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A MODIFIED QUASILINEARIZATION CONCEPT FOR SOLVING THE NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEM

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ABSTRACT

A new quasilinearization method, referred to as the Modified Quasilinearization Method, is proposed for numerically solving the nonlinear two-point boundary value problem with an undetermined terminal time. This method is an extension to previously proposed quasilinearization methods, in that the terminal boundary may be specified by a general function of the problem variables and time, rather than just by specific values of the variables. Moreover, for variable terminal-time problems the terminal time determination is made an integral part of the iterative method itself. In addition, a scheme is formulated and successfully implemented which significantly reduces the sensitivity of the convergence characteristics of the method to the required initially assumed parameters. Application of the proposed method to a two-dimensional, minimum-time, Earth-Mars transfer example reveals a significant reduction in computer time required for convergence, when compared to the previously proposed quasilinearization methods.

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SUMMARY

A new quasilinearization method, referred to as the Modified Quasilinearization Method, is proposed for numerically solving the nonlinear two-point boundary value problem. Application of the proposed method to a two-dimensional, minimum-time, Earth-Mars transfer reveals a significant reduction in computer time required for convergence, when compared to previously proposed quasilinearization methods. Moreover, the iterative method suggested substantially reduces the sensitivity of the convergence characteristics of the method to required initially assumed parameters.

For the case shown, the proposed method required 69 percent less computer time than the Generalized Newton-Raphson Method. Also, the suggested iteration method increases the convergence envelope, representing the initially assumed values, by approximately 350 percent.

INTRODUCTION

The optimization of spacecraft trajectories has been of considerable interest for a number of years, and significant progress has been made in building a capability for solving very complex trajectory problems. One class of trajectory optimization problems occurs when it is desired to determine the history of the variables which are capable of controlling the spacecraft state in such a manner that certain specified terminal constraints are satisfied while some index of performance is extremized. The calculus of variations is a classical tool for solving such problems, and with its use the optimization problem may be reduced to a two-point boundary value problem. Methods for solving the variational two-point boundary value problem are designated indirect methods. The convergence characteristics of these methods are extremely sensitive to initial Lagrange multiplier values. However, if convergence occurs it is rapid, and in some cases the convergence is quadratic. The indirect methods have been successfully investigated by Jurovics and McIntyre (ref. 1); Breakwell, Speyer, and Bryson (ref. 2); Jazwinski (ref. 3); and McGill and Kenneth (ref. 4).

The present investigation makes an extension to the quasilinearization concepts as presented by McGill and Kenneth (ref. 4). This extension places the quasilinearization approach in a more competitive position with the other indirect methods; first, by allowing the specification of the terminal boundary in terms of a general function of the problem variables and time rather than just specific values of the variables themselves, and second, by making the terminal time determination an integral part of the iteration process. This time determination is made without requiring any additional terms to be added to the existing differential equations, and no additional differential equations are needed.

In addition to the previously stated extensions, an iteration procedure is discussed which substantially reduces the convergence sensitivity to initially assumed parameters, thus making the method more competitive with the direct methods. This sensitivity reduction is made by requesting only a percentage of the terminal constraint dissatisfaction to be corrected on a given iteration.

The Modified Quasilinearization Method has been successfully implemented by solving a low constant thrust, two-dimensional, minimum-time, Earth-Mars transfer. The method is currently being applied to a rendezvous problem having time-dependent terminal constraints.

SYMBOLS

A	2n by 2n matrix of partial derivatives defined in equation (2)
a	additional state variable defined in equation (10)
B	2n vector of partial derivatives defined in equation (2)
C	2n - p vector of constant corrections
c	iteration factor
F	2n vector function representing the equations of motion and Euler-Lagrange equations
GM	gravitational constant of the sun
g	p vector of general initial boundary conditions
h	2n + 1 - p vector of general terminal boundary conditions
m	spacecraft mass
N	summation index
p	number of initially specified conditions
r	radial position
s	new independent variable defined in equation (10)

T thrust magnitude

t scalar independent variable time

u radial velocity

v tangential velocity

w $2n$ vector of dependent variables of the particular differential equation defined in equation (7)

Y $2n \times 2n - p$ matrix of solutions of equation (6)

y $2n$ vector of dependent variables of the homogeneous differential equation defined in equation (6)

z $2n$ vector composed of n state variables and n Euler-Lagrange variables

β control variable, thrust direction relative to the local horizontal

θ angular position

λ n vector of Euler-Lagrange variables

ρ scalar metric defined in equation (9)

Subscripts:

f terminal time

i index

n nth trajectory

o initial time

Superscripts:

i index

$(\bar{\ })$ assumed value

Operators:

$\dot{( )}$ total derivative with respect to time

$( )'$ total derivative with respect to s

$\delta( )$ variation operator

QUASILINEARIZATION CONCEPTS

The indirect methods proposed by Jurovics and McIntyre (ref. 1); Breakwell, Speyer, and Bryson (ref. 2); and Jazwinski (ref. 3) involve the integration of a set of nonlinear differential equations of motion. The coefficients for the linear adjoint or perturbation differential equations are formed with the variables generated by these nonlinear equations. The quasilinearization approach can be formulated by linearizing the differential equations of motion, then using the adjoint or perturbation functions in the same general manner as before. The coefficients used to generate a new nominal trajectory can be formed from the solution that corresponds to the previous nominal trajectory. This quasilinearization concept involves the solution of this sequence of linear differential equations, the solution of which converges, under appropriate conditions, to the solution of the desired nonlinear problem. Since the equations are linear, the terminal constraints can be satisfied on every iteration, if desired. However, the classical optimality condition is not satisfied until convergence has occurred; and even though the end points of the trajectory are satisfied, some care must be taken to insure that the trajectory shape is correct. Another characteristic of the quasilinearization techniques is that an initially assumed solution is required. However, if a reasonable estimate of this solution cannot be made, a starting solution (derived from integrating the nonlinear differential equations) may be adequate to result in convergence. This requires only that the initial values of the unknown variables be assumed, rather than the complete solution.

The problem is formulated in terms of an ordinary, first-order, nonlinear, vector differential equation

$$\dot{z} = F(z, t) \tag{1}$$

where z is a $2n$ vector composed of n state variables and n Euler-Lagrange variables, and t is the independent variable time. This equation may be expanded about the previous nominal trajectory, designated as the nth trajectory, and by ignoring the nonlinear terms yields

$$\dot{z} = Az + B \tag{2}$$

where

$$A = \left[\frac{\partial F}{\partial z}\right]_n \tag{3}$$

and

$$B = \dot{z}_n - Az_n \tag{4}$$

These terms are evaluated from the previous nominal trajectory.

Suppose that p of the initial values of z are specified, that is,

$$z_i\left(t_o\right) = z_{io} \qquad i = 1, \ldots, p \tag{5}$$

This implies that $2n - p$ initial values of z must be assumed along with an initial time t_o. The homogeneous part of equation (2) may be expressed as

$$\dot{y} = Ay \tag{6}$$

and hence is similar to the perturbation equation. Equation (6) may be integrated forward $2n - p$ times with each successive starting vector consisting of all zero elements except for the element that corresponds to one of the unknown initial conditions. This element is set equal to unity. This procedure leads to a $2n \times 2n - p$ matrix solution $Y\left(t, t_o\right)$.

The nonhomogeneous solution to equation (2) may be obtained as a solution to

$$\dot{w} = Aw + B \tag{7}$$

which generates a particular solution when integrated forward with the p known initial conditions and $2n - p$ assumed conditions. Now the general solution of the linear system, equation (2), becomes

$$z(t) = Y\left(t, t_o\right)C + w(t) \tag{8}$$

where C is a $2n - p$ vector of constants and $w(t)$ is the $2n$ vector of nonhomogeneous solutions to equation (7).

Since $2n + 1 - p$ conditions on the terminal value of z must be specified for a variable final-time problem, any $2n - p$ of these conditions may be selected, and the appropriate $2n - p$ members of equation (8) may be evaluated at the assumed terminal time. Then these equations are solved for the $2n - p$ constants C, which are corrections used to update the assumed initial conditions for the next iteration.

This procedure continues until a metric (which represents the maximum distance, over the complete independent variable range, between successive nominal trajectories) becomes less than some preselected value. This metric is defined by

$$\rho = \sum_{i=1}^{N} \max_{t} \left| z_{n+1}{}^{i}(t) - z_{n}{}^{i}(t) \right| \tag{9}$$

where n refers to the nth iteration. As this metric decreases, the optimal trajectory shape is converged upon for the assumed value of the terminal time. The one remaining unused terminal condition is used in a conventional scalar application of the Newton-Raphson iteration technique to produce a more accurate determination of terminal time. The above procedures are repeated until the time corrections are smaller than some preselected value.

The major objections to the Generalized Newton-Raphson Method are that the terminal time determination is very time-consuming (especially when a large error is made in the assumed terminal time) and the initial and terminal conditions must be simply specific values of the variables z, rather than general functions of these variables. The first objection has been partially removed by Long (ref. 5).

The method proposed by Long, designated as the Modified Generalized Newton-Raphson Method, involves a change of the independent variable $t = as$ where a is a constant and is considered a new state variable, and s is a new independent variable having values $0 \leq s \leq 1$. Now, the differential equation, equation (1), may be written

$$z' = \frac{dz}{ds} = aF(z, as) \tag{10}$$

The constant a is considered a new state variable and an additional differential equation $a' = 0$ may be added, but this is clearly not necessary since its solution is trivial. The value of a is initially assumed and corrected on each iteration and can be seen to be the desired terminal time.

The determination of terminal time now becomes an integral part of the iterative procedure; and its separate consideration, as required by the Generalized Newton-Raphson Method, is not necessary. However, a relatively complex term that corresponds to the new state variable a must be added to each differential equation. Moreover, the linear system from which the corrections are obtained is increased since the value of a must be iteratively determined.

MODIFIED QUASILINEARIZATION METHOD

The method proposed here, the Modified Quasilinearization Method, uses the quasilinearization concepts previously outlined, and removes both of the stated objections. The manner in which the terminal time is determined proves more satisfactory than in any other known quasilinearization method, and the generality of the terminal constraints makes the method more competitive with the adjoint and perturbation techniques proposed by Breakwell, Speyer, and Bryson (ref. 2), and by Jazwinski (ref. 3).

Equation (8) may be rewritten and evaluated at the terminal time to yield

$$Y\left(t_f, t_o\right)C = z\left(t_f\right) - w\left(t_f\right) \tag{11}$$

The right-hand side of this equation is the difference between the desired terminal value of z and the linear calculation of the terminal value of w. This difference is interpreted as the variation of $z\left(t_f\right)$ and is expressed as $\delta z\left(t_f\right)$. Now, if both sides of equation (11) are premultiplied by $\left[\frac{\partial h}{\partial z}\right]_f$, the resulting expression becomes

$$\left[\frac{\partial h}{\partial z}\right]_f Y\left(t_f, t_o\right)C = \left[\frac{\partial h}{\partial z}\right]_f \delta z\left(t_f\right) \tag{12}$$

where $\left[\frac{\partial h}{\partial z}\right]_f$ is a $2n + 1 - p \times 2n$ matrix describing the partial change of a general set of terminal boundary conditions $h\left(z_f, t_f\right)$ to a change in the terminal values of $z\left(t_f\right)$ itself. Since the right-hand side of equation (12) is $\delta h\left(t_f\right)$, a first-order expansion

$$dh = \delta h + \dot{h}\, dt_f \tag{13}$$

may be introduced to yield

$$dh = \left[\frac{\partial h}{\partial z}\right]_f Y\left(t_f, t_o\right)C + \dot{h}\, dt_f \tag{14}$$

where dh is a $2n + 1 - p$ vector of terminal constraint dissatisfaction and dt_f is the time correction to be made for the next iteration.

Equation (14) is analogous to the linear algebraic equation derived by Jazwinski (ref. 3), the difference being that in the present case the nonlinear differential equations of motion are linearized. When the optimization problem is reduced to a nonlinear two-point boundary value problem, p becomes equal to n, and the implementation of the two methods is similar. A detailed presentation of the numerical procedure described earlier is made by Lewallen (ref. 6).

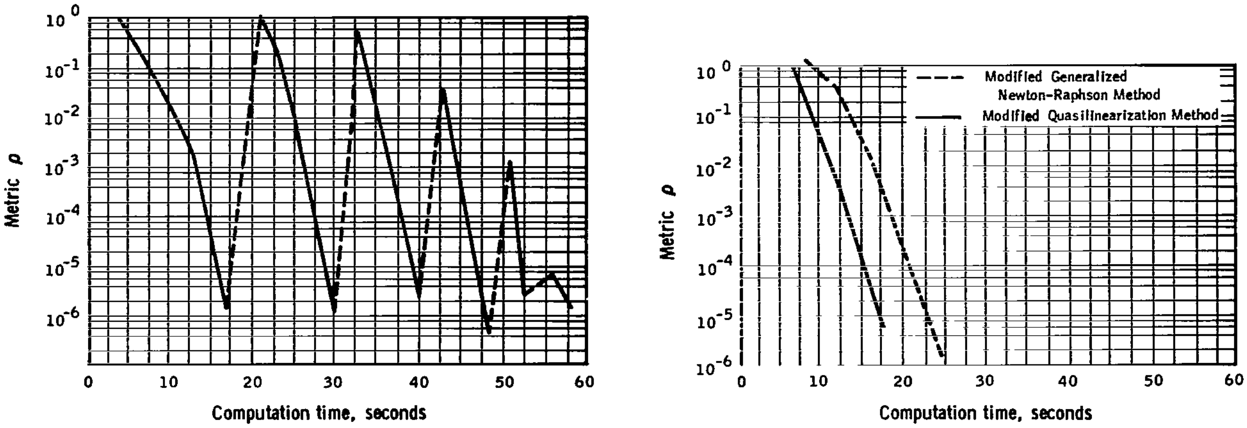
The iteration philosophy is that only a percentage of the terminal dissatisfaction be requested for correction on any given iteration, that is, $dh = -ch$ where $0 \leq c \leq 1$. It is expected that if a 100-percent correction is requested (normal iteration scheme) in cases where the linear representation is poor, the sequence of linear solutions will diverge. The less severe request of only a percentage correction is applied initially, and successively larger percentages are requested after each convergent iteration. Upon each divergent iteration, if any, the percentage correction may be reduced.

APPLICATION

The system model to be considered is that of minimizing the transfer time of a constant low-thrust rocket traveling between the circular and coplanar orbits of Earth and Mars. The associated differential and algebraic equations are outlined in the appendix. The single control variable is the thrust direction relative to the local horizontal.

Figure 1 illustrates how the metric ρ is reduced as a function of IBM 7094 computation time for a typical example using the Generalized Newton-Raphson Method, Modified Generalized Newton-Raphson Method, and Modified Quasilinearization Method. The cases shown are those for initial errors in the two Euler-Lagrange variables λ_{10} and λ_{20} and in the terminal time t_f of -10, -10 and 20 percent, respectively. The third Euler-Lagrange variable is initially set to negative unity for scaling purposes, and λ_4 is easily determined to be zero. The initial solution for these cases are obtained from a linear approximation of the solutions.

The Generalized Newton-Raphson Method, as shown in figure 1(a), requires convergence to an acceptable metric for an assumed terminal time. A time iteration is then made, and the previously reduced metric is degraded to some extent. This



(a) Generalized Newton-Raphson Method.

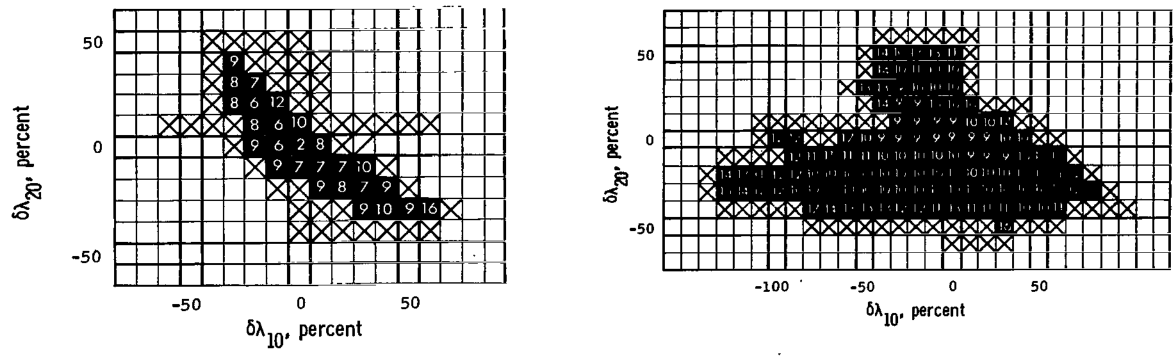
(b) Modified Generalized Newton-Raphson Method and Modified Quasilinearization Method.

Figure 1. - Metric ρ as a function of computation time using the linear initial solution, normal iteration scheme, and errors $\delta\lambda_{10}$ = -10 percent, $\delta\lambda_{20}$ = -10 percent, and δt_f = 20 percent.

degradation is indicated on the plots by dashed lines. The Modified Quasilinearization Method, for the case shown, is clearly more satisfactory in terms of required computational time, requiring 69 percent less than the time required for the Generalized Newton-Raphson Method and 28 percent less than the time required for the Modified Generalized Newton-Raphson Method. It should be noted that in the terminal phases of convergence, the convergence rate is almost quadratic.

Many cases have been investigated in an effort to determine how sensitive the method is to poor initial assumption for the Euler-Lagrange variables and terminal time. These numerical results are best illustrated by building envelopes of convergence, the boundary of which represents the last convergent trial. The errors that are discussed are the percentage errors or deviations from the values that result in an optimal solution.

Figure 2 illustrates the convergence envelopes for the cases where the optimal terminal time is initially assumed, and the initial solution is obtained by integrating the nonlinear differential equations of motion. A 100-percent correction in the terminal constraints is requested on every iteration in figure 2(a). In figure 2(b), a 50-percent correction is requested initially; and this factor is either increased or decreased by 10 percent on each successive iteration, depending on the convergence or divergence, respectively, of the previous trajectory. It is seen that when the initial value of the iteration factor is reduced from 100 to 50 percent, the convergence envelope size is increased by approximately 350 percent. The primary significance of this result is that, by reducing the initial iteration factor, the chances of obtaining a convergent solution with only one computer run are enhanced greatly. Further increase in the convergence envelope size is realized by decreasing the initial value of the iteration factor.



(a) Initial iteration factor, 100 percent.

(b) Initial iteration factor, 50 percent.

Note: The numbers indicate the iterations required for convergence.

Figure 2.- Convergence envelope for the Modified Quasilinearization Method using a nonlinear initial solution, terminal time error of zero percent, and initial iteration factors of 100 and 50 percent.

It might be speculated that, since the optimal terminal time is used as an initial condition for the first iteration, an easy case has been selected for illustration. Actually, cases were examined where terminal time errors were -20 and +20 percent. The convergence envelopes for -20 percent terminal time error were smaller than the envelopes shown in figure 2, but the envelopes for +20 percent terminal time error were larger as indicated in more detail by Lewallen (ref. 6). Reference 6 also shows that the Modified Quasilinearization Method compares very favorably with the method proposed by Jazwinski (ref. 3).

CONCLUSIONS

A theoretical extension of the quasilinearization concept, as applied to the nonlinear two-point boundary value problem, is possible which allows the terminal boundary to be specified as a general function of the problem variables. The numerical results indicate that for variable final time problems the suggested method of terminal time determination represents a significant reduction of computational time relative to previously published methods. Moreover, the proposed iteration philosophy is such that the convergence sensitivity of the method to the initially assumed parameters is greatly reduced, thus allowing convergence to occur for many cases which otherwise would have diverged.

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APPENDIX

The following application is formulated to illustrate the procedure discussed in the text. The problem is formulated in terms of the nonlinear differential equations of motion for the two-dimensional, Earth-Mars transfer under the influence of a constant low thrust

$$
\left.
\begin{aligned}
\dot{z}_1 &= \dot{u} = \frac{v^2}{r} - \frac{GM}{r^2} + \frac{T \sin \beta}{m} = F_1 \\
\dot{z}_2 &= \dot{v} = -\frac{uv}{r} + \frac{T \cos \beta}{m} = F_2 \\
\dot{z}_3 &= \dot{r} = u = F_3 \\
\dot{z}_4 &= \dot{\theta} = \frac{v}{r} = F_4
\end{aligned}
\right\} \qquad (15)
$$

where u and v are the radial and tangential velocities, respectively, and r and θ are the radial and angular displacements, respectively. The angle β is the angle the thrust vector T makes with the local horizontal, and m is the spacecraft mass.

The linear Euler-Lagrange differential equations must also be satisfied for a trajectory optimization problem

$$
\left.
\begin{aligned}
\dot{z}_5 &= \dot{\lambda}_1 = \left(\frac{v}{r}\right)\lambda_2 - \lambda_3 = F_5 \\
\dot{z}_6 &= \dot{\lambda}_2 = -\left(\frac{2v}{r}\right)\lambda_1 + \left(\frac{u}{r}\right)\lambda_2 - \left(\frac{1}{r}\right)\lambda_4 = F_6 \\
\dot{z}_7 &= \dot{\lambda}_3 = \left[\frac{v^2}{r^2} - \frac{2(GM)}{r^3}\right]\lambda_1 - \left(\frac{uv}{r^2}\right)\lambda_2 + \left(\frac{v}{r^2}\right)\lambda_4 = F_7 \\
\dot{z}_8 &= \dot{\lambda}_4 = 0 = F_8
\end{aligned}
\right\} \qquad (16)
$$

The classical optimality condition is used to eliminate the control variable terms in the differential equations of motion. This relation yields

$$\sin\beta = \frac{\lambda_1}{-\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad \cos\beta = \frac{\lambda_1}{-\sqrt{\lambda_1^2 + \lambda_2^2}} \tag{17}$$

The specified initial conditions are

$$\left.\begin{aligned} g_1 &= u(t_o) - u_o = 0 \\ g_2 &= v(t_o) - v_o = 0 \\ g_3 &= r(t_o) - r_o = 0 \\ g_4 &= \theta(t_o) - \theta_o = 0 \end{aligned}\right\} \tag{18}$$

where t_o is specified.

The specified terminal conditions are

$$\left.\begin{aligned} h_1 &= u(t_f) - u_f = 0 \\ h_2 &= v(t_f) - v_f = 0 \\ h_3 &= r(t_f) - r_f = 0 \end{aligned}\right\} \tag{19}$$

If it is desired to minimize terminal time, two additional terminal conditions are derived from the transversality conditions.

$$\left.\begin{aligned} h_4 &= \lambda_{4f} = 0 \\ h_5 &= 1 + \lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 + \lambda_4 F_4 = 0 \end{aligned}\right\} \tag{20}$$

The Lagrange multipliers are scaled by requiring that $\lambda_{30} = -1$; hence, the last terminal boundary condition h_5 is ignored. The boundary conditions become

$$\left.\begin{aligned}
g_1 &= u(t_o) - u_o = 0 & h_1 &= u(t_f) - u_f = 0 \\
g_2 &= v(t_o) - v_o = 0 & h_2 &= v(t_f) - v_f = 0 \\
g_3 &= r(t_o) - r_o = 0 & h_3 &= r(t_f) - r_f = 0 \\
g_4 &= \theta(t_o) - \theta_o = 0 & h_4 &= \lambda_4(t_f) = 0 \\
g_5 &= \lambda_3(t_o) + 1.0 = 0 & &
\end{aligned}\right\} \tag{21}$$

For the solution of the eight differential equations, 10 boundary conditions must be known. If the initial time is assumed, the above nine relations are adequate for solution.

The nonhomogeneous, linear, vector differential equation $\dot{z} = Az + B$ is composed of $n = 4$ linearized differential equations of motion (with the control terms eliminated by use of the optimality condition) and $n = 4$ linearized Euler-Lagrange differential equations. These equations are

$$\begin{aligned}
\dot{z}_{1_{n+1}} = \dot{u}_{n+1} &= \left(\frac{2v}{r}\right)_n v_{n+1} + \left[\frac{2(GM)}{r^3} - \frac{v^2}{r^2}\right]_n r_{n+1} - \left[\frac{T\lambda_2^{\,2}}{m\left(\lambda_1^{\,2} + \lambda_2^{\,2}\right)^{3/2}}\right]_n \lambda_{1_{n+1}} \\
&\quad + \left[\frac{T\lambda_1\lambda_2}{m\left(\lambda_1^{\,2} + \lambda_2^{\,2}\right)^{3/2}}\right]_n \lambda_{2_{n+1}} + \left(B_1\right)_n
\end{aligned} \tag{22}$$

$$\dot{z}_{2_{n+1}} = \dot{v}_{n+1} = -\left(\frac{v}{r}\right)_n u_{n+1} - \left(\frac{u}{r}\right)_n v_{n+1} + \left(\frac{uv}{r^2}\right)_n r_{n+1} + \left[\frac{T\lambda_1\lambda_2}{m\left(\lambda_1{}^2 + \lambda_2{}^2\right)^{3/2}}\right]_n \lambda_{1_{n+1}}$$

$$- \left[\frac{T\lambda_1{}^2}{m\left(\lambda_1{}^2 + \lambda_2{}^2\right)^{3/2}}\right]_n \lambda_{2_{n+1}} + \left(B_2\right)_n \tag{23}$$

$$\dot{z}_{3_{n+1}} = \dot{r}_{n+1} = u_{n+1} + \left(B_3\right)_n \tag{24}$$

$$\dot{z}_{4_{n+1}} = \dot{\theta}_{n+1} = \left(\frac{1}{r}\right)_n v_{n+1} - \left(\frac{v}{r^2}\right)_n r_{n+1} + \left(B_4\right)_n \tag{25}$$

$$\dot{z}_{5_{n+1}} = \dot{\lambda}_{1_{n+1}} = \left(\frac{\lambda_2}{r}\right)_n v_{n+1} - \left(\frac{v\lambda_2}{r^2}\right)_n r_{n+1} + \left(\frac{v}{r}\right)_n \lambda_{2_{n+1}}$$

$$- \lambda_{3_{n+1}} + \left(B_5\right)_n \tag{26}$$

$$\dot{z}_{6_{n+1}} = \dot{\lambda}_{2_{n+1}} = \left(\frac{\lambda_2}{r}\right)_n u_{n+1} - \left(\frac{2\lambda_1}{r}\right)_n v_{n+1} + \left[\frac{1}{r^2}\left(2v\lambda_1 - u\lambda_2 + \lambda_4\right)\right]_n r_{n+1}$$

$$- \left(\frac{2v}{r}\right)_n \lambda_{1_{n+1}} + \left(\frac{u}{r}\right)_n \lambda_{2_{n+1}} - \left(\frac{1}{r}\right)_n \lambda_{4_{n+1}} + \left(B_6\right)_n \tag{27}$$

$$\dot{z}_{7_{n+1}} = \dot{\lambda}_{3_{n+1}} = -\left(\frac{v\lambda_2}{r^2}\right)_n u_{n+1} + \left[\frac{1}{r^2}\left(2v\lambda_1 - u\lambda_2 + \lambda_4\right)\right]_n v_{n+1}$$

$$+ \left\{\frac{1}{r^3}\left[\frac{6(GM)\lambda_1}{r} - 2v^2\lambda_1 + 2uv\lambda_2 - 2v\lambda_4\right]\right\}_n r_{n+1}$$

$$+ \left[\frac{v^2}{r^2} - \frac{2(GM)}{r^3}\right]_n \lambda_{1_{n+1}} - \left(\frac{uv}{r^2}\right)_n \lambda_{2_{n+1}} + \left(\frac{v}{r^2}\right)_n \lambda_{4_{n+1}} + \left(B_7\right)_n \tag{28}$$

$$\dot{z}_{8_{n+1}} = \dot{\lambda}_{4_{n+1}} = \left(B_8\right)_n \tag{29}$$

where

$$\left.\begin{aligned}
\left(B_1\right)_n &= \left[-\frac{3(GM)}{r^2} - \frac{T}{m\sqrt{\lambda_1^{\,2} + \lambda_2^{\,2}}}\right]_n \\
\left(B_2\right)_n &= -\left(\frac{T\lambda_2}{m\sqrt{\lambda_1^{\,2} + \lambda_2^{\,2}}}\right)_n \\
\left(B_3\right)_n &= 0 \\
\left(B_4\right)_n &= \left(\frac{v}{r}\right)_n \\
\left(B_5\right)_n &= 0 \\
\left(B_6\right)_n &= -\left(\frac{\lambda_4}{r}\right)_n \\
\left(B_7\right)_n &= \left[\frac{6(GM)\lambda_1}{r^3} + \frac{v\lambda_4}{r^2}\right]_n \\
\left(B_8\right)_n &= 0
\end{aligned}\right\} \tag{30}$$

letting GM = gravitational constant for the Earth, T = spacecraft thrust, and $m = m_o - \dot{m}t$ = spacecraft mass.

These nonhomogeneous linear equations are integrated from t_o to $\bar{t}_f$ with the starting vector $z^T(t_o)_{n+1} = \left[u \;\; v \;\; r \;\; \theta \;\; \bar{\lambda}_1 \;\; \bar{\lambda}_2 \;\; \lambda_3 \;\; \bar{\lambda}_4\right]_{t_o}$, where the bar indicates an assumed value. The homogeneous linear equations (same as above except without the $(B_i)_n$, $i = 1, \ldots, 2n$ terms) are integrated from t_o to $\bar{t}_f$ with starting vectors

$$y_1(t_o)_{n+1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad y_2(t_o)_{n+1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad y_3(t_o)_{n+1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{31}$$

When the assumed terminal time $\bar{t}_f$ is reached, the algebraic equations that must be solved for the corrections to be applied to the initially assumed parameters $\bar{\lambda}_{10}$, $\bar{\lambda}_{20}$, $\bar{\lambda}_{40}$ and $\bar{t}_f$ are

$$\begin{bmatrix} \delta\lambda_{10} \\ \delta\lambda_{20} \\ \delta\lambda_{40} \\ \delta t_f \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \dot{u}_f \\ y_{21} & y_{22} & y_{23} & \dot{v}_f \\ y_{31} & y_{32} & y_{33} & \dot{r}_f \\ y_{81} & y_{82} & y_{83} & \lambda_{4f} \end{bmatrix}^{-1} \begin{bmatrix} du \\ dv \\ dr \\ d\lambda_4 \end{bmatrix}_f \tag{32}$$

where the elements in the matrix are evaluated at t_f. These corrections are applied, and a new nominal trajectory is integrated using $\dot{z} = Az + B$.

The numerical implementation used the following values on an Earth-Mars transfer example.

Astronomical unit (AU), meters	$0.14959870 \times 10^{12}$
Orbital radius of Earth, r_E, AU	0.10000000×10^{1}
Orbital radius of Mars, r_M, AU	0.15236790×10^{1}
Gravitational constant of the Sun, GM_S, m^3/sec^3	$0.13271504 \times 10^{21}$
Initial spacecraft mass, m_o, kilograms	0.67978852×10^{3}
Thrust, T, newtons	0.40312370×10^{1}
Mass rate, $\dot{m}$, kg/sec	$0.10123858 \times 10^{-4}$

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