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OPTIMUM PROCESSES IN SYSTEMS WITH DISTRIBUTED PARAMETERS AND SOME PROBLEMS OF INVARIANCE THEORY

by

A. I. Yegorov

Akademiia Nauk SSSR, Izvestiia, Seriia Mathematicheskaia, Vol. 29, No. 6, 1205-1260 (1965)

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THE INTERRELATION OF SMALL BODIES OF THE SOLAR SYSTEM

A. K. Terent'yeva

Translation of "K voprosu o vzaimosvyazi malykh tel solnechnoy sistemy"
Informatsionnyy Byulleten'
"Materialy Mezhdunarodnogo Geofizicheskogo Goda"
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The paper* investigates optimum processes in systems whose behavior is described by difference boundary problems for partial differential equations.

The majority of physical processes which are encountered by the engineers during their practical activity can be controlled; and, consequently, during their realization, one strives to attain the optimum (in some sense) alternative. The maximum principle of L. S. Pontryagin[1] appears as a mathematical method for the solution of problems of optimum control when the processes may be described by the ordinary differential equations. However, numerous control processes are described by partial differential equations with additional (boundary of initial) conditions. These equations may be of diverse type (equations of mass or heat exchange, equations of hydro or aero-dynamics, heat transfer, kinetics of chemical reactions, etc). If the behavior of the control system is described by equations among which there are some with partial derivatives, then it is called a system with distributed parameters[2]. In numerous simpler cases, such systems may be described by differential-differences equation; and, consequently, one can still apply the maximum principle[3].

The problems of optimum control of more complicated systems can not be solved directly by means of the maximum principle of L. S. Pontryagin (see article by Kharatishvili[4], pp. 516-518). Consequently, attempts have been made to generalize this principle in such a way that by its application one could investigate control processes of more complicated systems with distributed parameters[5,6,7,8,9,10,11,12,13,14,15]. In particular, the work[10] proposed a method based on the use of differential equations in Banach spaces. In numerous cases, such an approach allows the investigation of partial differential equations as if they were ordinary differential equations and solves the problem of optimum control using as the optimality criterion the functional

$$I = \int_{t_0}^{T} f(t, x(t), u(t))\, dt \quad . \tag{1}$$

In spite of obvious advantages, this method has also substantial short-comings since the introduction of Banach spaces requires additional limitations on the class of permissible controls which are not caused by the physical essence of the problem. In addition, the choice of the

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*The basic content of the work was presented at the seminar of L. S. Pontryagin on theory of optimum processes, 13 February 1964.

functional (1) as the optimality criterion for the problem with partial differential equations is not as successful as in the case of problems with ordinary differential equations. In particular, the indicated method does not solve the important practical problem in which the functional (1) is substituted by an integral evaluated over a surface bounding the domain with which the equations are under investigation.

Of definite interest is the method[13] based on the representation of control quantities by means of integral relationships. However, its justification can not be viewed as satisfactory. In addition, the application of this method to processes described by boundary problems with partial differential equations can not be considered sufficiently efficient because of the following reasons. First, the reduction of boundary problems to integral equations can not always be carried out though the problem can still be solvable by other methods. Second, it is always desirable to have optimality conditions expressed directly through quantities entering the equations and additional conditions.

In the present work, we used the method of solution which is to an equal degree applicable to the cases of hyperbolic, parabolic, and elliptic equations. Using this method, L. I. Rozopoer[16] studied the case when the control process is described by ordinary differential equation and finite differences equations. In subsequent works[17], we obtained invariance coefficients for systems relative to the external interactions whereas the starting point for investigation was a formula for functional increments found in the work of Rozopoer[16]. Analogous results (though only for special cases) were obtained also for systems with distributed parameters.

The paper consists of five sections. In Sections I and II we investigate various problems of optimum control of processes describable by boundary conditions for hyperbolic equations with data on characteristics. The necessary optimality conditions are formulated in the form of a maximum principle.

In Section III we establish connection of the investigated problem with the problems of variational calculus. It is shown that the Euler-Ostrogradskiy equations can be derived from the maximum principle if the control domain coincides with the entire space. If this domain is closed, then along the optimum surface one may find that the Legendre conditions are even not satisfied.

In Section IV we study problems of optimum control when processes are described by the boundary conditions for parabolic systems. We obtain a formula for the functional increment by means of which one finds the optimality conditions. These results are extended to analogous problems connected with the elliptical and hyperbolic systems.

Section V is devoted to the problems of the invariance theory. For linear equations we obtain the necessary and sufficient conditions for invariance relative to the external interaction whereas the criteria of invariance we choose functionals analogous to those investigated in Sections I through IV.

The author uses the occasion to express his gratitude to L. S. Pontryagin and the participants of his seminar for their interest in the present work. In addition, the author is sincerely grateful to V. G. Boltyanskiy, O. A. Oleynik, and Yu. V. Yegorov for very useful discussions of the results obtained in the paper.

Section I. OPTIMUM PROCESSES IN SYSTEMS WHOSE BEHAVIOR IS DESCRIBED BY HYPERBOLIC EQUATIONS

1. Formulation of the Problem. Optimality Conditions

Let the control process be described by a system of equations

$$z_{ixy} = f_i(x, y, z_1, \ldots, z_m, z_{1x}, \ldots, z_{mx}, z_{1y}, \ldots, z_{my}, v),$$

$$i = 1, \ldots, m, \tag{1.1}$$

where the functions f_i have within the domain $G\,(0 \le x \le X,\ 0 \le y \le Y)$ continuous derivatives of the first order in x and y and twice continuously differentiable over other arguments. As a class of permissible controls, we use the set of sectionally continuous functions $v = v(x, y)$ defined within the region G and with values within a certain bounded convex domain V (open or closed) of the r-dimensional euclidian space. We assume that the line of discontinuity of the permissible control is sufficiently smooth. We impose onto the function z_i defined by equations (1.1) certain boundary conditions (Goursat conditions)

$$z_i(0, y) = \varphi_i(y), \quad z_i(x, 0) = \psi_i(x), \quad i = 1, \ldots, m, \tag{1.2}$$

where φ_i and ψ_i are continuous, sectionally continuously differentiable functions defined within the domain G and satisfying matching conditions

$$\varphi_i(0) = \psi_i(0) .$$

Each permissible control may be associated with a unique solution

$$z(x, y) = \{z_1(x, y), \ldots, z_m(x, y)\}$$

of the problem (1.1)-(1.2) having derivatives z_{ixy} integrable over the domain G (see work of Budak and Gorbunov[18]). However, here one should distinguish two cases.

1) If the line of discontinuity of the function $v(x, y)$ is parallel to one of the coordinate axes, the boundary problem (1.1)-(1.2) decomposes into two analogous problems within the regions bordering one to another along that line. By solving consecutively these problems, we determine the solution of the original problem which may be continuous within the domain G and everywhere except along the points

of the discontinuity line of control $v(x, y)$, and will have continuous derivatives $z_{ix}(x, y)$, $z_{iy}(x, y)$, and $z_{ixy}(x, y)$ (see work of Budak and Gorbunov[18]).

2) Let the discontinuity line Γ of the function $v(x, y)$ have no common sections with the characteristics of the system (1.1) over any segment different from zero. Under the solution of the boundary problem (1.1)-(1.2) we understand the function $z(x, y)$ which satisfies the system of equations (1.1) in all points of the domain G not lying on Γ and the conditions (1.2) together with certain a priori given conditions of smoothness along Γ (see work of Yegorov[19]). Such a solution is uniquely determined; it is continuous within the domain G and has sectionally continuous derivatives z_{ix}, z_{iy}, and z_{ixy}.

Consequently, in what follows we will assume that to each permissible control one can associate a class of functions for which the boundary condition (1.1)-(1.2) can be uniquely solved.

Let A_i, $i = 1, \ldots, m$, be a given system of real numbers. Let us take an arbitrary permissible control $v(x, y)$ and denote by $z(x, y)$ the respective solution of the problem (1.1)-(1.2) and study the functional

$$S = \sum_{i=1}^{m} A_i z_i(X, Y) \quad , \tag{1.3}$$

where X and Y are constants entering into the definition of the domain G.

The problem is then: among all the permissible controls one is supposed to find such a control $v(x, y)$ (if it exists) that over the solution $z(x, y)$ of the Goursat problem corresponding to this control the functional S attains its largest (smallest) value.

We will call the permissible control realizing the minimum (maximum) of S the min-optimal (max-optimal) control according to S (see work of Rozopoer[16]).

Note that the problem (1.1)-(1.2) under consideration is of great theoretical and practical interest. The study of the solvability of this problem for various assumptions relative to the function f_i, φ_i, and ψ_i is the object of extensive literature[20, 21, 22, 23, 24, 25]. It is also known[26, 27, 28] that the study of sorption and desorption of gases, drying processes, and the like reduces to such a problem. The presence of control parameters in Equations (1.1) allows the control process, and in numerous cases the selection of the best operating conditions which (from the mathematical point of view) reduces to the calculation of the maximum or minimum of a certain function. In numerous cases the problem may be reduced to the study of the functional (1. 3). Let us investigate certain examples.

1) One is supposed to minimize the functional

$$I = \iint_G f_0 (x, y, z, z_x, z_y, v)\, dx\, dy \quad .$$

If we introduce a new variable z_0 by putting

$$z_{0xy} = f_0 (x, y, z, z_x, z_y, v), \quad z_0 (0, y) = z_0 (x, 0) = 0 , \qquad (1.4)$$

the problem reduces to the calculation of the minimum of the function $S = z_0 (X, Y)$ which represents a special case of the functional (1. 3) and is defined over the functions $z_0, \ldots , z_m$ specified by the totality of the relationships (1. 1)-(1. 2) and (1. 4).

2) One is supposed to minimize the functional $I = \Phi(z_1(X, Y), \ldots , z_m(X, Y))$, where Φ is a twice continuously differentiable function.

We introduce a new function $z_0 (x, y)$ by means of the equation

$$z_{0xy} = \sum_{i, k = 1}^{m} \frac{\partial^2 \Phi (z_1 (x, y), \ldots , z_m(x, y))}{\partial z_i \partial z_k} z_{ix} z_{ky}$$

$$+ \sum_{i = 1}^{m} \frac{\partial \Phi}{\partial z_i} f_i (x, y, z, z_x, z_y, v)$$

and additional conditions

$$z_0 (0, y) = \Phi(\varphi_1 (y), \ldots , \varphi_m(y)), \quad z_0 (x, 0) = \Phi (\psi_1 (x), \ldots , \psi_m(x)).$$

This reduces the problem to the study of the functional $S = z_0 (X, Y)$.

3) One is supposed to minimize the functional

$$I = \int_0^X F(x, z(x, Y), z_x(x, Y))\, dx .$$

We introduce the auxiliary function $z_0(x, y)$ by means of the equation

$$z_{0yz} = \sum_{i=1}^{m} \left[\frac{\partial F}{\partial z_i} z_{iy} + \frac{\partial F}{\partial z_{ix}} f_i(x, y, z, z_x, z_y, v) \right]$$

and additional conditions

$$z_0(0, y) = 0, \quad z_0(x, 0) = \int_0^x F(x, \psi(x), \psi'(x))\, dx .$$

4) In an analogous manner one investigates the problem concerning the minimization of the functional

$$I = \int_0^Y F(y, z(X, y), z_y(X, y))\, dy .$$

For the solution of the formulated optimum problem one introduces the auxiliary functions $u_1, \ldots, u_m$ by means of the equation

$$u_{ixy} = \frac{\partial H(x, y, p, v)}{\partial z_i} - \frac{d}{dx}\left(\frac{\partial H(x, y, p, v)}{\partial z_{ix}}\right) - \frac{d}{dy}\left(\frac{\partial H(x, y, p, v)}{\partial z_{iy}}\right) \tag{1.5}$$

and additional conditions

$$u_{1x}(x, Y) = - \frac{\partial H(x, Y, p, v)}{\partial z_{iy}} ,$$

$$u_{iy}(X, y) = - \frac{\partial H(X, y, p, r)}{\partial z_{ir}} , \quad u_i(X, Y) = A_i , \tag{1.6}$$

where A_i are constants entering into the definition of the functional S,

$$p = (z_1, \ldots, z_m, u_1, \ldots, u_m, z_{1x}, \ldots, z_{mx}, z_{1y}, \ldots, z_{my}) ,$$

$$H = \sum u_i f_i (x, y, z, z_x, z_y, v) \quad .$$

Conditions (1.6) represent ordinary linear differential equations with initial conditions.

In the general case, the right-hand side of the equations (1.5) contains functions z_{ixx}, z_{iyy}, v_x, and v_y. However, from the conditions imposed on Equations (1.1) and the permissible controls, it does not follow that such derivatives must exist. Consequently, we will assume in what follows that the function f_i may be represented in the form

$$f_i = \sum_{j,\,k=1}^{m} a_{ijk}(x, y, z)\, z_{kx} z_{jy} + \sum_{j=1}^{m} b_{ij}(x, y, z)\, z_{jx}$$

$$+ \sum_{j=1}^{m} c_{ij}(x, y, z)\, z_{jy} + d_i(x, y, z, v) \quad .$$

where the functions a_{ijk}, b_{ij}, c_{ij}, and d_i are continuously differentiable over x and y and are twice continuously differentiable over the other arguments. If it turns out that a_{ijk}, b_{ij}, and c_{ij} do depend on v, then one must require that the permissible controls have sectionally continuous derivatives $v_x(x, y)$ and $v_y(x, y)$.

While satisfying these conditions, the system of linear equations (1.5) has sectionally continuous coefficients and together with the additional conditions (1.6) determines uniquely $u_1(x, y), \ldots, u_m(x, y)$ for each permissible control. Consequently, in what follows we will assume that the function f_i and the permissible controls are such that the boundary problem (1.5)-(1.6) be unique solvable for each permissible control.

We will say that the permissible control v(x, y) satisfies the condition of maximum if the relationships

$$H(x, y, p(x, y), v(x, y)) \; ((=)) \sup_{v \in V} H(x, y, p(x, y), v) \;, \qquad (1.7)$$

is satisfied with $z(x, y)$ and $u(x, y)$ - the solutions of problems (1.1)-(1.2) and (1.5)-(1.6) corresponding to the control $v(x, y)$, while the symbol ((=)) denotes equality valid in all points of the domain G ($0 \leq x \leq X$, $0 \leq y \leq Y$) while there may exist sets of points lying over a finite number of lines with zero surface. The conditions of minimum is defined in an analogous way.

THEOREM 1 (the principle of maximum). <u>For the permissible control $b(x, y)$ to be min-optimal (max-optimal) according to S, it is necessary that it satisfies the condition of maximum (minimum).</u>

Although this theorem does not supply sufficient conditions for the existence of optimum control, it may be utilized for the practical solution of the optimum problem. As a matter of fact, the solution of this problem, according to the principle of maximum, leads to the need of determining $2n + 1$ unknowns z_i, u_i, and v from the $2n + 1$ Equations (1.1), (1.5), and (1.7). First 2n relationships represent second order differential equations whose solution in general generates 4n arbitrary functions. To eliminate them we have 4n additional conditions (1.2) and (1.6). This is sufficient to define, generally speaking, the separation of the solution of the problem (1.1)-(1.2) satisfying the conditions of the maximum principle. If it appears from the meaning of the problem that the optimum problem must have a mandatory solution, then at least one of the discovered isolated solutions must be the desired one.

2. The Formula for the Increments of the Functional S

To prove Theorem 1, we study the functional

$$I[p, v] = \iint_G \left[\sum_{i=1}^{m} u_i z_{ixy} - H(x, y, p, v) \right] dx\,dy \quad .$$

If v is a certain permissible control and $z = z(x, y)$ a solution of the problem (1.1)-(1.2) corresponding to this control, then the functional I is equal to zero for an arbitrary function $u = (u_1, \ldots, u_m)$.

Let $v = v(x, y)$ be a certain permissible control and $z(x, y)$ and $u(x, y)$ be the solution of the boundary conditions (1.1)-(1.2) and (1.5)-(1.6) corresponding to this control. Let us give the function v an arbitrary permissible increment Δv and let us denote by $z + \Delta z$ and $u + \Delta u$ the solution of the same problems but corresponding to the control $v + \Delta v$. It is clear that the functions Δz_i and Δu_i satisfy the conditions

$$\Delta z_{ixy} = \Delta \frac{\partial H}{\partial u_i} ,$$

$$\Delta u_{ixy} = \Delta \frac{\partial H}{\partial z_i} - \frac{d}{dx}\left(\Delta \frac{\partial H}{\partial z_{ix}}\right) - \frac{d}{dy}\left(\Delta \frac{\partial H}{\partial z_{iy}}\right) , \quad i = 1, \ldots , m, \tag{1.8}$$

and the additional conditions

$$\Delta z_i (0, y) = \Delta z_i (x, 0) = 0 , \tag{1.9}$$

$$\Delta u_{ix} (x, Y) = - \Delta \frac{\partial H (x, Y, p, v)}{\partial z_{iy}} , \quad \Delta u_{iy} (X, y) = - \Delta \frac{\partial H (X, y, p, v)}{\partial z_{ix}} , \tag{1.10}$$

$$\Delta u_i (X, Y) = 0, \quad i = 1, \ldots , m, \tag{1.11}$$

where

$$\Delta \frac{\partial H}{\partial p_i} = \frac{\partial H (x, y, p + \Delta p, v + \Delta v)}{\partial p_i} - \frac{\partial H (x, y, p, v)}{\partial p_i} . \tag{1.12}$$

Equations (1.10) are ordinary differential equations where with linear f_i functions

$$\Delta u_i (x, Y) \equiv 0, \qquad \Delta u_i (X, y) \equiv 0 \tag{1.13}$$

are their solutions which satisfy the additional conditions (1.11). Because of the uniqueness theorem, functions (1.13) form a unique solution of the boundary problem (1.10)-(1.11).

Consequently, in accordance with the above remark,

$$\Delta I = I [p + \Delta p, v + \Delta v] - I [p, v] = 0 . \tag{1.14}$$

On the other hand,

$$\Delta I = \iint_G \left\{ \sum_{i=1}^{m} \left[\Delta u_i \Delta z_{ixy} + u_i \Delta z_{ixy} + \Delta u_i z_{ixy}\right] - \left[H (x, y, p + \Delta p, v + \Delta v) - H (x, y, p, v) \right] \right\} dx\, dy . \tag{1.15}$$

The expression under the sign of the integral is transformed by means of the Green's formula (see work of Trikomi[29], p. 196):

$$\iint_G (qs_{xy} - sq_{xy})\, dx\, dy = \int_L (qs_y - sq_y)\, dy - (qs_x - sq_x)\, dx \ ,$$

where L is the contour limiting the domain G, and q and s - arbitrary functions having sectionally continuous derivatives of the first and second order. Since G is a rectangle, the Green's formula may be reduced to the form

$$\iint_G (qs_{xy} - sq_{xy})\, dx\, dy = \left\{ [q(x, y)\, s(x, y)]_{x \overset{X}{=} 0} \right\}_{y \overset{Y}{=} 0}$$
$$- \int_0^X (sq_x)_{y \overset{Y}{=} 0}\, dx - \int_0^Y (sq_y)_{x \overset{X}{=} 0}\, dy \ . \qquad (1.16)$$

Let us insert into this equality $q = \Delta u_i$ and $s = \Delta z_i$. Taking into account Equations (1.8) with additional conditions (1.9), (1.10), and (1.11), we obtain after elementary transformations

$$\iint_G \sum_{i=1}^{m} \Delta u_i\, \Delta z_{ixy}\, dx\, dy = \iint_G \sum_{i=1}^{m} \left[\Delta \frac{\partial H}{\partial z_i} \Delta z_i + \Delta \frac{\partial H}{\partial z_{ix}} \Delta z_{ix} \right.$$
$$\left. + \Delta \frac{\partial H}{\partial z_{iy}} \Delta z_{iy} \right] dx\, dy \ .$$

On the other hand, because of the first m equations of (1.8), we have:

$$\iint_G \sum_{i=1}^{m} \Delta u_i\, \Delta z_{ixy}\, dx\, dy = \iint_G \sum_{i=1}^{m} \Delta \frac{\partial H}{\partial u_i} \Delta u_i\, dx\, dy \ .$$

From the two last equations we obtain

$$\iint_G \sum_{i=1}^{m} \Delta u_i\, \Delta z_{ixy}\, dx\, dy = \frac{1}{2} \iint_G \sum_{i=1}^{4m} \Delta \frac{\partial H}{\partial p_i} \Delta p_i\, dx\, dy \ . \qquad (1.17)$$

We now substitute into Equation (1.16) $q = u_i$ and $s = \Delta z_i$. Then because of Equations (1.5) and (1.8) and the boundary conditions (1.6) and (1.9) we have

$$\iint_G \sum_{i=1}^{m} u_i \Delta z_{ixy}\, dx\, dy = -\sum_{i=1}^{m} A_i \Delta z_i (X, Y) + \iint_G \sum_{i=1}^{m} \left[\frac{\partial H}{\partial z_i} \Delta z_i \right.$$

$$\left. + \frac{\partial H}{\partial z_{ix}} \Delta z_{ix} + \frac{\partial H}{\partial z_{iy}} \Delta z_{iy} \right] dx\, dy \, . \qquad (1.18)$$

Since the function z_i forms the solution of the system of equations (1.1), we have

$$\iint_G \sum_{i=1}^{m} \Delta u_i \, z_{ixy}\, dx\, dy = \iint_G \sum_{i=1}^{m} \frac{\partial H}{\partial u_i} \Delta u_i \, dx\, dy \, . \qquad (1.19)$$

Using the Taylor formula, we obtain the equality

$$H(x, y, p + \Delta p, v + \Delta v) - H(x, y, p, v)$$

$$= \sum_{i=1}^{4m} \frac{\partial H(x, y, p, v + \Delta v)}{\partial p_i} - \Delta p_i$$

$$+ \frac{1}{2} \sum_{i,k=1}^{4m} \frac{\partial^2 H(x, y, p + \theta \Delta p, v + \Delta v)}{\partial p_i \partial p_k} \Delta p_i \Delta p_k$$

$$+ H(x, y, p, v + \Delta v) - H(x, y, p, v), \qquad 0 \le \vartheta \le 1 \, .$$

$$(1.20)$$

From Equations (1.14), (1.15), and (1.17)-(1.20), it follows that

$$\Delta I = -\sum_{i=1}^{m} A_i \Delta z_i (X, Y) - \iint_G [H(x, y, p, v + \Delta v)$$

$$- H(x, y, p, v)]\, dx\, dy + \frac{1}{2} \iint_G \sum_{i=1}^{4m} \left\{ \left[\frac{\partial H(x, y, p + \Delta p, v + \Delta v)}{\partial p_i} \right. \right.$$

$$\left. - \frac{\partial H(x, y, p, v + \Delta v)}{\partial p_i} \right] - \left[\frac{\partial H(x, y, p, v + \Delta v)}{\partial p_i} \right.$$

$$- \frac{\partial H(x, y, p, v)}{\partial p_i} \Bigg| \Delta p_i \, dx\, dy - \frac{1}{2} \iint_{G} \sum_{i, k = 1}^{4m} \frac{\partial^2 H(x, y, p + \theta \Delta p, v + \Delta v)}{\partial p_i \partial p_k}$$

$$\Delta p_i \Delta p_k dx\, dy \; .$$

Applying to the functionals $\partial H / \partial p_i$ the Taylor formula and taking into account the equality (1.14), we finally obtain

$$\Delta S = - \iint_G [H(x, y, p, v + \Delta v) - H(x, y, p, v)] \, dx\, dy - \eta \quad (1.21)$$

where

$$\Delta S = \sum A_i \Delta z_i (X, Y)$$

is the increment of the functional S, $\eta = \eta_1 + \eta_2$,

$$\eta_1 = \frac{1}{2} \sum_{i=1}^{4m} \iint_G \left[\frac{\partial H(x, y, p, v + \Delta v)}{\partial p_i} - \frac{\partial H(x, y, p, v)}{\partial p_i} \right] \Delta p_i \, dx\, dy \; ,$$

$$\eta_2 = \frac{1}{2} \sum_{i,k=1}^{4m} \iint_G \left[\frac{\partial^2 H(x, y, p + \theta \Delta p, v + \Delta v)}{\partial p_i \partial p_k} - \frac{\partial^2 H(x, y, p + \vartheta_1 \Delta p, v + \Delta v)}{\partial p_i \partial p_k} \right]$$

$$\cdot \; \Delta p_i \Delta p_k \, dx\, dy \; . \qquad (1.22)$$

3. The Estimate of the Residual Term η in the Formula (1.21)

To establish the necessary estimates of the quantity η we introduce auxiliary functions $\alpha_i(x, y)$ and $\beta_i(x, y)$, putting

$$\alpha_i = \Delta z_{ix} \; , \qquad \beta_i = \Delta z_{iy} \; .$$

Since the function f_i satisfies the Lipschitz condition, then from the first m equations of the system (1.8) and the condition (1.9) we obtain

$$|\alpha_i| \leq N\int_0^y \sum_{i=1}^{m} (|\Delta z_i| + |\alpha_i| + |\beta_i|)\, dy + N_1 \int_0^y \sum_{k=1}^{r} |\Delta v_k|\, dy ,$$

$$|\Delta z_i| \leq \int_0^x |\alpha_i|\, dx ,$$

$$|\beta_i| \leq N\int_0^x \sum_{i=1}^{m} (|\Delta z_i| + |\alpha_i| + |\beta_i|)\, dx + N_1 \int_0^x \sum_{k=1}^{r} |\Delta v_k|\, dx , \tag{1.23}$$

$$|\Delta z_i| \leq \int_0^y |\beta_i|\, dy ,$$

where N and N_1 are definite positive constants. Introducing the notation

$$\alpha = \sum_{i=1}^{m} |\alpha_i| , \quad \beta = \sum_{i=1}^{m} |\beta_i| , \quad \gamma = \sum_{i=1}^{m} |\Delta z_i| ,$$

$$\Delta v = \sum_{k=1}^{r} |\Delta v_k| , \tag{1.24}$$

we obtain from inequalities (1.23)

$$\alpha(x, y) \leq Nm\int_0^y \alpha(x, y)\, dy + Nm\int_0^\eta [\beta(x, \eta) + \gamma(x, \eta)]\, d\eta$$

$$+ N_1 m\int_0^Y \Delta v dy , \qquad \gamma \leq \int_0^x \alpha(x, y)\, dx ,$$

$$\beta(x, y) \leq Nm\int_0^x \beta(x, y)\, dx + Nm\int_0^\xi [\alpha(x, y) + \beta(x, y)]\, dx \tag{1.25}$$

$$+ N_1 m\int_0^X \Delta v dx , \qquad \gamma \leq \int_0^y \beta(x, y)\, dy ,$$

where

$$0 \leq x \leq \xi \leq X, \quad 0 \leq y \leq \eta \leq Y \quad .$$

From this, because of the known lemma (see article by Nemytskiy and Stepanov[30] p. 19), it follows that

$$\alpha(x, y) \leq M\int_0^{\eta} [\gamma(x, y) + \beta(x, y)]\, dy + M_1\int_0^{Y} \Delta v(x, y)\, dy \ ,$$

$$\beta(x, y) \leq P\int_0^{\xi} [\gamma(x, y) + \alpha(x, y)]\, dx + P_1\int_0^{X} \Delta v(x, y)\, dx \ ,$$

where M, M_1, P, P_1 are positive constants. Taking into account the estimates (1.25) for the function γ, we obtain

$$\alpha(x, y) \leq M_2\int_0^{\eta} \beta(x, y)\, dy + M_1\int_0^{Y} \Delta v(x, y)\, dy \quad ,$$

$$\beta(x, y) \leq P_2\int_0^{\xi} \alpha(x, y)\, dx + P_1\int_0^{X} \Delta v(x, y)\, dx \quad .$$

From this we find that

$$\begin{aligned} \alpha(\xi, \eta) \leq{}& M_3\int_0^{\eta}\int_0^{\xi}\alpha(x, y)dx\, dy + M_4\int_0^{Y}\int_0^{X} \Delta v(x, y)\, dx\, dy \\ &+ M_1\int_0^{Y} \Delta v(\xi, y)\, dy \quad , \\ \beta(\xi, \eta) \leq{}& P_3\int_0^{\eta}\int_0^{\xi} \beta(x, y)dx\, dy + P_4\int_0^{Y}\int_0^{X} \Delta v(x, y)\, dx\, dy \\ &+ P_1\int_0^{X} \Delta v(x, \eta)\, dx \quad . \end{aligned} \qquad (1.26)$$

Integrating the first of these inequalities over ξ between the limits of 0 and ξ and applying the above mentioned lemma, we obtain

$$\int_0^{\xi} \alpha(x, y)\, dx \le M_5 \int_0^X \int_0^Y \Delta v\,(x, y)\, dy\, dx \quad .$$

From this and the first inequality (1.26) we have

$$\alpha(x, y) \le M_6 \int_0^X \int_0^Y \Delta v\,(x, y)\, dy\, dx + M_7 \int_0^Y \Delta v\,(x, y)\, dy,$$

$$0 \le x \le X, \qquad 0 \le y \le Y \quad .$$

In an analogous manner we find:

$$\beta(x, y) \le P_6 \int_0^X \int_0^Y \Delta v\,(x, y)\, dx\, dy + P_7 \int_0^X \Delta v\,(x, y)\, dx \quad .$$

From this and the inequality (1.25) we obtain

$$\gamma\,(x, y) \le Q \int_0^X \int_0^Y \Delta v\,(x, y)\, dy\, dx \quad .$$

In this manner, because of (1.24), the inequalities

$$|\Delta z_i\,(x, y)| \le Q \iint_G \Delta v\,(x, y)\, dx\, dy \quad ,$$

$$|\Delta z_{ix}\,(x, y)| \le Q_1 \iint_G \Delta v\,(x, y)\, dx\, dy + R_1 \int_0^Y \Delta v\,(x, y)\, dy \quad , \tag{1.27}$$

$$|\Delta z_{iy}\,(x, y)| \le Q_2 \iint_G \Delta v\,(x, y)\, dx\, dy + R_2 \int_0^X \Delta v\,(x, y)\, dx \quad .$$

are valid for all x and y ($0 \le x \le X$, $0 \le y \le Y$).

Applying an analogous approach to the last m equations of the system (1.8), we obtain

$$|\Delta u_i\,(x, y)| \le Q_3 \iint_G \Delta v\,(x, y)\, dx\, dy \quad . \tag{1.28}$$

Since the functions $\partial H/\partial p_i$ satisfy the Lipschitz condition, we obtain from the first formula (1.22), because of the inequalities (1.27) and (1.28):

$$|\eta_1| \leq T\left(\iint_G \Delta v(x, y)\, dx\, dy\right)^2$$

$$+ T_1 \int_0^X \left|\int_0^Y \Delta v(x, y)\, dy\right|^2 dx + T_3 \int_0^Y \left|\int_0^X \Delta v(x, y)\, dx\right|^2 dy .$$

Consequently,

$$|\eta_1| \leq (T_1 XY + T_2 Y + T_3 X) \iint_G |\Delta v(x, y)|^2\, dx\, dy ,$$

and where T_i are definite positive constants.

The functions $\partial^2 H/\partial p_i \partial p_k$ are bounded in the G region. Consequently,

$$|\eta_2| \leq (T_4 XY + T_5 Y + T_6 X) \iint_G |\Delta v(x, y)|^2\, dx\, dy .$$

In this manner, the residual term in the formula (1.21) satisfies the inequality

$$|\eta| \leq (A \text{ mes } G + BX + CY) \iint_G |\Delta v(x, y)|^2\, dx\, dy , \qquad (1.29)$$

where A, B, and C are definite positive constants. If the function Δv differs from zero on the circle G_ϵ of radius ϵ, then from (1.29) it follows that

$$|\eta| \leq L\epsilon \iint_{G_\epsilon} \Delta v^2(x, y)\, dx\, dy , \qquad (1.30)$$

where L does not depend on ϵ.

4. Proof of Theorem 1. The Case of Linear Control System

From the formula (1.21) for the increments of the functional and the estimate (1.30) of the residual term within this formula, one can easily obtain the proof of Theorem 1.

As a matter of fact, let for the sake of definiteness, $v(x, y)$ be a control which is min-optimal according to S, and $z(x, y)$ and $u(x, y)$ - the solutions of boundary problems (1.1)-(1.2) and (1.5)-(1.6) corresponding to this control. Then for an arbitrary permissible increment $\Delta v(x, y)$ the inequality $\Delta S \geq 0$ is valid. Let us assume that there exists a point (ξ, η) within the domain G in which the maximum condition is not satisfied, i.e., there exists a control v^1 such that

$$H(\xi, \eta, p(\xi, \eta), v^1) > H(\xi, \eta, p(\xi, \eta), v(\xi, \eta)) \quad . \tag{1.31}$$

Since the functions $z(x, y)$ and $u(x, y)$ are continuous and z_x, z_y, and $v(x, y)$ are sectionally continuous, there exists a closed region $G^1 \in G$ containing the point (ξ, η) in which the left- and right-hand sides of the inequality (1.31) are not continuous and, consequently, are likewise not uniformly continuous. If (ξ, η) is the point of discontinuity of the control v, then it may be obviously related to the boundary of the domain G^1. It follows from the inequality (1.31) that one may specify a number $\delta > 0$ for which

$$H(x, y, p(x, y), v^1) - H(x, y, p(x, y), v(x, y)) > \delta \tag{1.32}$$

in all points $(x, y) \in G_\epsilon \subset G^1$, where G_ϵ - circle of radius ϵ. Let us introduce the control

$$v^2(x, y) = \begin{cases} v(x, y) \text{ at } (x, y) \overline{\in} G_\epsilon & , \\ v^1 \qquad \text{ at } (x, y) \in G_\epsilon & . \end{cases}$$

Then because of the relationships (1.21), (1.30), and (1.32)

$$\Delta S = - \iint_{G_\epsilon} \left| H(x, y, p(x, y), v^1) - H(x, y, p(x, y)) \right| dx\, dy - \eta$$

$$< - \iint_{G_\epsilon} \delta dx\, dy + |\eta| \leq - \iint_{G_\epsilon} \left\{ \delta - \epsilon L \left| \Delta v(x, y) \right|^2 \right\} dxdy \quad ,$$

where $\Delta v = v^1 - v(x, y)$. Since the function Δv is bounded, one may choose the number ϵ so small that the expression within the square brackets in the first part of the last inequality may be positive. Then ΔS will become negative which contradicts the assumption about the min-optimality according to S of the control $v(x, y)$. This proves the theorem.

Formula (1.21) for the increments of the functional together with the formulas (1.22) for the remainder term allow the establishment of a more general result for the linear boundary problem.

As a matter of fact, let the control process be described by the boundary problem

$$z_{ixy} = \sum_{k=1}^{m} \left| c_{ik}(x, y) z_{kx} + d_{ik}(x, y) z_{ky} + g_{ik}(x, y) z_k \right| + fi(v) ,$$

$$z_i(0, y) = \varphi_i(y), \qquad z_i(x, 0) = \psi_i(x), \qquad \varphi_i(0) = \psi(0), \tag{1.33}$$

$$i = 1, \ldots, m,$$

and let search for the control over which the functional S attains its minimum (maximum) value. In such a case

$$H(x, y, p, v) = \sum_{i,\, k=1}^{m} u_i \left| c_{ik} z_{kx} + d_{ik} z_{ky} + g_{ik} z_k \right|$$

$$+ \sum_{i=1}^{m} u_i f_i(v) ,$$

and the function u_i forms the solution of the boundary problem

$$\left.\begin{aligned}
u_{ixy} &= \sum_{k=1}^{m} \left| g_{ki} u_k - \frac{d}{dx}(c_{ki} u_k) - \frac{d}{dy}(d_{ki} u_k) \right| , \\
u_{ix}(x, Y) &= - \sum_{k=1}^{m} d_{ki}(x, Y)\, u_k(x, Y) , \\
u_{iy}(X, y) &= - \sum_{k=1}^{m} c_{ki}(X, y)\, u_k(X, y) , \\
u_i(X, Y) &= - A_i , \qquad i = 1, \ldots, m .
\end{aligned}\right\} \tag{1.34}$$

Since according to what was proved earlier

$$\Delta u_i\ (x,\ Y) = \Delta u_i\ (X,\ y) \equiv 0$$

(see formulas (1.13)) we find

$$\Delta u_i\ (x,\ y) \equiv 0\ .$$

Furthermore,

$$\frac{\partial H(x,\ y,\ p,\ v + \Delta v)}{\partial w_i} - \frac{\partial H(x,\ y,\ p,\ v)}{\partial w_i} = 0\ ,$$

$$w = (z_1, \ldots, z_m, \ldots, z_{1y}, \ldots, z_{my})\ .$$

Consequently, $\eta_1 = 0$. We now calculate η_2. We have

$$\frac{\partial^2 H\ (x,\ y,\ p,\ v)}{\partial w_i \partial w_k} \equiv 0\ , \quad i,\ k = 1, \ldots, 3m\ .$$

This means that

$$\eta_2 = \sum_{i=1}^{m} \sum_{k=1}^{3m} \iint_G \left[\frac{\partial^2 H\ (x,\ y,\ p + \theta \Delta p,\ v + \Delta v)}{\partial u_i \partial w_k} \right.$$

$$\left. - \frac{\partial^2 h\ (x,\ y,\ p + \theta_1 \Delta p,\ v + \Delta v)}{\partial u_i \partial w_k} \right] \Delta u_i \Delta w_k\ dx\ dy\ .$$

Since $\Delta u_i(x,\ y) \equiv 0$, it follows that $\eta_2 = 0$. Consequently, in the case under investigation, the formula (1.21) takes the form

$$\Delta S = -\iint_G \left| H\ (x,\ y,\ p,\ v + \Delta v) - H\ (x,\ y,\ p,\ v) \right| dx\ dy\ . \qquad (1.35)$$

By means of the last formula one can easily prove:

THEOREM 2. For the permissible control $v(x,\ y)$ of the boundary problem (1.33) to be locally min-optimal (max-optimal) according to S, it is necessary and sufficient that it satisfies the condition of maximum (minimum).

5. The Control of a System by Means of Boundary Conditions

Until now we assumed that the control is carried out only by means of the function v entering into Equation (1.1) or (1.33). The boundary values (1.2) of the function z_i were considered fixed. However, the method which is presented permits the solution of a more general problem.

Let the control process be described by the system of equations (1.1) while the boundary values of the function z_i are specified not by the conditions (1.2) but by means of the differential equations

$$z_{iy}(0, y) = \varphi_i(y, z_1, \ldots, z_m, v^1),$$

$$z_{ix}(x, 0) = \psi_i(x, z_1, \ldots, z_m, v^2) \tag{1.36}$$

and initial conditions

$$z_i(0, 0) = z_i^0, \qquad i = 1, \ldots, m, \tag{1.37}$$

where the functions φ_i and ψ_i are continuous over y and x and twice continuously differentiable over the remaining arguments. v_1 and v_2 are control parameters taking values from the domains V^1 and V^2 of s- and t-dimensional euclidian spaces, respectively.

The presence of parameters within Equations (1.36) permits the control of the process by means of boundary conditions. To the permissible controls within Equations (1.36) we relate also the sectionally continuous functions $v^1(y)$ and $v^2(x)$ with values in the regions V^1 and V^2 respectively. It is known (see, for instance, work of Sansone[31], pp. 16 and 17) that each pair of permissible controls $v^1(y)$ and $v^2(x)$ determines by means of Equations (1.36) and conditions (1.37) a unique pair of absolutely continuous functions $z(0, y)$ and $z(x, 0)$. We will understand in all what follows under permissible control within the boundary problem (1.1)-(1.36)-(1.37) the function

$$\omega(x, y) = (v(x, y), v^1(y), v^2(x)),$$

whose components are sectionally continuous functions with values in the domains V, V^1, and V^2, respectively. Consequently, to each permissible control $\omega(x, y)$ there corresponds a unique solution of the boundary problem (1.1)-(1.36)-(1.37) with the same smoothness conditions which we introduced for the boundary problem (1.1)-(1.2).

Let us introduce the notation

$$q = (z_1, \ldots, z_m, u_1, \ldots, u_m) ,$$

$$H_1 (y, q, v^1) = \sum_{i=1}^{m} u_i \varphi_i (y, z. v^1) ,$$

$$H_2 (x, q, v^2) = \sum_{i=1}^{m} u_i \psi_i (x, z, v^2) .$$

The function u_i is defined by means of Equation (1.5) and the additional conditions (1.6).

In the general form one is not able yet to solve the optimum problem with boundary conditions (1.36)-(1.37). However, it may be solved using the above described method if the following conditions

$$\left.\begin{array}{l} \left.\dfrac{\partial f_k (x, y, z, z_x, z_y, v)}{\partial z_{iy}}\right|_{y=0} = \dfrac{\partial \psi_k (x, z, v^2)}{\partial z_i} , \\ \left.\dfrac{\partial f_k (x, y, z, z_x, z_y, v)}{\partial z_{ix}}\right|_{x=0} = \dfrac{\partial \varphi_k (y, z, v^1)}{\partial z_i} , \\ k, i = 1, \ldots, m , \quad v \in V , \quad v^1 \in V^1 , \quad v^2 \in V^2 , \end{array}\right\} \qquad (1.38)$$

are satisfied.

Thus in what follows we assume that the conditions (1.38) are satisfied and, consequently, that for an arbitrary function $u(u_1, \ldots, u_m)$ the equalities

$$\begin{array}{l} \left.\dfrac{\partial H (x, y, p, v)}{\partial z_{iy}}\right|_{y=0} = \dfrac{\partial H_2 (x, q, v^2)}{\partial z_i} , \\ \left.\dfrac{\partial H (x, y, p, v)}{\partial z_{ix}}\right|_{x=0} = \dfrac{\partial H_1 (y, q, v^1)}{\partial z_i} , \end{array} \qquad (1.38')$$

are satisfied no matter what the values of v, v^1, and v^2 from the regions V, V^1, and V^2, respectively are.

We will say that the permissible control $\omega(x, y)$ in the boundary problem (1.1)-(1.36)-(1.37) satisfies the maximum conditions if

$$\left.\begin{array}{l} H(x, y, p(x, y), v(x, y)) \,((=))\, \sup\limits_{v \in V} H(x, y, p(x, y), v) , \\[2ex] H_1 (y, q(0, y), v^1(y)) \,(=)\, \sup\limits_{v^1 \in V^1} H_1 (y, q(0, y), v^1) , \\[2ex] H_2 (x, q(x, 0), v^2(x)) \,(=)\, \sup\limits_{v^2 \in V^2} H(x, q(x, 0), v) , \end{array}\right\} \quad (1.39)$$

where $z(x, y)$ and $u(x, y)$ are the solutions of the boundary problems (1.1)-(1.36)-(1.37) and (1.5)-(1.6) corresponding to the control $\omega(x, y) = (v(x, y), v^1(y), v^2(x))$ while the symbol (=) indicates an equality which is valid almost everywhere within the domain of the change of the argument. The conditions of minimum are defined in an analogous manner.

THEOREM 3. <u>For the permissible control $\omega(x, y)$ in the boundary condition (1.1)-(1.36)-(1.37) to be min-optimal (max-optimal) according to S, it is necessary that it satisfies the conditions of maximum (minimum)</u>.

The proof of this theorem is carried out following the same scheme as in the case of the proof of Theorem 1: one first finds a formula for the increment of the functional followed by the estimate of the residual term, and only then one proceeds to prove the theorem.

For the establishment of the formula for the increment of the functional, we take an arbitrary permissible control $\omega(x, y)$ and denote by $z(x, y)$ and $u(x, y)$ the solutions of the boundary problem (1.1)-(1.36)-(1.37) and (1.5)-(1.6) corresponding to this control. Then the equality

$$I\left|p, \omega\right| = \iint_G \left[\sum_{i=1}^m u_i z_{ixy} - H(x, y, p, v)\right] dx\, dy$$

$$+ \int_0^X \left[\sum_{i=1}^m u_i (x, 0)\, z_{ix} (x, 0) - H_2 (x, q(x, 0), v^2)\right] dx$$

$$+ \int_0^Y \left[\sum_{i=1}^m u_i (0, y)\, z_{iy} (0, y) - H_1 (y, q(0, y), v^1)\right] dy = 0$$

is valid.

Let us denote by $\Delta\omega$ the arbitrary permissible increment of the control $\omega(x, y)$ and by Δz and Δu the increments of the functional $z(x, y)$ and $u(x, y)$ corresponding to this control. It is clear that

$$\Delta I = I\left|p + \Delta p, \ \omega + \Delta\omega\right| - I\left|p, \ \omega\right| = 0 \ .$$

To transform ΔI we start from the equation

$$\iint_G p q_{xy} \, dy \, dx + \int_0^X p(x, 0) \, q_x(x, 0) \, dx + \int_0^Y p(0, y) \, q_y(0, y) \, dy$$

$$= \iint_G q p_{xy} \, dy \, dx - \int_0^X q(x, Y) \, p_x(x, Y) \, dx$$

$$- \int_0^Y q(X, y) \, p_y(X, y) \, dy + p(X, Y) \, q(X, Y)$$

$$+ p(0, 0) \, q(0, 0) \ , \qquad (1.40)$$

which is valid for arbitrary twice sectionally continuously differentiable functions p and q which equation may be derived from the Green's formula (1.16).

After putting into Equation (1.40) $p = \Delta u_i$ and $q = \Delta z_i$ and taking into account the conditions (1.38'), we find

$$\iint_G \sum_{i=1}^m \Delta u_i \Delta z_{ixy} dx \, dy + \int_0^X \sum_{i=1}^m \Delta u_i(x, 0) \, \Delta z_{ix}(x, 0) \, dx$$

$$+ \int_0^Y \sum_{i=1}^m \Delta u_i(0, y) \, \Delta z_{iy}(0, y) \, dy = \iint_G \sum_{i=1}^m \left[\Delta \frac{\partial H}{\partial z_i} \Delta z_i\right.$$

$$\left. + \Delta \frac{\partial H}{\partial z_{ix}} \Delta z_{ix} + \Delta \frac{\partial H}{\partial z_{iy}} \Delta z_{iy}\right] dx \, dy$$

$$- \int_0^X \sum_{i=1}^m \left\{\left[\Delta u_{ix}(x, Y) + \Delta \frac{\partial H(x, Y, p(x, Y), v)}{\partial z_{iy}}\right] \Delta z_i(x, Y)\right.$$

$$- \Delta \frac{\partial H (x, 0, p (x, 0), v)}{\partial z_{iy}} \Delta z_i (x, 0) \Big\} dx$$

$$- \int_0^Y \sum_{i=1}^m \Big\{ \Big[\Delta u_{iy} (X, y) + \Delta \frac{\partial H (X, y, p (X, y), v)}{\partial z_{ix}} \Big] \Delta z_i (X, y)$$

$$- \Delta \frac{\partial H (0, y, p (0, y), v)}{\partial z_{ix}} \Delta z_i (0, y) \Big\} dy$$

$$= \iint_G \sum_{i=1}^m \Big[\Delta \frac{\partial H}{\partial z_i} \Delta z_i + \Delta \frac{\partial H}{\partial z_{ix}} \Delta z_{ix} + \Delta \frac{\partial H}{\partial z_{iy}} \Delta z_{iy} \Big] dx\, dy$$

$$+ \int_0^X \sum_{i=1}^m \Delta \frac{\partial H_2 (x, q (x, 0), v^2)}{\partial z_i} \Delta z_i (x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^m \Delta \frac{\partial H_1 (y, q (0, y), v^1)}{\partial z_i} \Delta z_i (0, y)\, dy \quad .$$

On the other hand

$$\iint_G \sum_{i=1}^m \Delta u_i \Delta z_{ixy}\, dx\, dy + \int_0^X \sum_{i=1}^m \Delta u_i (x, 0) \Delta z_{ix} (x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^m \Delta u_i (0, y) \Delta z_{iy} (0, y)\, dy = \iint_G \sum_{i=1}^m \Delta \frac{\partial H}{\partial u_i} \Delta u_i\, dx\, dy$$

$$+ \int_0^X \sum_{i=1}^m \Delta \frac{\partial H_2 (x, q (x, 0), v^2)}{\partial u_i} \Delta u_i (x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^m \Delta \frac{\partial H_1 (y, q (0, y), v^1)}{\partial u_i} \Delta u_i (0, y)\, dx \quad .$$

From the two last equations we obtain

$$\iint_G \sum_{i=1}^{m} \Delta u_i \Delta z_{ixy}\, dx\, dy + \int_0^X \sum_{i=1}^{m} \Delta u_i(x, 0) \Delta z_{ix}(x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^{4m} \Delta u_i(0, y) \Delta z_{iy}(0, y)\, dy = \frac{1}{2}\Bigg[\iint_G \sum_{i=1}^{4m} \Delta \frac{\partial H}{\partial p_i} \Delta p_i\, dx\, dy$$

$$+ \int_0^X \sum_{i=1}^{2m} \Delta \frac{\partial H_2(x, q(x, 0), v^2)}{\partial q_i} \Delta q_i(x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^{2m} \Delta \frac{\partial H_1(y, q(0, y), v^1)}{\partial q_i} \Delta q_i(0, y)\, dy\Bigg] . \qquad (1.41)$$

Furthermore, using the same method as during the derivation of formulas (1. 18) and (1. 19), we find

$$\iint_G \sum_{i=1}^{m} u_i \Delta z_{ixy}\, dx\, dy + \int_0^X \sum_{i=1}^{m} u_i(x, 0) \Delta z_{ix}(x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^{m} u_i(0, y) \Delta z_{iy}(0, y)\, dy = - \sum_{i=1}^{m} A_i \Delta z_i(X, Y)$$

$$+ \iint_G \sum_{i=1}^{m} \left[\frac{\partial H}{\partial z_i} \Delta z_i + \frac{\partial H}{\partial z_{ix}} \Delta z_{ix} + \frac{\partial H}{\partial z_{iy}} \Delta z_{iy}\right] dx\, dy$$

$$+ \int_0^X \sum_{i=1}^{m} \frac{\partial H_2(x, q(x, 0), v^2)}{\partial z_i} \Delta z_i(x, 0)\, dx$$

$$+ \int_0^Y \sum_{i=1}^{m} \frac{\partial H_1(y, q(0, y), v^1)}{\partial z_i} \Delta z_i(0, y)\, dy \quad , \qquad (1.42)$$

$$\iint_G \sum_{i=1}^{m} \Delta u_i \, z_{ixy} \, dx \, dy + \int_0^X \sum_{i=1}^{m} \Delta u_i(x, 0) \, z_{ix}(x, 0) \, dx$$

$$+ \int_0^Y \sum_{i=1}^{m} \Delta u_i(0, y) \, z_{iy}(0, y) \, dy = \iint_G \sum_{i=1}^{m} \frac{\partial H}{\partial u_i} \Delta u_i \, dx \, dy$$

$$+ \int_0^X \sum_{i=1}^{m} \frac{\partial H_2(x, q(x, 0), v^2)}{\partial u_i} \Delta u_i(x, 0) \, dx$$

$$+ \int_0^Y \sum_{i=1}^{m} \frac{\partial H_1(y, q(0, y), v^1)}{\partial u_i} \Delta u_i(0, y) \, dy \quad . \tag{1.43}$$

Taking into account Equations (1.41), (1.42), and (1.43) and the fact that $\Delta I = 0$, we find using the same method as during the proof of Theorem 1 the formula for the increment of the functional

$$\Delta S = - \iint_G \left| H(x, y, p, v + \Delta v) - H(x, y, p, v) \right| dx \, dy$$

$$- \int_0^X \left| H_2(x, q(x, 0), v^2 + \Delta v^2) - H_2(x, q(x, 0), v^2) \right| dx$$

$$- \int_0^Y \left| H_1(y, q(0, y), v^1 + \Delta v^1) - H_1(y, q(0, y), v^1) \right| dy - \eta \, , \tag{1.44}$$

where $\eta = \eta_1 + \eta_2 + \eta_3$,

$$\eta_1 = \frac{1}{2} \iint_G \sum_{i=1}^{4m} \left\{ \left[\frac{\partial H(x, y, p, v + \Delta v)}{\partial p_i} - \frac{\partial H(x, y, p, v)}{\partial p_i} \right] \Delta p_i \right.$$

$$+ \sum_{k=1}^{4m} \left[\frac{\partial^2 H(x, y, p + \theta \Delta p, v + \Delta v)}{\partial p_i \partial p_k} \right.$$

$$\left. - \frac{\partial^2 H(x, y, p + \theta_1 \Delta p, v + \Delta v)}{\partial p_i \partial p_k} \right] \Delta p_i \Delta p_k \Bigg\} dx\, dy \quad ,$$

$$\eta_2 = \frac{1}{2} \int_0^X \sum_{i=1}^{2m} \left\{ \left[\frac{\partial H_2(x, q(x, 0), v^2 + \Delta v^2)}{\partial q_i} \right. \right.$$

$$\left. - \frac{\partial H_2(x, q(x, 0), v^2)}{\partial q_i} \right] \Delta q_i (x, 0)$$

$$+ \sum_{k=1}^{2m} \left[\frac{\partial^2 H_2(x, q(x, 0) + \theta_2 \Delta q, v^2 + \Delta v^2)}{\partial q_i \partial q_k} \right.$$

$$\left. - \frac{\partial^2 H_2(x, q(x, 0) + \theta_2 \Delta q, v^2 + \Delta v^2)}{\partial q_i \partial q_k} \right] \Delta q_i \Delta q_k \Bigg\} dx \quad ,$$

$$\eta_3 = \frac{1}{2} \int_0^Y \sum_{i=1}^{m} \left\{ \left[\frac{\partial H_1(y, q(0, y), v^1 + \Delta v^1)}{\partial q_i} \right. \right.$$

$$\left. - \frac{\partial H_1(y, q, (0, y), v^1)}{\partial q_i} \right] \Delta q_i (0, y)$$

$$+ \sum_{k=1}^{2m} \left[\frac{\partial^2 H_1(y, q(0, y) + \theta_4 \Delta q, v^1 + \Delta v^1)}{\partial q_i \partial q_k} \right.$$

$$\left. - \frac{\partial^2 H_1(y, q(0, y) + \theta_5 \Delta q, v^1 + \Delta v^1)}{\partial q_i \partial q_k} \right] \Delta q_i \Delta q_k \Bigg\} dy \; . \qquad (1.44')$$

Let us now estimate the residual term η in the formula (1.44). The magnitudes η_2 and $\eta/?/$ are defined by the values of the function z and u on the boundary of the domain G. From Equations (1.36) and the conditions (1.37) we obtain because of the Lipschitz conditions

$$\sum_{i=1}^{m} |\Delta z_i(0, y)| \le N \int_0^y \sum_{i=1}^{m} |\Delta z_i(0, y)| \, dy + P \int_0^Y \sum_{k=1}^{s} |\Delta v_k{}^1(y)| \, dy ,$$

$$\sum_{i=1}^{m} |\Delta z_i(x, 0)| \le N_1 \int_0^x \sum_{i=1}^{m} |\Delta z_i(x, 0)| \, dx + P_1 \int_0^X \sum_{k=1}^{t} |\Delta v_k{}^2(x)| \, dx .$$

From this, according to the lemma mentioned before, it follows that

$$|\Delta z_i(0, y)| \le M_0 \int_0^Y \sum_{i=1}^{s} |\Delta v_i^1(y)| \, dy ,$$

$$|\Delta z_i(x, 0)| \le M_1 \int_0^X \sum_{i=1}^{t} |\Delta v_i^2(x)| \, dx . \qquad (1.45)$$

We introduce the notations

$$\alpha(x, y) = \sum_{i=1}^{m} |\Delta z_{ix}| , \qquad \beta = \sum_{i=1}^{m} |\Delta z_{iy}| , \qquad \gamma = \sum_{i=1}^{m} |\Delta z_i| ,$$

$$|\Delta v| = \sum_{k=1}^{r} |\Delta v_k| , \qquad \Delta v^1 = \sum |\Delta v_k{}^1(y)| , \qquad |\Delta v^2| = \sum |\Delta v_k{}^2(x)| .$$

Since the functions f_i satisfy the Lipschitz conditions, then like in the case of the derivation of inequalities (1.23), we obtain here

$$\alpha(x, y) \le N_2 \int_0^y \alpha(x, y) \, dy + N_2 \int_0^\eta |\beta(x, y) + \gamma(x, y)| \, dy$$

$$+ N_3 \int_0^Y \Delta v(x, y) \, dy + N_4 \int_0^X \Delta v^2(x) \, dx + N_5 \Delta v^2(x) ,$$

$$\beta(x, y) \leq M_2 \int_0^x \beta(x, y)\, dx + M_2 \int_0^{\xi} \left| \alpha(x, y) + \gamma(x, y) \right| dx$$

$$+ M_3 \int_0^X \Delta v(x, y)\, dx + M_4 \int_0^Y \Delta v^1(y)\, dy + M_5 \Delta v^1(y) \;,$$

$$\gamma(x, y) \leq \int_0^x \alpha(x, y)\, dx \;, \quad \gamma(x, y) \leq \int_0^y \beta(x, y)\, dy \;,$$

where

$$0 \leq x \leq \xi \leq X \;, \quad 0 \leq y \leq \eta \leq Y \;.$$

From this we find:

$$\alpha(x, y) \leq N_6 \int_0^{\eta} \left| \beta(x, y) + \gamma(x, y) \right| dy + N_7 \int_0^Y \Delta v(x, y)\, dy$$

$$+ N_8 \int_0^X \Delta v^2(x)\, dx + N_9 \Delta v^2(x) \;,$$

$$\beta(x, y) \leq M_6 \int_0^{\xi} \left| \alpha(x, y) + \gamma(x, y) \right| dx + M_7 \int_0^X \Delta v(x, y)\, dx$$

$$+ M_8 \int_0^Y \Delta v^1(y)\, dy + M_9 \Delta v^1(y) \;.$$

Taking into account the estimates for the function γ, we find .

$$\alpha(x, y) \leq N_{10} \int_0^{\eta} \beta(x, y)\, dy + N_7 \int_0^Y \Delta v(x, y)\, dy$$

$$+ N_8 \int_0^X \Delta v^2(x)\, dx + N_9 \Delta v^2(x) \;, \qquad (*)$$

$$\beta(x, y) \le M_{10} \int_0^{\xi} \alpha(x, y)\, dx + M_7 \int_0^{X} \Delta v(x, y)\, dx$$

$$+ M_8 \int_0^{Y} \Delta v^1(y)\, dy + M_9 \Delta v^1(y) \quad . \qquad (*)$$

From these inequalities we obtain

$$\alpha(\xi, \eta) \le N_{11} \int_0^{\xi}\int_0^{\eta} \alpha(x, y)\, dx\, dy + N_{12} \int_0^{X}\int_0^{Y} \Delta v(x, y)\, dx\, dy$$

$$+ N_7 \int_0^{Y} \Delta v(\xi, y)\, dy + N_{13} \int_0^{Y} \Delta v^1(y)\, dy + N_8 \int_0^{X} \Delta v^2(x)\, dx$$

$$+ N_9 \Delta v^2(\xi) \quad ,$$

$$\beta(\xi, \eta) \le M_{11} \int_0^{\eta}\int_0^{\xi} \beta(x, y)\, dx\, dy + M_{12} \int_0^{X}\int_0^{Y} \Delta v(x, y)\, dx\, dy$$

$$+ M_7 \int_0^{X} \Delta v(x, \eta)\, dx + M_8 \int_0^{Y} \Delta v^1(y)\, dy + M_{13} \int_0^{X} \Delta v^2(x)\, dx$$

$$+ M_9 \Delta v^1(\eta) \quad .$$

Integrating the first of these inequalities over ξ between the limits of 0 and ξ and applying the above mentioned lemma, we find

$$\gamma \le \int_0^{\xi} \alpha(\xi, \eta)\, d\xi \le N_{14} \iint_G \Delta v(x, y)\, dx\, dy + N_{15} \int_0^{Y} \Delta v^1(y)\, dy$$

$$+ N_{16} \int_0^{X} \Delta v^2(x)\, dx \quad .$$

In an analogous manner, from the second inequality we obtain

$$\int_0^{\eta} \beta(\xi,\ \eta)\, d\eta \le M_{14} \iint_G \Delta v(x,\ y)\, dx\, dy + M_{15} \int_0^X \Delta v^2(x)\, dx$$

$$+ M_{16} \int_0^Y \Delta v^1(y)\, dy \quad .$$

From this and the inequalities (*), we find

$$|\Delta z_i(x,\ y)| \le N_{14} \iint_G \Delta v(x,\ y)\, dx\, dy + N_{15} \int_0^Y \Delta v^1(y)\, dy + N_{16} \int_0^X \Delta v^2(x)\, dx,$$

$$|\Delta z_{ix}(x,\ y)| \le N_{17} \iint_G \Delta v(x,\ y)\, dx\, dy + N_{18} \int_0^Y \Delta v(x,\ y)\, dy$$

$$+ N_{19} \int_0^X \Delta v^2 dx + N_{20} \int_0^Y \Delta v^1 dy + N_9 \Delta v^2(x) \quad ,$$

$$|\Delta z_{iy}(x,\ y)| \le M_{17} \iint_G \Delta v(x,\ y)\, dx\, dy + M_{18} \int_0^X \Delta v(x,\ y)\, dx$$

$$+ M_{19} \int_0^Y \Delta v^1 dy + M_{20} \int_0^X \Delta v^2 dx + M_9 \Delta v^1(y) \quad . \tag{1.46}$$

In an analogous manner we get

$$|\Delta u_i(x,\ y)| \le M_{21} \iint_G \Delta v(x,\ y)\, dx\, dy + M_{22} \int_0^Y \Delta v^1(y)\, dy + M_{23} \int_0^X \Delta v^2(x)\, dx \quad . \tag{1.47}$$

If $\Delta v_i^1(y) = \Delta v_i^2(x) \equiv 0$, while $\Delta v_i(x,\ y) \not\equiv 0$, then from (1.46) and (1.47) we obtain the inequalities (1.27) and (1.29). After establishing this fact, we go over directly to the proof of the theorem.

Let for the sake of definiteness the permissible control $\omega(x,\ y) = (v(x,\ y),\ v^1(y),\ v^2(x))$ be min-optimal according to S. Then during an arbitrary $\Delta\omega$ the inequality $\Delta S \geq 0$ is valid. Let us assume that the theorem is not true. Then within the closed domain G one can indicate either a surface domain G_1 within which the first equality (1.39) is not fulfilled, or a segment of a straight line located on the boundary of G over which one of the two last equalities (1.39) is not satisfied.

In the first of these cases one can find the permissible control $\tilde{v} \in V$ such that

$$H(x, y, p(x, y), \tilde{v}) - H(x, y, p(x, y), v) > 0 \quad \text{for} \quad (x, y) \in G_1 .$$

Then there exists a $\delta > 0$ such that

$$H(x, y, p(x, y), \tilde{v}) - H(x, y, p(x, y), v) > \delta$$

for $(x, y) \in G_\epsilon \subset G_1$, where G_ϵ - a circle of radius ϵ located adjoining the boundary in the interior of the domain G_1. Putting $\Delta v^1 = \Delta v^2 \equiv 0$ and repeating the same reasoning as the one carried during the proof of Theorem 1, we obtain by means of the estimates (1.46) and (1.47) that $\Delta S < 0$. However, this contradicts the condition and indicates that the first equality is satisfied under the conditions of maximum.

Let us investigate the second case. For definiteness we assume that the last equation (1.32) is not satisfied. There exists then a control $\tilde{v}^2 \in V^2$ and a segment l of the $y = 0$ boundary of the domain G such that

$$H_2(x, q(x, 0), \tilde{v}^2) - H_2(x, q(x, 0), v^2) > 0$$

for $x \in l$. Consequently, one can specify a number $\delta > 0$ such that

$$H_2(x, q(x, 0), \tilde{v}^2) - H_2(x, q, (x, 0) v^2) > \delta$$

for $x \in l_\epsilon \subset l$, where l_ϵ is a segment of length ϵ. Let us put

$$\Delta v_i = \Delta v_i^1 = 0$$

and study the auxiliary control

$$\bar{\omega}^1 (x, y) = (v, v^1, \bar{v}^2) \quad ,$$

where

$$\bar{v}^2 = \begin{cases} v^2 & \text{for } x \,\bar{\in}\, l_\epsilon , \\ \tilde{v}^2 & \text{for } x \in l_\epsilon . \end{cases}$$

Then the residual term η in Equation (1.44) coincides with η_2 (see Equation (1.44')) where

$$\Delta v^2 = \bar{v}^2 - v^2$$

and consequently, Δv^2 differs from zero only for $x \in l_\epsilon$.

Since the function $\partial H_2 / \partial q_i$ satisfies the Lifschitz conditions and $\partial^2 H_2 / \partial q_i \partial q_k$ are bounded, then because of the estimates (1.45) and (1.47) we obtain

$$|\eta| \leq M\epsilon \int_{l_\epsilon} \left(\sum_{k=1}^{t} |\Delta v_k{}^2(x)| \right)^2 dx ,$$

where M is a constant independent on ϵ. By means of this estimate we can easily establish that $\Delta S > 0$ and this contradicts the assumption about the min-optimality according to S of the control $\omega(x, y)$.

Theorem 3 is thus fully proved.

Let now the control process be described by the system of linear equations

$$z_{ixy} = \sum_{k=1}^{m} \left| c_{ik}(x, y) z_{kx} + d_{ik}(x, y) z_{ky} + g_{ik}(x, y) z_k \right| + f_i(v) ,$$

$$i = 1, \ldots, m , \tag{1.48}$$

with the additional conditions

$$z_{iy}(0, y) = \sum_{k=1}^{m} c_{ik}(0, y) z_k + \varphi_i(v^1) ,$$

$$z_{ix}(x, 0) = \sum_{k=1}^{m} d_{ik}(x, 0) z_k(x, 0) + \psi_i(v^2) ,$$

$$z_i(0, 0) = z_i{}^0 , \qquad i = 1, \ldots, m . \tag{1.50}$$

The special choice of the coefficients in systems (1.49) follows from the requirements (1.38). Like during the proof of Theorem 2, we find that in the case under investigation the residual term η in formula (1.45) is equal to zero and, consequently,

$$\Delta S = - \iint_G \left| H(x, y, p, v + \Delta v) - H(x, y, p, v) \right| dx\, dy$$
$$- \int_0^X \left| H_2(x, q, v^2 + \Delta v^2) - H_2(x, q, v^2) \right| dx$$
$$- \int_0^Y \left| H_1(y, q, v^1 + \Delta v^1) - H_1(y, q, v^1) \right| dy \quad .$$

From this formula follows the validity of the following theorem.

THEOREM 4. For a permissible control $\omega(x, y)$ in the boundary problem (1.48)-(1.49)-(1.50) to be locally min-optimal (max-optimal) according to the functional $S = \epsilon A_i z_i (X, Y)$ it is necessary and sufficient that it satisfies the conditions of maximum (minimum).

Section II. OTHER PROBLEMS OF OPTIMUM CONTROL FOR HYPERBOLIC SYSTEMS

We investigate the same problem concerning the minimization of the functional $S = \Sigma A_i z_i (X, Y)$ in which the control process is described by the boundary problem (1.1)-(1.36)-(1.37) where z_i^0, $i = 1, \ldots, m$ are given numbers. The permissible controls defined in Section I are omitted by the requirement that the respective numbers $z_i (X, Y)$ belong to a convex set D of the space of the variables $z_1, \ldots, z_m$. In this manner, in the problem under investigation, the permissible controls transfer by means of Equations (1.1) and (1.36) the point $(z_1^0, \ldots, z_m^0)$ into points of region D. In what follows, we will assume that the convex region D contains internal points and that it is closed.

For the solution of the problem like in the paper by Rozopoer[16] we introduce the function

$$A(z) = (A, z) = \sum A_i z_i$$

and denote by D* the set of points $z^* \in D^*$ at which

$$A(z^*) = \min_{z \in D} A(z)$$

If the set D* is not empty, then

$$A(z^*) \leq A(z), \quad z^* \in D^*, \quad z \in D,$$

and, consequently, the functional

$$S = \sum A_i z_i (X, Y) \quad ,$$

defined over the solutions of the boundary problem (1.1)-(1.36)-(1.37) cannot take values smaller than $A(z^*)$. If there exists a permissible control which transfers the point z^0 into an arbitrary point of the set D*, then such a control is a min-optimal according to S. In such a case, the problem reduces to the calculation of controls transferring z^0 into a given domain. Such a problem will not be investigated in what follows, i.e., we assume that there are no permissible controls transferring z^0 into D*.

1. The Necessary Optimality Conditions

We will say that the permissible control $\omega(x, y)$ satisfies the maximum condition relative to the given function $u(x, y)$ if the conditions (1.39) are satisfied, where $z(x, y)$ is the solution of the boundary problem (1.1)-(1.36)-(1.37).

THEOREM 5. If $\omega(x, y)$ is min-optimal control according to S and $z(x, y)$ - the corresponding solution of the problem (1.1)-(1.36)-(1.37), then there exists a vector function $u(x, y)$ relative to which the control $\omega(x, y)$ satisfies the maximum condition.

Let

$$\omega(x, y) = (v(x, y), v^1(y), v^2(x))$$

be a min-optimal control according to S and $z(x, y)$ - its corresponding solution of the boundary problem (1.1)-(1.36)-(1.37). We denote by D^- (D^+) that part of the domain D for which

$$A(z) \leqslant \Sigma A_i z_i(X, Y),\ z \in D^- (A(z) \geqslant \Sigma A_i z_i(X, Y),\ z \in D^+) .$$

The general part of these closed convex regions is the plane

$$\sum_{i=1}^{m} A_i (z_i - z_i(X, Y)) = 0 ,$$

which contains the point $z(X, Y)$. Since the control $\omega(x, y)$ is the min-optimal according to S, there do not exist permissible controls transferring the point z^0 into the domain D^-. Noting this fact, we introduce the variation of the control assuming that all the permissible controls are sectionally continuous.

We choose arbitrary points

$$(x_i, y_j),\ i, j \geqslant 0\ (x_0 = 0,\ y_0 = 0) ,$$

in the domain G and denote by G_{ij} the rectangle formed by the adjacent points (x_ν, y_μ) in which the corner of the lower left angle is represented by the point (x_i, y_j). We establish the square

$$I_{ij},\ x_{i+1} - \tau \leqslant x \leqslant x_{i+1},\ y_{j+1} - \tau \leqslant y \leqslant y_{j+1} ,$$

where the number τ is chosen so small that for the given set of points (x_i, y_j) these squares do not have common points*.

We take an arbitrary sectionally continuous vector-functions $\alpha_{ij}(x, y)$, $\beta_i(x)$ and $\gamma_j(y)$ defined for $x, y \in [0, 1]$ and taking values in the domains V, V^2, and V^1, respectively, of the changes of the control parameters v, v^2, and v^1. We introduce functions

$$v_b(x, y, \alpha_{ik}) = \begin{cases} v(x, y) & \text{for } (x, y) \bar{\in} I_{i-1, k-1} \\ \alpha_{ik}\left(\dfrac{x_i - x}{\tau}, \dfrac{y_k - y}{\tau}\right) & \text{for } (x, y) \in I_{i-1, k-1}, \end{cases}$$

$$v_b^1(y, \gamma_k) = \begin{cases} v^1(y) & \text{for } y \bar{\in} [y_k - \tau, y_k], \\ \gamma_k\left(\dfrac{y_k - y}{\tau}\right) & \text{for } y \in [y_k - \tau, y_k), \end{cases}$$

$$v_b^2(x, \beta_i) = \begin{cases} v^2(x) & \text{for } x \bar{\in} [x_i - \tau, x_i), \\ \beta_i\left(\dfrac{x_i - x}{\tau}\right) & \text{for } x \in [x_i - \tau, x_i). \end{cases}$$

The functions

$$\omega_b(x, y, \alpha_{ik}, \beta_i, \gamma_k) = (v_b(x, y, \alpha_{ik}), v_b^1(y, \gamma_k), v_b^2(x, \beta_i))$$

will be called the varied control, and Ω will denote the totality of all possible varied controls corresponding to all possible squares I_{ij} and all possible functions α_{ik}, β_i, and γ_k of the above mentioned type. We denote the solution of the boundary problem (1.1)-(1.36)-(1.37) corresponding to the control $\omega_b \in \Omega$ by $z(x, y, \omega_b)$. Then the function

$$\Delta z(x, y, \omega) = z(x, y, \omega_b) - z(x, y, \omega)$$

is the solution of the boundary problem

$$\Delta z_{ixy}(x, y, \omega) = \Delta \frac{\partial H}{\partial u_i}, \quad (x, y) \in G, \tag{2.1}$$

*In the case when the number of points (x_i, y_j) is finite, there are no doubts concerning the existence of such a τ.

$$\Delta z_{iy}(0, y, \omega) = \Delta \frac{\partial H_1}{\partial u_i}, \quad y \in |0, Y|, \quad \Delta z_i(x, 0, \omega)$$

$$= \Delta \frac{\partial H_2}{\partial u_i}, \quad x \in |0, X|, \tag{2.2}$$

$$\Delta z_i (0, 0, \omega) = 0; \quad i = 1, \ldots, m. \tag{2.3}$$

Since Equations (2.2) are ordinary differential equations, then, according to results of the paper by Rozopoer[16] it follows from (2.2) and (2.3) that:

$$\delta z_i (0, y, \omega) = \int_0^y \sum_{k=1}^{m} \frac{\varphi_i (y, z(0, y, \omega), \omega)}{\partial z_k} \delta z_k (0, y, \omega) dy$$

$$+ \sum_{j=1}^{k} R_i \left| y_j, \gamma_j \right|, \quad y_k < y < y_{k+1},$$

where

$$\delta z_i (x, 0, \omega) = \int_0^x \sum_{k=1}^{m} \frac{\psi_i(x, z(x, 0, \omega), \omega)}{\partial z} \delta z_k(x, 0, \omega) dx$$

$$+ \sum_{j=1}^{l} Q_i \left| x_j, \beta_j \right|, \quad x_l < x < x_{l+1},$$

$$\delta z_i = \lim_{\tau \to 0} \frac{\Delta z_i}{\tau}$$

$$R_i \left[y_j, \gamma_j \right] = \int_0^1 \left[\varphi_i(y_i, z(0, y_j, v^1), \gamma_j(y)) - \varphi_i(y_j, z(0, y_j, v^1), \ v^1(y)) \right] dy,$$

$$Q_i(x_j, \beta_j) = \int_0^1 \left[\psi_i(x_j, z(x_j, 0, v^2), \beta_j(x)) \right.$$

$$\left. - \psi_i(x_j, z(x_j, 0, v^2), \ v^2 (x)) \right] dx.$$

It was shown in the same paper that

$$\delta z_i(x, 0, \omega) = \sum_{j=1}^{l} \sum_{s=1}^{m} A_{is}(x, x_j) Q_s(x_j, \omega_b),$$

$$\delta z_i(0, y, \omega) = \sum_{j=1}^{k} \sum_{s=1}^{m} B_{is}(y, y_j) R_s(y_j, \omega_b); \tag{2.4}$$

where the matrices A_{is} and B_{is} do not depend on the choice of the functions β_i and γ_i.

From the relationships (2.1) it follows that

$$\Delta z_i(x, y, \omega) = \Delta z_i(x, 0, \omega) + \Delta z_i(0, y, \omega)$$

$$+ \int_0^x \int_0^y \sum_{s=1}^{3m} \frac{\partial f_i(x, y, w, v)}{\partial w_s} \Delta w_s dx dy + \sum_{j=1}^{l} \sum_{\nu=1}^{k} I_{ij\nu} [\omega_b] \tag{2.5}$$

$$+ E_i, \ (x, y) \in G_{l, k} - I_{l, k}.$$

Here we introduce the following notation:

$$I_{ijk} [\omega_b] = \int_{x_j - \tau}^{x_j} \int_{y_k - \tau}^{y_k} F_i(x, y, w, a_{jk}, v) dy \, dx,$$

$$E_i = \frac{1}{2} \sum_{s, q=1}^{3m} \int_0^x \int_0^y \frac{\partial^2 f_i(x, y, w + \theta \Delta w, v_b)}{\partial w_s \partial w_q} \Delta w_s \Delta w_q dy \, dx$$

$$+ \sum_{j=1}^{l} \sum_{\nu=1}^{k} \int_{x_j - \tau}^{x_j} \int_{y_\nu - \tau}^{y_\nu} \sum_{s=1}^{3m} \frac{\partial F_i(x, y, w, a_{i\nu}, v)}{\partial w_s} \Delta w_s dy \, dx,$$

where

$$F_i(x, y, w, a_{j\nu}, v) = f_i\left(x, y, w, a_{j\nu} \left(\frac{x_j - x}{\tau} \quad \frac{y_\nu - y}{\tau}\right)\right) - f_i(x, y, w, v).$$

In an analogous manner we find that

$$\Delta z_{ix}(x, y,\omega) = \Delta z_{ix}(x, 0,\omega) + \int_0^y \sum_{s=1}^{3m} \frac{\partial f_i(x, y, w, v)}{\partial w_s} \Delta w_s dy + \sum_{\nu=1}^{k} E_{i\nu} |\omega_b| + \overline{E}_i, \tag{2.6}$$

$$\Delta z_{iy}(x, y,\omega) = \Delta z_{iy}(0, y,\omega) + \int_0^x \sum_{s=1}^{3m} \frac{\partial f_i(x, y, w, v)}{\partial w_s} \Delta w_s dx + \sum_{j=1}^{l} F_{ij} |\omega_b| + \overline{F}_i \tag{2.7}$$

with $(x, y) \in G_{l,k} - I_{l,k}$,

where

$$E_{ip} = \begin{cases} 0 & \text{for } x_l < x < x_{l+1} - \tau, \\ \int_{y_p - \tau}^{y_p} F_i(x, y, w, a_{l,p}, v)\, dy & \text{for } x_{l+1} - \tau \leqslant x < x_{l+1}, \end{cases}$$

$$F_{ij} = \begin{cases} 0 & \text{for } y_k < y < y_{k+1} - \tau, \\ \int_{x_j - \tau}^{x_j} F(x, y, w, a_{jk}, v)\, dx & \text{for } y_{k+1} - \tau \leqslant y < y_{k+1}, \end{cases}$$

$$\overline{E}_i = \begin{cases} J_i = \frac{1}{2} \int_0^y \sum_{s,q=1}^{3m} \frac{\partial^2 f_i(x, y, w + \theta\Delta w, v_b)}{\partial w_s \partial w_q} \Delta w_s \Delta w_q dy & \text{for } x_l < x < x_{l+1} - \tau, \\ J_i + \sum_{p=1}^{k} \int_{y_p - \tau}^{y_p} \sum_{s=1}^{3m} \frac{\partial F_i(x, y, w, a_{l,p}, v)}{\partial w_s} \Delta w_s dy & \text{for } x_{l+1} - \tau \leqslant x < x_{l+1}, \end{cases}$$

$$\overline{F}_i = \begin{cases} L_i = \dfrac{1}{2} \displaystyle\sum_{s,q=1}^{3m} \int_0^x \dfrac{\partial^2 f_i(x, y, w + \theta\Delta w, v_b)}{\partial w_s \partial w_q} \Delta w_s \Delta w_q dx \\ \qquad\qquad\qquad\qquad \text{for } y_k < y < y_{k+1} - \tau, \\ L_i + \displaystyle\sum_{j=1}^{l} \int_{x_j-\tau}^{x_j} \sum_{s=1}^{3m} \dfrac{\partial F_i(x, y, w, a_{ij}, v)}{\partial w_s} \Delta w_s dx \\ \qquad\qquad\qquad\qquad \text{for } y_{k+1} - \tau \leqslant y < y_{k+1}. \end{cases}$$

According to what was proved earlier (see unequalities (1.46)) one can specify a positive number N such that

$$|\Delta w_i(x, y)| \leqslant N\tau ,$$

and consequently,

$$|\overline{E}_i| \leqslant N_1\tau^2 \qquad |E_i| \leqslant N_2\tau^2 \qquad |\overline{F}_i| \leqslant N_3\tau^2$$

and uniformly over x and y

$$\lim_{\tau \to 0} \frac{E_i}{\tau} = \lim_{\tau \to 0} \frac{\overline{E}_i}{\tau} = \lim_{\tau \to 0} \frac{F_i}{\tau} = 0.$$

Introducing the substitutions

$$\xi\tau = x_j - x, \quad \eta\tau = y_\nu - y$$

and going over to the limit we obtain

$$R_{ijp} \left|x_j, y_p, \omega_b\right| = \lim_{\tau \to 0} \frac{I_{ijp}}{\tau} = \int_0^1\int_0^1 \left|f_i(x_j, y_p, w(x_j, y_p), a_{jp}(\xi, \eta)) - f_i(x_j, y_p, w(x_j, y_p)v)\right| d\xi d\eta.$$

One can show that the totality of Equations (2.5), (2.6), and (2.7) is solvable and for all (x, y) not located on the lattice $x = x_j$, and $y = y_\nu$ there exist limits

$$\lim_{\tau \to 0} \frac{\Delta z_i(x, y, \omega)}{\tau} = \delta z_i,$$

$$\lim_{\tau \to 0} \frac{\Delta z_{ix}(x, y, \omega)}{\tau} = \delta z_{ix}, \qquad \lim_{\tau \to 0} \frac{\Delta z_{iy}(x, y, \omega)}{\tau} = \delta z_{iy},$$

where

$$\delta z_{ix} = \frac{\partial \delta z_i}{\partial x}, \quad \delta z_{iy} = \frac{\partial \delta z_i}{\partial y}.$$

Dividing these equations by τ going over to the limit for $\tau \to 0$, we find:

$$\left.\begin{aligned}
\delta z_i(x, y, \omega) &= \delta z_i(x, 0, \omega) + \delta z_i(0, y, \omega) \\
&\quad + \int_0^y \int_0^x \sum_{s=1}^{3m} \frac{\partial f_i(x, y, w, v)}{\partial w_s} \delta w_s dx\, dy \\
&\quad + \sum_{j=1}^{l} \sum_{p=1}^{k} R_{ijp} \left| x_j, y_p, \omega \right|, \\
\delta z_{ix}(x, y, \omega) &= \delta z_{ix}(x, 0, \omega) + \int_0^y \sum_{s=1}^{3m} \frac{\partial f_i(x, y, w, v)}{\partial w_s} \delta w_s dy, \\
\delta z_{iy}(x, y, \omega) &= \delta z_{iy}(0, y, \omega) + \int_0^x \sum_{s=1}^{3m} \frac{\partial f_i(x, y, w, v)}{\partial w_s} \delta w_s dx
\end{aligned}\right\} \quad (2.8)$$

for

$$x_l < x < x_{l+1} = X, \quad y_k < y < y_{k+1} = Y.$$

From the way the functions $\delta z_i(x, 0, \omega)$ and $\delta z_i(0, y, \omega)$ were defined it follows that

$$\delta z_i(x,0,\omega) = \delta z_i(0,y,\omega) = 0$$

for

$$0 \leqslant x \leqslant x_1, \quad 0 \leqslant y \leqslant y_1,$$

and, consequently, from (2.8) it follows that

$$\delta z_i(x,y,\omega) = \delta z_{ix}(x,y,\omega) = \delta z_{iy}(x,y,\omega) \equiv 0$$

for

$$0 \leqslant x < x_1, \quad 0 \leqslant y < y_1.$$

Further, from the relationship (2.4) and (2.8) we obtain

$$\delta z_i(x,y,\omega) = \sum_{s=1}^{m} A_{is}(x,x_1)\, Q_s(x_1\omega_b) + \int_{x_1}^{x}\int_{0}^{y} \sum_{s=1}^{3m} \frac{\partial f_i(x,y,w,v)}{\partial w_s}\, \delta w_s\, dy\, dx,$$

$$\delta z_{ix}(x,y,\omega) = \sum_{s=1}^{m} A_{is'}(x,x_1)\, Q_s\,(x_1\omega_b) + \int_{0}^{y} \sum_{s=1}^{3m} \frac{\partial f_i(x,y,w,v)}{\partial w_s}\, \delta w_s\, dy,$$

$$\delta z_{iy}(x,y,\omega) = \int_{x_1}^{x} \sum_{s=1}^{3m} \frac{\partial f_i(x,y,w,v)}{\partial w_s}\, \delta w_s\, dx$$

for

$$x_1 < x < x_2, \quad 0 < y < y_1 .$$

Solving this system, for instance, by the method of successive approximations, we find that the function δz_i may be represented in the form

$$\delta z_i(x,y,\omega) = \sum_{s=1}^{m} A^1_{is}\,(x,y,x_1)\, Q_s(x_1,\omega_b), \quad x_1 < x < x_2, \qquad 0 \leqslant y \leqslant y_1 \quad ,$$

where $A^1_{is}\,(x,y,x_1)$ is the fully defined function not depending on the choice of the functions α_{ij}, β_i, and γ_j. Continuing these deliberations, we define

$$\delta z_i(x, y, \omega) = \sum_{s=1}^{m} \sum_{j=1}^{l} A_{is}^{j}(x, y, x_j)\, Q_s(x_j, \omega_b),$$

$$0 \leq y \leq y_1, \quad x_l < x \leq x_{l+1} = X. \tag{2.9}$$

We find in an analogous way that

$$\delta z_i(x, y, \omega) = \sum_{s=1}^{m} \sum_{p=1}^{k} B_{is}^{p}(x, y, y_p)\, R_s(y_p, \omega_b),$$

$$0 \leq x \leq x_1, \quad y_k < y \leq y_{k+1} = Y. \tag{2.10}$$

From these relationships it follows in particular that

$$\delta z_i(x_l + 0, y, \omega) - \delta z_l(x_l - 0, y, \omega) = \sum_{s=1}^{m} \Big[A_{is}^{l}(x_l + 0, y, x_l)\, Q_s(x_l, \omega_l)$$

$$- A_{is}^{l-1}(x_l - 0, y, x_{l-1})\, Q_s(x_{l-1}, \omega_b)\Big],$$

$$\delta z_i(x, y_k + 0, \omega) - \delta z_i(x, y_k - 0, \omega) = \sum_{s=1}^{m} \Big[B_{is}^{k}(x, y_k + 0, y_k) R_s(y_k, \omega_b)$$

$$- B_{is}^{k-1}(x, y_k - 0, y_{k-1})\, R_s(y_{k-1}, \omega_b)\Big].$$

Consequently, functions $\delta z_i(x, y, \omega)$ which are defined by the formulas (2.9) and (2.10) are generally speaking discontinuous along the lines $x = x_l$ and $y = y_k$.

Continuing analogous deliberations, we get

$$\delta z_i(x, y, \omega) = \sum_{s=1}^{m} \sum_{j=1}^{l} \sum_{p=1}^{k} \Big[C_{ijps}(x, y, x_j, y_p)\, S(x_j, y_p, \omega_b)$$

$$+ D_{is}^{j}(x, y, x_j)\, Q_s(x_j, \omega_b) + F_{is}^{p}(x, y, y_p) R_s(y_p, \omega_b)\Big],$$

$$x_l < x \leq x_{l+1} = X, \quad y_k < y \leq y_{k+1} = Y.$$

Substituting in this equality $x = X$, and $y = Y$, we obtain finally

$$\delta z_i(X, Y, \omega) = \sum_{s=1}^{m} \sum_{j=1}^{l} \sum_{p=1}^{k} \Big| C_{ijps}(x_j, y_p)\, S(x_j y_p, \omega_b) + D^j_{is}(x_j)\, Q_s(x_j, \omega_b) + F^p_{is}(y_p)\, R_s(y_p, \omega b) \Big| , \tag{2.11}$$

where the constants C_{ijps}, D^j_{is}, and F^p_{is} do not depend on the choice of α_{ij}, β_i, and γ_j.

The point

$$z(X, Y) + \delta z(X, Y, \omega)$$

corresponding to the arbitrary variation $\omega_b \in \Omega$ of the control ω goes over a certain set Π within the space of the variables $z_1, \ldots, z_m$. In the same manner as it was done in the paper by Rozopoer[16] one can show that it is convex and that its arbitrary internal point cannot belong to the internal part of the set D^-. It follows from this that through the point $z(X, Y)$ one can draw the plane

$$\sum_{i=1}^{m} a_i(z_i - z_i(X, Y)) = 0, \tag{2.12}$$

dividing the sets Π and D where the signs of the coefficients a_i may be chosen in such a way the Π is in the half-space

$$\sum_{i=1}^{m} a_i(z_i - z_i(X, Y)) \geqslant 0.$$

Consequently, for an arbitrary ω_b

$$\sum a_i \delta z_i(X, Y, \omega) \geqslant 0,$$

i.e.,

$$\lim_{\tau \to 0} \frac{\Sigma\, a_i \Delta z_i(X, Y, \omega)}{\tau} \geqslant 0.$$

We introduce auxiliary functions u_i by means of Equations (1.5) and additional conditons

$$u_{ix}(x, Y) = -\frac{\partial H(x, Y, p(x, Y), v)}{\partial z_{iy}},$$

$$u_{iy}(X, y) = -\frac{\partial H(X, y, p(X, y), v)}{\partial z_{ix}}, \quad u_i(X, Y) = -a_i. \tag{2.13}$$

Using the same method which we applied above, we can obtain a formula for the increment of the functional

$$\overline{S} = \Sigma a_i z_i(X, Y)$$

in the form (1.44) and, consequently, the same method may be used to show that the conditions of the maximum (1.39) are necessary in order that the permissible control $\omega(x, y)$ realizes the minimum of the functional $\overline{S}$. However, $\overline{S}$ attains its minimum over the min-optimal control according to S.

Theorem 5 is thus fully proved.

It is obvious that the statement just proved remains valid even in the case when the control process is described by the boundary problem (1.1)-(1.2).

If the control process is described by a linear boundary problem (1.48)-(1.49)-(1.50), then one encounters as valid

THEOREM 6. Let $\mathbf{z}(x, y)$ be the solution of the boundary problem (1.48)-(1.49)-(1.50), corresponding to the control $\omega(x, y)$ and satisfying the condition $\mathbf{z}(X, Y) = \mathbf{z}^1$. Then, if $\omega(x, y)$ satisfies the condition of maximum (minimum) relative to the functions $u_i(x, y)$ taking the boundary values

$$u_i(X, Y) = -\lambda A_i - \mu B_i(z^1), \quad \mu \geqslant 0, \quad \lambda > 0,$$

where $B_i(z^1)$ are the coordinates of the normal, perpendicular to the D hyperplane, then the control $\omega(x, y)$ is min-optimal according to the functional

$$S = \sum_{i=1}^{m} A_i z_i(X, Y).$$

The proof of this theorem agrees almost completely with the proof of the corresponding theorem (see Theorem 4 in the paper by Rozopoer[16] for ordinary differential equations.

2. The Use of Theorem 5 for the Solution of Certain Specific Problems

The result just obtained does not, generally speaking, present a method for the establishment of the vector $u(x, y)$. However, in numerous particular cases this problem may be solved. Let us study some of these.

1) The point $z(X, Y)$ is located within the domain D. Then $a_i = A_i$, since an arbitrary plane in addition to the plane (2.12) crossing the point $z(X, Y)$ cuts the region D^- and, consequently, cannot separate D^- and Π.
2) The point $z(X, Y)$ belongs to the boundary of the domain D which is specified by the inequality $F(z) \leqslant 0$. Then the boundary is specified by the equation $F(z) = 0$. If the function $F(z)$ is differentiable, then the equation of the tangential plane through the point $z(X, Y)$ has the form:

$$\sum_{i=1}^{m} B_i(z_i - z_i(X, Y)) = 0, \quad B_i = \left[\frac{\partial F}{\partial z_i}\right]_{z=z(X, Y)}$$

Since the plane

$$\Sigma a_i(z_i - z_i(X, Y)) = 0$$

likewise crosses the point $z(X, Y)$, then

$$a_i = \lambda A_i + \mu B_i,$$

where, without a loss of generality, one can assume that

$$\lambda \geqslant 0, \quad \mu \geqslant 0 \ (\lambda^2 + \mu^2 \neq 0).$$

Since a_i is determined with accuracy up to a constant multiplier, then only one of the quantities λ and μ is independent. Since, according to the conditions of (2.13), $u_i(X, Y) = a_i$ and $F(z(X, Y)) = 0$, we obtain $m + 1$ relationships

$$u_i(X, Y) = -\lambda A_i - \mu B_i, \quad F(z(X, Y)) = 0, \quad i = 1, \ldots, m, \tag{2.14}$$

for the determination of $u_i(X, Y)$ and of the quantities λ or μ. Adding to (2.14) the conditions (1.37), we obtain 2m boundary conditions for the 2m functions $z_1, \ldots, z_m, u_1, \ldots, u_m$. These conditions, together with Equations (1.1), (1.5), (1.36), (1.39), and (2.13), form a "complete" system of relationships for the determination of the optimum control and the corresponding vector functions $z(x, y)$ and $u(x, y)$.

Suppose that we are required to determine the minimum of the functional

$$I = \int_0^X \int_0^Y f_0(x, y, z, z_x, z_y, v)\, dy\, dx$$

under conditions that the function $z(x, y)$ is the solution of the boundary problem (1. 1)-(1. 2), while the point $z(X, Y)$ belongs to a certain convex domain D of the space of variables $z_1, \ldots, z_m$. By introducing an auxiliary function z_0 through the relationships (1. 4), we reduce the problem to the calculation of the minimum. . . [Apparently one line is missing, Note of the Translator] . . . enters into the cylinder having a generatrix parallel to the axis z_0. Since the variable z_0 does not enter into the right-hand side of Equations (1. 1) and (1. 4), we find that $B_0 = 0$ in the relationships (2. 14). For the functional under investigation, $A_1 = \ldots = A_m = 0$, $A_0 = 1$, which means that it follows from (1. 5) and (1. 6) that $u_0(x, y) = -1$. In this manner, differential equations and boundary conditions of the problem under investigation take the form of the relationships

$$z_{ixy} = \frac{\partial H}{\partial u_i}, \quad z_i(0, y) = \varphi_i(y), \quad z_i(x, 0) = \psi_i(x),$$

$$u_{ixy} = \frac{\partial H}{\partial z_i} - \frac{d}{dx}\left(\frac{\partial H}{\partial z_{ix}}\right) - \frac{d}{dy}\left(\frac{\partial H}{\partial z_{iy}}\right), \quad u_{ix}(x, Y) = -\left.\frac{\partial H}{\partial z_{iy}}\right|_{y=Y},$$

$$u_{iy}(X, y) = -\left.\frac{\partial H}{\partial z_{ix}}\right|_{x=X}, \quad u_i(X, Y) = 0, \quad H = \sum_{i=1}^{m} u_i f_i - f_0,$$

from which the auxiliary Equations (1. 4) are excluded.

3. Generalization to the Case of an Arbitrary Number of Independent Variables

The formula for the increment of the functional S and the ensuring consequences may be generalized for the case when the control process is described by the Goursat problem with an arbitrary number of independent variables[18]. Nevertheless, to avoid a cluttering of the formulas with irrelevant details, we assume that the number of independent variables is equal to three.

Thus let the functions

$$z_i(x), \quad x = (x_1, x_2, x_3), \quad i = 1, \ldots, m,$$

be specified through the relations

$$\frac{\partial^3 z_i}{\partial x_1 \partial x_2 \partial x_3} = f_i\left(x, z_1, \ldots, z_m, \frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial z_m}{\partial x_3}, \ldots, \frac{\partial^2 z_m}{\partial x_2 \partial x_3}, v\right), \tag{2.15}$$

$$i = 1, \ldots, m, \quad 0 \leqslant x_k \leqslant X_k, \quad k = 1, 2, 3,$$

and the additional conditions

$$z_i(0, x_2, x_3) = \varphi_i^1(x_2, x_3), \quad z_i(x_1, 0, x_3) = \varphi_i^2(x_1, x_3),$$
$$z_i(x_1, x_2, 0) = \varphi_i^3(x_1, x_2), \tag{2.16}$$

where the functions f_i contain mixed derivatives of the z_j variables of an order not exceeding two. These functions are twice continuously differentiable over the totality of all arguments. The control$_k$ parameter obeys the same conditions as before. The functions φ_i^k are twice, sectionally, continuously, differentiable over their arguments and satisfy the natural matching conditions. Like in the previous cases we will assume that each permissible control has an associated class of functions within which the Goursat boundary problem can be uniquely solved.

As the criterion of optimality we chose the functional

$$S = \sum_{i=1}^{m} A_i z_i(X_1, X_2, X_3), \tag{2.17}$$

We introduce auxiliary variables u_i and the function

$$H(x, w, v) = \sum_{i=1}^{m} u_i f_i,$$

where

$$w = \left(u_1, \ldots, u_m, z_1, \ldots, z_m, \frac{\partial z_1}{\partial x_1}, \ldots, \frac{\partial z_m}{\partial x_3}, \ldots, \frac{\partial^2 z_m}{\partial x_2 \partial x_3}\right)$$

is a vector with a number of components equal to N. The function $u_i(x)$ is defined by means of the equations

$$\frac{\partial^3 u_i}{\partial x_1 \partial x_2 \partial x_3} = \frac{\partial H}{\partial z_i} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial H}{\partial z_{ix_k}}\right) + \frac{1}{2} \sum_{j \neq k}^{3} \frac{\partial^2}{\partial x_j \partial x_k}\left(\frac{\partial H}{\partial z_{ix_j x_k}}\right), \tag{2.18}$$

$i = 1, \ldots, m,$

and the auxiliary conditions

$$\begin{aligned}
\frac{\partial^2 u_i}{\partial x_1 \partial x_2} &= -\frac{\partial H}{\partial z_{ix_2}} + \frac{\partial}{\partial x_1}\left(\frac{\partial H}{\partial z_{ix_1 x_2}}\right) + \frac{\partial H}{\partial x_2}\left(\frac{\partial H}{\partial z_{ix_2 x_3}}\right) \quad \text{for } x_3 = X_3, \\
\frac{\partial^2 u_i}{\partial x_1 \partial x_3} &= -\frac{\partial H}{\partial z_{ix_2}} + \frac{\partial}{\partial x_1}\left(\frac{\partial H}{\partial z_{ix_1 x_2}}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial H}{\partial z_{ix_2 x_3}}\right) \quad \text{for } x_2 = X_2, \\
\frac{\partial^2 u_i}{\partial x_2 \partial x_3} &= -\frac{\partial H}{\partial z_{ix_1}} + \frac{\partial}{\partial x_2}\left(\frac{\partial H}{\partial z_{ix_1 x_2}}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial H}{\partial z_{ix_1 x_3}}\right) \quad \text{for } x_1 = X_1,
\end{aligned} \tag{2.19}$$

$$\left.\begin{aligned}
\frac{\partial u_i}{\partial x_1} &= \frac{\partial H}{\partial z_{ix_3 x_2}} \quad \text{for } x_2 = X_2, \quad x_3 = X_3; \\
\frac{\partial u_i}{\partial x_2} &= \frac{\partial H}{\partial z_{ix_1 x_3}} \quad \text{for } x_3 = X_3, \quad x_1 = X_1, \\
\frac{\partial u_i}{\partial x_3} &= \frac{\partial H}{\partial z_{ix_1 x_2}} \quad \text{for } x_1 = X_1, \quad x_2 = X_2,
\end{aligned}\right\} \tag{2.20}$$

$$u_i(X_1, X_2, X_3) = -A_i, \quad i = 1, \ldots, m. \tag{2.21}$$

Equations (2.20) are ordinary differential equations. Consequently, for each permissible control, they determine together with the conditions of (2.21) uniquely the functions $u_i(x_1, X_2, X_3)$, $u_i(X_1, x_2, X_3)$, and $u_i(X_1, X_2, x_3)$. We now solve Equations (2.19) with the additional conditions

$$\left.\begin{aligned}
u_i(x_1, x_2, X_3)\big|_{x_1 = X_1} &= u_i(X_1, x_2, X_3), \\
u_i(x_1, x_2, X_3)\big|_{x_2 = X_2} &= u_i(x_1, X_2, X_3),
\end{aligned}\right\} \quad \text{for } x_3 = X_3;$$

$$\left.\begin{aligned}
u_i(x_1, X_2, x_3)\big|_{x_1 = X_1} &= u_i(X_1, X_2, x_3), \\
u_i(x_1, X_2, x_3)\big|_{x_3 = X_3} &= u_i(x_1, X_2, X_3),
\end{aligned}\right\} \quad \text{for } x_2 = X_2;$$

$$\left.\begin{aligned}
u_i(X_1, x_2, x_3)\big|_{x_2 = X_2} &= u_i(X_1, X_2, x_3), \\
u_i(X_1, x_2, x_3)\big|_{x_3 = X_3} &= u_i(X_1, x_2, X_3),
\end{aligned}\right\} \quad \text{for } x_1 = X_1.$$

Because of the assumptions made above, the functions $u_i(x_1, x_2, X_3)$, $u_i(x_1, X_2, x_3)$, and $u_i(X_1, x_2, x_3)$ are uniquely determined. In this manner, the problem is in the last resort reduced to the Goursat problem: to find the solution of the system of Equations (2. 18) within the domain $0 \leq x_k \leq X_k$, satisfying the boundary conditions

$$\begin{aligned} &u_i(x_1, x_2, x_3)\,\big|_{x_2 = X_2} = u_i(x_1, X_2, x_3)\,, \\ &u_i(x_1, x_2, x_3)\,\big|_{x_3 = X_3} = u_i(x_1, x_2, X_3)\,, \qquad (2.\ 22) \\ &u_i(x_1, x_2, x_3)\,\big|_{x_1 = X_1} = u_i(X_1, x_2, x_3)\,, \quad i = 1, 2, \ldots, m. \end{aligned}$$

At the same time, one must keep in mind that the functions $\partial H/\partial w_k$ within the Equations (2. 18) and (2. 19) are differentiated over x_1, x_2, and x_3. This means that if one assumes that the class of permissible controls consists of sectionally-continuous functions, the following conditions must necessarily be satisfied: the right-hand side of these equations should not depend on the derivative functions v and on $z_{x_1x_1}$, $z_{x_2x_2}$, and $z_{x_3x_3}$. If the right-hand side of Equations (2. 18) and (2. 19) do depend on these quantities, then one must choose the functions $v(x)$ having sectionally continuous derivatives for the class of permissible controls.

Assuming that these conditions are satisfied, one can obtain the formula for the increment of the functional

$$\Delta S = -\int_0^{X_1}\int_0^{X_2}\int_0^{X_3} \Big| H(x, w, v + \Delta v) - H(x, w, v)\Big|\, dx_3 dx_2 dx_1 - \eta\ ,$$

using the same method as presented above, with

$$\eta = \eta_1 + \eta_2\ ,$$

$$\eta_1 = \frac{1}{2}\sum_{i=1}^{N}\int_0^{X_1}\int_0^{X_2}\int_0^{X_3}\left[\frac{\partial H(x, w, v + \Delta v)}{\partial w_i} - \frac{\partial H(x, w, v)}{\partial w_i}\right]\Delta w_i dx_3 dx_2 dx_1\ ,$$

$$\eta_2 = \frac{1}{2}\sum_{i,k=1}^{N}\int_0^{X_1}\int_0^{X_2}\int_0^{X_3}\left[\frac{\partial^2 H(x, w + \theta_1\Delta w,\ v + \Delta v)}{\partial w_i \partial w_k}\right.$$

$$\left. - \frac{|\partial^2 H(x, w + \theta_2\Delta w,\ v + \Delta v)}{\partial w_i \partial w_k}\right]\Delta w_i\, \Delta w_k\, dx_3\, dx_2\, dx_1\ .$$

From this formula one can derive the optimality conditions which may be formulated in the form of Theorems 1 and 2. If one assumes that the control is carried out by means of boundary conditions, one can obtain results which are analogous to Theorems 3 and 4.

4. The Control of a Process by Means of "Concentrated Controls"

We assumed, in all the problems investigated above, that all the components of the vector $v(x, y)$ are functions of two variables: x and y. However, the proposed method permits the solution of a problem where all the permissible controls $v(x, y)$ may be presented in the form

$$v(x, y) = (v^1(x),\ v^2(x, y),\ v^3(y))$$

(some components of this vector are functions of only a single independent variable x or y).

For definiteness, we investigate the problem of the minimization of the functional (1. 3) when the process is described by the boundary condition (1. 1)-(1. 2). The formula for the increment of the functional (1. 21) remains valid even in this case. Valid is also the estimate (1. 29) of the residual term of this formula. Consequently, the same method proves also

THEOREM 1'. For the permissible control $v(x, y) = (v^1(x), v^2(x, y), v^3(y))$ in the boundary problem (1. 1)-(1. 2) to be min-optimal according to the functional (1. 3), it is necessary that the condition

$$\iint \left[H(x, y, p(x, y), v(x, y) + \Delta v) - H(x, y, p(x, y), v(x, y) \right] dx\,dy \leq 0$$

be satisfied for an arbitrary permissible increment Δv, where $p(x, y)$ is a vector corresponding to the control $v(x, y)$ and which is fixed by Equations (1. 1), (1. 5), and the additional conditions (1. 2) and (1. 6).

If in particular the permissible control depends only on a single variable (for instance, on x) and in Equations (1. 1) we have

$$f_i(x, y, z, z_x, z_y, v) \equiv f_i^0(x, y, z, z_x, z_y) + f_i^1(x, v),$$

then the conditions (2. 23) take the form

$$\iint\limits_G \sum_{i=1}^{m} u_i(x, y)\left| f_i^1(x, v(x) + \Delta v) - f_i^1(x, v)\right| dx\,dy \leqslant 0 .$$

By introducing the notation

$$H^1(x, u(x), v) = \sum_{i=1}^{m} f_i^1(x, v) \int_0^Y u_i(x, y)\, dy ,$$

we obtain the optimality condition in the following form.

THEOREM 1''. For the permissible control $v(x)$ of the boundary problem (1. 1)-(1. 2) to be min-optimal (among controls depending only on x) according to the functional (1. 3), it is necessary that

$$H^1(x, u(x), v(x)) \;(=)\; \sup_{v \in V} H^1(x, u(x), v) ,$$

where the symbol (=) indicates equality valid for almost all x's of the segment $0 \leqslant x \leqslant X$.

Section III. THE VARIATIONAL CALCULUS AND THE PROBLEMS OF OPTIMUM CONTROL

The problems investigated in the present paper represent in essence problems of variational calculus. However, the classical methods can not be applied here since the control parameters may, in general, take values from a closed domain. In the case when the region of variation of the control parameters is open, one obtains from the principle of maximum the necessary conditions of the classical variational calculus for functionals with partial derivatives.

Let us search for the minimum of the functional

$$I = \int_0^X \int_0^Y f(x,\ y,\ z,\ z_x,\ z_y,\ v)\, dy\, dx,$$

which is defined over the functions $z = (z_1, \ldots, z_m)$ specified by the relationships

$$z_{ixy}(x,\ y) = v_i, \quad v = (v_1, \ldots, v_m), \quad z_i(0,\ y) = \varphi_i(y),$$

$$z_i(x,\ 0) = \psi_1(x), \quad i = 1, \ldots, m,$$

where the control parameter v is chosen from the class of all sectionally continuous vector functions.

Under the optimum control we understand the permissible controls found within the immediate vicinity of the function $z(x,\ y)$ corresponding to such a control. It is obvious that such a definition of the optimum control is a special case of the optimum control in the previous sense. Consequently, the principle of maximum remains valid and every optimum solution is also an extremal solution. The opposite is also valid: each extremal constitutes an optimum solution.

For the calculation of such a solution we introduce an auxiliary variable z_0:

$$z_{0xy} = f(x,\ y,\ z,\ z_x,\ z_y,\ v), \quad z_0(x,\ 0) = z_0(0,\ y) = 0,$$

and establish the function H:

$$H = u_0 f + \sum u_p v_p.$$

Then the auxiliary functions $u_i(x, y)$ are defined by means of the boundary problem

$$\left.\begin{array}{l} u_{ixy} = \dfrac{\partial f}{\partial z_i} u_0 - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial z_{ix}} u_0\right) - \dfrac{d}{dy}\left(\dfrac{\partial f}{\partial z_{iy}} u_0\right), \\ u_{ix}(x, Y) = -\left[\dfrac{\partial f}{\partial z_{iy}} u_0\right]_{y = Y}, \quad u_{iy}(X, y) = -\left[\dfrac{\partial f}{\partial z_{ix}} u_0\right]_{x = X}, \\ u_i(X, Y) = 0, \quad i = 1, \ldots, m, \quad u_0(x, y) = -1. \end{array}\right\} \tag{3.1}$$

From this we find that

$$H = \sum u_p v_p - f.$$

Since the function H reaches its maximum on the optimum control $v(x, y)$, we have

$$\left(\frac{\partial H}{\partial v_i}\right)_{v = v(x, y)} = \left(u_i - \frac{\partial f}{\partial v_i}\right)_{v = v(x, y)} = 0.$$

Consequently,

$$u_{ixy} = \frac{d^2}{dx\,dy}\left(\frac{\partial f}{\partial z_{ixy}}\right)$$

and because of Equation (3.1), we find that the solution $z(x, y)$ of the optimum problem under consideration satisfies the system of equations by Ostrogradskiy-Euler (see, for instance, paper by Akhiyezer[32] p. 122):

$$\frac{\partial f}{\partial z_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial z_{ix}}\right) - \frac{d}{dy}\left(\frac{\partial f}{\partial z_{iy}}\right) + \frac{d^2}{dx\,dy}\left(\frac{\partial f}{\partial z_{ixy}}\right) = 0.$$

As assumed earlier, the function f has a second continuous derivative with respect to the variables $v_1, \ldots, v_m$. Since the control $v(x, y)$ realizes the maximum of the function H, the quadratic form

$$\sum_{i, k = 1}^{m} \frac{\partial^2 H}{\partial v_i \partial v_k} \lambda_i \lambda_k = - \sum_{i, k = 1}^{m} \frac{\partial^2 f}{\partial v_i \partial v_k} \lambda_i \lambda_k$$

is nonpositive. Consequently, from the condition of the maximum (1.7), it follows that everywhere within the domain $G(0 \leq x \leq X,\ 0 \leq y \leq Y)$, except, perhaps, in points located on a finite number of lines with zero area, the inequality (Legendre condition)

$$\sum_{i,k=1}^{m} \frac{\partial^2 f(x, y, z, z_x, z_y, z_{xy})}{\partial v_i \partial v_k} \lambda_i \lambda_k \geqslant 0, \quad \sum_{i=1}^{m} \lambda_i^2 \neq 0 , \qquad (3.2)$$

representing the necessary condition for the function $z(x, y)$ to be extremal minimizing the functional I, is satisfied.

In the case when the domain of variation of the control parameter is closed, the derivatives $\partial H/\partial v_i$ can not become zero along the optimum trajectory $z(x, y)$ and, consequently, the condition (3.2) may even not be satisfied. As a confirmation of what was just said we investigate the simplest case.

Let the control process be described by the boundary problem

$$z_{xy} = v^2, \quad z(x, 0) = z(0, y) = 0, \quad 0 \leq x, y \leq 1,$$

where v - the control parameter, $|v| \leq 1$. As a criterion of optimality we use the functional

$$S = -\int_0^1\int_0^1 z_{xy}\, dx\, dy = - z(1, 1) \quad (f(x, y, z, z_x, z_y, v) \equiv - v^2).$$

It is easy to show that the min-optimal control according to S is $v(x, y) \equiv 1$, and consequently, during this control

$$\frac{\partial^2 f}{\partial v^2} < 0,$$

and the condition (3.2) is not satisfied.

Section IV. OPTIMUM PROCESSES IN SYSTEMS WHOSE BEHAVIOR IS DESCRIBED BY PARABOLIC EQUATIONS

1. The Formulation of the Problem. The Maximum Principle

Let E^n be a euclidian space of vectors $x = (x_1, \ldots, x_n)$ and G - a bounded region within E^n with a boundary Γ belonging to the class $A^{(2)}$ (see work of Miranda[33], p. 10), and $X_i(x)$ are the direction cosines of the external normal on the boundary Γ.

Let, furthermore, an elliptical operator $L = (L_1, \ldots, L_m)$ be defined within the domain G by the formula

$$L_i y = \sum_{p=1}^{m} \sum_{j,k=1}^{n} a_{jk}^{ip} \frac{\partial^2 y_p}{\partial x_j \partial x_k}, \tag{4.1}$$

where the functions $a_{jk}^{ip}(x_1, \ldots, x_n)$ within the domain $G + \Gamma$ belong to the class $C^{(2)}$. We denote by $M = (M_1, \ldots, M_m)$ the operator defined by the formula

$$M_i z = \sum_{p=1}^{m} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}^{pi} \frac{\partial z_p}{\partial x_k}\right) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(l_j^{pi} z_p\right), \quad i = 1, \ldots, m,$$

where

$$l_j^{pi} = -\sum_{k=1}^{n} \frac{\partial a_{jk}^{pi}}{\partial x_k}.$$

One can check directly that the equation

$$\sum_{i=1}^{m} \int_G (z_i L_i y - y_i M_i z)\, dx = \sum_{i,p=1}^{m} \sum_{j=1}^{n} \int_\Gamma \left[\sum_{k=1}^{n} a_{jk}^{ip}\left(z_i \frac{\partial y_p}{\partial x_k} - y_p \frac{\partial z_i}{\partial x_k}\right) + l_j^{ip} y_p z_i\right] X_j(x)\, d\sigma$$

is valid. In the same manner, used before for an elliptical type equation, we can transform this formula into the form

$$\sum_{i=1}^{m} \int_G (z_i L_i y - y_i M_i z)\, dx = \sum_{i=1}^{m} \int_\Gamma (z_i P_i y - y_i Q_i z)\, d\sigma \quad , \tag{4.2}$$

where

$$P_i y = \sum_{p=1}^{m} \left[a_l^{ip} \frac{dy_p}{dl_{ip}} + b_{ip} y_p \right], \quad Q_i z = \sum_{p=1}^{m} \left[a_\lambda^{pi} \frac{dz_p}{d\lambda_{ip}} + d_{ip} z_p \right]. \tag{4.3}$$

In formulas (4. 3) the directions l_{ip} are chosen arbitrarily provided $\cos(n, l_{ip}) > 0$ (n - external normal to Γ) and their direction cosines belong to the class $C^{(1)}$ on Γ. The directions λ_{ip} are chosen depending on the l_{ip}.

Let us assume that the coefficients within the operator L depend, in addition, on the variable t, $0 \leq t \leq T$, and we investigate the control systems whose behavior is described by a system of equations of the parabolic type

$$L_t y = f(t, x, y, y_x, u), \quad 0 \leq t \leq T, \quad x \in G \quad \left(L_{it} y = \frac{\partial y_i}{\partial t} - L_i y \right), \tag{4.4}$$

where the function $f = (f_1, \ldots, f_m)$ is continuous with respect to t and is twice continuously differentiable over the other arguments, while the parameter u takes the values within a certain convex (open or closed) region U of the p-dimentional euclidian space.

We assume further that the function $y(t, x) = (y_1, \ldots, y_m)$ is defined by the system of equations (4. 4) satisfying, in addition, the following conditions

$$\left. \begin{aligned} P_i(t, x) y &= \varphi_i(t, x, y, v), \quad x \in \Gamma, \quad 0 \leq t \leq T, \\ y(0, x) &= a(x), \quad x \in G, \end{aligned} \right\} \tag{4.5}$$

where the operators P_i are defined by the formulas (4. 3) in which the functions $a_l^{is}(t, x)$, $b_{ip}(t, x)$, and $a(x)$ are continuous, φ_i satisfy the same conditions as f_i, while the parameter v takes its value from a convex (open or closed) region V of a q-dimensional euclidian space.

The function $\omega(t, x) = (u(t, x), v(t, x))$ will be called the permissible control if all its components are sectionally continuous and $u(t, x)$ and $v(t, x)$ take values from the domains U and V, respectively.

In addition, we will assume that the discontinuity surface of the permissible control [Translator's note: one line seems to be missing.] ... or in the vicinity of its arbitrary point one can introduce a non-degenerate coordinate transformation

$$\tau = t, \quad \xi_i = \xi_i(t, x), \quad i = 1, \dots, n,$$

since the discontinuity surface goes over into a portion of the plane $\xi_n = 0$.

If the discontinuities of a certain permissible control satisfy the first condition, then the boundary probalem (4. 4)-(4. 5) corresponding to this control splits into several such problems located, however, in regions adjoining one another along the discontinuity surfaces of the control. In such a case the problem (4. 4)-(4. 5) has a unique continuous solution (see, for instance, work by Zagoriskiy [34]), and this solution is not subjected to any further additional smoothness condition over the discontinuity surfaces of the control.

If these surfaces satisfy the second condition, then one views as the solution of the problem (4.4)-(4.5) the vector function $y(t, x)$ satisfying the system of equations (4. 4), the conditions (4. 5), and certain smoothness conditions over the discontinuity surfaces of the control. In its most general form, this problem apparently was never studied although its particular cases were investigated in numerous papers[35, 36, 37, 38, 39, 40] which supply various existence and uniqueness theorems concerning the solutions. Consequently, we will assume everywhere in what follows that the given functions in (4. 4) and conditions (4. 5), in addition to the above listed properties, satisfy also the conditions under which to each permissible control there corresponds a unique solution of the problem (4. 4)-(4. 5).

Let $\omega(t, x)$ be a certain permissible control and $y(t, x)$ the corresponding solution of the problem (4. 4)-(4. 5), and let be given the functional

$$S = \sum_{i=1}^{m} \left[\int_G \alpha_i(x) y_i(T, x)\,dx + \int_0^T \int_G \beta_i(t, x) y_i(t, x)\,dx\,dt + \int_0^T \int_\Gamma \gamma_i(t, x) y_i(t, x)\,d\sigma\,dt \right], \tag{4. 6}$$

where α_i, β_i, and γ_i are given continuous functions.

Let us formulate the problem: among all the permissible controls one should find a control $\omega(t, x)$ (if it exists) such that the corresponding solution of the problem (4. 4)-(4. 5) realizes the minimum of the Functional S.

The permissible control $\omega(t, x)$ over which the functional S attains its maximum (minimum) value will be called the max-optimal (min-optimal) according to S. The functionals of a more general type will be studied at the end of the paragraph.

As it was mentioned above, the problem of optimum control processes described by parabolic equations are of definite theoretical and practical interest. Numerous papers[5,6,11] investigated certain problems for which the control is materialized by means of initial and boundary conditions and for the optimality criterion one chooses either the speed or the functional of the type

$$I = \int_0^1 \left|u(T, x) - u_0(x)\right|^2 dx + \gamma \int_0^T p^2(t)\, dt,$$

where $u_0(x)$ is a given function from $L_2(0, 1)$, $p(t)$ - control, and γ - a non-negative constant. We already investigated this problem for the case when the control of the process could be materialized simultaneously by means of controls entering into the equations as well as into the boundary conditions. It is clear that the functional

$$S_1 = \sum_{i=1}^{m} \int_0^T \int_G \left[\gamma_i(t, x)\, y_i(t, x) + \sum_{k=1}^{n} \alpha_{ik}(t, x) \frac{\partial y_i}{\partial x_k} + \beta_i(t, x) \frac{\partial y_i}{\partial t} \right] dx\, dt,$$

with α_{ik} and β_i - continuously differentiable function can be reduced to the form (4. 6).

To formulate the condition of optimality we introduce the auxiliary function $z(t, x) = (z_1, \ldots, z_m)$ by means of the boundary problem "adjoint" to (4. 4)-(4. 5):

$$M_{it} z = -\sum_{s=1}^{m} \left[\frac{\partial f_s(t, x, y, y_x, u)}{\partial y_t} z_s - \sum_{k=1}^{n} \frac{d}{dx_k} \left(\frac{\partial f_s(t, x, y, y_x, u)}{\partial y_{ix_k}} - z_s \right) \right]$$

$$+ \beta_i(t, x), \quad x \in G,$$

$$Q_i(t,x)z = \sum_{s=1}^{m}\left[\frac{\partial\varphi_s(t,x,y,v)}{\partial y_i} + \sum_{k=1}^{n}\frac{\partial f_s(t,x,y,y_x,v)}{\partial y_{ix_k}} X_k(x)\right] z_s \tag{4.8}$$

$$-\gamma_i(t,x), \quad x \in \Gamma, \quad z_i(T,x) = -\alpha_i(x), \quad x \in G, \quad i = 1, \ldots, m,$$

where

$$M_{it}z = \frac{\partial z_i}{\partial t} + M_i z,$$

are operators Q_i defined by the relationships (4. 3), the functions α_i, β_i, and γ_i enter into the definition of the functional S, and $X_k(x)$ - the directional cosines of the external normal to G at the boundary Γ. For the boundary problem (4. 7)-(4. 8) to be solvable it is necessary that the functions α_i and γ_i be connected with the matching conditions. In what follows we assume that these conditions are fulfilled.

We introduce the notation

$$w = \left(z_1, \ldots, z_m, y_1, \ldots, y_m, \frac{\partial y_1}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_n}\right),$$

$$p = (z_1, \ldots, z_m, y_1, \ldots, y_m), \quad H(t,x,w,u) = \sum_{i=1}^{m} z_i f_i(t,x,y,y_x,u),$$

$$h(t,x,p,v) = \sum_{i=1}^{m} z_i \varphi_i(t,x,y,v).$$

Then the boundary problems (4. 4)-(4. 5) and (4. 7)-(4. 8) may be written in the form

$$L_{it}y = \frac{\partial H(t,x,w,u)}{\partial z_i}, \quad y_i(0,x) = \alpha_i(x), \quad x \in G, \quad P_i y = \frac{\partial h(t,x,p,v)}{\partial z_i}, \tag{4.9}$$

$$x \in \Gamma,$$

$$\left.\begin{aligned} M_{it}z &= -\frac{\partial H(t,x,w,u)}{\partial y_i} + \sum_{k=1}^{n}\frac{d}{dx_k}\frac{\partial H(t,x,w,u)}{\partial y_{ix_k}} + \beta_i(t,x), \\ z_i(T,x) &= -\alpha_i(x), \quad Q_i z = \frac{\partial h(t,x,p,v)}{\partial y_i} + \sum_{k=1}^{n}\frac{\partial H(t,x,w,u)}{\partial y_{ix_k}} X_k(x) \\ &\qquad - \gamma_i(t,x), \quad x \in \Gamma. \end{aligned}\right\} \tag{4.10}$$

Using the formula (4.2) one can easily establish for arbitrary twice sectionally, continuously differentiable functions $y_i(t, x)$ and $z_i(t, x)$ that the Ostrogradskiy-Green formula

$$\sum_{i=1}^{m} \int_0^T \int_G (z_i L_{it} y + y_i M_{it} z)\, dx\, dt = -\sum_{i=1}^{m} \left[\int_0^T \int_\Gamma (z_i P_i y - y_i Q_i z)\, d\sigma\, dt - \int_G y_i z_i \Big|_{t=0}^{T} dx \right]. \qquad (4.11)$$

is valid. Let $\omega(r, x) = (u(t, x),\ v(t, x))$ be a certain permissible control and $y(t, x)$ and $z(t, x)$ be the corresponding solution of the boundary problems (4.9) and (4.10). We will say that the permissible control $\omega(t, x)$ satisfies the maximum condition if

$$H(t, x, w(t, x),\ u(t, x))\ ((=))\ \sup_{u \in U} H(t, x, w(t, x), u), \quad x \in G, \quad 0 \leq t \leq T,$$

$$h(t, x, p(t, x),\ v(t, x))\ (=)\ \sup_{v \in V} h(t, x, p(t, x), v), \quad x \in \Gamma, \quad 0 \leq t \leq T.$$

where the symbol ((=)) indicates an equality which is valid everywhere within the domain C $(0 \leq t \leq T,\ x \in G)$ with the exception that there may exist points located on a finite number of n-dimensional surfaces, the $(n+1)$-dimensional volume of which is equal to zero. The symbol (=) is defined in an analogous way except that instead of n and G one should take n-1 and Γ, respectively. The condition of minimum is defined in an analogous manner.

THEOREM 7 (the maximum principle). <u>For the permissible control $\omega(t, x) = (u(t, x),\ v(t, x))$ to be min-optimal (max-optimal) according to S, it is necessary that it satisfies the condition of maximum (minimum).</u>

This theorem, although it does not supply the sufficient conditions of optimality, may serve as a practical instrument for the calculation of the optimum controls and the corresponding solution of the boundary problem (4.4)-(4.5). One can become convinced in this fact by repeating the deliberations carried out in Section I.

2. The Formula for the Increment of the Functional S. The Proof of Theorem 7

Let (t, x) be an arbitrary permissible control and $y(t, x)$ and $z(t, x)$ the corresponding solution of the boundary problems (4.9) and (4.10). Then

$$I = \int_C \left[\sum_{i=1}^{m} z_i L_{it} y - H(t, x, w, u(t, x)) \right] dx\, dt$$

$$+ \int_\sigma \left[\sum_{i=1}^{m} z_i P_i y - h(t, x, p, v(t, x)) \right] d\sigma = 0,$$

where

$$C = (0 \leq t \leq T,\ x \in G), \quad \sigma = (0 \leq t \leq T,\ x \in \Gamma).$$

We take a certain permissible increment $\Delta\omega = (\Delta u, \Delta v)$ of the control $\omega(t, x)$ and denote by $y + \Delta y$ and $z + \Delta z$ the solutions of the same problems (4.9) and (4.10) but corresponding to the control $\omega + \Delta\omega$. Then

$$\Delta I = I\left[w + \Delta w,\ \omega + \Delta\omega\right] - I(w, \omega) = \int_C \left\{ \sum_{i=1}^{m} (\Delta z_i L_{it} \Delta y + \Delta z_i L_{it} y + z_i L_{it} \Delta y) \right.$$

$$\left. - \left| H(t, x, w + \Delta w,\ u + \Delta u) - H(t, x, w, u) \right| \right\} dx\, dt + \int_\sigma \left\{ \sum_{i=1}^{m} (\Delta z_i P_i \Delta y \right. \quad (4.12)$$

$$\left. + \Delta z_i P_i y + z_i P_i \Delta y) - \left| h(t, x, w + \Delta w, v + \Delta v) - h(t, x, w, v) \right| \right\} d\sigma = 0,$$

while functions Δy_i and z_i, $i = 1, \ldots, m$ form the solution for the respective boundary problems

$$\left. \begin{aligned} & L_{it} \Delta y = \Delta \frac{\partial H(t, x, w, u)}{\partial z_i}, \quad \Delta y_i(0, x) = 0, \quad x \in G, \\ & P_i \Delta y = \Delta \frac{\partial h(t, x, p, v)}{\partial z_i}, \quad x \in \Gamma, \end{aligned} \right\} \quad (4.13)$$

$$\left. \begin{aligned} & M_{it} \Delta z = -\Delta \frac{\partial H(t, x, w, u)}{\partial y_i} + \sum_{k=1}^{n} \frac{d}{dx_k} \left(\Delta \frac{\partial H(t, x, w, u)}{\partial y_{ix_k}} \right), \\ & \Delta z_i(T, x) = 0, \ x \in G, \quad Q_{it} \Delta z = \Delta \frac{\partial h(t, x, p, v)}{\partial y_i} \\ & \qquad + \sum_{k=1}^{n} \Delta \frac{\partial H(t, x, w, u)}{\partial y_{ix_k}} X_k(x), \quad x \in \Gamma, \end{aligned} \right\} \quad (4.14)$$

where

$$\Delta \frac{\partial H}{\partial w_k} = \frac{\partial H(t, x, w + \Delta w, u + \Delta u)}{\partial w_k} - \frac{\partial H(t, x, w, u)}{\partial w_k},$$

$$\Delta \frac{\partial h}{\partial p_k} = \frac{\partial h(t, x, p + \Delta p, v + \Delta v)}{\partial p_k} - \frac{\partial h(t, x, p, v)}{\partial p_k}.$$

Equality (4. 12) is now transformed by means of the formula (4. 11). Since the functions Δy and Δz are the solutions of the respective boundary problems (4. 13) and (4. 14), we have

$$\sum_{i=1}^{m} \left[\int_C \Delta z_i L_{it} \Delta y \, dx \, dt + \int_\sigma \Delta z_i P_i \Delta y \, d\sigma\right] = \sum_{i=1}^{m} \left\{\int_C \left[\Delta \frac{\partial H(t, x, w, u)}{\partial y_i} \Delta y_i\right.\right.$$

$$\left. - \sum_{k=1}^{n} \frac{d}{dx_k}\left(\Delta \frac{\partial H(t, x, w, u)}{\partial y_i}\right) \Delta y_i\right] dx \, dt$$

$$+ \int_\sigma \left[\Delta \frac{\partial h(t, x, p, v)}{\partial y_i}\right.$$

$$\left.\left. + \sum_{k=1}^{n} \Delta \frac{\partial H(t, x, w, v)}{\partial y_{ix_k}} X_k(x)\right] \Delta y_i d\sigma\right\}$$

$$= \sum_{i=1}^{m} \left\{\int_C \left[\Delta \frac{\partial H(t, x, w, u)}{\partial y_i} \Delta y_i\right.\right.$$

$$\left. + \sum_{k=1}^{n} \Delta \frac{\partial H(t, x, w, u)}{\partial y_{ix_k}} \Delta y_{ix_k}\right] dx \, dt$$

$$\left. + \int_\sigma \Delta \frac{\partial h(t, x, p, v)}{\partial y_i} \Delta y_i \, d\sigma\right\}.$$

On the other hand

$$\sum_{i=1}^{m}\left[\int_C \Delta z_i L_{it}\Delta y\, dx\, dt + \int_\sigma \Delta z_i P_i \Delta y\, d\sigma\right] = \sum_{i=1}^{m}\left[\int_C \Delta \frac{\partial H}{\partial z_i}\Delta z_i dx\, dt + \int_\sigma \Delta \frac{\partial h}{\partial z_i}\Delta z_i\, d\sigma\right].$$

Consequently,

$$\sum_{i=1}^{m}\left[\int_C \Delta z_i L_{it}\Delta y\, dx\, dt + \int_\sigma \Delta z_i P_i \Delta y\, d\sigma\right] = \frac{1}{2}\left[\sum_{i=1}^{N}\int_C \Delta \frac{\partial H}{\partial w_i}\Delta w_i\, dx\, dt + \sum_{i=1}^{2m}\int_\sigma \Delta \frac{\partial h}{\partial p_i}\Delta p_i\, d\sigma\right], \tag{4.15}$$

where $N = 2m + nm$ - the dimensionality of the vector w.

In an analogous manner we find

$$\sum_{i=1}^{m}\left[\int_C \Delta z_i L_{it} y\, dx\, dt + \int_\sigma \Delta z_i P_i y\, d\sigma\right] = \sum_{i=1}^{m}\left[\int_C \frac{\partial H}{\partial z_i}\Delta z_i dx\, dt + \int_\sigma \left.\frac{\partial h}{\partial z_i}\right| \Delta z_i d\sigma\right], \tag{4.16}$$

$$\sum_{i=1}^{m}\left[\int_C z_i L_{it}\Delta y\, dx\, dt + \int_\sigma z_i P_i \Delta y\, d\sigma\right] = -\sum_{i=1}^{m}\left[\int_G \alpha_i(x)\Delta y_i(T, x)\, dx + \int_C \beta_i(t, x)\Delta y_i(t, x)\, dx\, dt + \int_\sigma \gamma_i(t, x)\Delta y_i(t, x)\, d\sigma\right]$$
$$+\sum_{i=1}^{m}\left[\int_C \left(\frac{\partial H}{\partial y_i}\Delta y_i + \sum_{k=1}^{n}\frac{\partial H}{\partial y_{ix_k}}\Delta y_{ix_k}\right) dx\, dt + \int_\sigma \frac{\partial h}{\partial y_i}\Delta y_i\, d\sigma\right]. \tag{4.17}$$

The first sum on the right-hand side of the equality (4.17) represents the increment ΔS of the functional (4.6) during the transition from the control $\omega(t,x)$ to the control $\omega(t,x)+\Delta\omega$. Consequently, from the relationships (4.12), (4.15), (4.16), and (4.17) it follows that

$$\Delta S = -\int_C \Bigg[H(t,x,w+\Delta w,\ u+\Delta u) - H(t,x,w,u)$$

$$-\sum_{i=1}^{N}\left(\frac{\partial H}{\partial w_i}+\frac{1}{2}\Delta\frac{\partial H}{\partial w_i}\right)\Delta w_i\Bigg]\,dx\,dt - \int_\sigma \Bigg[h(t,x,p+\Delta p,\ v+\Delta v)$$

$$-h(t,x,p,v) - \sum_{i=1}^{2m}\left(\frac{\partial h}{\partial p_i}+\frac{1}{2}\Delta\frac{\partial h}{\partial p_i}\right)\Delta p_i\Bigg]\,d\sigma\ .$$

Applying to the functions h, H, $\partial H/\partial w_i$ and $\partial h/\partial p_i$ the Taylor formula and keeping within the expansion only the terms of the second order, one obtains, as it was done in § 1,

$$\Delta S = -\int_C \Big| H(t,x,w,u+\Delta u) - H(t,x,w,u)\Big|\,dx\,dt$$
$$-\int_\sigma \Big| h(t,x,p,v+\Delta v) - h(t,x,p,v)\Big|\,d\sigma - \eta\,, \tag{4.18}$$

where $\eta = \eta_1 + \eta_2$,

$$\eta_1 = \frac{1}{2}\sum_{i=1}^{N}\int_C \left(\frac{\partial H(t,x,w,u+\Delta u)}{\partial w_i} - \frac{\partial H(t,x,w,u)}{\partial w_i}\right)\Delta w_i\,dx\,dt$$
$$+\frac{1}{2}\sum_{i=1}^{2m}\int_\sigma \left(\frac{\partial h(t,x,p,v+\Delta v)}{\partial p_i} - \frac{\partial h(t,x,p,v)}{\partial p_i}\right)\Delta p_i\,d\sigma\ , \tag{4.19}$$

$$\eta_2 = \frac{1}{2}\left\{ \sum_{i,k=1}^{N} \int_C \left[\frac{\partial^2 H(t, x, w + \theta_1 \Delta w, u + \Delta u)}{\partial w_i \partial w_k} - \frac{\partial^2 H(t, x, w + \theta_2 \Delta w, v + \Delta v)}{\partial w_i \partial w_k}\right] \Delta w_i \Delta w_k \, dx\, dt \right. \tag{4.19}$$

$$\left. + \sum_{i,k=1}^{2m} \int_\sigma \left[\frac{\partial^2 h(t, x, p + \theta_3 \Delta p, v + \Delta v)}{\partial p_i \partial p_k} - \frac{\partial^2 h(t, x, p + \theta_4 \Delta p, v + \Delta v)}{\partial p_i \partial p_k}\right] \Delta p_i \Delta p_k \, d\sigma \right. .$$

To get the necessary estimates of the residual term η in formula (4.18), we reduce the boundary problem to a system of integro-differential equations (see, for instance, work of Zagorskiy[34], pp. 90-96):

$$\Delta y(t, x) = \int_0^t \int_G K_{11}\,(t, x, \tau, \xi)\, \Delta \frac{\partial H}{\partial z}\, d\xi\, d\tau + \int_0^t \int_\Gamma K_{12}\,(t, x, \tau, \xi)\, \psi(\tau, \xi)\, d_\xi \sigma\, d\tau, \quad x \in G, \tag{4.20}$$

$$\psi(t, X) = -\Delta \frac{\partial h}{\partial z} + \int_0^t \int_G K_{21}(t, X, \tau, \xi)\, \Delta \frac{\partial H}{\partial z}\, d\xi\, d\tau + \int_0^t \int_\Gamma K_{22}\,(t, X, \tau, \xi) \psi(\tau, \xi)\, d_\xi \sigma\, d\tau\,, \quad X \in \Gamma\,, \tag{4.21}$$

where

$$\Delta \frac{\partial H}{\partial z} = \left(\Delta \frac{\partial H}{\partial z_1}, \ldots, \Delta \frac{\partial H}{\partial z_m}\right), \quad \Delta \frac{\partial h}{\partial z} = \left(\Delta \frac{\partial h}{\partial z_1}, \ldots, \Delta \frac{\partial h}{\partial z_m}\right),$$

K_{ik} - the Green type matrix. Inserting the value $\psi(t, X)$ defined in (4.21) in the right-hand side of the same relationship and repeating successively a few times the same operation we obtain

$$\psi(t,X) = -\Delta\frac{\partial h}{\partial z} + \int_0^t\int_G K_n(t,X,\tau,\eta)\,\Delta\frac{\partial H}{\partial z}\,d\eta\,d\tau$$

$$+\int_0^t\int_\Gamma K^n(t,X,\tau,\eta)\,\psi(\tau,\eta)\,d_\eta\sigma\,d\tau \tag{4.22}$$

$$-\int_0^t\int_\Gamma\sum_{i=0}^{n-1} K^i(t,X,\tau,\eta)\,\Delta\frac{\partial h}{\partial z}\,d_\eta\sigma\,d\tau,$$

where

$$K_n(t,X,\tau,\eta) = K_{n-1}(t,X,\tau,\eta) + \int_\tau^t\int_\Gamma K_{n-1}(t,X,\alpha,\beta)\,K_0(\alpha,\beta,\tau,\eta)\,d_\beta\sigma\,d\alpha,$$

$$K^n(t,X,\tau,\eta) = \int_\tau^t\int_\Gamma K^{n-1}(t,X,\alpha,\beta)\,K^0(\alpha,\beta,\tau,\eta)\,d_\beta\sigma\,d\alpha,$$

$$K_0 = K_{21}, \quad K^0 = K_{22}, \quad n = 1, 2, \ldots .$$

The number n is chosen so large that the kernel K^n becomes bounded. This may be done because of the known estimates of the Green matrix and of its derivatives (see, for instance, work of Zagorskiy[34], p. 92). Then from (4.22) we get

$$w(t) \le P\int_0^t w(\tau)\,d\tau + \int_0^t\int_G Q_n(t,\tau,\eta)\left|\Delta\frac{\partial H}{\partial z}\right|d\eta\,d\tau$$

$$+\int_0^t\int_\Gamma R_n(t,\tau,\eta)\left|\Delta\frac{\partial h}{\partial z}\right|d_\eta\sigma\,d\tau \tag{4.23}$$

$$+\int_\Gamma\left|\Delta\frac{\partial h}{\partial z}\right|d\sigma,$$

where P - a definite positive constant,

$$w(t) = \int_\Gamma\left|\psi(t,X)\right|d\sigma, \qquad Q_n = \int_\Gamma\left|K_n(t,X,\tau,\eta)\right|d_X\sigma,$$

$$R_n = \int_\Gamma\sum_{i=0}^{n-1}\left|K^i(t,X,\tau,\eta)\right|d_X\sigma .$$

We introduce the notation

$$w_k(t) = \int_0^t w_{k-1}(\tau)\,d\tau, \quad w_0(t) = w(t), \quad Q_{nk} = \int_\tau^t Q_{nk-1}(t,\tau,\eta)\,d\tau,$$

$$Q_{n0} = Q_n, \quad R_{nk} = \int_0^t R_{nk-1}(t,\tau,\eta)\,d\tau, \quad R_{n1} = R_n(t,\tau,\eta) + 1, \tag{4.24}$$

$k = 2, 3, \ldots$.

Integrating the inequality (4. 23) successively, we find:

$$\begin{aligned} w_k(t) \leq P\int_0^t w_k(\tau)\,d\tau &+ \int_0^t \int_G Q_{nk}(t,\tau,\eta) \left|\Delta \frac{\partial H}{\partial z}\right| d\eta\,d\tau \\ &+ \int_0^t \int_\Gamma R_{nk}(t,\tau,\eta) \left|\Delta \frac{\partial h}{\partial z}\right| d\sigma\,d\tau . \end{aligned} \tag{4.25}$$

Let us choose the number k so large that the functions Q_{nk} and R_{nk} be bounded for $0 \leq \tau \leq t \leq T$, $x \in G$ and put

$$Q(t) = \int_0^t \int_G \left|\Delta \frac{\partial H}{\partial z}\right| \max_{0 \leq \theta \leq t} Q_{nk}(\theta,\tau,\eta)\,d\eta\,d\tau,$$

$$R(t) = \int_0^t d\tau \int_\Gamma \left|\Delta \frac{\partial h}{\partial z}\right| \max_{0 \leq \theta \leq t} R_{nk}(\theta,\tau,\eta)\,d\sigma.$$

Then from the inequality (4. 25) for $0 \leq \theta \leq t$ we get

$$w_k(\theta) \leq P\int_0^\theta w_k(\theta)\,d\theta + Q(t) + R(t).$$

We obtain from this using the known lemma (see work of Nemytskiy and Stepanon[30], p. 19) that

$$w_k(\theta) \leq A\left|Q(t) + R(t)\right|$$

for $0 \leq \theta \leq t \leq T$ and, consequently,

$$w_k(t) \leq A\left|Q(t) + R(t)\right|,$$

where A is a definite positive constant. Thus it follows from the relationships (4. 23), (4. 24), and (4. 25) that

$$w_k(t) \leq \int_0^t d\tau \left[\int_G M_1(t,\tau,\eta) \left| \Delta \frac{\partial H}{\partial z} \right| d\eta + \int_\Gamma N_1(t,\tau,\eta) \left| \Delta \frac{\partial h}{\partial z} \right| d\sigma \right], \quad (4.26)$$

where M_1 and N_1 are functions of the same type as Q_n and R_n, respectively.

Since the number n was chosen sufficiently large, we find from the relationships (4. 22) and (4. 26) that

$$|\psi(t, X)| \leq \int_0^T d\tau \left[\int_G M_2(t, \tau, X, \eta) \left| \Delta \frac{\partial H}{\partial z} \right| d\eta + \int_\Gamma N_2(t, \tau, X, \eta) \left| \Delta \frac{\partial h}{\partial z} \right| d_\eta \sigma \right], \quad (4.27)$$

where M_2 and N_2 - the scalar functions of the Green type functions.

The functions $\partial H/\partial z$ and $\partial h/\partial z$ are continuous over t and twice continuously differentiable over other arguments. Consequently, for each permissible control (4. 20) may be differentiated also with respect to $x_1, \ldots, x_n$ and by means of the above described method one can obtain the inequalities

$$\left| \Delta g_i(t, x) \right| \leq \int_C M_3(t, x, \tau, \eta) \sum_{s=1}^{r} \left| \Delta u_s(\tau, \eta) \right| d\tau\, d\eta + \int_\sigma N_3(t, x, \tau, \eta) \sum_{j=1}^{q} \left| \Delta v_j(\tau, \eta) \right| d_{\tau,\eta}\sigma, \quad (4.28)$$

where

$$g = \left(y_1, \ldots, y_m, \frac{\partial y_1}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_n} \right), \quad x \in G, \quad 0 \leq t \leq T.$$

Since the functions $\Delta z_i(t, x)$, $i = 1, \ldots, m$, form the solution of the boundary problem (4. 14), we find in an analogous manner that

$$\left|\Delta z_i(t,x)\right| \leq \int_C M_4(t,x,\tau,\eta) \sum_{s=1}^{r} \left|\Delta u_s(\tau,\eta)\right| d\eta\, d\tau$$

$$+ \int_\sigma N_4(t,x,\tau,\eta) \sum_{j=1}^{q} \left|\Delta v_j(\tau,\eta)\right| d_{\tau,\eta}\sigma\,, \tag{4.29}$$

$x \in G, \quad 0 \leq t \leq T, \quad i = 1, \ldots, m.$

Because of these inequalities, we find from (4.19):

$$\begin{aligned} |\eta_1| \leq B_1 \int_C \sum_{j=1}^{r} \left|\Delta u_j(t,x)\right| \Bigg\{ & \int_C M_{11}(t,x,\tau,\eta) \sum_{j=1}^{r} \left|\Delta u_j(\tau,\eta)\right| d\eta\, d\tau \\ & + \int_\sigma N_{11}(t,x,\tau,\eta) \sum_{k=1}^{q} \left|\Delta v_k\right| d_{\tau,\eta}\sigma \Bigg\} dx\, dt \\ + B_2 \int_\sigma \sum_{i=1}^{q} \left|\Delta v_i(t,x)\right| \Bigg\{ & \int_C M_{11}(t,x,\tau,\eta) \sum_{j=1}^{r} \left|\Delta u_j\right| d\eta\, d\tau \\ & + \int_\sigma N_{11}(t,x,\tau,\eta) \sum_{k=1}^{q} \left|\Delta v_k\right| d_{\tau,\eta}\sigma \Bigg\} d_{t,x}\sigma\,, \end{aligned} \tag{4.30}$$

where B_i - positive constants,

$$M_{11} = M_3 + M_4, \quad N_{11} = N_3 + N_4\,.$$

Since according to the condition

$$\frac{\partial^2 H}{\partial w_i \partial w_j} \quad \text{and} \quad \frac{\partial^2 h}{\partial p_i \partial p_j}$$

are bounded, then

$$
\begin{aligned}
|\eta_2| \le B_3 \int_C \Bigg| \int_C & M_{11}(t, x, \tau, \eta) \sum_{k=1}^{r} |\Delta u_k| \, d\tau \, d\eta \\
& + \int_\sigma N_{11}(t, x, \tau, \eta) \sum_{j=1}^{q} |\Delta v_j| \, d_{\tau,\eta}\sigma \Bigg|^2 dx \, dt \\
+ B_4 \int_\sigma \Bigg| \int_C & M_{11}(t, x, \tau, \eta) \cdot \sum_{k=1}^{r} |\Delta u_k| \, d\eta \, d\tau \\
& + \int_\sigma N_{11}(t, x, \tau, \eta) \sum_{j=1}^{q} |\Delta v_j| \, d_{\tau,\eta}\sigma \Bigg|^2 d_{t,x}\sigma \, .
\end{aligned} \tag{4.31}
$$

If one takes into account that the constant B may be chosen such that

$$
\sum_{k=1}^{r} \int_C |\Delta u_k(t, x)|^2 \, dx \, dt \le B^2 , \quad \sum_{j=1}^{q} \int_\sigma |\Delta v_j(t, x)|^2 \, d\sigma \le B^2 ,
$$

then from the inequalities (4. 30) and (4. 31) it follows that for the residual term η in the formula (4. 18) one has an estimate:

$$
\begin{aligned}
|\eta| \le \int_C \Bigg\{ \int_C & P(t, x, \tau, \eta) \sum_{k=1}^{r} |\Delta u_k| \, d\tau \, d\eta \\
& + \int_\sigma Q(t, x, \tau, \eta) \sum_{j=1}^{q} |\Delta v_j| \, d_{\tau,\eta}\sigma \Bigg\}^2 dx \, dt \\
+ \int_\sigma \Bigg\{ \int_C & P(t, x, \tau, \eta) \sum_{k=1}^{r} |\Delta u_k| \, d\tau d\eta \\
& + \int_\sigma Q(t, x, \tau, \eta) \sum_{j=1}^{q} |\Delta v_j| \, d_{\tau,\eta}\sigma \Bigg\}^2 d_{t,x}\sigma \, ,
\end{aligned} \tag{4.32}
$$

where the functions P and Q are of the same type as M_{11} and N_{11} .

Formula (4. 18) and inequality (4. 32) are analogous to the corresponding relationships in Section I. Consequently, the proof of Theorem 7 conincides almost literally to the proof of Theorem 1.

If the system of equations (4. 4) is linear and has the form

$$L_{it}\, y = \sum_{k=1}^{m} d_{ik}(t, x)\, y_k + f_i(v), \quad i = 1, \dots, m, \tag{4.33}$$

while the boundary conditions may be represented in the form

$$\begin{gathered} P_i(t, x)y = \sum_{k=1}^{m} c_{ik}(t, x)\, y_k + \varphi_i(v), \\ x \in \Gamma, \quad y(0, x) = a(x), \quad x \in G, \end{gathered} \tag{4.34}$$

then the following theorem is true:

THEOREM 8. If to each permissible control corresponds a unique solution of the boundary problem (4. 33)-(4. 34), then, for the control $\omega(t, x) = (u(t, x),\ v(t, x))$ to be min-optimal (max-optimal) according to the functional (4. 6), it is necessary and sufficient that it satisfies the conditions of maximum (minimum).

The proof of this theorem follows directly from the fact that in the case under consideration the formula (4. 18) for the increment of the functional S takes the form

$$\begin{aligned} \Delta S = & -\int_C \left| H(t, x, w, u + \Delta u) - H(t, x, w, u) \right| dx\, dt \\ & -\int_\sigma \left| h(t, x, p, v + \Delta v) - h(t, x, p, v) \right| d\sigma\ . \end{aligned} \tag{4.35}$$

3. Problems with other Optimality Criteria

The just obtained result may be applied to the solution of problems of optimum control with other optimality criteria.

Let, for instance, the control process be described by the boundary problem (4. 4)-(4. 5) for which the domain G is a rectangle $0 \le x_i \le X_i$ and let one choose as the optimality criterion the functional

$$S = \int_0^T \int_0^{X_1} \int_0^{X_2} f_0\ (t, x, y, y_x, u)\, dx_2 dx_1 dt. \tag{4.36}$$

We introduce an auxiliary variable y_0 by means of the relationship

$$\frac{\partial^3 y_0}{\partial x_1 \partial x_2 \partial t} = f_0\ (t, x, y, y_x, u), \quad y_0(x_1, x_2, 0) = y_0(x_1, 0, t) = y_0(0, x_2, t) = 0.$$

Then the problem reduces to the evaluation of the minimum of the functional $S = y_0(X_1, X_2, T)$. We form the function $\overline{H}$:

$$\overline{H}(t, x, w, u) = \sum_{i=1}^{m} z_i f_i(t, x, y, y_x, u) + z_0 f_0\ (t, x, y, y_x, u).$$

The function $z_i(t, x)$ is defined by means of the equations

$$M_{it}\, z = -\frac{\partial \overline{H}}{\partial y_i} + \sum_{k=1}^{2} \frac{d}{dx_k}\left(\frac{\partial \overline{H}}{\partial y_{ix_k}}\right), \quad i = 1, \ldots, m, \qquad \frac{\partial^3 z_0}{\partial x_1 \partial x_2 \partial t} = 0$$

and additional conditions (see formulas (2. 19) and (4. 10)):

$$Q_{it} z = \frac{\partial h(t, x, p, v)}{\partial y_i} + \sum_{k=1}^{2} \frac{\partial H(t, x, w, u)}{\partial y_{ix_k}} X_k(x), \quad x \in \Gamma\ ,$$

$$\frac{\partial z_0}{\partial t} = 0 \ \text{ for } \ x_1 = X_1\ , \quad x_2 = X_2\ , \quad \frac{\partial z_0}{\partial x_1} = 0 \ \text{ for } \ t = T, \quad x_2 = X_2\ ;$$

$$\frac{\partial z_0}{\partial x_2} = 0 \ \text{ for } \ t = T, \quad x_1 = X_1\ , \quad \frac{\partial^2 z_0}{\partial x_1 \partial x_2} = 0 \ \text{ for } \ t = T, \quad \frac{\partial^2 z_0}{\partial x_1 \partial t} = 0$$

$$\text{for } \ x_2 = X_2\ , \quad \frac{\partial^2 z_0}{\partial x_2 \partial t} = 0 \ \text{ for } \ x_1 = X_1\ , \quad z_0(X_1, X_2, T) = -1,$$

$$z_1(x_1, x_2, T) = 0, \quad i = 1, \ldots, m.$$

In this manner, $z_0(x_1, x_2, t) = -1$, and the function $\overline{H}$ takes form

$$\overline{H} = H(t, x, w, u) - f_0(t, x, y, y_x, u)\ ,$$

where

$$H = \sum_{i=1}^{m} z_0 f_0\ (t, x, y, y_x, u)\ .$$

Let us study the functional

$$I = \int_0^T\int_0^{X_1}\int_0^{X_2}\left[\sum_{i=1}^{m} z_i L_{it} y + z_0 \frac{\partial^3 z_0}{\partial x_1 \partial x_2 \partial t} - \overline{H}(t, x, w, u)\right] dx_2 dx_1 dt$$

$$+\int_0^T\int_\Gamma\left[\sum_{i=1}^{m} z_i P_i y - h(t, x, p, v)\right] d\sigma dt = I_1 + I_2 \quad ,$$

where

$$I_1 = \int_0^T\int_0^{X_1}\int_0^{X_2}\left[\sum_{i=1}^{m} z_i L_{it} y - H(t, x, w, u)\right] dx_2 dx_1 dt$$

$$+\int_0^T\int_\Gamma\left[\sum_{i=1}^{m} z_i P_i(t, x) y - h(t, x, p, v)\right] d\sigma dt \; ,$$

$$I_2 = \int_0^T\int_0^{X_1}\int_0^{X_2} z_0 \left[\frac{\partial^3 y_0}{\partial x_1 \partial x_2 \partial x_3} - f_0 \; (t, x, y, y_x, u)\right] dx_2 dx_1 dt \; .$$

By transforming the integrals I_1 and I_2 in the same manner as done in Sections I and IV, we obtain a formula for the increment of the functional (4. 36) in the following form:

$$\Delta S = -\int_0^T\int_0^{X_1}\int_0^{X_2} \left| \overline{H}(t, x, w, u + \Delta u) - \overline{H}(t, x, w, u) \right| dx_2 dx_1 dt$$

$$-\int_0^T\int_\Gamma \left| h(t, x, p, v + \Delta v) - h(t, x, p, v) \right| d\sigma \, dt - \eta \quad ,$$

where the residual term η is defined by formulas analogous to (4. 19).

Consequently, the necessary conditions of optimality for the problem under investigation may be formulated in the form of Theorem 7 where in the conditions for the maximum (minimum) of the function H is substituted by $\overline{H}$.

Using the results of Section I, one can analogously investigate other problems of optimum control processes where for the optimality criteria one utilizes various nonlinear functions. In particular, the results obtained may be applied to the study of problems investigated by Bellman and Osborn[5] and by Bulkovskiy Lerner[6].

4. Optimum Problems in the Theory of Elliptical Systems

The control problems are analogous to those which were investigated above and are encountered during the study of diffusion processes[3,8]. Nevertheless, one must investigate boundary problems for elliptical equations. Problems of such a type are encountered during the study of optimum thermal and electrical fields in various power devices.

At this point we will briefly outline the formulation of the maximum problem for electrical systems and derive the formula for the increment of the functional by means of which one finds the optimality conditions.

Thus, let us deal with an elliptical system of equations

$$Ly = f(x, y, y_x, u), \quad x = (x_1, \ldots, x_n) \in G, \tag{4.37}$$

where the operator L is defined by the formula (4. 1) and the function $f = (f_1, \ldots, f_m)$ is twice continuously differentiable over the totality of all its arguments. The control parameter u takes the values from a bounded domain U (closed or open) of an r-dimensional euclidian space.

Let further the function y(x) satisfy the boundary conditions

$$P_i(x)y = \varphi_i(x, y, v), \quad i = 1, \ldots, m, \quad x \in \Gamma, \tag{4.38}$$

where φ_i satisfies the same conditions as f_i, and the parameter v takes values of a bounded domain V of the q-dimensional euclidian space.

The permissible control $\omega(x) = (u(x), v(x))$ is defined in the same manner as in Part 1, and we assume that over the discontinuity surfaces of the control the desired function satisfies certain smoothness conditions[35]. We assume that we impose onto the known functions of the boundary problem, in addition to the above mentioned conditions, some additional limitations under which for each permissible control there exists a unique solution of the particular problem.

We formulate the problem: among all the permissible controls we determine the control $\omega(x)$ (if it exists) such that the corresponding solution y(x) of the boundary problem (4. 37)-(4. 38) realizes the minimum (maximum) of the functional

$$S = \sum_{i=1}^{m} \left[\int_G a_i(x)\, y_i(x)\, dx + \int_\Gamma \gamma_i(x)\, y_i(x)\, d\sigma \right], \tag{4.39}$$

where $a_i(x)$ and $\gamma_i(x)$ are given continuous functions.

We introduce the functions $H = \sum z_i f_i$ and $h = \sum z_i \varphi_i$. The function $z_i(x)$ is defined as the solution of the boundary problem

$$M_i z = \frac{\partial H}{\partial y_i} - \sum_{k=1}^{n} \frac{d}{dx_k}\left(\frac{\partial H}{\partial y_{ix_k}}\right) - a_i(x), \quad x \in G, \tag{4.40}$$

$$Q_i z = -\frac{\partial h}{\partial y_i} - \sum_{k=1}^{n} \frac{\partial H}{\partial y_{ix_k}} X_k(x) + \gamma_i(x), \quad x \in \Gamma \tag{4.41}$$

(the definition of the operators M_i and Q_i can be found at the beginning of the paragraph). If it appears that the right-hand side of Equation (4.40) contains the derivatives of $v_{x_1}(x), \ldots, v_{x_n}(x)$, then one demands from the permissible controls that they have sectionally continuous derivatives with sufficiently smooth discontinuity boundaries. Then the boundary problem (4.40)-(4.41) for each permissible control has a unique solution.

In the same manner as it was applied above, one can obtain a formula for the increment of the functional (4.39) in the following form:

$$\Delta S = -\int_G \left| H(x, w, u + \Delta u) - H(x, w, u) \right| dx$$
$$- \int_\Gamma \left| h(x, p, v + \Delta v) - h(x, p, v) \right| d\sigma - \eta , \tag{4.42}$$

where the residual term η is determined using formulas analogous to (4.19). If the boundary problem (4.37)-(4.38) is linear, then $\eta = 0$, and consequently, we have as valid:

THEOREM 9. <u>For the permissible control to be locally min-optimal (max-optimal) according to the functional (4.39) in the linear boundary problem (4.37)-(4.38) (functions f_i and φ_i are linear in y and y_x) it is necessary and sufficient that this control satisfies the conditions of maximum (minimum).</u>

In conclusion we note that the analogous problem (with analogous results) may be investigation also for the system of hyperbolic equations with initial and boundary conditions.

Let the control process be described by a system of equations with partial derivatives

$$Az = f(x_1, \ldots, x_k, z, u), \tag{5.1}$$

where A is a linear differential operator of parabolic, elliptic, or hyperbolic type, $z = (z_1, \ldots, z_m)$ is a vector characterizing the state of the system under control, and u is a vector characterizing the external interaction. Let also be given additional conditions which may contain vector v defining the external interaction on the system. We assume that the vector $\omega = (u, v)$ is subjected to the same conditions as the permissible control in the problems of optimum control discussed above while the additional conditions are such that to each vector ω corresponds a unique solution of Equations (5.1) with the same additional conditions.

Let, in addition, be given a certain functional $I[z]$ defined over the solutions of Equations (5.1). The basic problem of the invariance theory is to find conditions for which the functional I does not depend on the external interactions. In the paper by Rozonoer[17] it was shown that the invariance problem may be studied by the methods of the variational calculus in the case when the control processes are described by ordinary differential equations. Analogously, one may investigate the invariance problem also for the system with distributed parameters.

We investigated the control system whose behavior is described by the boundary problem (4.9) with certain smoothness conditions imposed on the discontinuity surfaces of the function u(t, x). We assume that with each permissible vector $\omega(t, x) = (u)t, x), v(t, x))$ is associated a respective unique solution of the same boundary problem and

$$\begin{aligned} f_i(t, x, y, y_x, u) &= \sum_{k=1}^{m} d_{ik}(t, x) y_k + g_i(t, x) u, \\ \varphi_i(t, x, y, v) &= p_i(t, x) v, \end{aligned} \tag{5.2}$$

where, for the purpose of simplification of subsequent formulas, u and v are viewed as scalar quantities.

As the functional I we utilized the expression (4.6) in which the time T and the domain G are viewed as fixed. The "adjoint" boundary problem (4.10) has in this case the form

$$M_{it}z = -\sum_{k=1}^{m} d_{ki}z_k + \beta_i(t, x), \quad z_i(T, x) = -\alpha_i(x), \quad x \in G,$$

$$Q_i z = -\gamma_i(t, x), \quad x \in \Gamma. \tag{5.3}$$

The formula (4.35) for the increment of the functional (4.6) takes the form

$$\Delta S = -\int_C \Delta u \left(\sum_{i=1}^{m} g_i z_i\right) dx dt - \int_\sigma \Delta v \sum p_i z_i d\sigma.$$

Consequently, if

$$\sum_{i=1}^{m} g_i(t, x) z_i(t, x) \equiv 0, \quad x \in G$$

$$\sum_{i=1}^{m} p_i(t, x) z_i(t, x) \equiv 0, \quad x \in \Gamma, \quad 0 \leq t \leq T, \tag{5.4}$$

the functional S does not depend on the external perturbation $\omega(t, x)$. By using the method of proving the opposite, it is easy to establish that these conditions are also necessary for the functional not to depend on ω (see, for instance, work of Rozonoer[17]). For the verification of the condition (5.4) one must find a solution of the boundary problem (5.3). However, for the special case presented below, one can establish the necessary and sufficient invariance conditions expressed through the coefficients of the equations of the boundary problem (4.9).

Let the control process be described by the equations

$$Ly_i = \sum_{k=1}^{m} d_{ik} y_k + g_i u \left(Ly_i \equiv \frac{\partial u_i}{\partial t} - \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 y_i}{\partial x_j \partial x_k}, \right.$$

$$\left. d_{ik}, \ g_i - \text{const}\right) \tag{5.5}$$

with the additional conditions

$$y_i(0, x) = \alpha_i(x), \quad x \in G, \quad Py_i = \psi_i(t, x), \quad x \in \Gamma; \tag{5.6}$$

here P is a linear differential operator defined over the boundary Γ, where the differentiation is carried out in a direction which is towards the outside relative to G.

Let us study the functional

$$S = \sum_{i=1}^{m} \left[\int_G \alpha_i(x) y_i(T, x)\, dx + \int_0^T\!\!\int_\Gamma \gamma_i(t, x) y_i(t, x)\, d\sigma dt \right]. \tag{5.7}$$

Then the functions $z_i(t, x)$, entering (5.4), are determined from the equations

$$Mz_i = -\sum_{k=1}^{m} d_{ki} z_k \left(Mz_i = \frac{\partial z_i}{\partial t} + \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(a_{jk} \frac{\partial z_i}{\partial x_k} \right) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (l_j z_i) \right) \tag{5.8}$$

with the additional conditions

$$\begin{aligned} &z_i(T, x) = -\alpha_i(x), \quad x \in G; \\ &Q(t, x)\, z_i = -\gamma_i(t, x), \quad x \in \Gamma, \quad 0 \leq t \leq T, \end{aligned} \tag{5.9}$$

where the operator Q is defined according to (4.3) as being conjugate to P.

We denote by D and D* the matrix of the coefficients d_{ik} and their conjugate matrix, respectively, and by (r, s) - the scalar product of the vectors r and s. Since in the case under investigation the boundary conditions (5.6) do not contain v, the conditions (5.4) may be written in the form

$$R(t, x) \equiv (z, g) = 0, \quad x \in G, \quad 0 \leq t \leq T.$$

Applying to this equality the operator M and taking into account that z satisfies the system of equations (5.8), we get

$$MR(t, x) = (Mz, g) = -(D^* z, g) = (z, Dg) = 0,$$

Analogously, we find that

$$M^k R(t, x) = (-1)^k (z, D^k g) = 0, \quad k = 0, 1, \ldots, m-1. \tag{5.10}$$

From this, because of the conditions (5.9), it follows that

$$M^k R(T, x) = -(-1)^k (\alpha(x), D^k g) = 0, \quad k = 0, 1, \ldots, m-1. \tag{5.11}$$

Putting in Equations (5.10) $x \in \Gamma$ and applying the operator Q, we obtain taking into account the conditions (5.9):

$$QM^k R(t, x) = (-1)^k (Qz, D^k g) = -(-1)^k (\gamma(t, x), D^k g) = 0, \quad k = 0, 1, \ldots, m-1. \tag{5.12}$$

The conditions (5.11) and (5.12) are necessary for the invariance of the functional (5.7) relative to the external interaction u in the boundary problem (5.5) - (5.6). Let us show that these conditions are also sufficient.

Since according to the condition even one of the vectors $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$ differs from the zero vector, there exist numbers $\lambda_0, \ldots, \lambda_{m-1}$ such that

$$\sum_{k=0}^{m-1} \lambda_k D^k g = 0.$$

Multiplying the k-th equation (5.10) by $(-1)^k \lambda_k$ and summing over all k, we obtain:

$$\sum_{k=0}^{m-1} (-1)^k \lambda_k M^k R(t, x) = 0, \quad x \in G, \quad 0 \leq t \leq T.$$

Introducing the notations

$$R_k = M^k R, \; k = 0, 1, \ldots, m-2,$$

we obtain from this equation and the conditions (5.10) and (5.11) a homogeneous boundary problem for the determination of R_k:

$$(-1)^{m-1}\lambda_{m-1}MR_{m-2} + (-1)^{m-2}\lambda_{m-2}R_{m-2} + \dots - \lambda_1 R_1 + \lambda_0 R_0 = 0$$

$$R_{m-2}(T,x) = 0, \; x \in G; \; QR_{m-2}(t,x) = 0, \quad x \in \Gamma,$$

$$MR_{m-3} - R_{m-2} = 0, \; R_{m-3}(T,x) = 0, \; x \in G; \; QR_{m-3}(t,x) = 0, \; x \in \Gamma,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$MR_0 - R_1 = 0, \; R_0(T,x) = 0, \; x \in G; \; QR_0(t,x) = 0, \; x \in \Gamma. \qquad (5.13)$$

Since we assume that the coefficients of the operators M and Q are sufficiently smooth, the boundary problem (5.13) has only a trivial solution (see, for instance, work of Zagorskey[34], pp. 97-103):

$$R_{m-2}(t,x) = R_{m-3}(t,x) = \dots = R_0(t,x) \equiv 0,$$

and from this follows the validity of the condition

$$\sum_{i=1}^{m} g_i(t,x) z_i(t,x) = 0, \quad x \in G.$$

This proves:

THEOREM 10. For the invariance of the functional (5.7) relative to the external interaction in the boundary problem (5.5)-(5.6) to be true, it is necessary and sufficient that the conditions

$$(\alpha(x), D^k g) = 0, \quad x \in G;$$

$$(\gamma(t,x), D^k g) = 0, \quad x \in \Gamma;$$

$$0 < t \leq T, \quad k = 0, 1, \dots, m-1.$$

be fulfilled.

From the method for the proof of this theorem is clear that the analogous results may be obtained for boundary problems which were investigated in Section I. In particular, for the boundary problem (1.33) one has to utilize the formula (1.35) for the increment of the functional (1.3) with the function u_i defined by means of the boundary problem (1.34).

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13. ABSTRACT

This article presents a study of optimum processes in systems whose behavior is described by distinguishable boundary problems for equations with partial derivatives. The method is the same as that used by Rozonoer in 1959 to investigate the case where the control process is described by ordinary differential and finite-difference equations.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimum processes						
Hyperbolic equations						
Lipschitz condition						
Inequalities						
Euclidian spaces						
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