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QUASILINEAR FORMULATION OF FINITE TRANSFORMATIONS

By

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SUMMARY

A set of quasilinear formulas are developed for translating changes $\Delta\alpha$ into a set ΔR, $\Delta\dot{R}$ which is related nonlinearly to α. This set of formulas is valid and numerically accurate for infinitesimal, as well as finite, $\Delta\alpha$.

1. Introduction

In differential correction schemes, the parameters which are generally corrected are not the coordinates themselves but usually functions of these coordinates chosen to avoid loss of significance in the differential correction matrix with time [1]. Such a set of parameters is given in Reference [2]. A modified set is presented in Reference [3]. This set corresponds to the one in use at present in the Goddard Minimum Variance Program.

In this program, as well as in other differential correction programs, the result of processing one or more observations is a vector $\Delta\alpha = [\Delta\alpha_1, \Delta\alpha_2, \Delta\alpha_3, \Delta\alpha_4, \Delta\alpha_5, \Delta\alpha_6]$ which has to be related to a change in position and velocity coordinates $\Delta R = [\Delta x, \Delta y, \Delta z, \Delta\dot{x}, \Delta\dot{y}, \Delta\dot{z}]$. References [2] and [3] present a linear relation of the form

$$\Delta R = S\,\Delta\alpha \tag{1}$$

where the matrix S is a function of position and velocity.

This relation is valid, strictly speaking, only for infinitesimal $\Delta\alpha$ and thus will lead to errors for finite $\Delta\alpha$. These errors are particularly severe for a least-squares approximation, but also cause difficulties in the minimum variance scheme when the vector $\Delta\alpha$ is at all sizeable. In addition, even small changes in several components of $\Delta\alpha$ are often completely "drowned" by the higher order changes induced, due to the nonlinearity, by the larger changes in other components. In either case, when $\Delta\alpha$ is recomputed from R after the transformation (1) has been applied, incorrect values of $\Delta\alpha$ have been applied, thus causing, at best, a serious delay in convergence and, at worst, preventing it altogether. As a result of these difficulties, Bailie [4] developed a set of formulas which guarantees the correct application of $\Delta\alpha$ for finite values of the vector. Experience has shown, however, that the formulas given in Reference [4] should be applied only for larger changes in $\Delta\alpha$, since for small values of $\Delta\alpha$ they produce sufficient numerical noise so as to prevent the fine adjustment of the orbit, once the residuals have been reduced to small values. It seems difficult to fix exact limits where one or the other of these transformations are to be used, and the use of double precision arithmetic in the finite transformation also does not seem particularly attractive. A formulation has therefore been developed which is valid for all admissible values of $\Delta\alpha$ and which, by its quasilinear form, will computationally (as well as analytically, of course) reduce to eqn. (1), for small values of $\Delta\alpha$.

2. <u>Notation</u>

R	position vector
$\dot{R}$	velocity vector
$H = R \times \dot{R}$	angular momentum vector
$d = R \cdot \dot{R}$	dot product
$r = [R \cdot R]^{1/2}$	magnitude of position vector
$v = [\dot{R} \cdot \dot{R}]^{1/2}$	magnitude of velocity vector
$'$	transformed value
μ	gravitational constant
α, β, δ	coefficients defined in equation 14
$\Delta\alpha_i$	i^{th} component of $\Delta\alpha$ vector ($i = 1, - - -, 6$)
ΔR_i	vector change in R due to a change in the i^{th} component of $\Delta\alpha$
a	semi-major axis

3. Analysis

The transformations are:

A. A rotation of the position vector about the velocity vector through an angle $\Delta\alpha_1$.

In reference [4], this rotation is accomplished by:

$$R' = R \cos \Delta\alpha_1 + \dot{R} \frac{d}{v^2} (1 - \cos \Delta\alpha_1) - \frac{H}{v} \sin \Delta\alpha_1 \tag{2}$$

It is clear in this case that for small $\Delta\alpha_1$, the computation of $1 - \cos \Delta\alpha_1$ may lead to a loss in significance unless special precautions are taken. If R' is written as:

$$R' = R + \Delta R_1 \tag{3}$$

and the identity

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

is used, ΔR is given by:

$$\Delta R = -2\left[R - \frac{d}{v^2} \dot{R}\right] \sin^2 \frac{\Delta\alpha_1}{2} - \frac{H}{v} \sin \Delta\alpha_1 \tag{4}$$

B. A rotation of the velocity vector about the position vector through an angle $\Delta\alpha_2$.

From Reference [4]

$$\dot{R}' = \dot{R} \cos \Delta\alpha_2 + \frac{d}{r^2} R'(1 - \cos \Delta\alpha_2) + \frac{H'}{r} \sin \Delta\alpha_2$$

as above.

$$\dot{R}' = \dot{R} + \Delta\dot{R}_2 \tag{5}$$

$$\Delta\dot{R} = -2\left[\dot{R} - \frac{d}{r^2} R'\right] \sin^2 \frac{\Delta\alpha_2}{2} + \frac{H'}{r} \sin \Delta\alpha_2 \tag{6}$$

C. A rotation of both R and $\dot{R}$ about H through an angle $\Delta\alpha_3$.

$$\begin{aligned} R'' &= R' \cos \Delta\alpha_3 + \frac{H' x R'}{h} \sin \Delta\alpha_3 \\ \dot{R}'' &= \dot{R}' \cos \Delta\alpha_3 + \frac{H' x \dot{R}'}{h} \sin \Delta\alpha_3 \end{aligned} \tag{7}$$

Replacing $1 - \cos \Delta\alpha_3$ as above and eliminating the two cross products

$$\begin{aligned} \Delta R_3 &= -R'\left(\frac{d}{h} \sin \Delta\alpha_3 - 2 \sin^2 \frac{\Delta\alpha_3}{2}\right) + \dot{R}'\left(\frac{r^2}{h} \sin \Delta\alpha_3\right) \\ \Delta\dot{R}_3 &= -R'\left(\frac{v^2}{h} \sin \Delta\alpha_3\right) + \dot{R}'\left(\frac{d}{h} \sin \Delta\alpha_3 - 2 \sin^2 \frac{\Delta\alpha_3}{2}\right) \end{aligned} \tag{8}$$

where

$$\mathbf{R}'' = \mathbf{R}' + \Delta\mathbf{R}_3$$

$$\dot{\mathbf{R}}'' = \dot{\mathbf{R}}' + \Delta\dot{\mathbf{R}}_3 \tag{9}$$

D. The remaining three parameters are the instantaneous values of

$$\alpha_4 = d$$

$$\alpha_5 = \frac{1}{a}$$

$$\alpha_6 = r$$

The transformations due to changes in these parameters are best handled simultaneously. In order to simplify the notation, R in the following replaces $\mathbf{R}''$ and the final transform is denoted by $\mathbf{R}'$. The equations to be satisfied are:

$$\mathbf{R}' \cdot \dot{\mathbf{R}}' = \mathbf{R} \cdot \dot{\mathbf{R}} + \Delta\alpha_4 \tag{10}$$

$$\frac{2}{[\mathbf{R}' \cdot \mathbf{R}']^{1/2}} - \frac{\dot{\mathbf{R}}' \cdot \dot{\mathbf{R}}'}{\mu} = \frac{2}{[\mathbf{R} \cdot \mathbf{R}]^{1/2}} - \frac{\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}}{\mu} + \Delta\alpha_5 \tag{11}$$

$$[\mathbf{R}' \cdot \mathbf{R}']^{1/2} = [\mathbf{R} \cdot \mathbf{R}]^{1/2} + \Delta\alpha_6 \tag{12}$$

where

$$R' = R + \Delta R$$
$$\dot{R}' = \dot{R} + \Delta\dot{R} \tag{13}$$

In order to leave $\Delta\alpha_1$ and $\Delta\alpha_2$ invariant, ΔR and $\Delta\dot{R}$ must be coplanar with R and $\dot{R}$, and in order to leave $\Delta\alpha_3$ invariant, $\Delta\dot{R}$ must be parallel to $\dot{R}$. Hence

$$\Delta R = \alpha R + \beta\dot{R}$$
$$\Delta\dot{R} = \delta\dot{R} \tag{14}$$

Substitution from eqns. (14) and (12) into eqn. (11) leads to a single quadratic equation for δ:

$$\frac{2}{r + \Delta\alpha_6} - \frac{(\dot{R} + \delta\dot{R}) \cdot (\dot{R} + \delta\dot{R})}{\mu} = \frac{2}{r} - \frac{v^2}{\mu} + \Delta\alpha_5 \tag{15}$$

This equation is solved for δ, the sign of the root is selected so that δ vanishes whenever $\Delta\alpha_5 = \Delta\alpha_6 = 0$.

$$\delta = \left[1 - \frac{\mu}{v^2}\left(\Delta\alpha_5 + \frac{2\,\Delta\alpha_6}{(r + \Delta\alpha_6)r}\right)\right]^{1/2} - 1 \tag{16}$$

Written in quasilinear form:

$$\delta = \frac{L_1}{(1 + L_1)^{1/2} + 1} \tag{17}$$

where

$$L_1 = - \frac{\mu}{v^2}\left(\Delta\alpha_5 + \frac{2\,\Delta\alpha_6}{(r + \Delta\alpha_6)r}\right) \tag{18}$$

Substitution from eqn. (14) into eqns. (10) and (12) leads to a linear and a quadratic equation between α and β respectively. After some algebra, the solution can be represented in quasilinear form as:

$$\alpha = \frac{L_2}{(1 + L_2)^{1/2} + 1} \tag{19}$$

where

$$L_2 = \frac{\Delta\alpha_6(2r + \Delta\alpha_6)v^2(1+\delta)^2 + \delta(2+\delta)d^2 - \Delta\alpha_4(2d + \Delta\alpha_4)}{h^2(1+\delta)^2} \tag{20}$$

$$\beta = \frac{1}{v^2}\left(\frac{\Delta\alpha_4 - \delta d}{1+\delta} - \alpha d\right) \tag{21}$$

The major terms in L_1, L_2, β are thus proportional to $\Delta\alpha_4$, $\Delta\alpha_5$, $\Delta\alpha_6$ and will vanish identically when $\Delta\alpha_4 = \Delta\alpha_5 = \Delta\alpha_6 = 0$ and, for small values of $\Delta\alpha$, the higher order terms appear as coefficients, thus insuring the accurate computation of ΔR. No assumption of smallness of $\Delta\alpha$ has been made in the derivation of these formulas; they are therefore valid for any admissible set of $\Delta\alpha$.

4. References

[1] Pines, S., Payne, M. and Wolf, H.; "Asymptotically Singular Differential Correction Matrices," WADD report.

[2] Pines, S., Wolf, H., Woolston, D. and Squires, R.; "The Goddard Minimum Variance Program," Analytical Mechanics Associates, Inc. report to Goddard Space Flight Center.

[3] Boyce, W.M. and Pines, S.; "Analytic Derivatives Over Extended Time Arcs," NASA Manned Spacecraft Center Memorandum.

[4] Bailie, A.; "Goddard Variational Parameters," Analytical Mechanics Associates, Inc. report to Goddard Space Flight Center.