,700}{(\tilde{\nu} - 60\,830)^2 + 188\,400}$	$0 \leq \tilde{\nu} \leq \infty$
D	0.02 .10	$0 \leq \tilde{\nu} \leq 60\,830$ $60\,830 < \tilde{\nu} \leq \infty$
E	$6.574\times10^{-7}\,\tilde{\nu} + 0.0052$	$0 \leq \tilde{\nu} \leq \infty$
F	$-6.574\times10^{-7}\,\tilde{\nu} + 0.1048$ 0	$0 \leq \tilde{\nu} \leq 159\,400$ $159\,400 < \tilde{\nu} \leq \infty$
G	2.00 .02	$0 \leq \tilde{\nu} \leq 20\,000$ $20\,000 < \tilde{\nu} \leq \infty$
H	$\dfrac{636.6\, b_c}{(\tilde{\nu} - 60\,830)^2 + b_c^2}$ where $b_c = 90\,000/T^{1/2}$	$0 \leq \tilde{\nu} \leq \infty$

[a] Profiles are given in figs. 2 and 5 to 8.

TABLE II. - INTEGRATED INTENSITIES CALCULATED BY THREE METHODS FOR PROFILES A TO F WITH A COLD WALL[a]

Method of calculation	Temperature distribution					
	1		4		7	
	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent
			Profile A			
Exact	1.29×10^6	---	5.73×10^5	---	3.30×10^5	---
One wave-number group[b] a_{Pl}	1.32	2	6.70	17	3.90	18
One wave-number group,[b] one origin group a_e	1.15	-11	5.73	0	3.54	7
			Profile B			
Exact	5.91×10^5	---	3.03×10^5	---	1.59×10^5	---
One wave-number group[b] a_{Pl}	1.21×10^6	105	5.43	79	2.40	51
One wave-number group,[b] one origin group a_e	5.09×10^5	-14	3.03	0	1.57	-1
			Profile C			
Exact	2.23×10^5	---	1.18×10^5	---	6.28×10^4	---
One wave-number group[b] a_{Pl}	1.24×10^6	455	5.62	378	2.52×10^5	301
One wave-number group,[b] one origin group a_e	1.96×10^5	-12	1.18	0	6.26×10^4	0
			Profile D			
Exact	1.08×10^6	---	5.73×10^5	---	3.15×10^5	---
One wave-number group[b] a_{Pl}	1.36	27	6.70	17	3.46	10
One wave-number group,[b] one origin group a_e	1.29	19	5.73	0	2.99	-5
			Profile E			
Exact	1.21×10^6	---	6.05×10^5	---	3.08×10^5	---
One wave-number group[b] a_{Pl}	1.45	20	6.44	6	3.11	1
One wave-number group,[b] one origin group a_e	1.39	15	6.06	0	2.96	-4
			Profile F			
Exact	1.26×10^6	---	6.35×10^5	---	3.65×10^5	---
One wave-number group[b] a_{Pl}	1.32×10^6	5	6.72	6	3.87	6
One wave-number group,[b] one origin group a_e	9.93×10^5	-21	6.36	0	3.79	4

[a] Cold, black wall is at distance L along ray of 50 cm.

[b] Wave-number group extended 0 to 400 000 cm^{-1}.

TABLE III. - INTEGRATED INTENSITIES CALCULATED BY THREE METHODS FOR PROFILES A TO F WITH A HOT WALL[a]

Method of calculation	Temperature distribution			
	1		7	
	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent
	Profile A			
Exact	2.63×10^{6}	---	3.30×10^{5}	---
One wave-number group[b] a_{Pl}	1.76	-33	3.90	18
One wave-number group,[b] one origin group a_e	2.22	-15	3.55	8
	Profile B			
Exact	3.91×10^{6}	---	1.73×10^{5}	---
One wave-number group[b] a_{Pl}	2.60	-34	2.45	42
One wave-number group,[b] one origin group a_e	3.49	-11	1.68	-3
	Profile C			
Exact	4.36×10^{6}	---	7.99×10^{4}	---
One wave-number group[b] a_{Pl}	2.49	-43	2.57×10^{5}	222
One wave-number group,[b] one origin group a_e	4.18	-4	7.80×10^{4}	-2
	Profile D			
Exact	1.47×10^{6}	---	3.22×10^{5}	---
One wave-number group[b] a_{Pl}	1.48	1	3.48	8
One wave-number group,[b] one origin group a_e	1.83	24	3.04	-6
	Profile E			
Exact	1.53×10^{6}	---	3.14×10^{5}	---
One wave-number group[b] a_{Pl}	1.66	9	3.14	0
One wave-number group,[b] one origin group a_e	1.84	20	3.00	-4
	Profile F			
Exact	2.72×10^{6}	---	3.65×10^{5}	---
One wave-number group[b] a_{Pl}	1.70	-37	3.88	6
One wave-number group,[b] one origin group a_e	2.03	-25	3.79	4

[a] Hot, black wall is at distance L along ray of 50 cm and is at same temperature as adjacent gas.

[b] Wave-number group extended 0 to 400 000 cm^{-1}.

TABLE IV. - INTEGRATED INTENSITIES CALCULATED BY FIVE METHODS FOR PROFILES G AND H WITH A COLD WALL[a]

[Temperature distribution designated 8.]

Method of calculation	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent
Profile G		
Exact	4.17×10^5	----
One wave-number group[b] a_{Pl}	7.53×10^1	-100
Two wave number groups[c] a_{Pl}	4.17×10^5	0
One wave-number group,[b] one origin group a_e	1.63×10^4	-96
Two wave-number groups,[c] one origin group a_e	4.17×10^5	0
Profile H		
Exact	5.09×10^4	----
One wave-number group[b] a_{Pl}	5.00×10^5	881
Three wave-number groups[d] a_{Pl}	1.24×10^5	143
One wave-number group,[b] one origin group a_e	6.88×10^4	35
Three wave-number groups,[d] one origin group a_e	6.49×10^4	27

[a] Cold wall is at distance L along ray of 50 cm.

[b] Wave-number group extended 0 to 400 000 cm^{-1}.

[c] Wave-number groups extended 0 to 20 000 and 20 000 to 400 000 cm^{-1}.

[d] Wave-number groups extended 0 to 45 830, 45 830 to 75 830, and 75 830 to 400 000 cm^{-1}.

TABLE V. - INTEGRATED INTENSITIES CALCULATED BY FIVE METHODS FOR BINARY MIXTURE A, BINARY MIXTURE B, AND TERNARY MIXTURE WITH A COLD WALL[a]

Method of calculation	Temperature distribution			
	8		9	
	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent	Integrated intensity, I, W/(cm^2)(steradian)	Error from exact I, percent
	Binary mixture A			
Exact	2.71×10^5	---		
One wave-number group[b] a_{Pl}	1.93	-29		
Three wave-number groups[c] a_{Pl}	3.32	23		
Three wave-number groups,[c] one origin group a_e	1.89	-30		
Three wave-number groups,[c] two origin group a_e	2.59	-5		
	Binary mixture B			
Exact	1.55×10^5	---		
One wave-number group[b] a_{Pl}	7.75	401		
Three wave-number groups[c] a_{Pl}	3.03	96		
Three wave-number groups,[c] one origin group a_e	1.84	19		
Three wave-number groups,[c] two origin group a_e	1.98	28		
	Ternary mixture			
Exact			5.68×10^5	---
One wave-number group[b] a_{Pl}			1.77	-69
Three wave-number groups[c] a_{Pl}			6.48	14
Three wave-number groups,[c] one origin group a_e			2.88	-49
Three wave-number groups,[c] two origin group a_e			4.69	-17

[a] Cold wall is at distance L along ray of 50 cm for binary mixtures and 150 cm for ternary mixture.

[b] Wave-number group extended 0 to 400 000 cm^{-1}.

[c] Wave-number groups extended 0 to 100 000, 100 000 to 200 000, and 200 000 to 400 000 cm^{-1}.

TABLE VI. - LINE POSITIONS AND LINE STRENGTH PARAMETERS FOR GASES a AND b OF MIXTURES

Gas a		Gas b	
Line position, $\tilde{\nu}_o$, cm^{-1}	Line strength parameter, J	Line position, $\tilde{\nu}_o$, cm^{-1}	Line strength parameter, J
9 994	1.2022	33 606	0.7986
19 362	1.9862	56 418	.2602
37 438	2.3076	73 998	2.8975
72 834	.3478	82 341	.4347
90 324	.0792	125 624	.4061
119 202	1.5181	146 670	3.1317
123 356	.7789	159 466	.4395
129 578	.8320	176 833	1.3123
131 880	.1995	177 817	.8202
137 408	.1505	206 413	.5250
150 498	.3421	214 598	1.4836
182 760	.7748	260 527	.5263
206 455	.6532	274 945	.7483
220 791	.4057	282 562	1.2592
308 420	.1390	285 207	.4235
338 452	.2008	291 665	2.2219
343 178	.0245	309 732	.0092
347 055	.0956	339 333	1.4360
355 507	.9856	388 435	.5223
366 994	.0218	388 579	2.0375

TABLE VII. - LINE POSITIONS AND LINE STRENGTH PARAMETERS FOR GASES d AND e OF MIXTURES

Gas d		Gas e	
Line position, $\tilde{\nu}_o$, cm^{-1}	Line strength parameter, J	Line position, $\tilde{\nu}_o$, cm^{-1}	Line strength parameter, J
5 499	0.1767	69 134	0.4401
44 935	.5048	76 387	.0758
83 586	.2458	104 551	3.0978
84 226	1.7403	108 506	.2534
91 882	1.4745	128 839	.1772
98 932	.4911	164 442	2.9064
100 973	3.2213	195 023	.7689
116 644	1.1733	195 235	.6671
118 050	1.4220	217 783	1.9388
125 507	1.1864	218 185	2.1993
128 079	1.1195	218 264	1.8557
154 744	3.2730	246 373	.3219
186 333	.1222	272 675	.9993
195 654	2.1874	283 060	.8711
235 237	.7292	306 927	.4269
310 601	.3628	337 134	.6405
321 790	1.6439	374 029	1.5674
331 851	1.1472	380 010	.5573
359 631	1.1099	385 080	1.3798
375 420	.9530	398 209	.9496

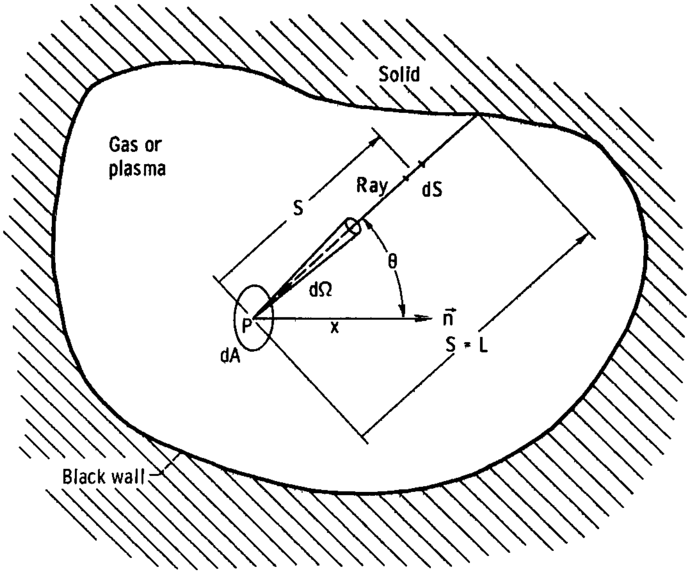

Figure 1. - Quantities and elements in the analysis of intensity and radiant flux density.

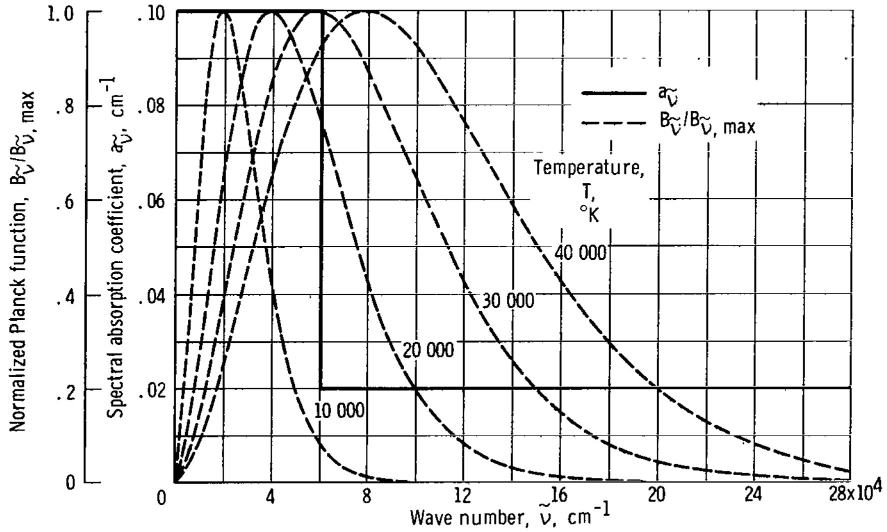

Figure 2. - Spectral absorption coefficient designated profile A. Normalized Planck functions for four temperatures are given for reference.

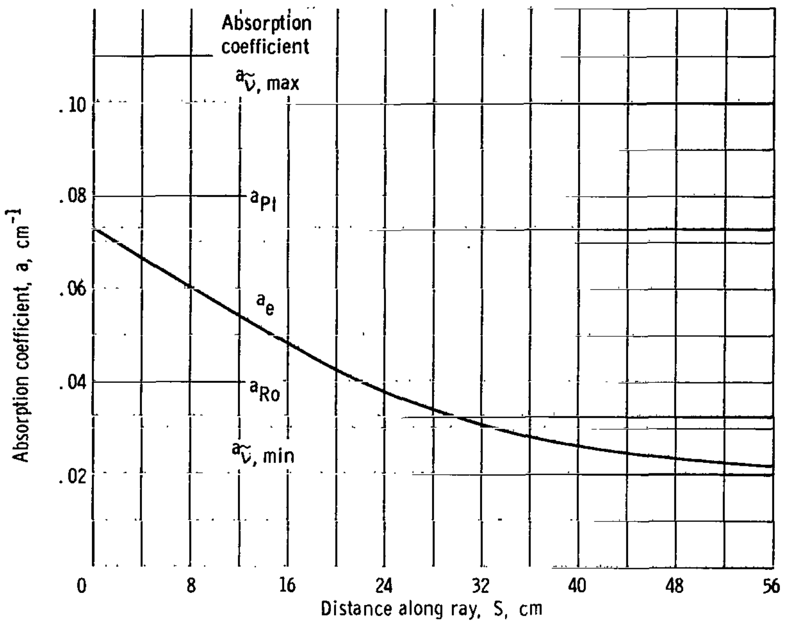

Figure 3. - Various mean absorption coefficients for profile A, as well as minimum and maximum spectral absorption coefficients. Temperature, 20 000° K.

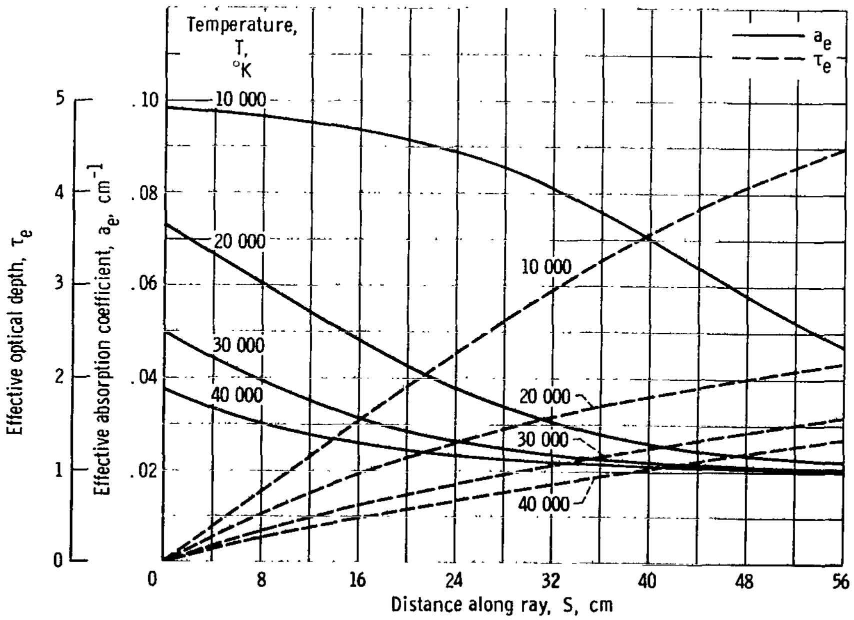

Figure 4. - Effective absorption coefficient and effective optical depth for profile A at four temperatures.

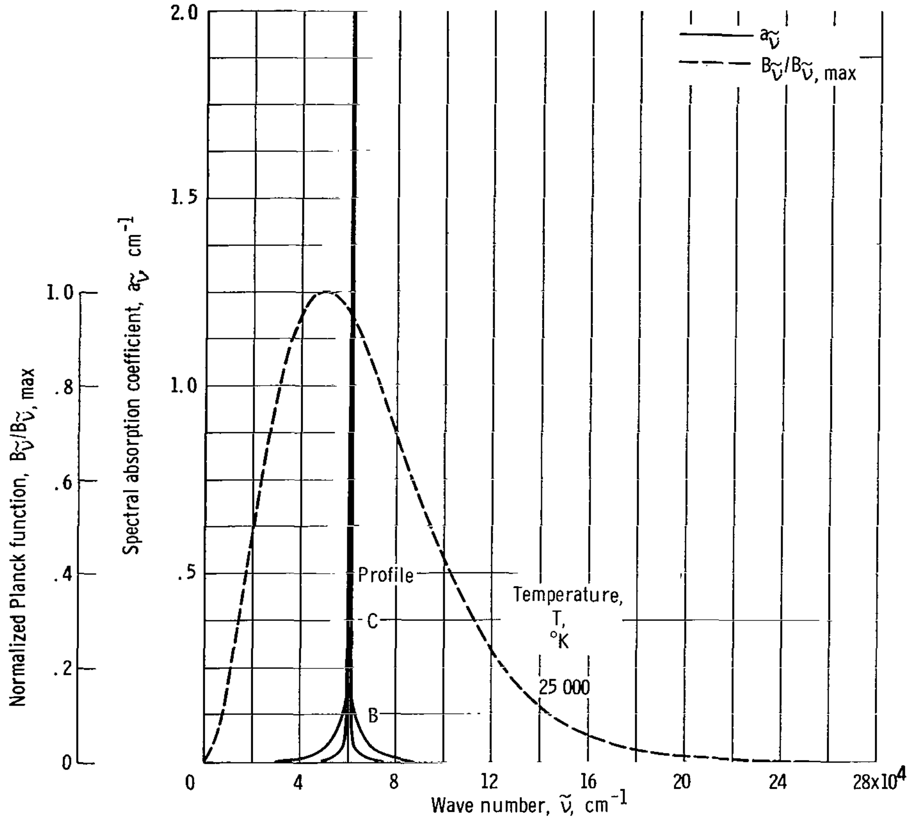

Figure 5. - Lorentz-line spectral absorption coefficients designated profiles B and C. Normalized Planck function is given for reference and is for 25 000° K.

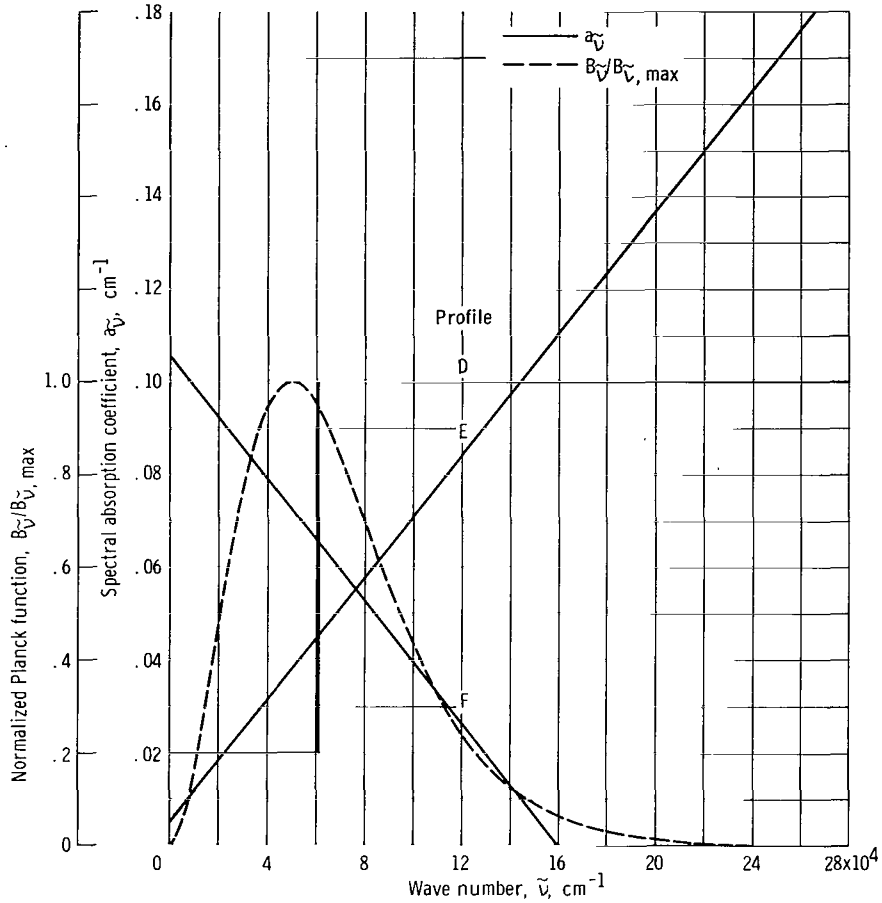

Figure 6. - Spectral absorption coefficients designated profiles D, E, and F. Normalized Planck function is given for reference and is for 25 000° K.

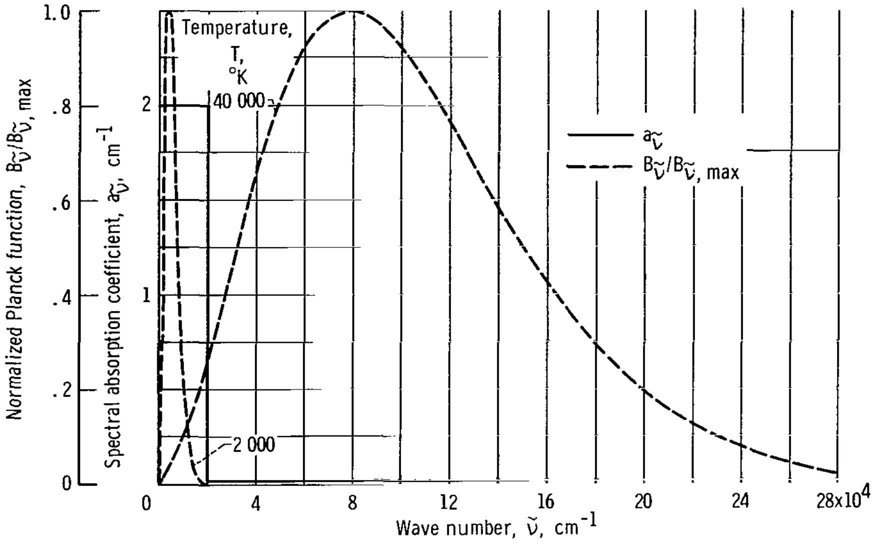

Figure 7. - Spectral absorption coefficient designated profile G. Normalized Planck functions for two temperatures are given for reference.

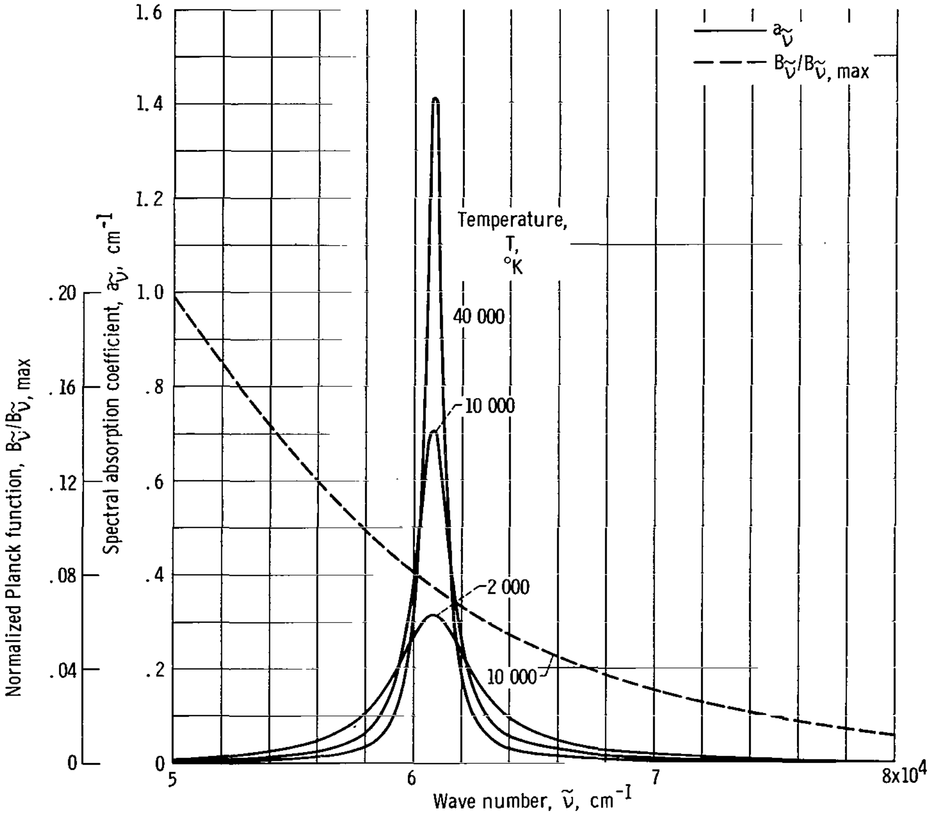

Figure 8. - Variable-width Lorentz-line spectral absorption coefficient designated profile H. Normalized Planck function is given for reference.

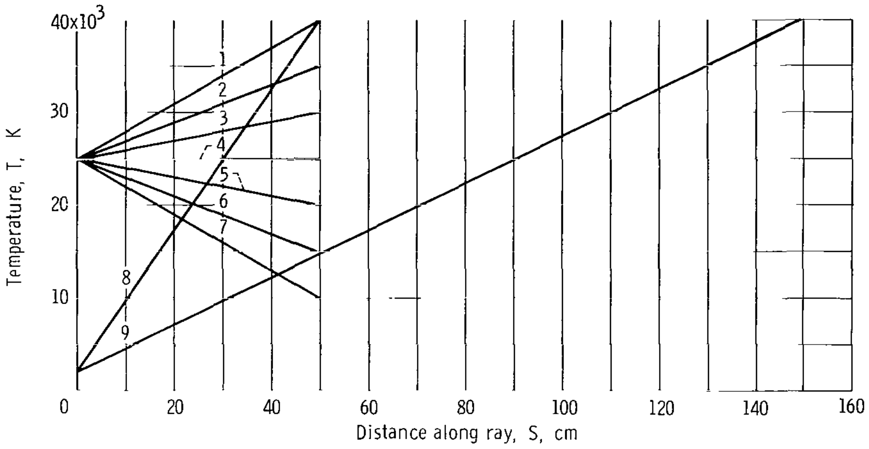

Figure 9. - Temperature distributions designated 1 through 9.

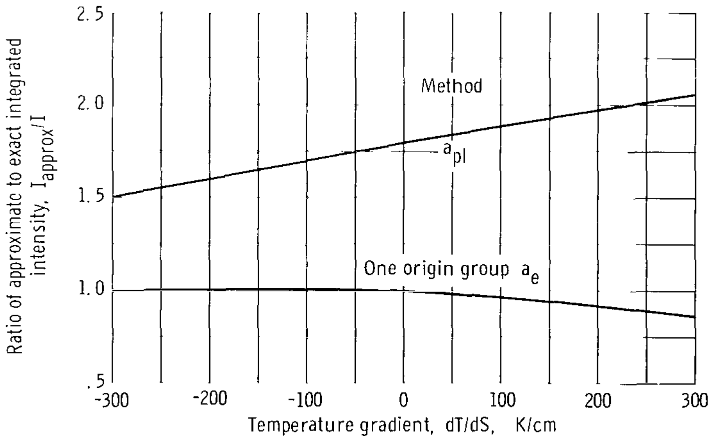

Figure 10. - Ratios of approximate to exact integrated intensities for profile B. Range of temperature gradients correspond to temperature distributions 1 through 7; cold wall at distance L along ray of 50 centimeters. One wave-number group was used.

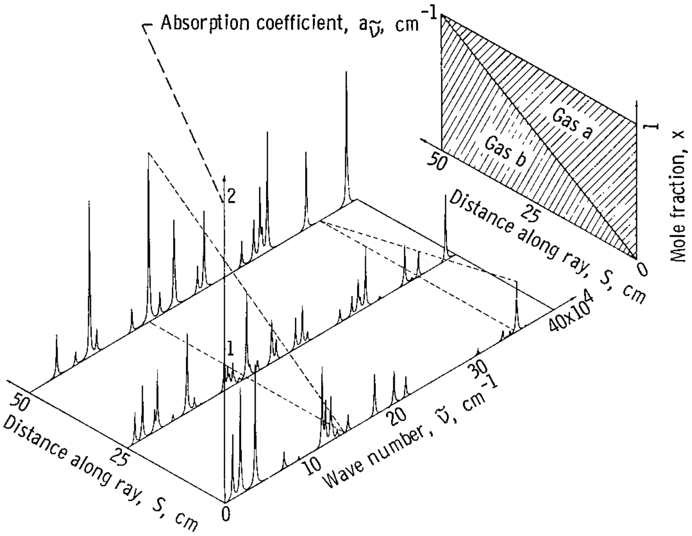

Figure 11. - Spectral absorption coefficient of binary mixture A. Mole fractions of gases a and b are given at right.

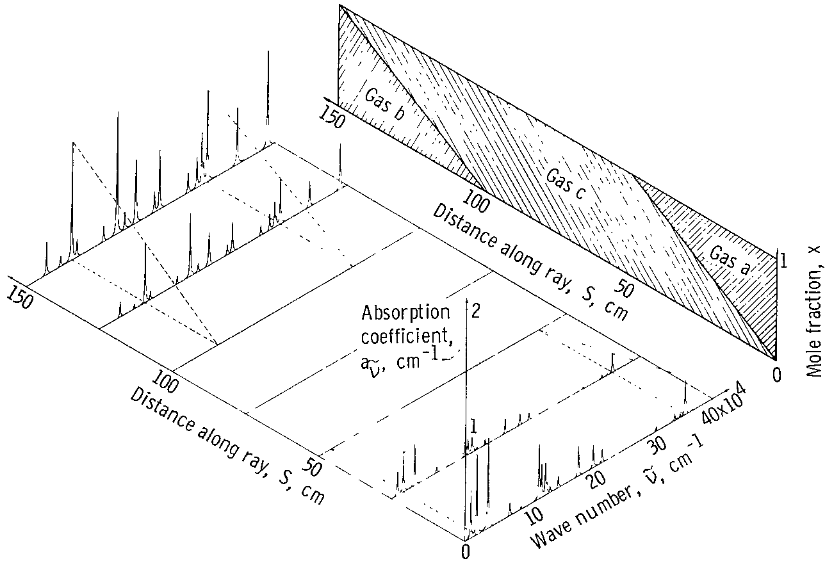

Figure 12. - Spectral absorption coefficient of ternary mixture. Mole fractions of gases a, b, and c are given at right. Note that gas c is transparent.

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> —NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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