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METHODS OF REGULARIZATION FOR COMPUTING ORBITS IN CELESTIAL MECHANICS

by E. Stiefel, M. Rössler, J. Waldvogel, and C. A. Burdet

Prepared by
SWISS FEDERAL INSTITUTE OF TECHNOLOGY
Zurich 6, Switzerland

for

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PREFACE

The gravitational attraction of a celestial body on a particle increases beyond all limits whenever the particle approaches the attracting center and finally collides with it. Consequently the differential equations of motion present singularities at collision; the art of removing such singularities by appropriate transformations of the coordinates and of time is called <u>regularization</u>.

Several methods for regularizing the <u>2-dimensional</u> motion of a particle, subjected to gravitational forces, are known. In 1895 <u>T.N. Thiele</u> achieved simultaneous regularization of two attracting centers and in 1915 <u>G.D. Birkhoff</u> found a simpler method for reaching the same goal. A remarkable regularization of the plane motion of a particle about a single attracting center was published by <u>T.Levi-Civita</u> in 1906. He introduced parabolic coordinates in the plane of motion and used the eccentric anomaly in place of time as the independent variable. This procedure has the desirable property of transforming the equations of pure Kepler motion into <u>linear</u> differential equations, thus permitting easy integration and a simple theory of perturbations.

Several authors have proposed to take advantage of this fact for establishing analytical as well as numerical methods in celestial mechanics. In particular, this was discussed in the spring of 1964 during a symposium at the research institute at Oberwolfach, Germany [16]. It was generally felt that such a theory would have only a doubtful value if restricted to 2-dimensional motion. Happily, <u>P. Kustaanheimo</u> succeeded at the end of the session in constructing a <u>3-dimensional</u> generalization of Levi-Civita's transformation by replacing complex variables by spinors. In the paper [3] we reformulated this in terms of matrices, discussed the analytical and geometric properties of the transformation and outlined the perturbation theory. This opened the way for further generalizations, for example the construction of a 3-dimensional transformation of Birkhoff's type [17].

Other 3-dimensional regularizations were known before, but as far as we know they have not the property of generating linear differential equations. We mention in this connection only the ingenious work of <u>K.F. Sundman</u> who established in 1913 his famous result on forever convergent expansions in the problem of the three bodies.

In 1965 the <u>National Aeronautics and Space Administration</u> of the U.S.A. suggested that we study the problem of regularization with the 3-dimensional case as the principal area of research, furnish additional knowledge of possible types of trajectories and improve methods for numerical integration of trajectories.

This research was organized as a cooperative project of <u>NASA</u> and the <u>Swiss Federal Institute of Technology</u>. It is my pleasant duty to express our thanks to both organizations and to <u>IBM</u> for sponsoring this work. We are also indebted to

NASA's representatives Dr. E.D. Geissler, Dr. H.A. Sperling and Commodore C. Dearman for their interest, comments and helpful assistance.

It should be mentioned in this connection that this report is intimately connected with research work done by NASA scientists. For instance R.A. Broucke [18] of the Jet Propulsion Laboratory has developed a perturbation theory of the osculating orbit based on [3], which is somewhat different from the theory contained in this report (cf. section 1.4); R.F. Arenstorf [19] and H.A. Sperling [20] of Marshall Space Flight Center have published remarkable contributions to the theory and application of regularization.

NASA's scientific support has created wider interest in celestial mechanics at our university and, in particular, Mr. P. Sturzenegger and Mr. B. Stanek have facilitated our work by investigating some special problems and by carrying out computations. We are very obliged to them and also to Mrs. S. Eisner who, with everlasting energy, took care of all the little details involved in printing and publishing this report.

Finally we want to thank Mr. A. Schai, director of our computing center; he was always ready to help us and to put our programs on the Control Data 1604-A computer with high priority.

Zurich, September 1966. E. Stiefel

How to read this report

1. A reader only interested in perturbations and practical computations will skip the more theoretical investigations on simultaneous regularization of two attracting centers (sections 1.1.2, 1.2.2 and chapter 3).
2. References to literature are in square brackets.
3. We have the custom to list on the left-hand border of an equation the numbers of the previous formulae needed for proving that equation. For instance

   (1,98) $$(a + b)^2 = a^2 + 2ab + b^2 \qquad (1,99)$$

   means more explicitly: "from formula (1,98) it follows that $(a + b)^2 = a^2 + 2ab + b^2$ and this result is the new formula (1,99) ".

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1. PRINCIPLES OF REGULARIZATION

by E. Stiefel

The motion of heavenly bodies may be predicted using the theory of classical celestial mechanics. This theory leads to a set of differential equations, whose solution provides the equation of the respective orbits of the various bodies. The standard classical methods of solution of these equations is very successful if the various bodies considered remain well apart from each other as they move in their orbits. However these methods become cumbersome and inaccurate if the bodies are involved in a near-collision, and break down alltogether if an actual collision is involved. A very important practical problem for instance concerns the motion of a space vehicle as it moves from the earth to the moon. This is in a state of near-collision both at the beginning and at the end of its orbit.

The intention of this report is to introduce and investigate numerical as well as analytical methods, which deal with this problem taking into account this somewhat shifted point of view. Such methods should be able to compute an orbit during and beyond collision, and transformed into perturbation methods they should converge rapidly also for orbits of arbitrary high eccentricity. This implies the introduction of regularized coordinates and a regularizing time. Furthermore the classical orbital elements (inclination, longitude of node, pericenter, etc.) are not unambiguously defined as the eccentricity of the orbit approaches 1 (the major axis a remaining bounded). For this reason, and in order to provide a convenient general theory, we introduce also regularized elements in this paper.

We emphasize the practical computational aspects and avoid lengthy theories by using sources already available in the literature. The report should be readable however without consulting such sources too much.

At the end of the paper the general properties of regularized methods are listed. Their advantages and disadvantages in the light of our experience, are discussed.

1.1 Motion in a plane

A particle of mass m is subjected to the gravitational force of a central body M located at the origin of a x_1, x_2-plane. (Fig. 1.1). A possible path is a Kepler ellipse focused at the origin; if the eccentricity of this ellipse is close to 1, the orbit is very close to a straight line segment. In the limiting case, the orbit is a straight line segment, the particle moving forwards and backwards on this line, its position vector making a sharp bend of angle 2π at the origin. In order to remove this singular behaviour, generalized coordinates u_1, u_2 are introduced by mapping the physical x-plane ($x = x_1 + i x_2$) onto a parametric u-plane ($u = u_1 + i u_2$) in such a way that the image of the particle moves on a straight line always in the same direction going beyond the origin after collision and making no turns at collision. Thus the angle 2π in the physical plane should become only π in the parametric plane. In general regularizing transformations must have the basic property that angles at attracting centers are halved.

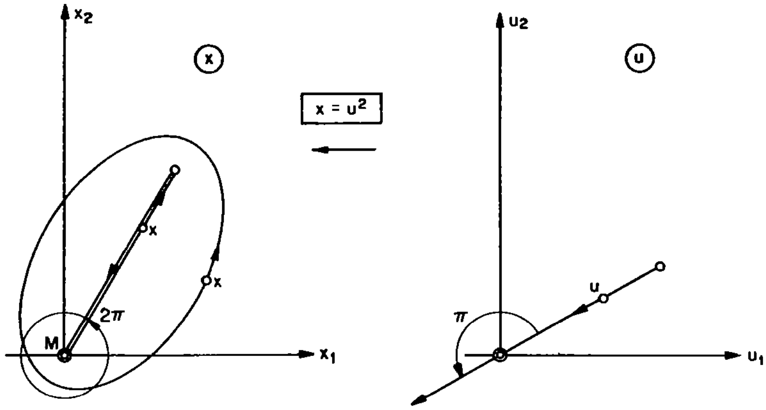


Fig. 1.1. Regularizing Transformation.

The kinetic energy T of the particle is a quadratic form in the generalized velocities $\dot{u}_j$ with coefficients depending on the position of the image point. If our mapping $x = x(u)$ is conformal at points not occupied by attracting matter, this form is reduced to a sum of squares, thus ensuring that each Lagrangian equation contains only one acceleration $\ddot{u}_j$. We take advantage of this fact by restricting ourselves to conformal transformations. The complex variable x is then an analytical function $x(u)$ of the complex argument u. We use the Cauchy-Riemann equations

$$\frac{\partial x_1}{\partial u_1} = \frac{\partial x_2}{\partial u_2} , \qquad \frac{\partial x_1}{\partial u_2} = - \frac{\partial x_2}{\partial u_1} , \tag{1,1}$$

and we introduce the functional determinant

$$D = \left| \frac{dx}{du} \right|^2 = \sum_{(j)} \left(\frac{\partial x_i}{\partial u_j} \right)^2 , \tag{1,2}$$

where i is either 1 or 2. Denoting differentiation with respect to the time t by a dot, the velocity v of the particle in the physical plane is given by

$$v^2 = |\dot{x}|^2 = \left| \frac{dx}{du} \right|^2 |\dot{u}|^2 = D \cdot \sum \dot{u}_j^2 \tag{1,3}$$

and its kinetic energy by

$$T = \tfrac{1}{2} v^2 = \tfrac{1}{2} D \sum \dot{u}_j^2 . \tag{1,4}$$

(The mass m of the particle is assumed to be $= 1$; in our subsequent working the magnitude of this mass is irrelevant because it cancels out of the equations of motion). The forces acting on the particle are supposed to have a potential that splits up into a conservative potential $\mathcal{U}(x_i)$ (eventually singular at centers of attraction) and a perturbing potential $V(x_i, t)$ regular at those centers and eventually depending explicitly on time. The Lagrangian equations of motion with respect to the generalized coordinates u_j are then

$$\frac{d}{dt}(D\dot{u}_j) - \frac{1}{2}\frac{\partial D}{\partial u_j}\sum \dot{u}_k^2 + \frac{\partial}{\partial u_j}(\mathcal{U} + V) = 0 , \tag{1,5}$$

where the potentials $\mathcal{U}, V$ are written as functions of u_k and t before differentiation. If we go from the parametric plane to the physical plane by our transformation, we have in general conservation of angles, excepting that at the image points of attracting centers angles are doubled. Such points are unconformal and the coefficient of the highest derivative in (1,5) (that is the determinant D) vanishes there, thus producing a singularity of the differential equation. In order to avoid this phenomenon a regularizing time - also called fictitious time s - is introduced by the relations

See Errata

$$\frac{ds}{dt} = \frac{1}{D} , \quad \frac{d}{dt} = \frac{1}{D}\frac{d}{ds} , \quad t = \int D \, ds . \tag{1,6}$$

We denote differentiation with respect to s by an accent and obtain the following modified forms of (1,5)(1,3):

$$u_j'' - \frac{v^2}{2}\frac{\partial D}{\partial u_j} + D\frac{\partial \mathcal{U}}{\partial u_j} = - D\frac{\partial V}{\partial u_j} , \tag{1,7}$$

where v^2 is given by

$$v^2 = \frac{1}{D}\sum u_j'^2 . \tag{1,8}$$

On the right-hand side of (1,7) appear the perturbing forces

$$q_j = - \frac{\partial V}{\partial u_j}$$

in the parametric plane. They may be computed from the perturbing forces p_i in the physical plane by the formulae

$$p_i = - \frac{\partial V}{\partial x_i} , \quad q_j = - \frac{\partial V}{\partial u_j} = - \sum_{(i)} \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial u_j} ,$$

$$q_j = \sum_{(i)} \frac{\partial x_i}{\partial u_j} p_i . \tag{1,9}$$

(1,7) becomes

$$u_j'' - \frac{v^2}{2}\frac{\partial D}{\partial u_j} + D\frac{\partial \mathcal{U}}{\partial u_j} = D q_j . \tag{1,10}$$

This equation (1,10), derived above for the case in which the perturbing forces can be derived from a potential, is also valid if this is not possible, so long as the

q_j are computed using (1,9). The last step of regularization is the elimination of v^2 by the vis viva integral

$$\frac{v^2}{2} + \mathcal{U} = h + W . \tag{1,11}$$

h is the constant of energy and

$$W = \int \sum p_i \, dx_i = \int \sum q_j \, du_j \tag{1,12}$$

the work done by the perturbing forces. The result is

$$u_j'' + \frac{\partial}{\partial u_j} [D(\mathcal{U} - h)] = D q_j + \frac{\partial D}{\partial u_j} W . \tag{1,13}$$

This system of differential equations is perfectly <u>regular</u> if the pole of $\mathcal{U}$ at an attracting center is compensated by an appropriate zero of D.

A few remarks concerning <u>initial conditions</u> are in order. We have

$$\dot{x}_1 = \frac{\partial x_1}{\partial u_1} \dot{u}_1 + \frac{\partial x_1}{\partial u_2} \dot{u}_2 , \qquad \dot{x}_2 = \frac{\partial x_2}{\partial u_1} \dot{u}_1 + \frac{\partial x_2}{\partial u_2} \dot{u}_2 .$$

By solving for $\dot{u}_1, \dot{u}_2$ and taking into account (1,1) the formulae

$$\dot{u}_1 = \frac{1}{D} \left(\frac{\partial x_1}{\partial u_1} \dot{x}_1 + \frac{\partial x_2}{\partial u_1} \dot{x}_2 \right) , \qquad \dot{u}_2 = \frac{1}{D} \left(\frac{\partial x_1}{\partial u_2} \dot{x}_1 + \frac{\partial x_2}{\partial u_2} \dot{x}_2 \right)$$

are obtained; thus from (1,6)

$$u_j' = \sum_{(i)} \frac{\partial x_i}{\partial u_j} \dot{x}_i \tag{1,14}$$

This enables us to compute at instant $t = s = 0$ the velocities u_j' in the parametric plane from the given velocities $\dot{x}_i$ in the physical plane. Denoting values at this instant $t = s = 0$ by the subscript o, we have also

$$h = \frac{v_o^2}{2} + \mathcal{U}_o , \qquad W = \int_o^s q_j \, du_j . \tag{1,15}$$

Sometimes it is pratical to introduce a <u>scaling factor</u> $\lambda(u_j)$ in the definition of the fictitious time:

$$dt = \lambda D \, ds . \tag{1,16}$$

This slightly more general regularization leads to the following basic and final set of formulae.

Notations

	physical space	parametric space
coordinates	x_i	u_j
velocity	v	
time	$t,\ \frac{d()}{dt} = (\dot{\ })$	$s,\ \frac{d()}{ds} = ()'$
conservative potential	$\mathcal{U}(x_i)$	$\mathcal{U}(u_j)$
perturbing forces	p_i	q_j
work	$W = \int \sum p_i\, dx_i$	$W = \int \sum q_j\, du_j$

Transformations

coordinates $\quad x_i = x_i(u_j), \quad D = \sum_{(j)} \left(\frac{\partial x_i}{\partial u_j}\right)^2 \quad$ (for any i) $\quad$ (1,17)

time $\quad dt = \lambda D\, ds \quad$ (1,18)

velocity $\quad \dot{x}_i = \frac{1}{\lambda D} \sum_{(j)} \frac{\partial x_i}{\partial u_j} u_j', \quad u_j' = \lambda \sum_{(i)} \frac{\partial x_i}{\partial u_j} \dot{x}_i \quad$ (1,19)

$$v^2 = \frac{1}{\lambda^2 D} \sum u_j'^2 \tag{1,20}$$

perturbing force $\quad q_j = \sum_{(i)} \frac{\partial x_i}{\partial u_j} p_i\ . \quad$ (1,21)

Equations of motion

$$\frac{1}{\lambda} \frac{d}{ds}\left(\frac{u_j'}{\lambda}\right) - \frac{v^2}{2} \frac{\partial D}{\partial u_j} + D \frac{\partial \mathcal{U}}{\partial u_j} = D q_j \tag{1,22}$$

or

$$\frac{1}{\lambda} \frac{d}{ds}\left(\frac{u_j'}{\lambda}\right) + \frac{\partial}{\partial u_j}[D(\mathcal{U} - h)] = D q_j + \frac{\partial D}{\partial u_j} W \tag{1,23}$$

$$h = \frac{v_0^2}{2} + \mathcal{U}_0\ . \tag{1,24}$$

(v_0, $\mathcal{U}_0$ = initial velocity and potential).

1.1.1 Transformation of Levi-Civita. In the sequel of the paper we consider only gravitational forces described by Newton's law of attraction. We begin with the simplest case of a single attracting center located at the origin of physical x-plane; if the classical equations of motion are used, the attractive force becomes infinite, if the particle is at the origin. Levi-Civita [1] has developed in a famous paper a method for removing this singularity by introducing the parametric u-plane and using the simplest mapping of the u-plane onto the x-plane satisfying the requirement to double angles at the origin and be conformal elsewhere. This transformation is (Fig. 1.1)

$$x = u^2; \qquad x_1 = u_1^2 - u_2^2, \qquad x_2 = 2u_1 u_2. \tag{1,25}$$

The distance r of the particle from the origin of the physical plane is

$$r = |x| = |u|^2 = \sum u_j^2, \tag{1,26}$$

and from (1,2) we obtain

$$D = \left|\frac{dx}{du}\right|^2 = 4|u|^2 = 4r,$$

$$\frac{\partial D}{\partial u_j} = 4\frac{\partial r}{\partial u_j} = 4\frac{\partial}{\partial u_j}\sum u_k^2 = 8u_j.$$

With the choice $\lambda = \frac{1}{4}$ of the scaling factor the equations (1,23) of motion become

$$4u_j'' + \frac{\partial}{\partial u_j}(r\mathcal{U}) - 2hu_j = rq_j + 2u_j W.$$

For the Newtonian gravitation the product $(r\mathcal{U})$ is a constant; thus the equations are reduced to

$$4u_j'' - 2hu_j = rq_j + 2u_j W, \tag{1,27}$$

and in particular the Kepler motion about the attracting center is given by the differential equations

$$4u_j'' - 2hu_j = 0 \tag{1,28}$$

because no perturbing forces are acting. These equations are not only regular at the origin but also linear with constant coefficients. This brings out the deeper reason for the fact, that regularization is not only useful for collision orbits but also for orbits of modest eccentricity. If h is negative the motion (1,28) is a harmonic oscillation. The orbit of the image-point in the u-plane is an ellipse centered at the origin and mapped onto an ellipse of the physical plane focused at the central body.

1.1.2 Birkhoff's Transformation. For the transfer orbit of a vehicle from earth to moon a simultaneous regularization at both attracting centers is needed. This was performed by Birkhoff [2]. In order to facilitate the generalization to 3-dimensional motion, we give a somewhat modified account of his lines of approach to the problem. The orbit of the moon about the earth is assumed to be a perfect circle.

A rotating coordinate system y_1, y_2 is introduced (Fig. 1.2) in such a way that earth and moon occupy fixed places on the y_1-axis, the origin being their center of gravity. The problem of computing the orbit of a particle of negligible mass in this force field is known as the restricted circular problem. We are still restricted of course to planar orbits in the y-plane. By convenient choice of the units of mass, time and distance we may assume that

1. The total mass of earth and moon $= 1$.
2. The distance of the moon from the earth $= 1$.
3. The gravitational constant $= 1$.

Denoting the mass of the moon by μ we find this body at $(1-\mu, 0)$ and the earth of mass $(1-\mu)$ at $(-\mu, 0)$. The angular velocity of the rotating system is $= 1$ as follows from the third law of Kepler. Finally we denote by r_1, r_2 the distances of the moving particle from the earth and the moon respectively.

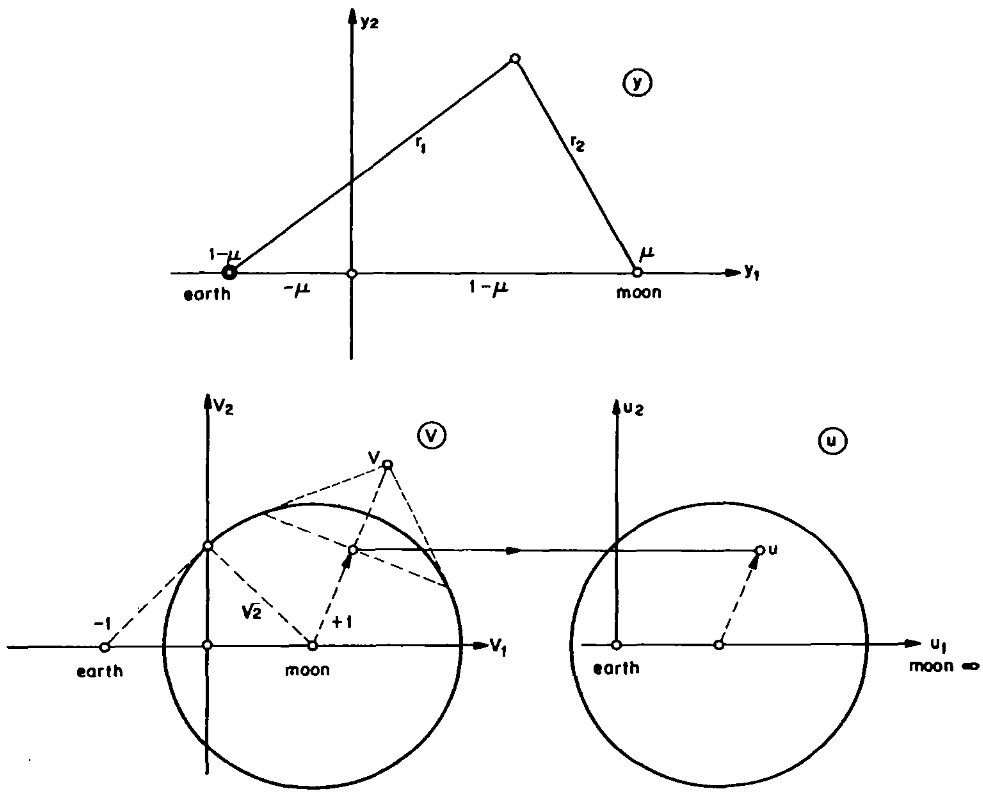


Fig. 1.2. Birkhoff's Transformation.

In the problem at hand the conservative potential $\mathcal{U}$ is composed of the two gravitational potentials and the potential of the centrifugal force:

$$\mathcal{U} = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}(y_1^2 + y_2^2)$$

or up to a non-essential additive constant

$$\mathcal{U} = -(1-\mu)\left(\frac{1}{r_1} + \frac{1}{2} r_1^2\right) - \mu\left(\frac{1}{r_2} + \frac{1}{2} r_2^2\right) ; \tag{1,29}$$

the perturbing force is the Coriolis force

$$P_1 = 2\dot{y}_2 , \qquad P_2 = -2\dot{y}_1 . \tag{1,30}$$

The key for achieving the desired regularization is the remark that Levi-Civita's transformation (section 1.1.1) has not only regularizing properties at the origin but also at infinity. It is therefore sufficient to throw the earth into the origin and the moon into infinity by appropriate and elementary conformal transformations. The following chain of mappings is proposed. (The v-plane (listed first in the table) is the parametric plane corresponding to the regularized equations of motion, the y-plane (listed at the foot of the table) is the physical plane of Fig. 1.2).

space	coordinates	abscissa of earth	abscissa of moon	Transformation
1	v_j	$-\frac{1}{2}$	$\frac{1}{2}$	
				$V_j = 2v_j$
2	V_j	-1	1	
				Inversion
3	u_j	0	∞	
				Levi-Civita (KS)
4	x_i	0	∞	
				Inversion
5	Y_i	-1	1	
				$y_1 = \frac{1}{2}Y_1 + (\frac{1}{2}-\mu)$
6	y_i	$-\mu$	$1-\mu$	

(1,31)

By inversion is understood a transformation by reciprocal radii. The center of inversion is at the point $(1, 0)$ and the radius of inversion is $\sqrt{2}$. (This statement is valid for the transformation $2 \longrightarrow 3$ as well as for $4 \longrightarrow 5$; Fig. 1.2 illustrates the mapping $2 \longrightarrow 3$). The transformations $1 \longrightarrow 2$ and $5 \longrightarrow 6$ are only unimportant adjustments; the essential transformations $2 \longrightarrow 5$ are conveniently expressed in complex notation by

$$u - 1 = \frac{2}{\overline{V} - 1} , \qquad x = u^2, \qquad Y - 1 = \frac{2}{\overline{x} - 1} ,$$

where $\overline{V}$ is the complex conjugate of V ($V = V_1 + i V_2$). These give for transformations $2 \longrightarrow 5$,

$$Y = \frac{1}{2}\left(V + \frac{1}{V}\right) \tag{1,32}$$

and so, the complete transformation $1 \longrightarrow 6$ is

$$y = \tfrac{1}{2} Y + (\tfrac{1}{2} - \mu) = \tfrac{1}{4}\left(V + \frac{1}{V}\right) + (\tfrac{1}{2} - \mu),$$

$$y = \tfrac{1}{2}\left(v + \frac{1}{4v}\right) + (\tfrac{1}{2} - \mu). \tag{1,33}$$

In real notation this may be written

$$\begin{aligned} y_1 &= (\tfrac{1}{2} - \mu) + \tfrac{1}{2}\left(v_1 + \frac{\tfrac{1}{4} v_1}{v_1^2 + v_2^2}\right), \\ y_2 &= \qquad\qquad\;\; \tfrac{1}{2}\left(v_2 - \frac{\tfrac{1}{4} v_2}{v_1^2 + v_2^2}\right). \end{aligned} \tag{1,34}$$

The distances r_1, r_2 have the following expressions:

$$r_1 = |y + \mu| = \tfrac{1}{2}\left|v + \frac{1}{4v} + 1\right| = \tfrac{1}{2}\frac{|v + \frac{1}{2}|^2}{|v|}, \tag{1,35}$$

$$r_2 = |y + \mu - 1| = \tfrac{1}{2}\left|v + \frac{1}{4v} - 1\right| = \tfrac{1}{2}\frac{|v - \frac{1}{2}|^2}{|v|}. \tag{1,36}$$

The absolute value in the numerator of (1,35) is the distance of the image of the particle in the parametric plane from the image of the earth. For establishing the equations of motion the scheme (1,17) - (1,24) is applied.

$$D = \left|\frac{dy}{dv}\right|^2 = \tfrac{1}{4}\left|1 - \frac{1}{4v^2}\right| = \tfrac{1}{4}\frac{|v - \frac{1}{2}||v + \frac{1}{2}|}{|v|^4} = \frac{r_1 r_2}{|v|^2},$$ [illegible]

$$D = \frac{r_1 r_2}{v_1^2 + v_2^2}. \tag{1,37}$$

For the computation of the Coriolis forces q_j in the parametric space the abbreviations

$$b_{ik} = \frac{\partial y_i}{\partial v_k} \tag{1,38}$$

are introduced. With $\lambda = 1$ we have from (1,19)(1,21)(1,30)

$$\dot{y}_i = \frac{1}{D}\sum b_{ik} v_k',$$

$$q_j = \sum b_{ij} p_i = 2(b_{1j}\dot{y}_2 - b_{2j}\dot{y}_1),$$

$$q_j = \frac{2}{D}\sum_{(k)}(b_{1j} b_{2k} - b_{2j} b_{1k}) v_k'. \tag{1,39}$$

Because the Coriolis forces do no work, the equations (1,23) of motion are finally

$$v_j'' + \frac{\partial}{\partial v_j}[D(\mathcal{U} - h)] = 2\sum_{(k)}(b_{1j} b_{2k} - b_{2j} b_{1k}) v_k'. \tag{1,40}$$

Here the expression

$$D\,\mathcal{U} = \frac{1}{v_1^2+v_2^2}\left[-(1-\mu)(r_2+\tfrac{1}{2}r_1^3 r_2) - \mu\,(r_1+\tfrac{1}{2}r_1 r_2^3)\right] \tag{1,41}$$

has no longer singularities at the attracting centers; thus equations (1,40) are perfectly regular. From (1,20)(1,11) we obtain the energy relation

$$h = \frac{1}{2D}\sum v_j'^{\,2} + \mathcal{U} \tag{1,42}$$

(observe $W=0$) and after integration of the differential equations (1,40) the physical time t is given by (1,18)

$$t = \int_0^s D\,ds\ . \tag{1,43}$$

By Birkhoff's transformation a new singularity is produced at the origin of the parametric v-plane. This can be seen from (1,41). This event does not generate a serious danger because this origin corresponds to the point at infinity of the physical plane. T.N.Thiele removed also this singularity by substituting for V in (1,32) an exponential not attaining the value 0. His transformation is

$$V = e^{iz}\,, \qquad Y = \cos z\,, \qquad y = (\tfrac{1}{2}-\mu) + \tfrac{1}{2}\cos z\ .$$

It is worthy of note that the right-hand sides of (1,40) can be simplified with the help of the Cauchy-Riemann equations for the analytical function $y(v)$. These expressions are reduced for $j=1,2$ to $(2D\,v_2')$ and $(-2D\,v_1')$ respectively. But we do not take advantage of this fact because it is no longer true for the 3-dimensional motion of a particle.

1.2 Motion in 3-dimensional space

In this section we consider the motion of a particle moving in the 3-dimensional physical space referred to rectangular coordinates x_1, x_2, x_3. It turns out that a generalization of the methods of section 1.1 to 3-dimensional motion is impossible if only three generalized coordinates u_1, u_2, u_3 are introduced. But almost all such methods have their adequate generalization if we are allowed to fix the position of our particle by 4 parameters u_1, u_2, u_3, u_4 related by a non-holonomic condition. Thus the parametric space will be a 4-dimensional space.

1.2.1 The KS-Transformation. This is the generalization of Levi-Civita's transformation described in section 1.1.1. The 4 parameters u_j are introduced by the following definitions:

$$\boxed{\begin{aligned} x_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ x_2 &= 2(u_1 u_2 - u_3 u_4) \\ x_3 &= 2(u_1 u_3 + u_2 u_4) \end{aligned}} \tag{1,44}$$

For $u_3 = u_4 = 0$ this coincides indeed with (1,25). As in (1,26) the distance r of the particle from the origin of the physical space is given by

$$r = \sum u_j^2 , \tag{1,45}$$

where summation goes from 1 to 4. This follows from (1,44) by explicit verification. Also we have

$$r + x_1 = 2(u_1^2 + u_4^2) , \qquad r - x_1 = 2(u_2^2 + u_3^2) . \tag{1,46}$$

This furnishes the following two alternatives for the computation of the u_j from the x_i :

$$\begin{aligned} u_1^2 + u_4^2 &= \tfrac{1}{2}(r + x_1) , & u_2^2 + u_3^2 &= \tfrac{1}{2}(r - x_1) , \\ u_2 &= \frac{x_2 u_1 + x_3 u_4}{r + x_1} , & u_1 &= \frac{x_2 u_2 + x_3 u_3}{r - x_1} , \\ u_3 &= \frac{x_3 u_1 - x_2 u_4}{r + x_1} , & u_4 &= \frac{x_3 u_2 - x_2 u_3}{r - x_1} . \end{aligned} \tag{1,47}$$

The second and third line are obtained by solving the second and third equation (1,44) with respect to u_2, u_3 or with respect to u_1, u_4. The u_j are of course only determined after choice of one among them, but this is irrelevant for our purposes.

The transformation (1,44) has been studied in the article [3] and many conformal properties have been recorded. It follows from these considerations that the basic formulae (1,17) - (1,24) are applicable with the only modification that summation runs from 1 to 3 in the physical space and from 1 to 4 in the parametric space. With $\lambda = \frac{1}{4}$ we obtain immediately

(1,17) $$D = \sum_{(j)} \left(\frac{\partial x_i}{\partial u_j}\right)^2 = 4(u_1^2 + u_2^2 + u_3^2 + u_4^2) = 4r , \qquad \frac{\partial D}{\partial u_j} = 8 u_j ,$$

(1,18) $$dt = r \, ds , \tag{1,47a}$$

(1,19) $$\begin{aligned} u_1' &= \tfrac{1}{2}(u_1 \dot{x}_1 + u_2 \dot{x}_2 + u_3 \dot{x}_3) , \\ u_2' &= \tfrac{1}{2}(-u_2 \dot{x}_1 + u_1 \dot{x}_2 + u_4 \dot{x}_3) , \\ u_3' &= \tfrac{1}{2}(-u_3 \dot{x}_1 - u_4 \dot{x}_2 + u_1 \dot{x}_3) , \\ u_4' &= \tfrac{1}{2}(u_4 \dot{x}_1 - u_3 \dot{x}_2 + u_2 \dot{x}_3) . \end{aligned} \tag{1,48}$$

(1,20) $$v^2 = \frac{4}{r} \sum u_j'^2 , \tag{1,49}$$

(1,21) $$\begin{aligned} q_1 &= 2(u_1 p_1 + u_2 p_2 + u_3 p_3) , \\ q_2 &= 2(-u_2 p_1 + u_1 p_2 + u_4 p_3) , \\ q_3 &= 2(-u_3 p_1 - u_4 p_2 + u_1 p_3) , \\ q_4 &= 2(u_4 p_1 - u_3 p_2 + u_2 p_3) . \end{aligned} \tag{1,50}$$

Let us assume now that our particle is subjected to the gravitational attraction of a body located at the origin and to some unspecified perturbing forces. Thus the potential is

$$\mathcal{U} = -\frac{M}{r} , \tag{1,51}$$

where M is the product of the gravitational constant with the mass of the central body. Our automatic formula generator goes on as follows:

(1,22) $$4u_j'' + \left(\frac{2M}{r} - v^2\right)u_j = rq_j , \tag{1,52}$$

(1,23) $$4u_j'' - 2h\,u_j = rq_j + 2\,W u_j , \tag{1,53}$$

(1,24) $$h = \frac{v_0^2}{2} + \mathcal{U}_0 = \frac{v_0^2}{2} - \frac{M}{r_0} , \qquad W = \int_0^s \sum q_j\, du_j . \tag{1,54}$$

The set (1,48) of equations implies

$$u_4 u_1' - u_3 u_2' + u_2 u_3' - u_1 u_4' = 0 . \tag{1,55}$$

This is the non-holonomic condition mentioned at the beginning of this section. Equation (1,49) transforms the vis viva integral (1,11) into

$$2\sum u_j'^2 = M + r(h + W). \tag{1,56}$$

This set of formulae is, in itself, a collection of guiding rules for the numerical computation of an orbit. Let us call it

<u>First procedure</u>

(Perturbed motion of a particle about a central body; computation of the parameters u_j as functions of the fictitious time s.)

<u>Initial conditions</u>. Compute initial position and velocity of the particle from (1,47) and (1,48), also h from (1,54). $W_0 = 0$.

<u>Differential equations</u>. Integrate the system of 10 simultaneous equations of first order

$$\begin{aligned} &4u_j'' - 2h\,u_j = rq_j + 2Wu_j , \quad j = 1,2,3,4, \\ &t' = r , \qquad W' = \sum q_j u_j' . \end{aligned} \tag{1,57}$$

At each step r, x_i, q_j are computed from (1,45)(1,44)(1,50), the perturbing forces p_i in the physical space being known from other sources.[1] (1,55) and (1,56) are used as checks.

As far as the author knows, this simple procedure has never been used for explicit numerical computations. It will be modified and refined in sections 1.3 and 1.4 for <u>elliptic</u> initial conditions but it is possibly successful for <u>hyperbolic</u>, <u>parabolic</u> or <u>near-parabolic</u> initial conditions. In such a case we advocate to compute the <u>perturbations</u>[2]

$$\Delta u_j = u_j - u_{jK} , \quad \Delta r = r - r_K , \quad \Delta t = t - t_K \tag{1,58}$$

---

[1] If no collision occurs, the perturbing forces must not remain finite at the origin as was assumed in section 1.1.

[2] Throughout the paper the subscript K indicates values corresponding to the unperturbed Kepler motion.

of the coordinates of distance and of time. From (1,45) it follows

$$\Delta r = \sum (2u_{jK} + \Delta u_j)\Delta u_j \,, \tag{1,59}$$

thus (1,57) can be transformed into

$$4(\Delta u_j)'' - 2h\,\Delta u_j = (r_K + \Delta r)\,q_j + 2W(u_{jK} + \Delta u_j)\,, \tag{1,60}$$

$$(\Delta t)' = \Delta r \quad , \qquad W' = \sum q_j\,(u_{jK}' + (\Delta u_j)')\,, \tag{1,61}$$

where Δr is given by (1,59). This arrangement of the rules for computation avoids the loss of significant figures by subtraction of almost equal numerical values. The computation of the unperturbed Kepler orbit is described by equations (1,76) (1,87)(1,88) in section 1.3.

The equations (1,52) of motion have not been taken into account in our first procedure. They have the advantage that they avoid the computation of the work W but they suffer from the fact that both quantities $2M/r$ and v^2 are infinite at collision. Nevertheless these equations are very useful for the discussion of the osculating Kepler orbit in section 1.4.

1.2.2 The B_3-Transformation. The generalization of Birkhoff's transformation (section 1.1.2) to 3-dimensional motion is immediate. The y-coordinate system is supplemented by a y_3-axis perpendicular to the plane of Fig. 1.2 and the particle is allowed to move in space, r_1, r_2 denoting as before its distances to the attracting centers (earth and moon). The potential (1,29) is modified by a term containing y_3 and becomes

$$\mathcal{U} = -(1-\mu)\left(\frac{1}{r_1} + \frac{1}{2}r_1^2\right) - \mu\left(\frac{1}{r_2} + \frac{1}{2}r_2^2\right) + \frac{1}{2}y_3^2 \,. \tag{1,62}$$

Again 4 generalized coordinates V_j are introduced for describing the motion of the particle and the chain (1,31) of transformations is applied with the only modification that the mapping of space 3 onto space 4 is performed by the KS-transformation just discussed. Hence the spaces 1 through 3 have four dimensions and the remaining spaces 4 through 6 only three. The inversion $2 \longrightarrow 3$ for instance is given by the formulae

$$u_1 - 1 = \frac{2(V_1-1)}{(V_1-1)^2 + V_2^2 + V_3^2 + V_4^2}\,, \qquad u_k = \frac{2V_k}{(V_1-1)^2 + V_2^2 + V_3^2 + V_4^2}\,, \qquad k = 2,3,4. \tag{1,63}$$

The composition of the 5 transformations of table (1,31) is a little tedious because complex notation is no longer available. The final result is

$$\begin{aligned} y_1 &= (\tfrac{1}{2} - \mu) + \tfrac{1}{2}\left[V_1 + \frac{V_1(V_4^2 + \frac{1}{4})}{V_1^2 + V_2^2 + V_3^2}\right], \\ y_2 &= \tfrac{1}{2}\left[V_2 + \frac{V_2(V_4^2 - \frac{1}{4}) - V_3V_4}{V_1^2 + V_2^2 + V_3^2}\right], \\ y_3 &= \tfrac{1}{2}\left[V_3 + \frac{V_3(V_4^2 - \frac{1}{4}) + V_2V_4}{V_1^2 + V_2^2 + V_3^2}\right]. \end{aligned} \tag{1,64}$$

For $v_3 = v_4 = 0$ this reduces to the previous transformation (1,34). Because inversions are conformal mappings in 4-dimensional as well as in 3-dimensional spaces, the prescriptions (1,17) - (1,24) for establishing the equations of motion still hold true; one obtains

(1,17) $$D = \sum_{(j)} \left(\frac{\partial y_i}{\partial v_j}\right)^2 = \frac{r_1 r_2}{v_1^2 + v_2^2 + v_3^2} \ , \tag{1,65}$$

where

$$r_1 = \frac{1}{2}\,\frac{(v_1+\frac{1}{2})^2 + v_2^2 + v_3^2 + v_4^2}{\sqrt{v_1^2+v_2^2+v_3^2}} \ , \qquad r_2 = \frac{1}{2}\,\frac{(v_1-\frac{1}{2})^2 + v_2^2 + v_3^2 + v_4^2}{\sqrt{v_1^2+v_2^2+v_3^2}} \ .$$

By choosing $\lambda = 1$, we have for the fictitious time s

(1,18) $$dt = D\,ds\ . \tag{1,66}$$

The transportation of the Coriolis forces into the parametric space u_j leads to exactly the same results as before, namely

$$q_j = \frac{2}{D}\sum_{(k)} (b_{1j} b_{2k} - b_{2j} b_{1k})\, v_k' \ , \qquad b_{ik} = \frac{\partial y_i}{\partial v_k} \ , \tag{1,67}$$

where k is now running from 1 to 4 and the equations of motion (1,23) are literally the same as in (1,40), that is

$$v_j'' + \frac{\partial}{\partial v_j}[D(\mathcal{U}-h)] = 2\sum_{(k)} (b_{1j} b_{2k} - b_{2j} b_{1k})\, v_k' \ , \quad j = 1,2,3,4, \tag{1,68}$$

and

$$h = \frac{1}{2D}\sum v_j'^2 + \mathcal{U}\ . \tag{1,69}$$

The initial position of the particle in the parametric space can be computed by making use of table (1,31) in the reverse order, using the formulae (1,47) for the inverse KS-transformation. Initial velocities are taken from (1,19)

$$v_k' = \sum_{(i)} b_{ik}\, \dot{x}_i \ . \tag{1,70}$$

After integration of the equations (1,68) the physical time is computed from

$$t = \int D\,ds \ . \tag{1,71}$$

The foregoing brief description of the B_3-transformation is adequate for our purposes. A thorough analysis with detailed proofs is given in [4]. Further information is contained in chapter 3 of this report (Waldvogel); there the B_3-transformation is established for the more general elliptic restricted problem, where the moon is allowed to move on an elliptic Kepler orbit.

1.3 Kepler motion

1.3.1 The unperturbed motion of a particle about a central body is governed by the equations (1,53)(1,54)

$$u_j'' + \left(\frac{M}{2r_0} - \frac{v_0^2}{4}\right) u_j = 0\,, \qquad j = 1,2,3,4\,, \tag{1,72}$$

where r_0, v_0 are respectively the initial distance and the velocity in the physical space. If the coefficient of u_j in (1,72) is positive, we may introduce a frequency ω by

$$\omega^2 = \frac{M}{2r_0} - \frac{v_0^2}{4} \tag{1,73}$$

and write our equations

$$u_j'' + \omega^2 u_j = 0\,. \tag{1,74}$$

Thus the motion of the image of the particle in the parametric space is a harmonic oscillation and its orbit is an ellipse centered at the origin. This orbit is mapped by the KS-transformation (1,44) onto a Kepler ellipse in the physical space and if the image makes one revolution in the u-space, the particle itself makes two in the physical space. Its velocity v is determined by (1,49)

$$v^2 = \frac{4}{r}\sum u_j'^2\,, \qquad r = \sum u_j^2\,; \tag{1,75}$$

r, given by (1,45), is the distance of the particle from the origin of the physical space during its flight. By integration of the equations of motion we obtain

$$u_j = \alpha_j \cos\omega s + \beta_j \sin\omega s\,, \qquad u_j' = \omega(-\alpha_j \sin\omega s + \beta_j \cos\omega s)\,. \tag{1,76}$$

s is the fictitious time satisfying $dt = r\,ds$ and α_j, β_j are constants which are computed from the initial conditions as follows

$$\alpha_j = (u_j)_0\,, \qquad \beta_j = \frac{1}{\omega}(u_j')_0\,. \tag{1,77}$$

Obviously the 8 parameters α_j, β_j characterize the motion of the particle; we call them the regularized elements of the orbit. From (1,55) it follows at instant $s = 0$

$$\alpha_4\beta_1 - \alpha_3\beta_2 + \alpha_2\beta_3 - \alpha_1\beta_4 = 0\,. \tag{1,78}$$

Furthermore (1,56) can be written for $s = 0$

$$2\omega^2\sum\beta_j^2 = M + r_0 h = M + r_0\left(\frac{v_0^2}{2} - \frac{M}{r_0}\right) = M - 2\omega^2 r_0\,,$$

where (1,54) and (1,73) are used. Thus it follows from (1,75) and (1,77)

$$2\omega^2\sum(\alpha_j^2 + \beta_j^2) = M\,. \tag{1,79}$$

The 9 parameters $\omega, \alpha_j, \beta_j$ are thus related by the two identities (1,78)(1,79). This remark reduces the number of independent parameters to 7 exceeding by one the classical number. This stems from the fact that the mapping of an orbit from physical into parametric space is not unique.

We shall next compute distance r and time t in the physical space.

(1,75)(1,76) $$r = \left(\sum \alpha_j^2\right)\cos^2\omega s + \left(\sum \beta_j^2\right)\sin^2\omega s + 2\left(\sum \alpha_j\beta_j\right)\sin\omega s\cos\omega s\,,$$

$$r = \frac{1}{2}\sum(\alpha_j^2+\beta_j^2) + \frac{1}{2}\cos 2\omega s\sum(\alpha_j^2-\beta_j^2) + \sin 2\omega s\sum \alpha_j\beta_j\,, \tag{1,80}$$

$$t = \int r\,ds = \frac{s}{2}\sum(\alpha_j^2+\beta_j^2) + \frac{1}{4\omega}\sin 2\omega s\sum(\alpha_j^2-\beta_j^2) + \frac{1}{2\omega}(1-\cos 2\omega s)\sum\alpha_j\beta_j\,. \tag{1,81}$$

These formulae together with (1,76) and (1,44) determine a given Kepler motion explicitly.

We now proceed to establish some connections with the classical theory and its notations. The time T of revolution in the physical space is attained for $\omega s = \pi$, thus

(1,81)(1,79) $$T = \frac{\pi}{2\omega}\sum(\alpha_j^2+\beta_j^2) = \frac{\pi M}{4\omega^3}\,. \tag{1,82}$$

If a denotes the semi-major axis of the Kepler ellipse in the physical space, we have from Kepler's third law

$$T = \frac{2\pi}{\sqrt{M}}\,a^{3/2}\,,$$

and confrontation with (1,82) furnishes

$$a = \frac{M}{4\omega^2}\,. \tag{1,83}$$

By inserting this into (1,79) we obtain the important result

$$a = \frac{1}{2}\sum(\alpha_j^2+\beta_j^2)\,. \tag{1,84}$$

The mean angular velocity μ of the particle is

(1,82) $$\mu = \frac{2\pi}{T} = \frac{4\omega}{\sum(\alpha_j^2+\beta_j^2)}\,. \tag{1,85}$$

By inserting the value (1,73) of ω into (1,83), we obtain a well-known relation of classical celestial mechanics,

$$\frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{M}\,, \tag{1,86}$$

which holds true at any point of the Kepler orbit.

These formulae are a little simplified if the initial position of the particle is the <u>pericenter</u> of the Kepler orbit. Denoting by e, E <u>eccentricity</u> and <u>eccentric anomaly</u> comparison of (1,80) with the classical formula

$$r = a(1 - e\cos E)$$

leads to the result

$$e = -\frac{1}{2a}\sum(\alpha_j^2-\beta_j^2)\,, \qquad E = 2\omega s\,, \qquad \sum \alpha_j\beta_j = 0\,. \tag{1,87}$$

It still remains to consider the cases where the coefficient of u_j in (1,72) is negative or vanishing. If the former event occurs, we have equations of the type

$$u_j'' - \omega^2 u_j = 0\,, \quad u_j = \alpha_j\, e^{\omega s} + \beta_j\, e^{-\omega s}\,,$$

and the orbit is hyperbolic. A vanishing coefficient leads to

$$u_j'' = 0\,, \quad u_j = \alpha_j s + \beta_j\,.$$

This orbit is a straight line in the parametric space and a parabola in the physical space.

1.3.2 Variation of the elements under the influence of perturbing forces. Returning to the general elliptic case we may write equations (1,53)(1,54)

$$u_j'' + \omega^2 u_j = F_j\,, \tag{1,88}$$

$$\omega^2 = \frac{M}{2r_0} - \frac{v_0^2}{4}\,, \qquad F_j = \frac{1}{4}(r q_j + 2\dot{W} u_j)\,. \tag{1,89}$$

This system is integrated by the familiar method of variation of constants. We put

$$u_j = \alpha_j(s)\cos\omega s + \beta_j(s)\sin\omega s\,, \qquad u_j' = \omega(-\alpha_j(s)\sin\omega s + \beta_j(s)\cos\omega s)\,, \tag{1,90}$$

thus introducing varying elements $\alpha_j(s)$, $\beta_j(s)$. They must satisfy the differential equations

$$\alpha_j' = -\frac{1}{\omega}F_j\sin\omega s\,, \qquad \beta_j' = \frac{1}{\omega}F_j\cos\omega s\,. \tag{1,91}$$

In order to rewrite the energy equation (1,56), we use (1,54) and (1,45) namely

$$h = \frac{v_0^2}{2} - \frac{M}{r_0} = -2\omega^2\,,$$

$$r = \sum(\alpha_j\cos\omega s + \beta_j\sin\omega s)^2\,,$$

or

(1,80) $$r = \frac{1}{2}\sum(\alpha_j^2+\beta_j^2) + \frac{1}{2}\cos 2\omega s\sum(\alpha_j^2-\beta_j^2) + \sin 2\omega s\sum\alpha_j\beta_j\,, \tag{1,92}$$

$$\sum u_j'^2 = \omega^2\sum(-\alpha_j\sin\omega s + \beta_j\cos\omega s)^2\,;$$

thus

(1,56) $$2\omega^2\sum(-\alpha_j\sin\omega s + \beta_j\cos\omega s)^2 = M - 2\omega^2\sum(\alpha_j\cos\omega s + \beta_j\sin\omega s)^2 + rW\,,$$

or

$$rW = 2\omega^2\sum(\alpha_j^2+\beta_j^2) - M\,. \tag{1,93}$$

We now collect the formulae of this section and section 1.2.1. This collection is our

Second procedure

(Perturbed motion of a particle about a central body; elliptical initial conditions. Variation of elements.)

Data. M - product of gravitational constant and mass of the central body located at the origin of a cartesian system x_1, x_2, x_3.

p_i - components of the perturbing force (per unit of mass of the particle).

At instant $t=0$ the position x_i and velocities $\dot{x}_i$ of the particle are given.

Initial conditions. At instant $t=0$ compute the initial values of the generalized coordinates u_1, u_2, u_3, u_4 of the particle by either of the two sets

$$
\begin{aligned}
u_1^2+u_4^2 &= \tfrac{1}{2}(r+x_1), & u_2^2+u_3^2 &= \tfrac{1}{2}(r-x_1), \\
u_2 &= \frac{x_2u_1+x_3u_4}{r+x_1}, & u_1 &= \frac{x_2u_2+x_3u_3}{r-x_1}, \qquad r=\sqrt{\sum x_i^2}, \\
u_3 &= \frac{x_3u_1-x_2u_4}{r+x_1}, & u_4 &= \frac{x_3u_2-x_2u_3}{r-x_1}.
\end{aligned}
$$

Take the left- (right-) hand set if $x_1 \geq 0$ $(x_1<0)$ and choose u_4 (u_3) arbitrarily. At instant $t=0$ compute also

$$
\begin{aligned}
u_1' &= \tfrac{1}{2}(u_1\dot{x}_1+u_2\dot{x}_2+u_3\dot{x}_3), \\
u_2' &= \tfrac{1}{2}(-u_2\dot{x}_1+u_1\dot{x}_2+u_4\dot{x}_3), \\
u_3' &= \tfrac{1}{2}(-u_3\dot{x}_1-u_4\dot{x}_2+u_1\dot{x}_3), \\
u_4' &= \tfrac{1}{2}(u_4\dot{x}_1-u_3\dot{x}_2+u_2\dot{x}_3).
\end{aligned}
\qquad
\omega^2 = \frac{M}{2r_0}-\frac{v_0^2}{4},
$$

(r_0, v_0 - initial distance and velocity)

The initial values $(\alpha_j)_0, (\beta_j)_0$ of the elements α_j, β_j are now given by

$$(\alpha_j)_0 = u_j, \qquad (\beta_j)_0 = \frac{1}{\omega}u_j'.$$

Furthermore at instant $t=0$ we have the initial values

$$t_0=0, \quad W_0=0.$$

Differential equations. (argument s)

$$
\begin{aligned}
\alpha_j' &= -\frac{1}{\omega}F_j\sin\omega s, \qquad \beta_j' = \frac{1}{\omega}F_j\cos\omega s, \\
t' &= r, \qquad W' = \sum q_j u_j', \qquad j=1,2,3,4.
\end{aligned}
\tag{1,94}
$$

At each step of integration compute

$$
\begin{aligned}
u_j &= \alpha_j\cos\omega s+\beta_j\sin\omega s, & x_1 &= u_1^2-u_2^2-u_3^2+u_4^2, \\
u_j' &= \omega(-\alpha_j\sin\omega s+\beta_j\cos\omega s), & x_2 &= 2(u_1u_2-u_3u_4), \\
& & x_3 &= 2(u_1u_3+u_2u_4), \\
& & r &= u_1^2+u_2^2+u_3^2+u_4^2,
\end{aligned}
\tag{1,95}
$$

$$\begin{aligned} q_1 &= 2(\ u_1 p_1 + u_2 p_2 + u_3 p_3)\ , \\ q_2 &= 2(-u_2 p_1 + u_1 p_2 + u_4 p_3)\ , \\ q_3 &= 2(-u_3 p_1 - u_4 p_2 + u_1 p_3)\ , \\ q_4 &= 2(\ u_4 p_1 - u_3 p_2 + u_2 p_3)\ , \qquad F_j = \tfrac{1}{4}(r q_j + 2 W u_j)\ . \end{aligned} \tag{1,95}$$

<u>Checks.</u>

$$\alpha_4 \beta_1 - \alpha_3 \beta_2 + \alpha_2 \beta_3 - \alpha_1 \beta_4 = 0\ ,$$

$$r W = 2\omega^2 \sum (\alpha_j^2 + \beta_j^2) - M\ .$$

<u>1.3.3 Perturbations of the elements</u>. If the perturbing force is small compared with the central attraction, it is advisable to establish a companion procedure computing the perturbations

$$\Delta\alpha_j = \alpha_j - \alpha_{jK}\ , \quad \Delta\beta_j = \beta_j - \beta_{jK}\ , \quad \Delta r = r - r_K\ , \quad \Delta t = t - t_K \tag{1,96}$$

of the elements, of distance and of time. As always the subscript K indicates values corresponding to the unperturbed Kepler motion.

$$\alpha_{jK} = (\alpha_j)_0 = (u_j)_0\ , \qquad \beta_{jK} = (\beta_j)_0 = \frac{1}{\omega}(u_j')_0\ .$$

r_K, t_K are given by (1,80)(1,81) if the initial values $(\alpha_j)_0$, $(\beta_j)_0$ of the elements are inserted. From (1,92) it follows

$$\Delta r = \sum(\bar\alpha_j \Delta\alpha_j + \bar\beta_j \Delta\beta_j) + \cos 2\omega s \sum(\bar\alpha_j \Delta\alpha_j - \bar\beta_j \Delta\beta_j) + \sin 2\omega s \sum(\bar\alpha_j \Delta\beta_j + \bar\beta_j \Delta\alpha_j)$$

where $\bar\alpha_j$ for instance is an abbreviation for the arithmetic mean of the perturbed and unperturbed elements of the α-type.

<u>Companion procedure</u>

Substitute for the differential equations (1,94) the following routine.

<u>Differential equations.</u>

$$(\Delta\alpha_j)' = -\frac{1}{\omega} F_j \sin \omega s\ , \quad (\Delta\beta_j)' = \frac{1}{\omega} F_j \cos \omega s\ ,$$

$$(\Delta t)' = \Delta r\ , \quad W' = \sum q_j u_j'\ .$$

(Initial conditions $(\Delta\alpha_j)_0 = 0$, $(\Delta\beta_j)_0 = 0$, $(\Delta t)_0 = 0$, $W_0 = 0$).

At each step of integration compute

$$\Delta r = \sum(\bar\alpha_j \Delta\alpha_j + \bar\beta_j \Delta\beta_j) + \cos 2\omega s \sum(\bar\alpha_j \Delta\alpha_j - \bar\beta_j \Delta\beta_j) + \sin 2\omega s \sum(\bar\alpha_j \Delta\beta_j + \bar\beta_j \Delta\alpha_j)\ ,$$

where

$$\bar\alpha_j = (\alpha_j)_0 + \tfrac{1}{2}\Delta\alpha_j\ , \qquad \bar\beta_j = (\beta_j)_0 + \tfrac{1}{2}\Delta\beta_j\ ,$$

$$\alpha_j = (\alpha_j)_0 + \Delta\alpha_j\ , \qquad \beta_j = (\beta_j)_0 + \Delta\beta_j\ ;$$

$$t = \Delta t + \tfrac{s}{2}\sum[(\alpha_j)_0^2 + (\beta_j)_0^2] + \frac{1}{4\omega}\sin 2\omega s \sum[(\alpha_j)_0^2 - (\beta_j)_0^2] + \frac{1}{2\omega}(1 - \cos 2\omega s)\sum(\alpha_j)_0(\beta_j)_0$$

and proceed with (1,95).

In order to avoid loss of significant figures, the energy-check should be modified as follows. Because there is no work done by perturbing forces during the pure Kepler motion, we have $rW = \Delta(rW)$, hence

$$rW = 4\omega^2 \sum (\bar{\alpha}_j \Delta\alpha_j + \bar{\beta}_j \Delta\beta_j) . \tag{1,97}$$

This companion procedure is the basic tool for the numerical experiments outlined in chapter 2 of this report (Rössler). A final remark should be added concerning dissipative perturbing forces such as drag for example. In these cases, the velocities of the particle in physical space are also needed. These are given by (1,19) namely

$$\begin{aligned} \dot{x}_1 &= \frac{2}{r}(u_1 u_1' - u_2 u_2' - u_3 u_3' + u_4 u_4') , \\ \dot{x}_2 &= \frac{2}{r}(u_1 u_2' + u_2 u_1' - u_3 u_4' - u_4 u_3') , \\ \dot{x}_3 &= \frac{2}{r}(u_1 u_3' + u_3 u_1' + u_2 u_4' + u_4 u_2') . \end{aligned} \tag{1,98}$$

1.3.4 Ejection orbits. It must be stressed that the frequency ω depends on the initial conditions; ω should be known with high accuracy as will be shown in section 1.7. If the particle is starting at instant $t=0$ at the origin (thus coinciding with the central body) this frequency appears in undeterminate form

(1,73) $$\omega^2 = \frac{M}{2r_0} - \frac{v_0^2}{4} ,$$

because r_0 vanishes and v_0 is infinite. In this case we give only the direction of the initial velocity vector $\dot{x}_i$ but we give also the numerical value of either ω, the constant h of energy or the semi-major axis a_0 of the unperturbed orbit, these quantities being connected by

(1,54)(1,83)(1,89) $$\omega^2 = -\frac{h}{2} = \frac{M}{4a_0} . \tag{1,99}$$

The unperturbed orbit in physical space is a segment of straight line and from the given data the coordinates x_i^* of the apocenter are at once obtained as well as the corresponding parameters u_j^* by (1,47). In the parametric space the apocenter is associated with the value $s = \frac{\pi}{2\omega}$, thus we have from (1,76)

$$(\beta_j)_0 = u_j^* ;$$

the $(\alpha_j)_0$ vanish.

If the particle starts not exactly at the origin but near the origin, ω is only poorly determined, thus ω should also be given in advance and again the initial velocity-vector only by its direction. The velocity u_j' at the initial instant is then determined by (1,48) up to a proportionality factor. This factor may be computed from the law of energy (1,56)

$$2\sum u_j'^2 = M + r_0 h = M - 2r_0\omega^2 . \tag{1,100}$$

Nevertheless initial position and velocity must be given with high accuracy if (1,48) is applied.

1.4 . The osculating Kepler motion

We return now to the equation (1,52) of a perturbed Kepler motion

$$u_j'' + \left(\frac{M}{2r} - \frac{v^2}{4}\right) u_j = \frac{r}{4} q_j \, . \tag{1,101}$$

The osculating Kepler motion at an arbitrary instant t is by definition the pure Kepler motion constructed with the actual values of the coordinates u_j and velocities u_j' at time t as initial conditions [1]. The semi-major axis of the osculating orbit is a function a of t or s and is obtained from (1,86)

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{M} \, . \tag{1,102}$$

Thus (1,101) can be transformed into

$$u_j'' + \frac{M}{4a} u_j = \frac{r}{4} q_j \, . \tag{1,103}$$

The variation of a , as time goes on, is intimately connected with the work W done by the perturbing forces. We obtain explicitly this dependence of a on W from the vis viva integral (1,11). This gives

$$\frac{v^2}{2} - \frac{M}{r} = h + W \, , \qquad \frac{1}{a} = -\frac{2}{M}(h + W) \, . \tag{1,104}$$

A disadvantage of (1,103) is the variability of the coefficient of u_j . This can be avoided by introducing a <u>new fictitious time</u> σ defined by the differential relation

$$ds = \sqrt{\frac{a}{a_0}} \, d\sigma \, , \qquad dt = \sqrt{\frac{a}{a_0}} \, r \, d\sigma \, . \tag{1,105}$$

a_0 is the semi-axis of the osculating orbit at the initial instant $t = s = \sigma = 0$, and may be obtained from (1,102)

$$(1{,}89) \qquad \frac{1}{a_0} = \frac{2}{r_0} - \frac{v_0^2}{M} = \frac{4}{M} \omega^2 \, , \tag{1,106}$$

where ω is the frequency used throughout section 1.3. The substitution (1,105) transforms the equations of motion into

$$u_j'' - \frac{a'}{2a} u_j' + \frac{M}{4a_0} u_j = \frac{a}{a_0} \frac{r}{4} q_j \, ,$$

where accents denote differentiation with respect to σ. a' may be eliminated by

[1] This is to say if the perturbing force is switched off at instant t, the particle moves onward on the osculating orbit.

differentiation of (1,104)

$$\frac{a'}{a^2} = \frac{2}{M} W' - \frac{2}{M} \sum q_k u_k' = \frac{1}{2 a_0 \omega^2} \sum q_k u_k' , \qquad (1,12)$$

thus

$$u_j'' + \omega^2 u_j = \frac{1}{4} \frac{a}{a_0} \left(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k' \right) . \qquad (1,107)$$

The right-hand sides of these equations can be considered as perturbations, because they are proportional to the perturbing forces. It still remains to express a by quantities attached to the parametric space. With σ as independent variable equation (1,49) is transformed into

$$v^2 = \frac{4}{r} \frac{a_0}{a} \sum u_j'^2 ,$$

thus

$$\frac{1}{a} = \frac{2}{r} - \frac{4}{Mr} \frac{a_0}{a} \sum u_j'^2 = \frac{2}{r} - \frac{1}{\omega^2 a r} \sum u_j'^2 \qquad (1,102)$$

and by solving with respect to a

$$a = \frac{1}{2} \left(r + \frac{1}{\omega^2} \sum u_j'^2 \right) . \qquad (1,108)$$

As in section 1.3 the equations (1,107) are integrated by variation of constants. We agree however to <u>denote the new fictitious time again by</u> s and we put therefore

$$u_j = \alpha_j \cos \omega s + \beta_j \sin \omega s , \quad u_j' = \omega (- \alpha_j \sin \omega s + \beta_j \cos \omega s) . \qquad (1,109)$$

The α_j, β_j are functions of s and are the <u>elements of the osculating Kepler motion</u>. Its semi-axis is

$$a = \frac{1}{2} \Big[\sum (\alpha_j \cos \omega s + \beta_j \sin \omega s)^2 + \sum (- \alpha_j \sin \omega s + \beta_j \cos \omega s)^2 \Big] = \frac{1}{2} \sum (\alpha_j^2 + \beta_j^2) , \qquad (1,108)(1,45) \quad (1,110)$$

as could be expected from (1,84).

<u>Third procedure</u>
(Osculating orbit.)

<u>Data and initial conditions</u> as in second procedure.

Compute also $a_0 = \dfrac{M}{4 \omega^2}$.

See errata.

<u>Differential equations.</u>
(argument s)

$$\alpha_j' = - \frac{1}{\omega} F_j \sin \omega s , \quad \beta_j' = \frac{1}{\omega} F_j \cos \omega s ,$$

$$t' = \sqrt{\frac{a}{a_0}} \, r .$$

u_j, u_j', x_i, r, q_j as in second procedure. Compute at each step also

$$a = \frac{1}{2} \sum (\alpha_j^2 + \beta_j^2) , \quad F_j = \frac{1}{4} \frac{a}{a_0} \left(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k' \right) .$$

<u>Check.</u> $\alpha_4 \beta_1 - \alpha_3 \beta_2 + \alpha_2 \beta_3 - \alpha_1 \beta_4 = 0$.

As in (1,96) we establish a companion routine by computing only perturbations with respect to the pure Kepler motion. Let $\Delta t = t - t_K$ be the perturbation of time, where t_K is the time passed during the motion of the particle on the unperturbed Kepler orbit up to the position corresponding to a given value of s. According to our third procedure we have

$$t' = \sqrt{\frac{a}{a_0}}\, r$$

and in particular on the unperturbed Kepler orbit ($a = a_0 =$ const.)

$$t_K' = r_K ,$$

thus

$$(\Delta t)' = \sqrt{\frac{a}{a_0}}\, r - r_K , \tag{1,111}$$

r_K and t_K being determined by the formulae (1,80)(1,81) of the pure Kepler motion.

Companion procedure

Data and initial conditions as in second procedure,

$$a_0 = \frac{M}{4\omega^2} , \quad (\Delta\alpha_j)_0 = 0 , \quad (\Delta\beta_j)_0 = 0 , \quad (\Delta t)_0 = 0 .$$

Differential equations. (argument s)

$$(\Delta\alpha_j)' = -\frac{1}{\omega} F_j \sin\omega s , \quad (\Delta\beta_j)' = \frac{1}{\omega} F_j \cos\omega s ,$$

$$(\Delta t)' = \sqrt{\frac{a}{a_0}}\, r - r_K .$$

At each step of integration compute

$$r_K = a_0 + \tfrac{1}{2}\cos 2\omega s \sum\left[(\alpha_j)_0^2 - (\beta_j)_0^2\right] + \sin 2\omega s \sum (\alpha_j)_0 (\beta_j)_0 ,$$

$$t_K = a_0 s + \frac{1}{4\omega}\sin 2\omega s \sum\left[(\alpha_j)_0^2 - (\beta_j)_0^2\right] + \frac{1}{2\omega}(1-\cos 2\omega s)\sum (\alpha_j)_0 (\beta_j)_0 ,$$

$$\alpha_j = (\alpha_j)_0 + \Delta\alpha_j , \quad \beta_j = (\beta_j)_0 + \Delta\beta_j , \quad t = t_K + \Delta t .$$

u_j, u_j', x_i, r, q_j as in second procedure. Compute at each step also

$$a = \tfrac{1}{2}\sum(\alpha_j^2 + \beta_j^2) , \quad F_j = \frac{1}{4}\frac{a}{a_0}\left(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k'\right) .$$

Check. $\alpha_4\beta_1 - \alpha_3\beta_2 + \alpha_2\beta_3 - \alpha_1\beta_4 = 0$.

We should not forget to adapt the rules (1,98) for the velocities to the modified definition of fictitious time:

(1,105)

$$\begin{aligned} \dot{x}_1 &= \frac{2}{r}\sqrt{\frac{a_0}{a}}\,(u_1u_1' - u_2u_2' - u_3u_3' + u_4u_4') , \\ \dot{x}_2 &= \frac{2}{r}\sqrt{\frac{a_0}{a}}\,(u_1u_2' + u_2u_1' - u_3u_4' - u_4u_3') , \\ \dot{x}_3 &= \frac{2}{r}\sqrt{\frac{a_0}{a}}\,(u_1u_3' + u_3u_1' + u_2u_4' + u_4u_2') . \end{aligned} \tag{1,112}$$

The obvious advantage of the third procedure is that it avoids the computation of the work W done by the perturbing forces; moreover, operating with the familiar osculating orbit facilitates the comparison of classical and regularized computations. But it should be mentioned however that the companion routine suffers a little from loss of significant figures because on the right-hand side of (1,111) the difference of two almost equal quantities appears. Our numerical experiments however convinced us that this is not a serious danger.

1.5 Analytical theory of perturbations[1)]

1.5.1 First-order perturbations. The methods and procedures outlined above are valid for any particle subjected to elliptical initial conditions and moving under the influence of a central attraction and perturbing forces. There is no necessity to assume that the perturbing force is small compared with the central attraction.

On the contrary, this section is devoted to the study of perturbing forces which are infinitesimally small; this is to say a theory of first-order perturbations is developed. As the left-hand sides of the differential equations (1,107) are already linear, no linearization is needed; this is in contrast to the classical theories of first-order coordinate perturbations [6] which are based on the non-linear differential equations of the Kepler motion. As in classical theories the restriction to first order is performed by evaluating the right-hand sides of (1,107) no longer on the actual orbit, but on the unperturbed Kepler orbit which osculates at time $t = 0$; thus

$$u_j'' + \omega^2 u_j = \frac{1}{4}\left(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k'\right)_K . \tag{1,113}$$

As always, the subscript K indicates values to be taken on the unperturbed orbit. The ratio a/a_0 disappears because $a_K = a_0$. The right-hand sides of these equations

$$F_j = \frac{1}{4}\left(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k'\right)_K \tag{1,114}$$

are known functions of the regularizing time s ; therefore the differential equations for the elements (as recorded in the third procedure) can be integrated by quadratures:

$$\alpha_j = -\frac{1}{\omega}\int F_j(s)\sin\omega s\, ds\ , \qquad \beta_j = \frac{1}{\omega}\int F_j(s)\cos\omega s\, ds\ . \tag{1,115}$$

For first-order perturbations (1,111) is approximated by

$$\begin{aligned}(\Delta t)' &= \sqrt{1+\frac{\Delta a}{a_0}}\, r - r_K \sim \left(1+\frac{1}{2}\frac{\Delta a}{a_0}\right)(r_K + \Delta r) - r_K \sim \Delta r + \frac{1}{2}\frac{\Delta a}{a_0} r_K\ ,\\ \Delta t &= \int\left(\Delta r + \frac{r_K}{2a_0}\Delta a\right) ds\ ,\end{aligned} \tag{1,116}$$

---

1) More details on first- and higher-order perturbations are contained in [5].

thus avoiding loss of significant figures. This implies the computation of

(1,110) $$\Delta a \sim \sum[(\alpha_j)_0 \Delta\alpha_j + (\beta_j)_0 \Delta\beta_j] \tag{1,117}$$
and
(1,92) $$\Delta r \sim \Delta a + \cos 2\omega s \sum[(\alpha_j)_0 \Delta\alpha_j - (\beta_j)_0 \Delta\beta_j] + \sin 2\omega s \sum[(\alpha_j)_0 \Delta\beta_j + (\beta_j)_0 \Delta\alpha_j] . \tag{1,118}$$

Finally r_K is taken from (1,80) or from (1,45).

Fourth procedure

(First-order perturbations of elements and of time; osculating Kepler orbit.)

Initial conditions. As in second procedure.

Computation of the unperturbed motion (osculating at instant $t = 0$). For sake of simplicity of notation the subscript K is suppressed.

$$a_0 = \frac{M}{4\omega^2} ,$$

$$u_j = (\alpha_j)_0 \cos \omega s + (\beta_j)_0 \sin \omega s , \quad u_j' = \omega[-(\alpha_j)_0 \sin \omega s + (\beta_j)_0 \cos \omega s], \tag{1,119}$$

Perturbing forces

$$\begin{aligned} x_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 , & q_1 &= 2(u_1 p_1 + u_2 p_2 + u_3 p_3) , \\ x_2 &= 2(u_1 u_2 - u_3 u_4) , & q_2 &= 2(-u_2 p_1 + u_1 p_2 + u_4 p_3) , \\ x_3 &= 2(u_1 u_3 + u_2 u_4) , & q_3 &= 2(-u_3 p_1 - u_4 p_2 + u_1 p_3) , \\ r &= u_1^2 + u_2^2 + u_3^2 + u_4^2 . & q_4 &= 2(u_4 p_1 - u_3 p_2 + u_2 p_3) . \end{aligned} \tag{1,120}$$

$$t_K = a_0 s + \frac{1}{4\omega} \sin 2\omega s \sum[(\alpha_j)_0^2 - (\beta_j)_0^2] + \frac{1}{2\omega} (1 - \cos 2\omega s) \sum (\alpha_j)_0 (\beta_j)_0 , \tag{1,121}$$

$$F_j = \frac{1}{4}(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k') .$$

Perturbations of elements.

$$\Delta\alpha_j = -\frac{1}{\omega}\int_0^s F_j \sin \omega s \, ds , \quad \Delta\beta_j = \frac{1}{\omega}\int_0^s F_j \cos \omega s \, ds .$$

Perturbation of semi-major axis

$$\Delta a = \sum[(\alpha_j)_0 \Delta\alpha_j + (\beta_j)_0 \Delta\beta_j] .$$

Perturbation of distance

$$\Delta r = \Delta a + \cos 2\omega s \sum[(\alpha_j)_0 \Delta\alpha_j - (\beta_j)_0 \Delta\beta_j] + \sin 2\omega s \sum[(\alpha_j)_0 \Delta\beta_j + (\beta_j)_0 \Delta\alpha_j] .$$

Perturbation of time

$$\Delta t = \int(\Delta r + \frac{r}{2a_0} \Delta a) \, ds .$$

Elements of the osculating orbit at instant s .

$$\alpha_j = (\alpha_j)_0 + \Delta\alpha_j , \quad \beta_j = (\beta_j)_0 + \Delta\beta_j , \quad t = t_K + \Delta t .$$

Position u_j, x_j of the particle from (1,95).

An account on numerical experiments is given in chapter 2 (Rössler). In the sequel the integrals (1,115)(1,116) are computed by Fourier expansion, therefore some remarks about the periodicity of our functions are in order. A function $f(s)$ is called symmetric or skew-symmetric if

$$f(s+\frac{\pi}{\omega}) = f(s) \qquad \text{or} \qquad f(s+\frac{\pi}{\omega}) = -f(s)$$

respectively. As can be seen from (1,120) the parametric coordinates u_j are skew-symmetric but the physical coordinates x_i are symmetric. Let us assume temporarily that the p_i in (1,120) are any functions depending only on the position x_i of the particle in the physical space; thus they are symmetric functions. The corresponding functions q_j are skew-symmetric as well as the perturbing functions F_j. But it should be stressed that the integrands

$$(\Delta\alpha_j)' = -\frac{1}{\omega} F_j \sin \omega s \,, \qquad (\Delta\beta_j)' = \frac{1}{\omega} F_j \cos \omega s \tag{1,122}$$

are symmetric and have therefore by definition the period $\frac{\pi}{\omega}$.

1.5.2 Three-body problem. We consider now the motion of a particle of negligible mass in the force-field of two heavy bodies moving about each other on perfect Kepler orbits. As always the first body - referred to as central body - is at the origin of the x_i-system and its gravitational parameter (product of mass and gravitational constant) is denoted by M. The second body of gravitational parameter $\bar{M}$ moves on the relative Kepler orbit, assumed to be an ellipse. Let $\bar{a}$ be its semi-major axis and

$$\bar{\mu} = (M+\bar{M})^{1/2}\, \bar{a}^{-3/2} \tag{1,123}$$

the mean angular velocity of this second body, also called perturbing body. $\bar{M}$ should be small with respect to M. In our first-order theory the path of the particle is also a pure Kepler ellipse, as far as the computation of the perturbing forces is concerned. In order to compute these forces, the position of the particle will be fixed by its fictitious time s and the position of the perturbing body by the physical time t. Furthermore s, t are considered to be independent variables, since the forces p_i exerted by the perturbing body on the particle are defined indeed for two arbitrarily chosen positions of these two bodies.

For a fixed position of the particle the p_i are periodic functions of the mean anomaly ($\bar{\mu} t$) with the period 2π; therefore we may expand them into a Fourier-series:

$$p_i = \sum_{n=-\infty}^{+\infty} p_{in} \operatorname{cis} n\bar{\mu} t \tag{1,124}$$

(In order do avoid accumulated exponents, we use the notation $\operatorname{cis}\alpha = \cos\alpha + i\sin\alpha$). The Fourier coefficients p_{in} are determined uniquely by the position of the particle; they are symmetric functions $p_{in}(s)$ of s in the sense of the preceding definition. By inserting (1,124) into (1,120) the Fourier coefficients of the integrands (1,122) are obtained and from our discussion above it follows that these coefficients are again symmetric functions of s .

In order to simplify notation let f stand for any of the 8 integrands (1,122). The Fourier expansion of the integrands has now the typical form

$$f = \sum_{-\infty}^{+\infty} f_n(s)\ \text{cis}\ n\bar{\mu}t\ , \tag{1,125}$$

where $f_n(s)$ is of period $\frac{\pi}{\omega}$ with respect to its argument s.

But during the actual motion of the particle and the perturbing body the variables s, t are not independent but correlated by the fact that t is also the Kepler-time t_K of the particle corresponding to the value of s under consideration. By writing (1,121) in the concentrated form

$$\begin{gathered} t = a_0 s + \lambda_1 + \lambda_2 \cos 2\omega s + \lambda_3 \sin 2\omega s\ , \\ \lambda_1 = \frac{1}{2\omega}\sum (\alpha_j)_0 (\beta_j)_0\ , \quad \lambda_2 = -\lambda_1\ , \quad \lambda_3 = \frac{1}{4\omega}\sum\left[(\alpha_j)_0^2 - (\beta_j)_0^2\right], \end{gathered} \tag{1,126}$$

equation (1,125) is transformed into

$$f = \sum_{-\infty}^{+\infty} \text{cis}\ n\bar{\mu}a_0 s\left[f_n(s)\ \text{cis}\ n\bar{\mu}\left(\lambda_1 + \lambda_2\cos 2\omega s + \lambda_3 \sin 2\omega s\right)\right].$$

The expression in brackets is a symmetric function of s of period $\frac{\pi}{\omega}$ and may therefore be expanded in a Fourier-series of the type

$$[\ ] = \sum_{\nu=-\infty}^{+\infty} f_{n\nu}\ \text{cis}\ 2\nu\omega s\ ,$$

hence

$$f = \sum_{(n,\nu)} f_{n\nu}\ \text{cis}\,(2\nu\omega + n\bar{\mu}a_0)s\ , \tag{1,127}$$

where the coefficients $f_{n\nu}$ are constants. Any integrand (1,122) has such an expansion and by integration it follows finally

$$\int f\,ds = \text{const} + f_{00}s + \sum{}' \frac{f_{n\nu}}{i(2\nu\omega + n\bar{\mu}a_0)}\ \text{cis}\,(2\nu\omega + n\bar{\mu}a_0)s\ , \tag{1,128}$$

the accent indicating the omission of $(n,\nu) = (0,0)$. The constant must be determined in such a way that the whole expression vanishes for $s = 0$. This finishes the computation of the perturbations $\Delta\alpha_j$ of the elements and by further integration the perturbation Δt of time is obtained, as was described in our fourth procedure.

We proceed to discuss briefly the event of vanishing denominators in (1,128). We have then

$$\frac{\nu}{n} = -\frac{a_0}{2\omega}\bar{\mu} = -\frac{\bar{\mu}}{\mu}\ ,$$

where μ is the mean angular velocity of the particle, determined by (1,85) and (1,84). A vanishing denominator thus occurs if and only if the mean motion of the particle and the perturbing body have a ratio that is a rational number. Such a situation is known in classical celestial mechanics as <u>resonance</u>.

In practice however, the Fourier expansions should not be carried out as described above. The following method is better adapted to automatic computation. An auxiliary variable s_1 is introduced defined by

$$s_1 = \bar{\mu} a_0 s,$$

hence

(1,127) $$f = \sum_{(n,\nu)} f_{n\nu} \operatorname{cis}(2\nu\omega s + n s_1), \tag{1,129}$$

(1,126) $$t = \frac{1}{\bar{\mu}} s_1 + \lambda_1 + \lambda_2 \cos 2\omega s + \lambda_3 \sin 2\omega s. \tag{1,130}$$

Evidently, the integrands f can be considered as functions of the two <u>independent</u> variables s, s_1, because any choice of s determines the position of the particle and then an arbitrary value of s_1 yields a corresponding value (1,130) of time and consequently a position of the perturbing body. The development (1,129) is then obtained by tabulating the 8 integrands f at equally spaced values of s, s_1 and by puting into action a standard automatic routine for double harmonic analysis.

By introducing s_1 also in the final result (1,128), the result

$$\int f ds = const + f_{00} s + \sum{}' \frac{f_{n\nu}}{i(2\nu\omega + n\bar{\mu} a_0)} \operatorname{cis}(2\nu\omega s + n s_1) \tag{1,128a}$$

is obtained. The term $f_{00} s$ is the <u>secular perturbation</u> and the sum is a double Fourier-series with respect to s, s_1.

In chapter 2 of this report, Dr. <u>Rössler</u> has worked out an ALGOL-program for computing first-order perturbations, based on the preceding analysis. In order to obtain consistent algorithms, he introduces also regularized elements $\bar{\alpha}_j, \bar{\beta}_j$ for the motion of the perturbing body. Furthermore he uses instead of s, s_1 two modified independent variables intimately related to the eccentric anomalies of the particle and the perturbing body.

<u>P.A. Hansen</u> [7] was the first to appreciate the advantages of a Fourier expansion with respect to the eccentric anomaly of the particle instead of using its mean anomaly as independent variable as was customary in the works of his predecessors. The introduction of s_1 is due to him. Therefore there are some points of contact between Hansen's methods and ours. Hansen's procedures are very accurate and have been widely applied; they can however not handle the problem at hand. Our main goal has been indeed to establish a perturbation theory remaining valid for near-collisions with the central body, that is to say for elliptic orbits with eccentricity only slightly inferior to 1 or even $=1$. The numerical experiments described in chapter 2 indicate that this goal has been successfully attained.

1.6 Secular perturbations

The investigations of section 1.5 have clearly indicated that the theory of the osculating orbit and its perturbations, based on regularized elements, proceeds along the same lines as in the classical theories of <u>Lagrange</u>, <u>Leverrier</u> and their successors.

In this section we discuss some aspects of literal developments of perturbing functions and of secular perturbations. We do not attempt to present a complete theory but restrict ourselves to some examples of relative simplicity. The subscript K, denoting quantities attached to an unperturbed Kepler motion, is suppressed in this section and by α_j, β_j we understand the constant elements of such a motion. With this convention the equations (1,114) and (1,122) of our first-order theory can be written

$$F_j = \frac{1}{4}\left(r q_j + \frac{1}{\omega^2} u_j' \sum q_k u_k'\right), \tag{1,131}$$

where

$$u_j = \alpha_j \cos \omega s + \beta_j \sin \omega s, \quad u_j' = \omega(-\alpha_j \sin \omega s + \beta_j \cos \omega s), \tag{1,132}$$

$$r = \sum u_j^2, \tag{1,133}$$

$$(\Delta\alpha_j)' = -\frac{1}{\omega} F_j \sin \omega s, \quad (\Delta\beta_j)' = \frac{1}{\omega} F_j \cos \omega s. \tag{1,134}$$

We remember the significance of our notations:

u_1, u_2, u_3, u_4 – coordinates of the particle in the parametric space,

q_j – perturbing forces in the parametric space,

accent indicates differentiation with respect to s,

$\Delta\alpha_j$, $\Delta\beta_j$ – perturbations of the elements and

ω is defined by (1,73)

$$\omega^2 = \frac{M}{2r_0} - \frac{v_0^2}{4},$$

where M is the gravitational parameter of the central mass and r_0, v_0 initial position and velocity of the particle.

1.6.1 Conservative perturbing potential. Let us assume now that the perturbing forces p_i in the physical space may be calculated from a conservative potential $V(x_i)$ which depends only on the position of the particle. Taking into account our KS-transformation (1,44) the perturbing potential becomes a function $V(u_j)$ in the parametric space; if we replace u_j, using expression (1,132), this function is further transformed into a function $V(\alpha_j, \beta_j; s)$ of s, where the α_j, β_j should be treated as parameters independent of s. As was established after formula (1,8) we have

$$q_j = -\frac{\partial}{\partial u_j} V(u_j),$$

thus

$$\frac{\partial}{\partial s} V(\alpha_j, \beta_j; s) = \sum_{(k)} \left[\frac{\partial}{\partial u_k} V(u_j)\right] u_k' = -\sum q_k u_k' \; ; \tag{1,135}$$

the last expression appears in (1,131). From (1,132) we also obtain the partial derivatives of V with respect to the elements α_j and β_j :

$$\frac{\partial}{\partial \alpha_j} V(\alpha_j, \beta_j; s) = \left[\frac{\partial}{\partial u_j} V(u_j)\right] \frac{\partial u_j}{\partial \alpha_j} = -q_j \cos \omega s, \quad \frac{\partial V}{\partial \beta_j} = -q_j \sin \omega s. \tag{1,136}$$

By collecting (1,131) through (1,136) we have for instance

$$
\begin{aligned}
(\Delta\alpha_j)' &= -\frac{1}{4\omega}\left(r q_j \sin\omega s + \frac{\sin\omega s}{\omega^2}\, u_j' \sum q_k u_k'\right) \\
&= \frac{1}{4\omega}\left(r\,\frac{\partial V}{\partial\beta_j} + \frac{\sin\omega s}{\omega^2}\, u_j' \,\frac{\partial V}{\partial s}\right) \\
&= \frac{1}{4\omega}\left[r\,\frac{\partial V}{\partial\beta_j} + \frac{\sin\omega s}{\omega}\left(-\alpha_j \sin\omega s + \beta_j\cos\omega s\right)\frac{\partial V}{\partial s}\right] \\
&= \frac{1}{8\omega}\left[2r\,\frac{\partial V}{\partial\beta_j} + \frac{1}{\omega}\left(-\alpha_j + \alpha_j\cos 2\omega s + \beta_j \sin 2\omega s\right)\frac{\partial V}{\partial s}\right],
\end{aligned}
$$

similarly

$$
(\Delta\beta_j)' = -\frac{1}{8\omega}\left[2r\,\frac{\partial V}{\partial\alpha_j} + \frac{1}{\omega}\left(\beta_j + \beta_j\cos 2\omega s - \alpha_j\sin 2\omega s\right)\frac{\partial V}{\partial s}\right].
$$

This set of rules for computing the perturbations of the α_j, β_j is analogous to the canonical equations for the perturbations of the classical elements. We adapt these rules to the more familiar classical notation by allowing the particle to start from its pericenter and introducing the eccentric anomaly $E = 2\omega s$. Hence

$$
r = a(1 - e\cos E), \qquad \omega^2 = \frac{M}{4a}
$$

follows from (1,83). a is the semi-major axis and e the eccentricity; it follows that

$$
\begin{aligned}
\frac{d(\Delta\alpha_j)}{dE} &= \frac{a}{2M}\left[a(1-e\cos E)\frac{\partial V}{\partial\beta_j} + (-\alpha_j + \alpha_j\cos E + \beta_j\sin E)\frac{\partial V}{\partial E}\right], \\
\frac{d(\Delta\beta_j)}{dE} &= -\frac{a}{2M}\left[a(1-e\cos E)\frac{\partial V}{\partial\alpha_j} + (\beta_j + \beta_j\cos E - \alpha_j\sin E)\frac{\partial V}{\partial E}\right].
\end{aligned}
\tag{1,137}
$$

In order to compute the integrals of the right-hand sides, V is expanded into a Fourier-series with respect to E; this implies a literal development, this is to say that the coefficients of the expansion must be given as explicit algebraic expressions in the elements α_j, β_j and s or E; otherwise their partial derivatives are not available. An analogous analysis can be carried out in the case in which the perturbing potential is not conservative but depends explicitly on time.

1.6.2 Secular perturbations. Let us now investigate the secular perturbations of first-order in the problem of the three bodies. As in section 1.5.2, a bar over a symbol denotes a quantity attached to the perturbing body. If no resonance occurs, the secular influence of the moving and perturbing body is equivalent to the influence of the Gaussian ring obtained by distributing the mass of the perturbing body over its elliptical orbit proportionally to the Kepler time on this orbit. The potential of this ring at a given point in the physical space is the integral

$$V = -\frac{\bar{M}}{\bar{T}} \int \frac{dt}{\rho} \ , \qquad (1,138)$$

where ρ is the distance from the given point to the general point of the ring, $\bar{M}$ the gravitational parameter of the perturbing body and $\bar{T}$ its time of revolution. (Fig. 1.3). The perturbing potential V is conservative and thus our rules (1,137) are applicable. If

$$V = V_0 + V_1 \cos E + V_2 \sin E + \cdots \qquad (1,139)$$

is the Fourier expansion of this potential, we need only the first three coefficients V_0, V_1, V_2, because we are only concerned with secular perturbations and are therefore only interested in the constant terms in the Fourier-series of the right-hand sides of formulae (1,137).

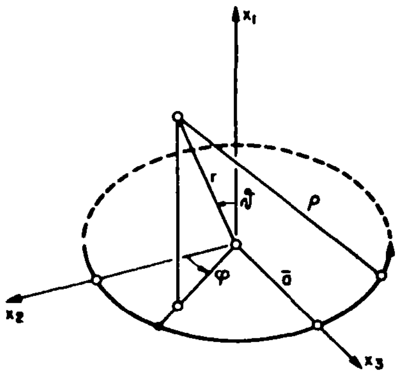


Fig. 1.3. Gaussian ring.

The further investigations of this section are restricted to a circular motion of the perturbing body. (Fig. 1.3). The circle of radius $\bar{a}$ is assumed to be in the x_2, x_3-plane and the position of the particle is described by polar coordinates r, ϑ, φ. In this special case the potential V of the circular ring is given by the Legendre expansion

$$V = -\frac{\bar{M}}{\bar{a}} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(\frac{r}{\bar{a}}\right)^{2n} P_{2n}(\cos\vartheta) \ , \qquad (1,140)$$

where P_{2n} is the Legendre polynomial of degree $(2n)$. Because both sides are harmonic functions, it is sufficient to verify this formula for $\vartheta = 0$ (particle on the x_1-axis). It is then reduced to

$$V = -\frac{\bar{M}}{\bar{a}} \sum \binom{-\frac{1}{2}}{n} \left(\frac{r}{\bar{a}}\right)^{2n} = -\frac{\bar{M}}{\bar{a}} \left[1 + \left(\frac{r}{\bar{a}}\right)^2\right]^{-1/2} = -\frac{\bar{M}}{\sqrt{\bar{a}^2 + r^2}} \ .$$

The last expression is undoubtedly the value of the ring-potential at a point on the x_1-axis. The series (1,140) is convergent in the interior of a sphere having the circular ring as its equator. In order to transfer V into the parametric space, we use the explicit formula

$$r^{2n} P_{2n}(\cos\vartheta) = \sum_{k=0}^{n} \frac{(-1)^k}{4^k} \binom{2n}{2k} \binom{2k}{k} (r^2 - x_1^2)^k x_1^{2(n-k)} . \tag{1,141}$$

From the KS-transformation the following expressions are obtained

(1,44) $$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2 ,$$

(1,47) $$r + x_1 = 2(u_1^2 + u_4^2) , \quad r - x_1 = 2(u_2^2 + u_3^2) ,$$

hence

$$r^{2n} P_{2n} = \sum (-1)^k \binom{2n}{2k} \binom{2k}{k} (u_1^2 + u_4^2)^k (u_2^2 + u_3^2)^k (u_1^2 - u_2^2 - u_3^2 + u_4^2)^{2(n-k)} . \tag{1,142}$$

The zonal harmonics $r^{2n} P_{2n}$ are thus homogeneous polynomials [1] of degree $(4n)$ in the parameters u_j. The formulae (1,140) and (1,142) establish the perturbing potential in the parametric space. According to the computational program outlined in the first lines of section 1.6.1 it still remains to introduce the elements α_j, β_j. This is achieved by formula (1,76) adapted to the eccentric anomaly $E = 2\omega s$.

$$u_j = \alpha_j \cos \frac{E}{2} + \beta_j \sin \frac{E}{2} . \tag{1,143}$$

The equations (1,137)(1,140)(1,142)(1,143) furnish all the necessary tools for computing the secular perturbations due to a perturbing body moving on a circular orbit.

1.6.3 An example. In order to give an example of explicitly computed secular perturbations, we truncate the series (1,140) after $n = 1$. This is only reasonable if the particle does not closely approach the perturbing body. With this approximation we obtain from (1,142)

$$V = \frac{\bar{M}}{\bar{a}} \left\{ -1 + \frac{1}{2\bar{a}^2} \left[(u_1^2 - u_2^2 - u_3^2 + u_4^2)^2 - 2(u_1^2 + u_4^2)(u_2^2 + u_3^2) \right] \right\}$$

and by (1,143) this becomes a Fourier polynomial in E.

Working with this perturbing potential Dr. Rössler has computed the secular perturbations; by introducing new quantities, connected with the classical orbital elements, he obtains a rather simple result.

Let (Fig. 1.4) A, C be pericenter and apocenter of the orbit of our particle and B, D the endpoints of the minor axis. The corresponding values of the eccentric anomaly are in that order

$$E: \quad 0^\circ, \quad 180^\circ, \quad 90^\circ, \quad 270^\circ,$$

---

[1] It can be proved that they satisfy the 4-dimensional Laplace equation as does any harmonic function in the physical space if transfered into the parametric space.

consequently these 4 points have the parametric coordinates (1,143)

$$u_j : \quad \alpha_j , \quad \beta_j , \quad \frac{\sqrt{2}}{2}(\alpha_j+\beta_j) , \quad \frac{\sqrt{2}}{2}(-\alpha_j+\beta_j) .$$

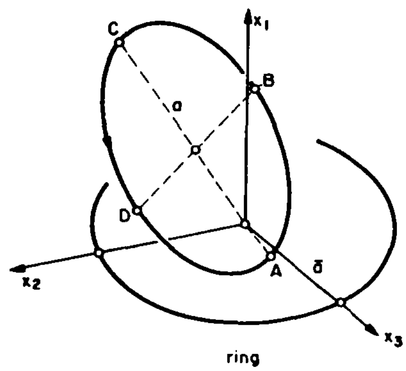


Fig. 1.4. Approximate secular perturbations.

By straightforward arithmetic the altitudes $x_{1A}, x_{1B}, x_{1C}, x_{1D}$ are obtained from the KS-transformation (1,44). In particular it turns out that

$$x_{1A} - x_{1C} = \alpha_1^2 - \alpha_2^2 - \alpha_3^2 + \alpha_4^2 - \beta_1^2 + \beta_2^2 + \beta_3^2 - \beta_4^2 , \tag{1,144}$$

$$x_{1B} - x_{1D} = 2(\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3 + \alpha_4\beta_4) . \tag{1,145}$$

The shape of the Kepler orbit may be determined by its axis and its eccentricity

(1,84) $$a = \frac{1}{2}(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2+\beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2) , \tag{1,146}$$

(1,87) $$e = -\frac{1}{2a}(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2-\beta_1^2-\beta_2^2-\beta_3^2-\beta_4^2) , \tag{1,147}$$

its position in space by the two "inclinations"

$$\sigma = \frac{1}{2a}(x_{1A} - x_{1C}) , \qquad \tau = \frac{1}{2a}(x_{1B} - x_{1D}) . \tag{1,148}$$

With this notation the final result is as follows

$$\begin{aligned} \Delta\alpha_j &= -\frac{3}{8}\frac{\bar{M}}{M}\left(\frac{a}{\bar{a}}\right)^3 \left\{(\sigma e \mp 1)\tau\alpha_j + \left[\frac{2}{3} + e + \frac{1}{3}e^2 \pm \sigma(1+3e+e^2) - \sigma^2 e\right]\beta_j\right\} E , \\ \Delta\beta_j &= \frac{3}{8}\frac{\bar{M}}{M}\left(\frac{a}{\bar{a}}\right)^3 \left\{(\sigma e \mp 1)\tau\beta_j + \left[\frac{2}{3} - e + \frac{1}{3}e^2 \mp \sigma(1-3e+e^2) + \sigma^2 e\right]\alpha_j\right\} E . \end{aligned} \tag{1,149}$$

The upper sign must be taken for $j = 1, 4$ and the lower for $j = 2, 3$. The verification of this result is a little tedious but straightforward, the identities (1,78) and (1,87)

$$\alpha_4\beta_1 - \alpha_3\beta_2 + \alpha_2\beta_3 - \alpha_1\beta_4 = 0, \qquad \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 = 0$$

being used several times. As always the elements α_j, β_j are computed according to the rules "initial conditions" of our second procedure (section 1.3.2).

In chapter 2 (cf. 2.2.5) Dr. Rössler describes four different methods for computing the motion of a satellite about the earth, taking into account the perturbations by the moon and he discusses also their accuracy. The four methods are:

First method (cf. 2.2.5.2). Companion of the second procedure (section 1.3.3).

Second method (cf. 2.2.5.3). Companion of the third procedure (section 1.4). Computation of the special perturbations of the elements of the osculating orbit by numerical integration of the corresponding differential equations.

Third method (cf. 2.2.5.4). Analytical first-order perturbations of the elements by double harmonic analysis (fourth procedure, section 1.5.1). In particular secular perturbations.

Fourth method (cf. 2.2.5.5). Secular perturbations according to the formulae (1,149).

The orbit of the satellite under consideration has eccentricity 0.5 and high inclination with respect to the ecliptic; the very small difference between the results of the second and third methods is due to the perturbations of higher order, the fourth method gives the perturbations of the elements with an error of only about 4%. The reason for this is not the high eccentricity or large inclination but is simply the truncation of the Legendre series. (The ratio $a : \bar{a}$ is $1 : 6$.).

We have not established a companion formula to (1,149) for the perturbation of time. According to our fourth procedure, to do so would require as a prerequisite the computation of

(1,117) $$\Delta a = \sum \left[(\alpha_j)_0 \, \Delta\alpha_j + (\beta_j)_0 \, \Delta\beta_j \right] . \qquad (1,150)$$

In the three-body problem the $\Delta\alpha_j$ and $\Delta\beta_j$ appear as series of the type (1,128), but if these series are inserted into (1,150) the secular terms cancel out because of the well-known fact that there is no secular first-order perturbation of the axis of the osculating orbit. Thus

$$\Delta a = \sum_{(n,\nu)} a_{n\nu} \operatorname{cis}(2\nu\omega + n\bar{\mu}a_0)s \, , \qquad (1,151)$$

with unspecified coefficients $a_{n\nu}$. For the evaluation of the secular perturbation (1,116) of time the constant term a_{00} of this series is needed; this term is determined by the initial conditions at instant $s = 0$:

$$\Delta a = 0 \, , \qquad a_{00} = -\sum_{(n,\nu)}{}' a_{n\nu} \, . \qquad (1,152)$$

Therefore all the coefficients $a_{n\nu}$ with $(n,\nu) \neq (0,0)$ should be known and consequently also all the Fourier coefficients of the expansion (1,139) of the perturbing potential are required. We recall the fact that three of these coefficients were sufficient for establishing the secular perturbations of the elements. This complication makes it impossible for us to establish a formula for the perturbation of time which is as simple as (1,149). A similar complication occurs in the classical theory if the perturbation of the mean anomaly is wanted.

Not only perturbations by a third body can be computed by our analytical theory, but also perturbations of other types as for instance that generated by the asphericity of the earth. But in that case convergence is not so rapid because the perturbing potential is no longer regular at the origin (center of the earth) as is assumed in section 1.1.

1.6.4 An ejection orbit. In order to demonstrate the merits of regularization, we compute in this section explicitly the secular perturbations of an ejection orbit. (Fig. 1.5). A particle is ejected from the origin A into the x_1, x_2-plane.

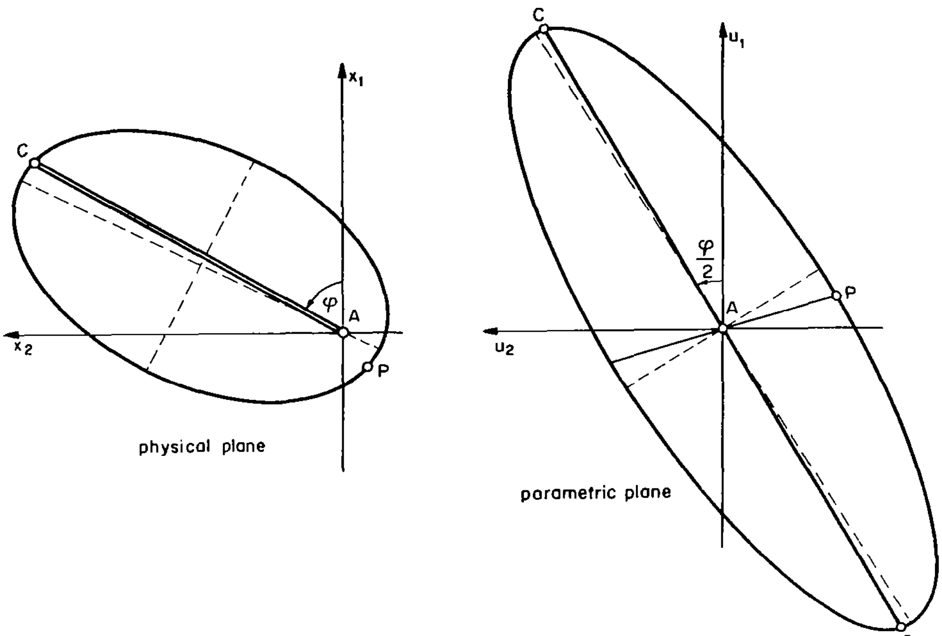


Fig. 1.5. Secular perturbation of an ejection orbit.

Under the influence of the attraction of the central body (located at A) its unperturbed orbit is a segment AC with apocenter at C. Let φ be the angle between this segment and the x_1-axis and $2a = 1$ the distance AC. The perturbing Gaussian ring is still a circle in the x_2, x_3-plane with radius $\bar{a}$. The unperturbed as well as the perturbed orbit are in the x_1, x_2-plane; therefore it is sufficient to take only this plane and the u_1, u_2-plane of the parametric space into consideration. The correspondence between these two planes is given by Levi-Civita's transformation (1,25)

$$x_1 = u_1^2 - u_2^2, \qquad x_2 = 2u_1 u_2 ,$$

or in complex notation (1,153)

$$x = u^2 \qquad (x = x_1 + i x_2 , \quad u = u_1 + i u_2) .$$

The orbit in the parametric space is thus the straight line CC building the angle

$\frac{1}{2}\varphi$ with the u_1-axis and the parametric coordinates of the upper point C are

$$u_1^* = \cos\frac{\varphi}{2}\ ,\qquad u_2^* = \sin\frac{\varphi}{2}\ .$$

As was pointed out in section 1.3.4 the elements of the unperturbed orbit follow at once from this information [1)]

$$\alpha_1 = 0\ ,\quad \alpha_2 = 0\ ,\quad \beta_1 = \cos\frac{\varphi}{2}\ ,\quad \beta_2 = \sin\frac{\varphi}{2}\ . \tag{1,154}$$

Furthermore we have according to the definitions (1,146)(1,147)(1,148)

$$a = \frac{1}{2}\ ,\quad e = 1\ ,\quad \sigma = -\cos\varphi\ ,\quad \tau = 0\ . \tag{1,155}$$

The perturbations (1,149) are now reduced to

$$\begin{aligned} \Delta\alpha_1 &= K\,(2 - 5\cos\varphi - \cos^2\varphi)\cos\frac{\varphi}{2}\ , & \Delta\beta_1 &= 0\ , \\ \Delta\alpha_2 &= K\,(2 + 5\cos\varphi - \cos^2\varphi)\sin\frac{\varphi}{2}\ , & \Delta\beta_2 &= 0 \end{aligned} \tag{1,156}$$

with

$$K = -\frac{3}{8}\,\frac{\bar{M}}{M}\left(\frac{a}{\bar{a}}\right)^3 E\ . \tag{1,157}$$

As time goes on, the osculating Kepler orbit is thus given by (1,76)

$$\begin{aligned} u_1 &= K\,(2 - 5\cos\varphi - \cos^2\varphi)\cos\frac{\varphi}{2}\cos\frac{E}{2} + \cos\frac{\varphi}{2}\sin\frac{E}{2}\ , \\ u_2 &= K\,(2 + 5\cos\varphi - \cos^2\varphi)\sin\frac{\varphi}{2}\cos\frac{E}{2} + \sin\frac{\varphi}{2}\sin\frac{E}{2}\ . \end{aligned} \tag{1,158}$$

In the u_1, u_2-plane the point C and the point P with coordinates

$$K(2 - 5\cos\varphi - \cos^2\varphi)\cos\frac{\varphi}{2}\ ,\qquad K(2 + 5\cos\varphi - \cos^2\varphi)\sin\frac{\varphi}{2}$$

are endpoints of conjugate diameters of the ellipse. In <u>Fig</u>. <u>1.5</u> the values

$$\varphi = 60°\ ,\qquad K = -0.15$$

are adopted. The ellipse in the u_1, u_2-plane is constructed from the conjugate diameters. The endpoints of its major and minor axis are mapped onto the apo- and pericenter of the osculating Kepler ellipse in the physical x_1, x_2-plane.

This example is also computed in chapter 2 (cf. 2.2.6) by using the fourth procedure and double harmonic analysis. The Fourier expansion of $(\Delta\alpha_1)$ is printed out. Furthermore the same investigations are carried out for a circular unperturbed orbit in the x_1, x_2-plane (cf. 2.2.7). <u>The rate of convergence of the Fourier-series is about the same for this circular orbit as for the ejection orbit</u>.

1.7 On stability and convergence

In this section some remarks are added concerning the numerical stability of the integration of the differential equations and the convergence of the Fourier-series; we do not attempt to establish a complete analysis of this kind of problem.

[1)] The elements $\alpha_3, \alpha_4, \beta_3, \beta_4$ vanish for all the orbits under consideration.

1.7.1 Stability of pure Kepler motion. The regularized differential equations are

(1,74) $$u_j'' + \omega^2 u_j = 0, \quad j = 1,2,3,4, \tag{1,159}$$

where accents mean differentiation with respect to the fictitious time s defined by

(1,47a)(1,45) $$dt = r\,ds, \quad r = \sum u_j^2 . \tag{1,160}$$

The four unknown functions $u_j(s)$ are subjected to given initial conditions at instant $s = 0$:

$$u_j(0) = (u_j)_0 , \quad u_j'(0) = (u_j')_0 . \tag{1,161}$$

We shall now discuss the influence of errors $\Delta(u_j)_0$, $\Delta(u_j')_0$ in the initial values on the calculated motion of the particle, assuming ω fixed and exactly known in advance. Such errors generate errors

(1,77) $$\Delta\alpha_j = \Delta(u_j)_0 , \quad \Delta\beta_j = \frac{1}{\omega}\Delta(u_j')_0 \tag{1,162}$$

of the regularized elements and thus also errors

(1,76) $$\Delta u_j = (\Delta\alpha_j)\cos\omega s + (\Delta\beta_j)\sin\omega s \tag{1,163}$$

of the solutions of our differential equations. It follows

$$|\Delta u_j(s)| \leq |\Delta\alpha_j| + |\Delta\beta_j| . \tag{1,164}$$

Therefore the $|\Delta u_j|$ are at any time smaller than a given quantity ε provided the errors of the elements are suitably small:

$$|\Delta\alpha_j| < \frac{\varepsilon}{2} , \quad |\Delta\beta_j| < \frac{\varepsilon}{2} .$$

Thus we have the result that the differential system (1,159) has the property of strict stability.

Errors of the coordinates u_j may occur at any step of numerical integration and such erroneous values are then used as initial conditions for the next step. Because the true motion is strictly stable, as integration proceeds such errors do not carry the calculated position of the particle too far away from its true position. Thus the numerical integration of (1,159) is numerically stable. The classical equations of Kepler motion do not share this property, because they are not strictly but only orbitally stable.[1] In this section we do not discuss the accumulation of truncating or rounding-off errors. Chapter 4 will be devoted to some

[1] The reader will recall that strict stability is a much stronger condition than the more usual orbital stability. Orbital stability requires only that if slightly perturbed, the particle follows an orbit which is very close to the unperturbed orbit, but it may at a later time be at a position on this orbit quite different from the corresponding position on the unperturbed orbit. Strict stability requires in addition, that at a later time these positions are close to each other.

aspects of this more difficult realm of problems.

It must be recalled however that the frequency ω is determined by the initial conditions

(1,73) $$\omega^2 = \frac{M}{2r_0} - \frac{v_0^2}{4} . \tag{1,165}$$

Consequently it may happen that a slightly erroneous but constant value of ω is used at every step of integration. Instead of the true coordinates

$$u_j(s) = \alpha_j \cos \omega s + \beta_j \sin \omega s$$

the modified values

$$u_j^*(s) = \alpha_j \cos(\omega + \Delta\omega)s + \beta_j \sin(\omega + \Delta\omega)s \tag{1,166}$$

are thus computed, assuming for the sake of simplicity the initial values (1,161) to be accurate. In order to facilitate the discussion we introduce a variation Δs defined by

$$\frac{\Delta s}{s} = \frac{\Delta \omega}{\omega} . \tag{1,167}$$

Then we have

$$u_j^*(s) = \alpha_j \cos \omega(s + \Delta s) + \beta_j \sin \omega(s + \Delta s)$$

or

$$u_j^*(s) = u_j(s + \Delta s) . \tag{1,168}$$

This equation shows that the orbit is not changed at all, but the calculated position of the particle on its orbit moves away from its true position on this orbit; this phenomenon is of unstable character since Δs is proportional to s. More precisely it follows from (1,167) that the relative error of s is equal to the relative error of ω. The motion (1,159) is thus orbitally stable but not strictly stable. Therefore ω should be given with very high accuracy. By virtue of equation (1,83) this is equivalent with an accurate value of a. As in the classical theory the semi-major axis a is the most important orbital element.

In practice we are faced of course with a superposition of the two phenomena discussed above. Any errors of the position of the particle and its velocities in physical space produce indeed errors of the elements α_j as well as an error of ω. Nevertheless it must be stressed that after choice of a fixed value of ω the numerical integration of (1,159) proceeds with perfect numerical stability as was pointed out in the preceding discussion. This integration is thus reproduceable even if different numerical techniques or different automatic computers are used.

It still remains to discuss the influence of erroneous initial values on the physical time t if time is computed by

$$t = \int r\, ds . \tag{1,169}$$

As at the beginning of this section we assume a fixed and accurate value of the frequency ω. As can be seen from (1,81) errors $\Delta\alpha_j$, $\Delta\beta_j$ produce a secular perturbation

$$\Delta t = \frac{s}{2} \sum \left[\Delta(\alpha_j^2) + \Delta(\beta_j^2) \right] \tag{1,170}$$

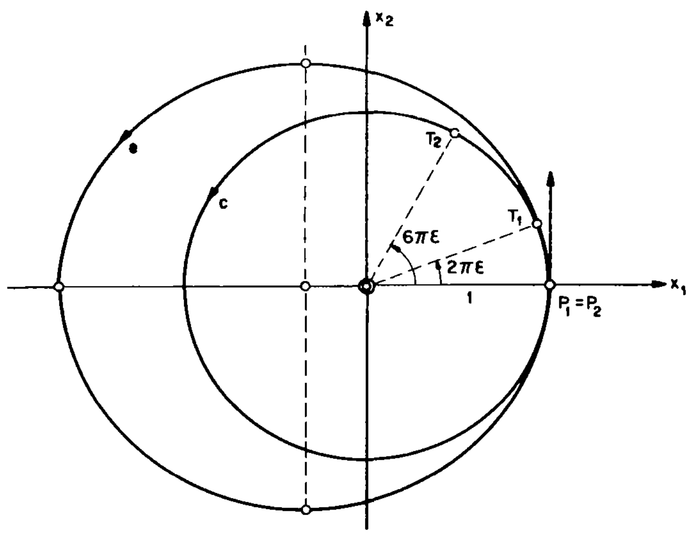
of the time, hence the computation of physical time is <u>unstable</u>.

We illustrate this phenomenon by the following very simple example of planar motion. (<u>Fig. 1.6</u>). The initial position of the particle is the point $(1,0)$ of the x_1, x_2-coordinate system and the initial velocity is $(0,1)$. By puting $M = 1$ we obtain as orbit of the particle the circle c and the motion of the particle is determined by

$$\varphi = t , \tag{1,171}$$

where φ is the true anomaly (polar angle) and t the physical time. From (1,165) it follows

$$\omega = \frac{1}{2} . \tag{1,172}$$



<u>Fig. 1.6</u>. Stability.

Let us assume now that an error ε occurs in the vertical component of the initial velocity, such that the initial position $(1,0)$ remains as before but the initial velocity $(0, 1+\varepsilon)$ is used. According to our assumptions the differential equations (1,159) are integrated with the <u>true value</u> (1,172) of ω but under the erroneous initial conditions

$$(x_1)_0 = 1 , \quad (x_2)_0 = 0 ; \qquad (\dot{x}_1)_0 = 0 , \quad (\dot{x}_2)_0 = 1+\varepsilon . \tag{1,173}$$

In order to obtain the results of this integration we compute the corresponding elements α_j, β_j . From the rules "initial conditions" of the second procedure (section 1.3.2) we have

$$(u_1)_0 = 1, \quad (u_2)_0 = 0 \; ; \qquad (u_1')_0 = 0, \quad (u_2')_0 = \frac{1+\varepsilon}{2},$$
$$\alpha_1 = 1, \quad \alpha_2 = 0 \; ; \qquad \beta_1 = 0, \quad \beta_2 = 1+\varepsilon, \tag{1,174}$$

and thus the errors of the elements are

$$\Delta\alpha_1 = 0, \quad \Delta\alpha_2 = 0 \; ; \qquad \Delta\beta_1 = 0, \quad \Delta\beta_2 = \varepsilon . \tag{1,175}$$

The motion of the particle in the parametric plane is now

(1,76) $$u_1 = \cos \omega s, \quad u_2 = (1+\varepsilon)\sin \omega s$$

and for the special value $s = 2\pi$ we obtain $u_1 = -1, \; u_2 = 0,$ hence

(1,44) $$x_1 = 1, \quad x_2 = 0 .$$

The particle is again at its initial position, this is to say at point P_1 of Fig. 1.6. The corresponding value of physical time is

(1,81) $$t = \pi\left[1 + (1+\varepsilon)^2\right] \sim 2\pi(1+\varepsilon).$$

At this instant the anomaly of the particle on its true orbit is $\varphi = 2\pi(1+\varepsilon)$ as follows from (1,171) and the corresponding point is denoted by T_1 in Fig. 1.6. After one revolution we have thus the error $2\pi\varepsilon$ in the true anomaly. After many revolutions this error is multiplied by the number of revolutions and this result demonstrates clearly the instability of the computation of motion.

In contrast to these considerations let us discuss now what happens if the motion is determined by integration of the classical equations of celestial mechanics. The erroneous initial conditions (1,173) put the particle on the elliptic orbit e of Fig. 1.6. Its semi-major axis a is determined by

(1,86) $$\frac{1}{a} = 2-(1+\varepsilon)^2 \sim 1-2\varepsilon, \qquad a \sim 1+2\varepsilon$$

and the corresponding revolution time is according to Kepler's third law

$$T = 2\pi a^{3/2} \sim 2\pi(1+2\varepsilon)^{3/2} \sim 2\pi(1+3\varepsilon) .$$

After this time the particle is again at initial position $P_2 = P_1$ but on its true orbit it is at position T_2 corresponding to the value $\varphi = 2\pi(1+3\varepsilon)$ of the true anomaly. In this case we have therefore after one revolution the error $6\pi\varepsilon$ in the true anomaly.

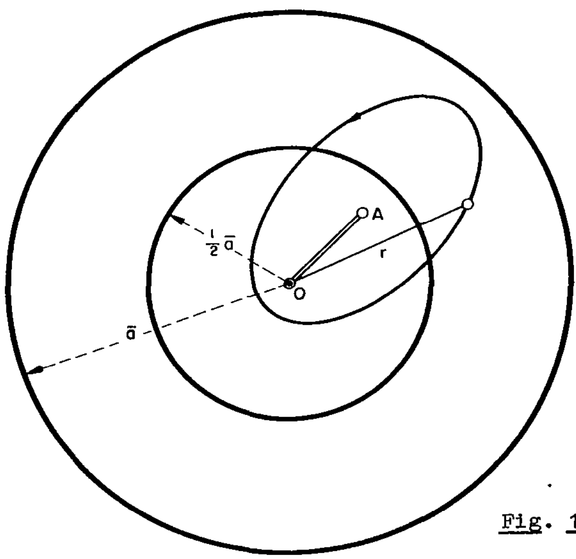
We may thus establish the following conclusion. In this example the regularized method is characterized by a mild instability, due to the underlying correct value of ω; but the classical method has a sharp instability, the ratio of the two instabilities being about $1:3$.

As above we may venture to predict now the accumulation of truncation- and rounding-off errors during a numerical integration. If regularized methods are used the errors of a single integration-step deteriorate as always the accuracy of the initial conditions for the next step. But because the same fixed value of ω is used at each step, we may hope that the accumulation of errors is governed by the mild instability and is thus more favourable than for the classical differential equations. This prediction is corroborated by the numerical experiments in chapter 4 of this report.

We may summarize these considerations as follows. Our regularized methods are characterized by a neat separation of the computation of the orbit from the determination of the position of the particle in its orbit. This separation may be considered to be an advantage since it has the tendancy to stabilize the computation.

Our discussion of stability brings out the deeper reason for our attitude in preferring the companions of the second and third procedure (cf. 1.3.2 and 1.4) to the procedures themselves; in the companions the dominant part of the physical time t (that is the Kepler-time t_K) is computed by an explicit formula and not by numerical integration.

1.7.2 Convergence of Fourier expansions. We now proceed to discuss a very simple example which demonstrates the advantage of expansion with respect to the eccentric anomaly in contrast to expansion with respect to the mean anomaly. We restrict ourselves to plane motion of the particle (Fig. 1.7). As always the central mass M is located at the origin and r is the distance of the particle from the origin.



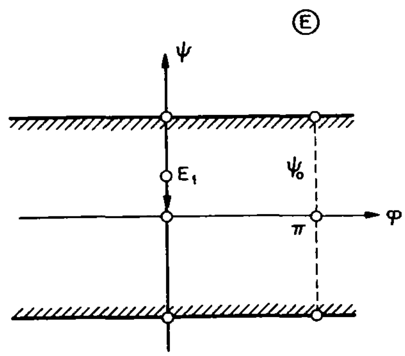


Fig. 1.7. Convergence of Fourier expansion.

Let furthermore the particle be subjected to a conservative perturbing potential $V(r)$ which depends only on the distance r and is an analytic function of the complex variable r regular for all values of r satisfying

$$|r| < \bar{a} . \qquad (1,176)$$

This situation occurs for instance if the perturbing potential is generated by a circular Gaussian ring (cf. 1.6.2) of radius $\bar{a}$ lying in the plane of motion and centered at the origin.

On a Kepler ellipse with semi-major axis a and eccentricity e, the distance of the particle is

$$r = a(1 - e\cos E), \tag{1,177}$$

hence the potential is transformed into an analytic function of the complex variable E which is the eccentric anomaly. We shall now discuss the domain of regularity of this function. We put

$$E = \varphi + i\psi \tag{1,178}$$

and we have accordingly

$$1 - e\cos E = 1 - e(\cos\varphi \cdot Ch\,\psi - i\sin\varphi \cdot Sh\,\psi),$$

where Ch, Sh are the hyperbolic functions. Thus

$$\begin{aligned}|1 - e\cos E|^2 &= 1 - 2e\cos\varphi \cdot Ch\,\psi + e^2(\cos^2\varphi \cdot Ch^2\psi + \sin^2\varphi \cdot Sh^2\psi)\\ &= 1 - 2e\cos\varphi \cdot Ch\,\psi + e^2(Ch^2\psi - \sin^2\varphi).\end{aligned}$$

This expression attains its maximum value for $\varphi = \pi$, this value being

$$1 + 2e\,Ch\,\psi + e^2 Ch^2\psi = (1 + e\,Ch\,\psi)^2,$$

hence

$$|1 - e\cos E| \leq 1 + e\,Ch\,\psi$$

and

$$|r| \leq a(1 + e\,Ch\,\psi). \tag{1,179}$$

Let now ψ_0 be the solution of the equation

$$a(1 + e\,Ch\,\psi_0) = \bar{a}, \qquad Ch\,\psi_0 = \frac{1}{e}\left(\frac{\bar{a}}{a} - 1\right). \tag{1,180}$$

This value ψ_0 does exist as a real and positive quantity if

$$\frac{1}{e}\left(\frac{\bar{a}}{a} - 1\right) > 1, \qquad \bar{a} > a(1 + e),$$

this is to say if the apocenter (and consequently the Kepler ellipse) is well inside the circle of radius $\bar{a}$ described above. Assume now $|\psi| < \psi_0$. From this hypothesis and from (1,179)(1,180) it follows $|r| < \bar{a}$. Thus the potential V is regular in the interior of the horizontal strip $|\psi| < \psi_0$ of the complex E-plane. (Fig. 1.7). Since V is a periodic function of E with real period 2π, the Fourier expansion of V with respect to E converges in the interior of this strip and in particular it converges uniformly for all real values of the eccentric anomaly E.

Let us consider now the family of orbits contained in the interior of a concentric circle of radius $\frac{1}{2}\bar{a}$. For any orbit of this family we have

$$a(1+e) < \frac{1}{2}\bar{a}\,, \quad \operatorname{ch}\psi_0 = \frac{1}{e}\left(\frac{\bar{a}}{a} - 1\right) > 2 + \frac{1}{e} \geq 3$$

and

$$\psi_0 > 1.76 \;.$$

For all the orbits of the family the function V is thus regular in the common strip $|\psi| \leq 1.76$ regardless of the eccentricity of the orbit and even for a collision orbit (the segment OA in Fig. 1.7). The rate of convergence of the Fourier-series of V with respect to E is determined by the breadth of this strip; hence the convergence is uniform with respect to the individuals of our family including the collision orbit with $e = 1$.

The situation is different if the mean anomaly m is used as independent variable, defined by the Kepler equation

$$m = E - e \sin E \,. \tag{1,181}$$

In order to establish V as a function of m, this equation must be solved with respect to E. This operation produces new singularities namely branch points in the complex m-plane determined by

$$\frac{dm}{dE} = 1 - e \cos E = 0 \,.$$

One solution E_1 of this equation is a point on the imaginary axis of the E-plane and the corresponding branch point m_1 is also on the imaginary axis of the m-plane. If the eccentricity e increases and approaches its limit 1, the points E_1, m_1 approach the real axis of their planes. Because m_1 is a singularity of V (considered as function of m), the Fourier expansion of V with respect to m will converge very poorly for highly eccentric orbits of our family and we have no longer uniform convergence in our family.

As we can see from this example, the convergence with respect to m is extremely sensitive to the eccentricity of the orbit, whereas the expansion with respect to E does not suffer from this disadvantage.

More information about the rate of convergence of such Fourier-series is available by consulting the theory of asymptotic behaviour of the Fourier coefficients of analytic functions.

1.8 Conclusions

We list here some characteristic properties of the regularizing methods that are presented in this report. We also compare these methods with some classical procedures. Only the KS-regularization (cf. 1.2.1) is considered.

1.8.1 General theoretical aspects.

- Regularized methods are not sensitive to the eccentricity of the (unperturbed) orbit, they remain efficient for collision orbits without loss of accuracy or convergence.

- The differential equations of a pure Kepler motion are linear. This incorporates the theory of perturbed motions into the well-explored realm of forced oscillations with non-linear restoring forces; discussion of stability and error propagation is thus facilitated.

- Because the coefficients of these linear differential equations are constant, the methods of "perturbations of coordinates" and "perturbations of elements" are practically equivalent in contrast to the classical approach.

- The regularized orbital elements are unambiguously defined even for a colliding osculating orbit and determine this orbit unambiguously. They obey a simple set of differential equations. But since there are 8 such elements and since the fictitious time s is introduced, a system of 9 or 10 first order equations must be integrated. The classical theory uses only 6.

1.8.2 General perturbations (Double Fourier expansion).

- In all our experiments the rate of convergence of the Fourier-series was not appreciably influenced by the eccentricity of the osculating orbit; in particular it was for ejection orbits as well as for nearly circular orbits.

- However the formal apparatus is slightly more complicated than in the classical Lagrange theory. In particular, the theory of the osculating orbit was developed only for the case of a finite semi-major axis. (No parabolic or hyperbolic osculating orbits).

1.8.3 Numerical aspects.

- The use of the fictitious time s causes a modification of the step length of integration which gives a "slow motion picture" of the particle's motion in the vicinity of most sharp bends in the orbit and, in particular, when the particle is near to the attracting center. This property is advantageous for the computation of transfer orbits from one celestial body to another.

- However, because the physical time t appears as a function of the independent variable s, the computation of particle's position at a given time t is only feasible by interpolation.

- In our numerical experiments we always used the Runge-Kutta method for integration of differential equations. Error propagation was more favorable by far for the regularized computation of the coordinates x_i (pure Kepler motion) than for the integration of the classical equations

$$\ddot{x}_i = -\frac{M}{r^3}\, x_i$$

(Cowell's method). With high probability this statement will remain true for perturbed orbits and if elements instead of coordinates are used.

- Consequently a larger step may be used than for classical integration. This advantage outweighs the increase of numerical labor due to the transformation of coordinates and time and the higher number of differential equations required by regularized methods.

- Therefore regularized methods may be more economical than classical ones, in particular if there is high eccentricity. This prediction was corroborated by experiments of Dr. Rössler (cf. 2.1.6). He computed the perturbations Δx_i of the coordinates:

1. By our second procedure (cf. 1.3.2).
2. By Encke's method [6, page 176].

- There are more refined methods for numerical integration than Runge-Kutta (for instance Fehlberg's method). If they need the derivatives of the perturbing forces, regularized methods are not advantageous, since the transformations involved in regularization complicate the computation of such derivatives.

2. COMPUTATIONAL PROGRAMS FOR SPECIAL AND GENERAL PERTURBATIONS WITH REGULARIZED VARIABLES

by M. Rössler

2.1 The program NUMPER ("numerical perturbations")

This program (Appendix 2.1) is a synthesis of the companion procedures of the second and third procedure described in sections 1.3.3 and 1.4. As perturbing force only the gravitational influence of a third body is taken into account; for other perturbing forces a special subroutine must be built in by the user. The motion of the perturbing body is either assumed to be an unperturbed Kepler ellipse or it can be given by an ephemeris. In the latter case interpolation is carried out by Lagrange's formula. Numerical integration is performed by the Runge-Kutta method.

2.1.1 List of symbols. The program is written in ALGOL 60, therefore some modifications of the symbols used in chapter 1 are needed.

real

TO = instant of physical time attached to the given initial conditions.

H = total energy h of the particle per unit of mass at time TO (only needed for ejection or near-ejection (cf. 1.3.4)).

M = gravitational parameter of the central body (product of gravitational constant and mass).

X1,X2,X3 = coordinates of the particle in physical space.

R = distance of the particle from the central body in physical space.

V1,V2,V3 = components of velocity of the particle in physical space.

V = magnitude of velocity of the particle in physical space.

OM = ω (cf. (1,73)).

C1 = $\frac{1}{2}\sum\left((\alpha_j)_0^2+(\beta_j)_0^2\right)$, C2 = $\frac{1}{2}\sum\left((\alpha_j)_0^2-(\beta_j)_0^2\right)$, C3 = $\sum(\alpha_j)_0(\beta_j)_0$,

where $(\alpha_j)_0$ and $(\beta_j)_0$ are the elements of the initial osculating Kepler orbit (cf. second procedure of chapter 1).

MP = gravitational parameter of the perturbing body.

XP1,XP2,XP3 = coordinates of the perturbing body.

RP = distance of the perturbing body from the central body.

VP1,VP2,VP3 = components of velocity of the perturbing body.

VP = magnitude of velocity of the perturbing body.

OMP = angular velocity of the perturbing body to be computed by the following modification of formula (1,73)

```
OMP↑2 = (M+MP)/RP/2-VP*VP/4 ,
```

where RP and VP are initial distance and velocity.

[CP1 = $\frac{1}{2}\sum(\bar\alpha_j^2+\bar\beta_j^2)$, CP2 = $\frac{1}{2}\sum(\bar\alpha_j^2-\bar\beta_j^2)$, CP3 = $\sum \bar\alpha_j \bar\beta_j$,
where $\bar\alpha_j$ and $\bar\beta_j$ are the elements of the Kepler orbit of the perturbing body.]

The symbols in square brackets above and in what follows are only needed if the perturbing body moves in a pure Kepler orbit.

{TBEG = initial instant of the ephemeris of the perturbing body.
DTTAB = step of the ephemeris.
TFL = scaling factor for adaption of the unit of length in the ephemeris to the unit of length in the program. (The coordinates XP1,XP2,XP3 are obtained by multiplying the rectangular coordinates of the ephemeris by this factor.)}

The symbols in curly brackets above and in what follows are only needed if the motion of the perturbing body is given by an ephemeris.

P1,P2,P3 = components of the perturbing force in physical space.

SUM = $\sum q_j u_j'$, where q_j and u_j' are the components of the perturbing force and the velocity in parametric space.

A = semi-major axis of osculating orbit (only needed if the third procedure is used (cf. 1.4)).

DR = Δr = perturbation of the distance of the particle from the central body (only needed if the second procedure is used (cf. 1.3.3)).

DS = step of Runge-Kutta integration (fictitious time).

TMAX = integration limit (physical time).

S = fictitious (regularized) time of the particle.

SP = fictitious time of the perturbing body.

T = physical time.

integer

N = number of differential equations to be integrated (for N = 10 the companion procedure of the second procedure is carried out, and for N = 9 the companion of the third procedure).

{NTAB = number of entries in the ephemeris, diminished by one.
NDEG = degree of the Lagrangian interpolation polynomials.}

NOUT : after NOUT Runge-Kutta steps the physical time, coordinates and velocities and the perturbed elements of the particle are computed and printed out.

boolean

NEARCENTRE : if true, the particle is assumed to start very near to the origin or exactly at the origin (cf. 1.3.4), then the value of H is needed, and V1,V2,V3 may be put in with an arbitrary scaling factor, so that they indicate only the direction of initial velocity,
if false, normal initial conditions as described in the second procedure.

array

ALO,BEO[1:4] = $(\alpha_j)_0$, $(\beta_j)_0$ = elements of the initial osculating Kepler orbit.

AL,BE[1:4] = α_j, β_j = perturbed elements (varying with time).

U,DUDS[1:4] = u_j, u_j' = parametric coordinates and velocities of the particle.

ALP,BEP[1:4] = $\bar{\alpha}_j$, $\bar{\beta}_j$ = elements of the perturbing body.

UP[1:4] = parametric coordinates of the perturbing body.

TAB[1:3,0:NTAB] = coordinates $\bar{x}_i(t_k)$ of the perturbing body taken from the ephemeris.

LAM[0:NDEG] = λ_k = coefficients of Lagrange's interpolation formula.

FCT[1:3] = $\bar{x}_i(t)$ = interpolated coordinates of the perturbing body.

DEL[1:N] = either ($\Delta\alpha_j$, $\Delta\beta_j$, Δt, W) if the second procedure (N = 10) is used

or ($\Delta\alpha_j$, $\Delta\beta_j$, Δt) if the third procedure (N = 9) is used,

where $\Delta\alpha_j$, $\Delta\beta_j$ = perturbations of the elements, Δt = perturbation of time and W = work done by the perturbing force.

Q[1:4] = components of the perturbing force in parametric space.

G[1:N] = right-hand sides of the differential equations.

2.1.2 Underlying formulae.

2.1.2.1 Initial conditions of the particle at instant T = TO :

a) NEARCENTRE = false (normal initial conditions).

Given: initial position X1,X2,X3 and initial velocity V1,V2,V3. We compute immediately ω and the elements $(\alpha_j)_0$, $(\beta_j)_0$, choosing $u_4 = 0$ or $u_3 = 0$ (cf. (1,47)), thus

```
R := SQRT(X1*X1+X2*X2+X3*X3);                                   (2,1)
if X1>0 then
   ALO[1] := SQRT((R+X1)/2);     ALO[2] := X2*ALO[1]/(R+X1);
   ALO[3] := X3*ALO[1]/(R+X1);   ALO[4] := 0;
        else                                                    (2,2)
   ALO[2] := SQRT((R-X1)/2);     ALO[1] := X2*ALO[2]/(R-X1);
   ALO[3] := 0;                  ALO[4] := X3*ALO[2]/(R-X1);
V := SQRT(V1*V1+V2*V2+V3*V3);                                   (2,3)
OM := SQRT(M/R/2-V*V/4);
BEO[1] := ( ALO[1]*V1+ALO[2]*V2+ALO[3]*V3)/OM/2;
BEO[2] := (-ALO[2]*V1+ALO[1]*V2+ALO[4]*V3)/OM/2;
BEO[3] := (-ALO[3]*V1-ALO[4]*V2+ALO[1]*V3)/OM/2;                (2,4)
BEO[4] := ( ALO[4]*V1-ALO[3]*V2+ALO[2]*V3)/OM/2;
```

b) NEARCENTRE = true (start near the origin or exactly at the origin).

Given: H,X1,X2,X3 and V1,V2,V3 down to an arbitrary scaling factor. We compute

```
R as in (2,1);    OM := SQRT(-H/2);    V as in (2,3);
if R=0 then
   ALO[1] := ALO[2] := ALO[3] := ALO[4] := 0;
   if V1>0 then
       BEO[1] := SQRT((V+V1)*M/V)/OM/2; BEO[2] := V2*BEO[1]/(V+V1);
       BEO[3] := V3*BEO[1]/(V+V1);      BEO[4] := 0;
           else
```

```
        BEO[2] := SQRT((V-V1)*M/V)/OM/2; BEO[1] := V2*BEO[2]/(V-V1);
        BEO[3] := 0;                     BEO[4] := V3*BEO[2]/(V-V1);
        else
     ALO[1],ALO[2],ALO[3],ALO[4] according to (2,2);
     true magnitude of velocity VC from
     VC := SQRT(2*M/R-4*OM*OM);
     BEO[1],BEO[2],BEO[3],BEO[4] according to (2,4), but with the
     true velocities V1/V*VC,V2/V*VC,V3/V*VC instead of V1,V2,V3.
```

In all cases we also compute

```
C1 := ( ALO[1]↑2+ALO[2]↑2+ALO[3]↑2+ALO[4]↑2+BEO[1]↑2+BEO[2]↑2
       +BEO[3]↑2+BEO[4]↑2)/2;
C2 := ( ALO[1]↑2+ALO[2]↑2+ALO[3]↑2+ALO[4]↑2-BEO[1]↑2-BEO[2]↑2          (2,5)
       -BEO[3]↑2-BEO[4]↑2)/2;
C3 := ALO[1]*BEO[1]+ALO[2]*BEO[2]+ALO[3]*BEO[3]+ALO[4]*BEO[4];
```

2.1.2.2 Perturbing body on a Kepler orbit:

Elements of the orbit as in 2.1.2.1 a), but replace X1,X2,X3 by XP1,XP2,XP3; V1,V2,V3 by VP1,VP2,VP3; ALO[1:4],BEO[1:4] by ALP[1:4],BEP[1:4]; OM by OMP and M by M+MP. Finally compute CP1,CP2,CP3 as in (2,5), but replace ALO[1:4],BEO[1:4] by ALP[1:4],BEP[1:4].

Computation of the coordinates of the perturbing body at any time T:

```
Solve the following Kepler equation with respect to SP
T-TO = SP*CP1+SIN(2*OMP*SP)/OMP/2*CP2+(1-COS(2*OMP*SP))/OMP/2*CP3;
(In the program the solution of this equation is performed by Newton's method,
taking as initial guess  SP := (T-TO)/CP1-CP3/CP1/OMP/2).
for  J := 1,2,3,4  do  UP[J] := ALP[J]*COS(OMP*SP)+BEP[J]*SIN(OMP*SP);
XP1 := UP[1]↑2-UP[2]↑2-UP[3]↑2+UP[4]↑2;
XP2 := 2*(UP[1]*UP[2]-UP[3]*UP[4]);
XP3 := 2*(UP[1]*UP[3]+UP[2]*UP[4]);
```

2.1.2.3 Perturbing body given by ephemeris:

Lagrange interpolation coefficients $\lambda_k = (-1)^n\binom{n}{k}$, where n = NDEG, k running from 0 to NDEG. In the program these coefficients are computed by recursion. At a given instant T the coordinates FCT[1:3] of the perturbing body are computed by Lagrange's formula; the program chooses the tabular values to be used for this purpose.

2.1.2.4 Right-hand sides G[1:N] of the differential equations:

(For any value of the independent variable S and the corresponding array DEL[1:N]).

```
T := TO+C1*S+C2*SIN(2*OM*S)/OM/2+C3*(1-COS(2*OM*S))/OM/2+DEL[9];
for this time T compute the position XP1,XP2,XP3 of the perturbing body ac-
cording to section 2.1.2.2 or 2.1.2.3.
Perturbed elements:         AL[J] := ALO[J]+DEL[J];
                            BE[J] := BEO[J]+DEL[J+4];    (J := 1,2,3,4)
Parameters of the particle: U[J] := AL[J]*COS(OM*S)+BE[J]*SIN(OM*S);
Parametric velocities:      DUDS[J] := OM*(-AL[J]*SIN(OM*S)+BE[J]*COS(OM*S));
Distance of the particle from the central body:  R := U[1]↑2+U[2]↑2+U[3]↑2
                                                      +U[4]↑2;
```

```
Coordinates of the particle: X1 := U[1]↑2-U[2]↑2-U[3]↑2+U[4]↑2;
     X2 := 2*(U[1]*U[2]-U[3]*U[4]);  X3 := 2*(U[1]*U[3]+U[2]*U[4]);
Computation of the perturbing force:
   DEN1 := ((X1-XP1)↑2+(X2-XP2)↑2+(X3-XP3)↑2)↑1.5;
   DEN2 := (XP1↑2+XP2↑2+XP3↑2)↑1.5;
   in physical space:     P1 := -MP*((X1-XP1)/DEN1+XP1/DEN2);
                          P2 := -MP*((X2-XP2)/DEN1+XP2/DEN2);
                          P3 := -MP*((X3-XP3)/DEN1+XP3/DEN2);
   in parametric space: Q[1] := 2*( U[1]*P1+U[2]*P2+U[3]*P3);
                        Q[2] := 2*(-U[2]*P1+U[1]*P2+U[4]*P3);
                        Q[3] := 2*(-U[3]*P1-U[4]*P2+U[1]*P3);
                        Q[4] := 2*( U[4]*P1-U[3]*P2+U[2]*P3);
```
Computation of SUM = $\sum q_j u_j'$:
```
   SUM := Q[1]*DUDS[1]+Q[2]*DUDS[2]+Q[3]*DUDS[3]+Q[4]*DUDS[4];
if N=10 (companion of the second procedure) then
   G[J]   := -(R*Q[J]+2*DEL[10]*U[J])/OM/4*SIN(OM*S);
   G[J+4] :=  (R*Q[J]+2*DEL[10]*U[J])/OM/4*COS(OM*S);      (J := 1,2,3,4)
   Computation of the perturbation of distance DR:
      DAL2  :=  (2*ALO[1]+DEL[1])*DEL[1]+(2*ALO[2]+DEL[2])*DEL[2]
               +(2*ALO[3]+DEL[3])*DEL[3]+(2*ALO[4]+DEL[4])*DEL[4];
      DBE2  :=  (2*BEO[1]+DEL[5])*DEL[5]+(2*BEO[2]+DEL[6])*DEL[6]
               +(2*BEO[3]+DEL[7])*DEL[7]+(2*BEO[4]+DEL[8])*DEL[8];
      DALBE :=  ALO[1]*DEL[5]+BEO[1]*DEL[1]+DEL[1]*DEL[5]+ALO[2]*DEL[6]
               +BEO[2]*DEL[2]+DEL[2]*DEL[6]+ALO[3]*DEL[7]+BEO[3]*DEL[3]
               +DEL[3]*DEL[7]+ALO[4]*DEL[8]+BEO[4]*DEL[4]+DEL[4]*DEL[8];
      DR := (DAL2+DBE2)/2+(DAL2-DBE2)/2*COS(2*OM*S)+DALBE*SIN(2*OM*S);
   G[9]  := DR;
   G[10] := SUM;
else (companion of the third procedure)
   semi-major axis A of the osculating orbit:
   A := (AL[1]↑2+AL[2]↑2+AL[3]↑2+AL[4]↑2+BE[1]↑2+BE[2]↑2+BE[3]↑2+BE[4]↑2)/2;
   G[J]   := -A/C1*(R*Q[J]+DUDS[J]*SUM/OM/OM)/OM/4*SIN(OM*S);
   G[J+4] :=  A/C1*(R*Q[J]+DUDS[J]*SUM/OM/OM)/OM/4*COS(OM*S);    (J := 1,2,3,4)
   G[9]   := SQRT(A/C1)*R-(C1+C2*COS(2*OM*S)+C3*SIN(2*OM*S));
```

2.1.2.5 Differential equations:

```
DEL[J]' = G[J];     (J = 1,...,N)
```

(where the accent means differentiation with respect to S).

Integration is performed by a Runge-Kutta subroutine.

2.1.2.6 Output formulae:

T,X1,X2,X3,AL[1:4],BE[1:4] as computed in 2.1.2.4.

Velocities of the particle in physical space (if R≠0):

```
if N=10 then  V1 := 2/R*(U[1]*DUDS[1]-U[2]*DUDS[2]-U[3]*DUDS[3]+U[4]*DUDS[4]);
              V2 := 2/R*(U[1]*DUDS[2]+U[2]*DUDS[1]-U[3]*DUDS[4]-U[4]*DUDS[3]);
              V3 := 2/R*(U[1]*DUDS[3]+U[2]*DUDS[4]+U[3]*DUDS[1]+U[4]*DUDS[2]);
```

else compute V1,V2,V3 as for N=10, but with the factor 2/R*SQRT(C1/A) instead of 2/R.

if N=10 then the left- and right-hand sides of the equation (1,97)

```
R*DEL[10] =      2*OM↑2*((2*ALO[1]+DEL[1])*DEL[1]+(2*ALO[2]+DEL[2])*DEL[2]
    +(2*ALO[3]+DEL[3])*DEL[3]+(2*AlO[4]+DEL[4])*DEL[4]+(2*BEO[1]+DEL[5])*DEL[5]
    +(2*BEO[2]+DEL[6])*DEL[6]+(2*BEO[3]+DEL[7])*DEL[7]+(2*BEO[4]+DEL[8])*DEL[8])
```

are computed and printed out as check.

2.1.3 Input and output. Because ALGOL 60 does not include input and output, the following description refers to our experiments on a Control Data 1604-A computer [8].

2.1.3.1 Input:

At first the units of length, mass and time must be chosen; they are arbitrary. The input is listed on punched cards in the following sequence, with the values being legal ALGOL numbers (arbitrary signed or unsigned, decimal or exponent notation), except for the boolean variable NEARCENTRE, where the value must be a plus (=false) or a minus (=true) sign. Each value must be followed by a comma; the number of values per card, the length of the numbers, and the number of spaces are arbitrary.

Symbol used in the program	input
N	Set =10, if companion of the second procedure is desired, set = 9, if companion of the third procedure is desired.
NEARCENTRE	Set true or false according to the rules outlined in the list of symbols.
TO	Initial time.
[H	Value of initial energy, only to be set if NEARCENTRE=true.]
M	Gravitational parameter of the central body.
X1,X2,X3	Initial coordinates of the particle at time TO.
V1,V2,V3	Components of initial velocity of the particle at time TO. (Observe modification indicated in the list of symbols if NEARCENTRE=true).
MP	Gravitational parameter of the perturbing body.
NTAB	Set =0, if the perturbing body is moving in a pure Kepler orbit with given initial data. Set =NTAB (as described in list of symbols), if the motion of the perturbing body is taken from an ephemeris.
if NTAB=0 then	
[XP1,XP2,XP3	Initial coordinates of the perturbing body at time TO.
VP1,VP2,VP3	Components of velocity of the perturbing body at time TO.]
else	
[NDEG	Degree of the Lagrange polynomials for interpolation in the ephemeris.
TBEG	Initial instant of the ephemeris.
DTTAB	Step of the ephemeris.
TFL	Value of scaling factor.

TAB[1,0],TAB[2,0],TAB[3,0] ⋮ TAB[1,NTAB],TAB[2,NTAB],TAB[3,NTAB]	Taken from the ephemeris.
DS	Step of integration.
NOUT	Set according to the list of symbols.
TMAX	Approximate last time of wanted particle position.

Remarks:

a) Choice of DS: An appropriate step τ in physical time is chosen, and DS computed from $DS = \frac{1}{r}\tau$, where r is the medium distance expected during the unperturbed motion of the particle.

b) If initial data are of parabolic or hyperbolic type, the machine gives a red light.

c) If the information delivered by the ephemeris is not sufficient to carry out the Lagrange interpolation, the machine gives a red light.
Therefore at least $\frac{1}{2}$ NDEG tabular values should be available before the start of particle TO and after its wanted end position TMAX.

2.1.3.2 Output: (Appendix 2.2)

For checking purposes some of the input data as well as some other important quantities are printed out immediately in the following order.

1.) if N=10 (second procedure) then the basic rule of regularization is printed out

DT = R*DS,

else (third procedure) the corresponding rule

DT = SQRT(A/AO)*R*DS,

is listed.

2.) TO and M are printed out.

3.) Information concerning the particle (referred to as "satellite"): initial coordinates and velocities and perhaps energy (different versions depending upon, whether NEARCENTRE=true or =false), semi-major axis, eccentricity and period of revolution corresponding to the unperturbed orbit.

4.) Information concerning the perturbing body:

if NTAB=0 (pure Kepler orbit) then mass, initial coordinates and velocities, semi-major axis, eccentricity, period of revolution,

else (ephemeris) mass, ephemeris adapted to the unit of length used in the program.

5.) Step of integration DS and value of NOUT.

The results of the integration are listed as follows

1st column: physical time T.

2nd column: physical coordinates X1,X2,X3 of the particle.

3rd column: components of velocity V1,V2,V3 of the particle. (If collision occurs, the components indicate only the direction of velocity, because the magnitude of the velocity is infinite.)

4th column: perturbed elements ALPHA[J].

5th column: perturbed elements BETA[J]. (J = 1,2,3,4)

If N=10 (second procedure), a 6th column is printed out containing in the first line the quantity rW = R*DEL[10] of equation (1,97) and in the second line the right-hand side of that equation. This is the energy check.

2.1.4. Description of the program NUMPER. We give a rough description of the parts of the program. The following numbers of the parts correspond to the numbers on the left-hand border of Appendix 2.1.

part 1: Declarations of the quantities under consideration. NFCT is later replaced by 3.

part 2: procedure REGEL: computation of the regularized initial elements taking into account the different modifications (NEARCENTRE = true or = false), computation of the auxiliary quantities C1,C2,C3.
The same procedure is used for computing the elements of the perturbing body if assumed to move in a Kepler orbit.

part 3: Read in of most of the data. Activation of procedure REGEL with respect to the particle.

part 4: Some declarations; CS and SN are symbols for cosine and sine, VF is an auxiliary variable.
procedure LAINTAB determines the set of tabular values of the ephemeris to be chosen for interpolation at a given time T and carries out this interpolation. We do not explain this procedure in detail because it is a standard interpolation routine.

part 5: procedure RK1ST is the standard Runge-Kutta routine of fourth order. H is the step.

part 6: procedure F is the computation of the right-hand sides G[1:N] of the differential equations. This procedure runs until the end of part 11.

part 7: Coordinates of the perturbing body if assumed to move on a Kepler ellipse. This part includes the solution of the Kepler equation by Newton's method.

part 8: Coordinates of the perturbing body if an ephemeris is used; procedure LAINTAB is activated.

part 9: Computation of the coordinates of the particle and of the perturbing force in physical and parametric space.

part 10: Right-hand sides G[1:10] of the differential equations, if N = 10 (2^{nd} procedure).

part 11: Right-hand sides G[1: 9] of the differential equations, if N = 9 (3^{rd} procedure).

part 12: Read in of the remaining data concerning the perturbing body. Computation of either the elements of the perturbing body (activation of REGEL) or the LAM[0:NDEG].

part 13: Computation of the output data: physical time, coordinates and velocities of the particle, values of the elements at the time under consideration, as was explained in 2.1.3.2.

part 14: Integration loop.

part 15: Information if errors occur.

Remarks:

For input and output the special procedures READ and OUTPUT and the declaration format, which are not included in ALGOL 60, are used repeatedly as is the custom on our Control Data 1604-A system. Details about these input - output facilities can be found in the reference [8]. Appropriate adaptions must be made if the program is used on another computer.

2.1.5 First numerical example: Perturbations of a highly eccentric satellite orbit by the moon. (Appendix 2.2).

2.1.5.1 Program:

The following version of program NUMPER (cf. 2.1) was used.

N = 10 (companion of the second procedure (cf. 1.3.3)),
NEARCENTRE = true (start of the particle near the centre of the earth),
NTAB $\neq$ 0 (motion of the moon given by ephemeris).

2.1.5.2 Configuration (Fig. 2.1):

Attracting centre = earth, at the origin of the x_1, x_2, x_3 -system,
particle: unperturbed orbit = ellipse in the x_1, x_3 -plane with high eccentricity,
perturbing body = moon, orbit taken from the ephemeris [9].

2.1.5.3 Units:

Length: 1 km, mass: 1 kg, time: 1 mean solar day.
The gravitational parameters are M = $2.965621833 \cdot 10^{15}$, MP = $3.637460852 \cdot 10^{13}$.

2.1.5.4 Initial coordinates of the particle:

TO = 0,
H = $\cdot\, 10^{10}$, corresponding to the semi-major axis 148 281.09165 .
(X1,X2,X3) = (10 000,0,0),
direction of initial velocity (V1,V2,V3) = (0,0,1).
This initial position is the pericentre of the unperturbed orbit. The eccentricity is 0.932560518 and the period of revolution 6.58795532 .

2.1.5.5 Ephemeris of the moon:

The x_1, x_2 -plane is the equator of the earth corresponding to the epoch 1966.0 . The ephemeris gives XP1,XP2,XP3 with an accuracy of 7-8 decimals and with a time-step of 0.5 days. The unit of length of the ephemeris is the mean radius of the earth, thus TFL = 6 367.672608 .
We choose NTAB = 32, NDEG = 6, TBEG = -3, DTTAB = 0.5 .
The initial time TO = 0 is the date 243 8941.0 J.D. (= Jan. 4.0, 1966) of the original ephemeris.

2.1.5.6 Parameters of integration:

DS = 10^{-6} (approximately 45 steps per revolution),
NOUT = 1 ,
TMAX = 10 (approximately 1.5 revolutions).

2.1.5.7 Remarks:

For this satellite the influence of the moon is the most important perturbation. The unperturbed orbit is well outside the atmosphere and in the interior of the

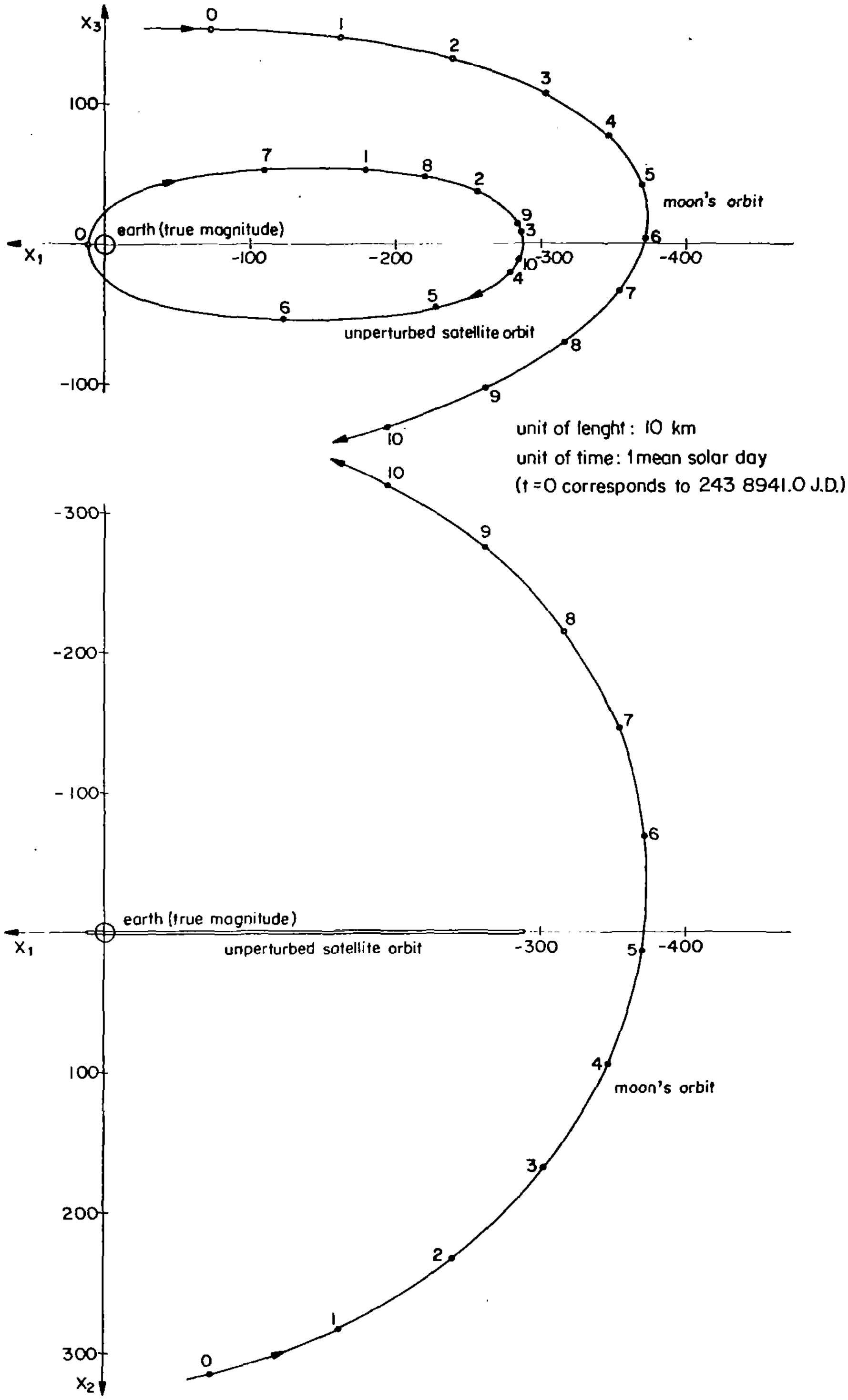


Fig. 2.1. First example. Unperturbed orbits.

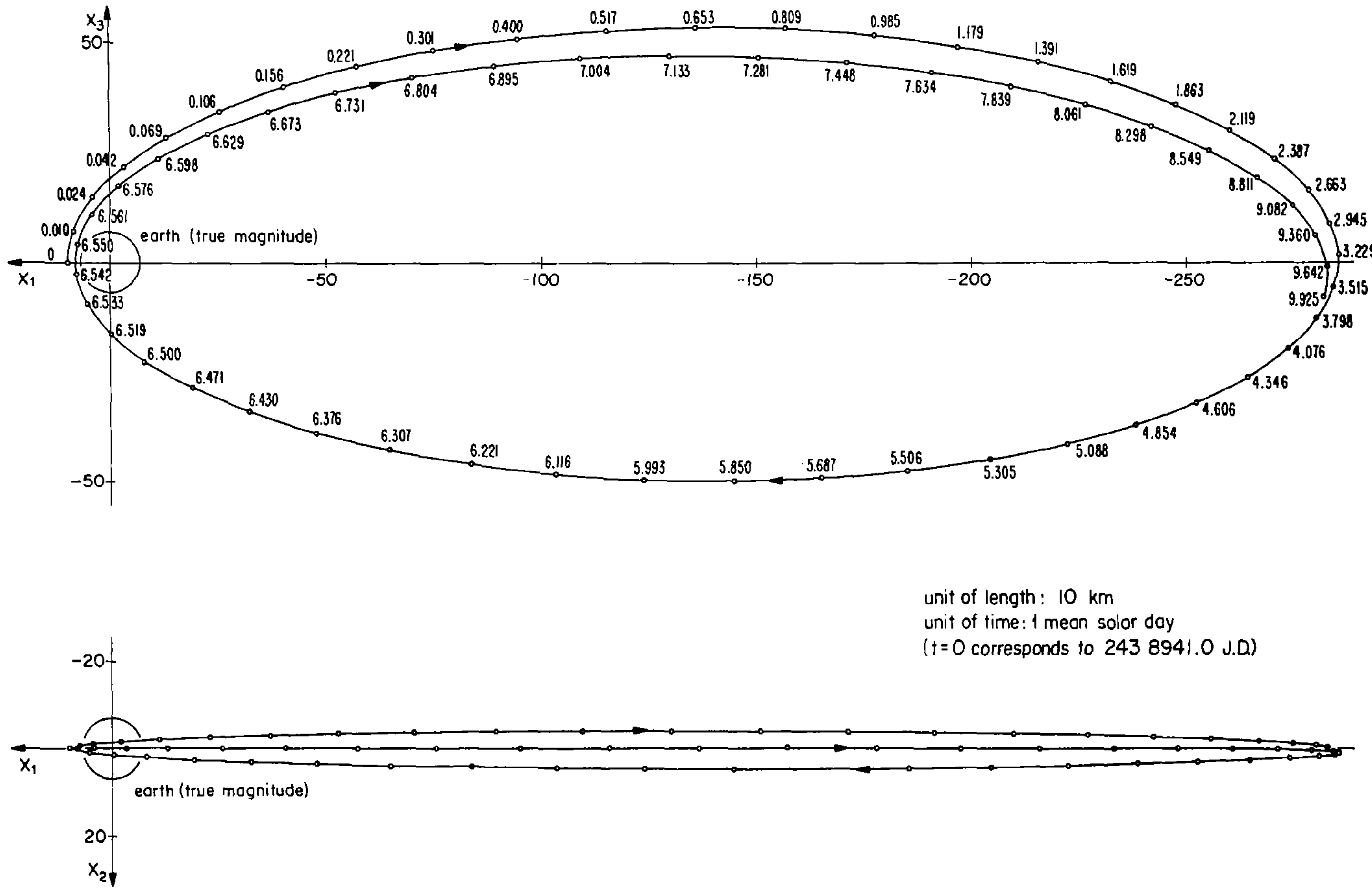


Fig. 2.2. First example. Perturbed satellite orbit.

moon's orbit. In Fig. 2.1 corresponding positions of the particle and of the moon are indicated (step = 1 day of physical time).

2.1.5.8 Discussion of the numerical results:

The results are listed in Appendix 2.2. The perturbed orbit is plotted in Fig. 2.2 with the points indicating the equidistant values of S. The physical time corresponding to each point is indicated. From this plot the automatic regulation of the step length performed by the fictitious time can be seen (near the peri- and apocentre the points are much denser than elsewhere).
A smaller integration step does not pay off, because the error produced by the ephemeris then becomes dominant. However, the integration with a smaller step would improve the balance of the energy equation.
After the first revolution, the satellite has lost about 1.9% of its initial energy, causing its pericentre to move closer to the earth.

2.1.6 Comparison with the classical method of Encke. In order to explain briefly Encke's method, we introduce the following notations:

x_i, r = coordinates and distance of the particle in the perturbed orbit,

x_{iK}, r_K = coordinates and distance of the particle in the unperturbed Kepler orbit,

$\Delta x_i = x_i - x_{iK}$, $\Delta r = r - r_K$ = perturbations,

$\bar{x}_i$, $\bar{r}$ = coordinates and distance of the perturbing body,

M, $\bar{M}$ = gravitational parameters of the central and the perturbing body respectively,

ϱ = distance of the particle from the perturbing body.

The classical differential equations for the Δx_i are

$$\Delta\ddot{x}_i = M\left(\frac{x_{iK}}{r_K^3} - \frac{x_{iK}+\Delta x_i}{(r_K+\Delta r)^3}\right) - \bar{M}\left(\frac{x_{iK}+\Delta x_i - \bar{x}_i}{\varrho^3} + \frac{\bar{x}_i}{\bar{r}^3}\right) , \qquad (2,6)$$

and the independent variable is the physical time t .
In the following examples either a constant step was chosen or an automatic step regulation was put into operation. Integration was performed by writing the differential equations (2,6) as a system of six simultaneous equations of first order and using the same Runge-Kutta method as in the program NUMPER.

2.1.6.1 Second numerical example:

Perturbations of a highly eccentric satellite orbit by the moon. Units: km, kg, day. $M = 2.9800083\cdot10^{15}$, $\bar{M} = 3.6656343\cdot10^{13}$. Initial conditions: satellite on the positive x_3-axis at distance 10 000 km, initial velocity parallel to x_2-axis of magnitude 750 000 km per day (eccentricity of the unperturbed orbit $e \approx 0.89$). Moon on a circular orbit of radius 384 400 km in the x_1, x_2-plane; initial position on the positive x_1-axis.
For $t = 3.1841455$ days (about one revolution) the following results were obtained.

Table 2.1. Comparison of NUMPER with Encke's method for highly eccentric orbit.

method	number of steps	step length DS, Δt resp.		x_1	x_2	x_3
NUMPER	2	DS = $2\cdot10^{-5}$		60.00	35 379.12	-33 888.55
	8	$5\cdot10^{-6}$		80.98	35 400.52	-33 911.38
	40	$1\cdot10^{-6}$		80.99	35 400.52	-33 911.34
	200	$2\cdot10^{-7}$		80.99	35 400.52	-33 911.34
Encke constant step	16	Δt = 0.2		26.47	62 365.72	-37 286.08
	64	0.05		15.58	22 144.64	-36 560.63
	319	0.01		81.03	35 439.95	-33 960.81
	1593	0.002		80.99	35 400.74	-33 911.40
Encke regulated step	46	0.152 > Δt	> 0.0047	81.00	35 404.30	-33 915.00
	103	0.152	0.00119	80.99	35 400.64	-33 911.33
	272	0.038	0.00059	80.99	35 400.50	-33 911.35
	356	0.025	0.00040	80.99	35 400.52	-33 911.35

Conclusions: The Encke-method with constant step needs at least 1593 steps to obtain the accuracy of 8 steps of the regularizing method and can therefore not be recommended. With automatic step regulation the corresponding number of Encke-steps is reduced to about 100. Although one step of the regularizing method needs about 3 times as much computing time as an Encke-step, the regularization does accelerate greatly the computation of the orbit.

2.1.6.2 Third numerical example:

Perturbations of an almost circular satellite orbit with high inclination. Units and masses as in the second example. Initial conditions: satellite on the positive x_3-axis at distance 75 000 km, initial velocity parallel to x_2-axis of magnitude 200 000 km per day (eccentricity $\approx$ 0.007). Moon as in the second example. For t = 3.0176050 days (about one revolution) the following results were obtained.

Table 2.2. Comparison of NUMPER with Encke's method for a nearly circular orbit.

method	number of steps	step length DS, Δt resp.	x_1	x_2	x_3
NUMPER	2	DS = $2\cdot10^{-5}$	-6.31	75 162.85	-7 502.45
	8	$5\cdot10^{-6}$	4.37	75 171.71	-7 510.35
	40	$1\cdot10^{-6}$	4.34	75 171.72	-7 510.34
	200	$2\cdot10^{-7}$	4.34	75 171.72	-7 510.34
Encke constant step	16	Δt = 0.2	4.33	75 173.26	-7 508.02
	61	0.05	4.34	75 171.72	-7 510.33
	302	0.01	4.34	75 171.72	-7 510.34
	1509	0.002	4.34	75 171.72	-7 510.34

Conclusions: Because the unperturbed orbit is almost a circle, an automatic regulation of the Encke-steps would not give an improvement worth mentioning. Therefore a constant step was chosen, making the Encke-method as fast as possible. Four Encke steps give about the same accuracy as one step of the regularizing method, and the corresponding machine times are almost the same.

Remark: The program NUMPER will also be used in sections 2.2.5.2 and 2.2.5.3.

2.2 The program ANPER ("analytical perturbations")

This program (Appendix 2.3) computes the first-order perturbations of the elements α_j, β_j $(j = 1,2,3,4)$ (osculating Kepler orbit) and of the physical time t according to section 1.5 (fourth procedure) of this report. It takes only into account the perturbations by a third body assumed to move on a pure Kepler orbit. The perturbations are evaluated by double harmonic analysis.

2.2.1 The independent variables. Instead of the variables s, s_1 used in the theoretical analysis of section 1.5, we introduce modified variables which are better adapted to numerical computation. Let E be the *eccentric anomaly of the particle* on its unperturbed orbit and E_0 the initial value of E. By taking (1,87) into account, we have

$$E = E_0 + 2\omega s \quad . \tag{2,7}$$

Thus the definition of s_1 in section 1.5.2 is modified to read

$$s_1 = \frac{\bar{\mu}\, a_0}{2\omega} (E - E_0) \quad .$$

a_0 is related to the mean angular velocity μ of the particle by (1,84)(1,85); therefore

$$s_1 = \frac{\bar{\mu}}{\mu} (E - E_0) \quad . \tag{2,8}$$

As can be seen from (1,129), the integrands f now have the period 2π in both of the variables E and s_1. This property still holds true if any constant is added to s_1; thus instead of s_1 we may use the variable

$$E_1 = \frac{\bar{\mu}}{\mu} E + c \quad , \tag{2,9}$$

where c is a constant to be determined in the following. We introduce for this purpose the mean anomalies $m, \bar{m}$ of our bodies as well as their initial values $m_0, \bar{m}_0$. We can write

$$\bar{m} = \bar{m}_0 + \bar{\mu} t = \bar{m}_0 + \bar{\mu} \frac{m - m_0}{\mu} = \bar{m}_0 + \frac{\bar{\mu}}{\mu}(E - e \sin E) - \frac{\bar{\mu}}{\mu} m_0 \ ,$$

$$\bar{m} = \left(\bar{m}_0 - \frac{\bar{\mu}}{\mu} m_0\right) - c + E_1 - \frac{\bar{\mu}}{\mu} e \sin E \quad . \tag{2,10}$$

e is the eccentricity of the particle's orbit, and Kepler's equation has been inserted. We choose now $c = \bar{m}_0 - \frac{\bar{\mu}}{\mu} m_0$; then (2,9)(2,10) are reduced to

$$E_1 = \frac{\bar{\mu}}{\mu} E + \left(\bar{m}_0 - \frac{\bar{\mu}}{\mu} m_0\right) \quad , \tag{2,11}$$

$$\bar{m} = E_1 - \frac{\bar{\mu}}{\mu} e \sin E \quad . \tag{2,12}$$

As follows from the last equation (2,12), this choice has the advantage that E_1 is, apart from a pure periodic term, the mean anomaly of the perturbing body. As in the theoretical section 1.5 the dynamic situation is determined by the two independent variables E, E_1, because any choice of E determines the position of the particle and then any chosen value of E_1 yields by (2,12) the mean anomaly of the perturbing body. With respect to either of the two variables E, E_1 the fundamental period is 2π and is divided into $2N$ equal parts for performing the harmonic analysis.

2.2.2 The elements. In order to facilitate the comparison of the regularized computations with classical results, we introduce the elements corresponding to the pericentres of the two orbits; however, the initial positions of the two bodies remain general and are allowed to be different from the pericentres. From (1,76) and (1,87) the coordinates of the particle at instant $t = s = 0$ are obtained as follows

$$\begin{aligned} (u_j)_0 &= (\alpha_j)_R \cos\frac{E_0}{2} + (\beta_j)_R \sin\frac{E_0}{2} \quad , \\ (u_j')_0 &= \omega\cdot\left[-(\alpha_j)_R \sin\frac{E_0}{2} + (\beta_j)_R \cos\frac{E_0}{2}\right] \quad , \end{aligned} \tag{2,13}$$

where $(\alpha_j)_R, (\beta_j)_R$ are the elements corresponding to the pericentre of the osculating orbit at instant $t = 0$ (the subscript R is meant to signify "reduced to the pericentre"). By solving (2,13) with respect to the reduced elements we have

$$(\alpha_j)_R = (u_j)_0 \cos\frac{E_0}{2} - \frac{1}{\omega}(u_j')_0 \sin\frac{E_0}{2} \quad , \qquad (\beta_j)_R = (u_j)_0 \sin\frac{E_0}{2} + \frac{1}{\omega}(u_j')_0 \cos\frac{E_0}{2} \quad , \tag{2,14}$$

and

(1,87) $$\sum_{j=1}^{4} (\alpha_j)_R (\beta_j)_R = 0 \quad .$$

The same reduction is performed for the perturbing body by introducing the reduced elements

$$(\bar\alpha_j)_R = (\bar u_j)_0 \cos\frac{\bar E_0}{2} - \frac{1}{\bar\omega}(\bar u_j')_0 \sin\frac{\bar E_0}{2} \quad , \qquad (\bar\beta_j)_R = (\bar u_j)_0 \sin\frac{\bar E_0}{2} + \frac{1}{\bar\omega}(\bar u_j')_0 \cos\frac{\bar E_0}{2} \quad . \tag{2,15}$$

ANPER computes the perturbations $\Delta\alpha_j, \Delta\beta_j$ of the reduced elements $(\alpha_j)_R, (\beta_j)_R$ according to the formulae (cf. fourth procedure, section 1.5.1.)

$$\Delta\alpha_j = -\frac{1}{2\omega^2}\int_{E_0}^{E} F_j \sin\frac{E}{2}\, dE \quad , \qquad \Delta\beta_j = \frac{1}{2\omega^2}\int_{E_0}^{E} F_j \cos\frac{E}{2}\, dE \quad .$$

It computes also the perturbation of time (cf. (1,116))

$$\Delta t = \frac{1}{2\omega}\int_{E_0}^{E}\left(\Delta r + \frac{r_K}{2a_0}\cdot\Delta a\right) dE \quad .$$

Any of the eight perturbations of the elements appears in the form of a double Fourier polynomial with a secular term

$$c \cdot (E - E_0) + \sum_{\nu=0}^{N-1} \sum_{n=-N+1}^{N-1} \left[a_{\nu n} \cos(\nu E + nE_1) + b_{\nu n} \sin(\nu E + nE_1) \right] \qquad ; \qquad (2,16)$$

the coefficients c, $a_{\nu n}$, $b_{\nu n}$ are printed out. This formula is analogous to (1,128a). The perturbation of time Δt appears in a more complicated form

$$\begin{aligned} & c_1 (E - E_0) + c_2 (E \cos E - E_0 \cos E_0) + c_3 (E \sin E - E_0 \sin E_0) \\ & + \sum_{\nu=0}^{N-1} \sum_{n=-N+1}^{N-1} \left[a_{\nu n} \cos(\nu E + nE_1) + b_{\nu n} \sin(\nu E + nE_1) \right] \quad . \end{aligned} \qquad (2,17)$$

As above, $2N$ is the number of grid points of the harmonic analysis. All these nine perturbations vanish for $t = 0$, that is to say for $E = E_0$.
ANPER performs also the summation of the Fourier-series for a given value of E, and the perturbed elements $(\alpha_j)_R + \Delta\alpha_j$, $(\beta_j)_R + \Delta\beta_j$ as well as the perturbed time $t_K + \Delta t$ are printed out. The coordinates of the particle - if needed - could be computed by hand as follows

$$u_j = \left[(\alpha_j)_R + \Delta\alpha_j \right] \cdot \cos\frac{E}{2} + \left[(\beta_j)_R + \Delta\beta_j \right] \cdot \sin\frac{E}{2} \qquad , \quad (j = 1,2,3,4) \quad .$$

the coordinates x_i in the physical space are then determined by (1,44).

<u>2.2.3 Rules for the user</u>. ANPER is written in ALGOL 60. We do not describe this program in detail as we did for NUMPER but restrict ourselves to recording the in- and output specifications. Again the special procedures READ and OUTPUT, the declaration <u>format</u> and furthermore the procedures BINWRITE and BINREAD for handling the tapes are used. They are not included in ALGOL 60, but only defined on our Control Data 1604-A system [8]; appropriate adaptions must be made if the program is used on another computer.

<u>2.2.3.1 Input</u>:
Units of length, mass and time are arbitrary.
Before going to an electronic machine the user has to compute:
- Initial coordinates and velocities of the particle at time $t = 0$ in physical space, (ev. given by classical orbital elements),
- initial position and velocities of the particle in parametric space as in the second procedure (section 1.3.2),
- the value E_0 of the eccentric anomaly corresponding to the initial position from the classical formulae of Kepler motion,
- initial values of the elements $(\alpha_j)_R$, $(\beta_j)_R$ from (2,14);
- the same work has to be carried out with respect to the perturbing body ($\bar{E}_0$, $(\bar{\alpha}_j)_R$, $(\bar{\beta}_j)_R$).

The input is listed on punched cards in the following sequence; the values must be legal ALGOL numbers, each of them followed by a comma. The number of values per card, the length of the numbers, and the number of spaces are arbitrary.

Symbol used in the program	Symbol used in the underlying theory 2.2.1,2.2.2.
M	M = gravitational parameter of the central body (product of mass and gravitational constant).
EO	$E_\circ$ = eccentric anomaly of the particle at initial time $t=0$.
ALO[1],ALO[2],ALO[3],ALO[4] BEO[1],BEO[2],BEO[3],BEO[4]	$\left.\begin{matrix}(\alpha_j)_R\\(\beta_j)_R\end{matrix}\right\}$ = reduced elements of the particle at the pericentre of the unperturbed orbit.
MS	$\bar{M}$ = gravitational parameter of the perturbing body.
ESO	$\bar{E}_\circ$ = eccentric anomaly of the perturbing body at initial time $t=0$.
ALS[1],ALS[2],ALS[3],ALS[4] BES[1],BES[2],BES[3],BES[4]	$\left.\begin{matrix}(\bar{\alpha}_j)_R\\(\bar{\beta}_j)_R\end{matrix}\right\}$ = reduced elements of the perturbing body at the pericentre of the orbit.
JKMAX	N, ($2N$ is the number of points on the two orbits used for the harmonic analysis).
TF,TFT	Scaling factors for the listing of the Fourier coefficients; every coefficient of the perturbation of an element is multiplied by TF, every coefficient of the perturbation of time is multiplied by TFT, when it is printed out. (The largest coefficients should have the order of magnitude 10^{11}.)
I	Summation of the Fourier-series: number of summations to be carried out.
$\underbrace{\text{E,E,...,E}}_{\text{(I values)}}$	Values of E for which the summation is desired.

2.2.3.2 Output: (Appendix 2.4)

a) For checking purposes at the beginning of the computation:

The following information is printed out:

M,EO,ALO[1],ALO[2],ALO[3],ALO[4],BEO[1],BEO[2],BEO[3],BEO[4],

AO = semi-major axis of particle's osculating orbit at instant $t=0$,

EXZO = eccentricity of particle's osculating orbit at instant $t=0$,

formula for computing the unperturbed Kepler time t_K (denoted by T),

MS,ESO,ALS[1],ALS[2],ALS[3],ALS[4],BES[1],BES[2],BES[3],BES[4],

AS = semi-major axis of the orbit of the perturbing body,

EXZS = eccentricity of the orbit of the perturbing body,

formula for computing the Kepler time,

JKMAX,

equation (2,11) (with the numerical values of $\frac{\bar{\mu}}{\mu}$ and $\bar{m}_\circ - \frac{\bar{\mu}}{\mu} m_\circ$).

b) Investigation of resonance: (cf. 1.5.2)

For any value of the subscript ν (formulae (2,16)(2,17)) the value of $\nu + n\frac{\bar{\mu}}{\mu}$ which is nearest to 0 is printed out (n is the second summation index and μ, $\bar{\mu}$ are the mean angular motions). However, the information is suppressed if this minimum of $\left|\nu + n\frac{\bar{\mu}}{\mu}\right|$ is larger than for a preceding value of ν.

c) Fourier-series of the perturbations:

In Appendix 2.4 the perturbations $\Delta\alpha_j, \Delta\beta_j, \Delta t$ of the elements and of the time are denoted by D ALPHA 1,....,D ALPHA 4,D BETA 1,....,D BETA 4,DT. Perturbation of the elements: after D ALPHA (or D BETA) the chosen scaling factor TF is printed out. It follows the secular term; in the list of the Fourier coefficients the first and second columns indicate the values of ν and n, while the third and fourth columns contain the cosine and sine coefficients (cf. (2,16)).

Perturbation of the time: after DT the scaling factor TFT is printed out. The secular terms appear in the form (2,17), and the periodic terms are printed out according to the same pattern as for the perturbation of the elements.

d) Summation of the Fourier-series:

In the first column the chosen values of the independent variable E are listed again. The second column contains the unperturbed values of the elements and the Kepler time t_K, the third column the perturbations of the elements and of the time, and the fourth column the perturbed values of the elements and of the time.

2.2.4 Remarks. Concerning an appropriate choice for the number N used for the harmonic analysis we may give the following rough guess. Let a be the semi-major axis of the orbit of the particle, ϱ the minimal distance between the two orbits and d the number of wanted significant decimals of the perturbations; then choose at least

$$N = \frac{d}{\left|\log \frac{a+\varrho}{a}\right|} ,$$

where $\log$ is the Briggsian logarithm.

2.2.5 Fourth numerical example: Perturbations computed by four different methods.

2.2.5.1 Configuration: (Fig. 2.3)

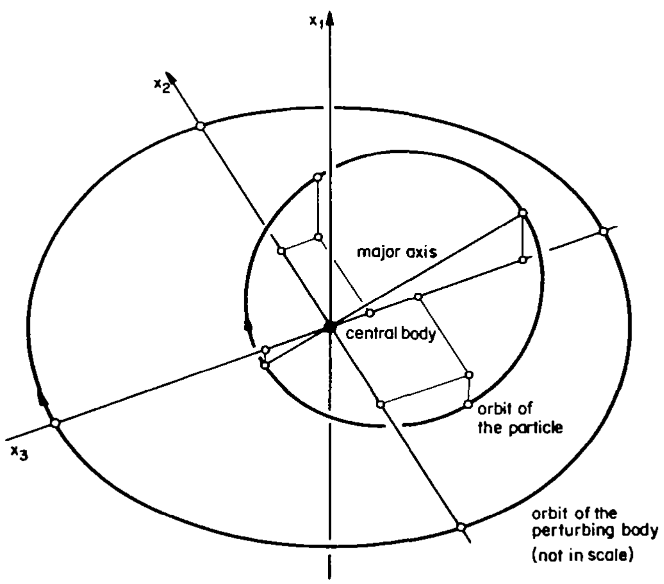


Fig. 2.3. Fourth example. Configuration.

Central body: at the origin, gravitational parameter $M = 1$.

Perturbing body: $\overline{M} = 0.01$. The orbit is a circle in the x_2, x_3-plane of radius $\overline{a} = 18$. Initial position (0,0,18), initial velocity (0, 0.23687784006 ,0).

The corresponding elements are $\overline{\alpha}_1 = 3$, $\overline{\alpha}_2 = 0$, $\overline{\alpha}_3 = 3$, $\overline{\alpha}_4 = 0$, $\overline{\beta}_1 = 0$, $\overline{\beta}_2 = 3$, $\overline{\beta}_3 = 0$, $\overline{\beta}_4 = -3$ and coincide with their reduced values $(\overline{\alpha}_j)_R$, $(\overline{\beta}_j)_R$; therefore $\overline{E}_0 = 0$.

Particle: the unperturbed orbit is an ellipse with the semi-major axis $a = 3$, eccentricity $e = 0.5$, inclination 39.7° to the x_2, x_3-plane, start at the pericentre ($E_0 = 0$). Initial position $(-\frac{1}{2}, 0, \sqrt{2})$, initial velocity $\left(\frac{2\sqrt{2}\sqrt{3}}{9}, \frac{\sqrt{2}\sqrt{3}}{3}, \frac{\sqrt{3}}{9}\right)$. The unperturbed elements are $\alpha_1 = \frac{\sqrt{2}}{2}$, $\alpha_2 = 0$, $\alpha_3 = 1$, $\alpha_4 = 0$, $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = -\frac{\sqrt{2}}{2}$, $\beta_4 = -\sqrt{2}$ and coincide with the reduced elements $(\alpha_j)_R$, $(\beta_j)_R$; furthermore the two parameters σ, τ introduced in section 1.6.3 are $\sigma = -\frac{1}{3}$, $\tau = \frac{\sqrt{2}}{3}$.

2.2.5.2 First method. Companion of the second procedure (section 1.3.3):

Program NUMPER.

Input data: N = 10, NEARCENTRE = false, TO = 0, NTAB = 0, DS = 0.1 $\sqrt{3}$ (corresponding to a step 0.1 of E) (approximately 63 steps per revolution), NOUT = 1, TMAX = 500 (about 15 revolutions of the particle, and about 1 revolution of the perturbing body).

The purpose of this computational example is to discuss the goodness of the energy balance (cf. 1.3.3, 2.1.2.6 and 2.1.3.2). In Fig. 2.4 the quantity

$$(1,97) \qquad r\cdot W - 4\omega^2 \sum \left(\left[(\alpha_j)_0 + \tfrac{1}{2}\Delta\alpha_j\right] \Delta\alpha_j + \left[(\beta_j)_0 + \tfrac{1}{2}\Delta\beta_j\right] \Delta\beta_j \right)$$

is plotted against $E = 2\omega s$. At the end $E = 96$ we read for this quantity the value $3.61 \cdot 10^{-11}$, the corresponding value of rW is $-3.07681 \cdot 10^{-5}$, and thus the relative error of the energy check is about 10^{-6}. This is a satisfactory result.

2.2.5.3 Second method. Companion of the third procedure (section 1.4):

Program NUMPER.

Input data: N = 9, NEARCENTRE = false, TO = 0, NTAB = 0, DS = 0.1 $\sqrt{3}$ (corresponding to a step 0.1 of E) (approximately 63 steps per revolution), NOUT = 5, TMAX = 500 (about 15 revolutions of the particle, and about 1 revolution of the perturbing body).

The results of this computation are displayed in two ways. First, the perturbations $\Delta\alpha_j$, $\Delta\beta_j$, Δt corresponding to $E = 80$ (about 13 revolutions of the particle) are tabulated in Table 2.3 under the heading NUMPER. Second, the perturbations $\Delta\alpha_2$ and $\Delta\beta_4$ are plotted in Fig. 2.5 and Fig. 2.6 against E.

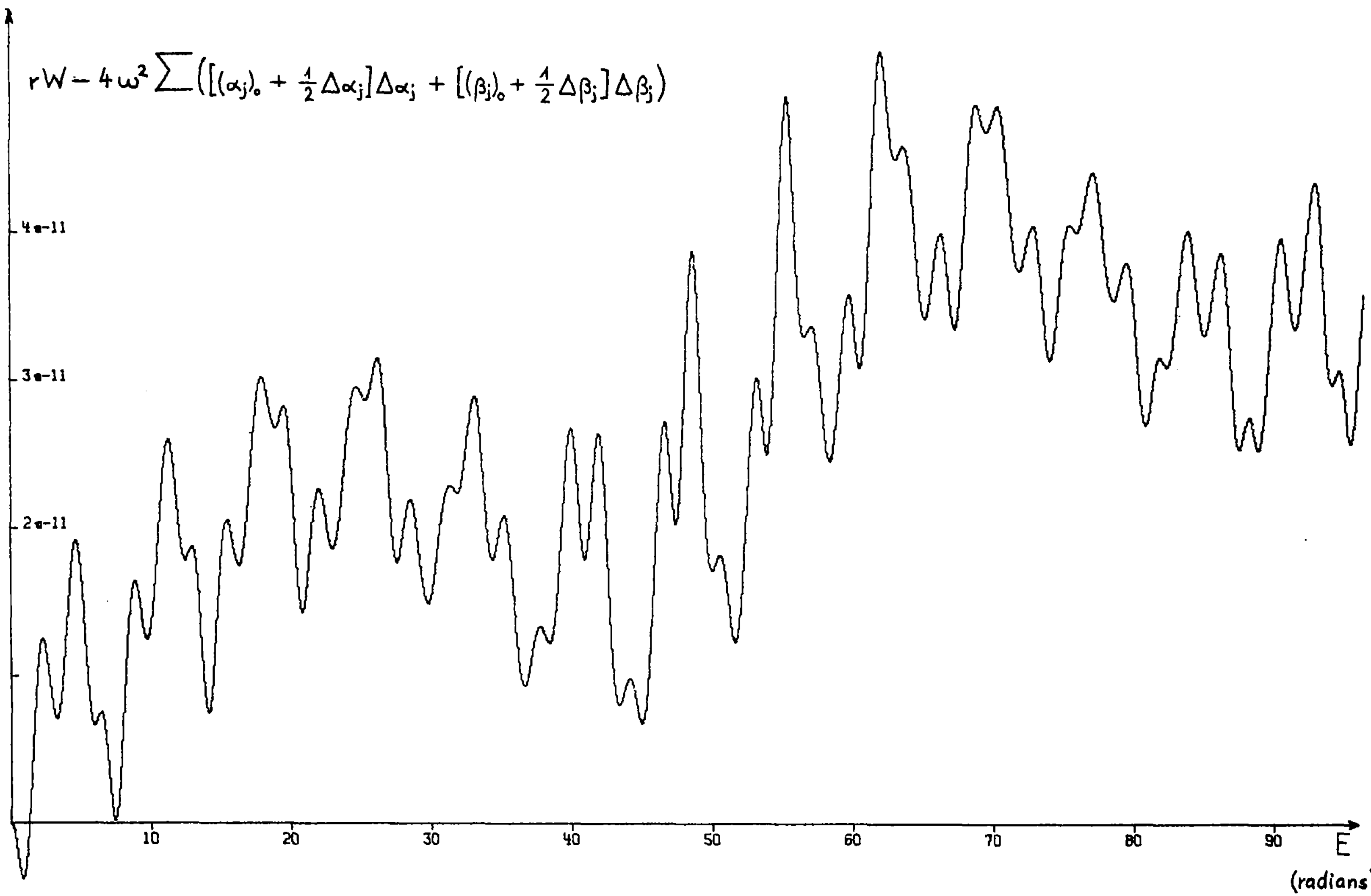


Fig. 2.4. Fourth example. Energy balance.

Table 2.3. Comparison of special perturbations and first-order general perturbations.

NUMPER		ANPER	
E	$= 80$	E	$= 80$
$\Delta\alpha_1$	$= 0.00072460$	$\Delta\alpha_1$	$= 0.00073163$
$\Delta\alpha_2$	$= -0.00255181$	$\Delta\alpha_2$	$= -0.00254940$
$\Delta\alpha_3$	$= 0.00169407$	$\Delta\alpha_3$	$= 0.00169465$
$\Delta\alpha_4$	$= 0.00033265$	$\Delta\alpha_4$	$= 0.00031953$
$\Delta\beta_1$	$= -0.00055131$	$\Delta\beta_1$	$= -0.00055084$
$\Delta\beta_2$	$= 0.00049547$	$\Delta\beta_2$	$= 0.00049473$
$\Delta\beta_3$	$= 0.00045080$	$\Delta\beta_3$	$= 0.00045364$
$\Delta\beta_4$	$= 0.00137048$	$\Delta\beta_4$	$= 0.00136828$
Δt	$= 0.021224$	Δt	$= 0.021180$

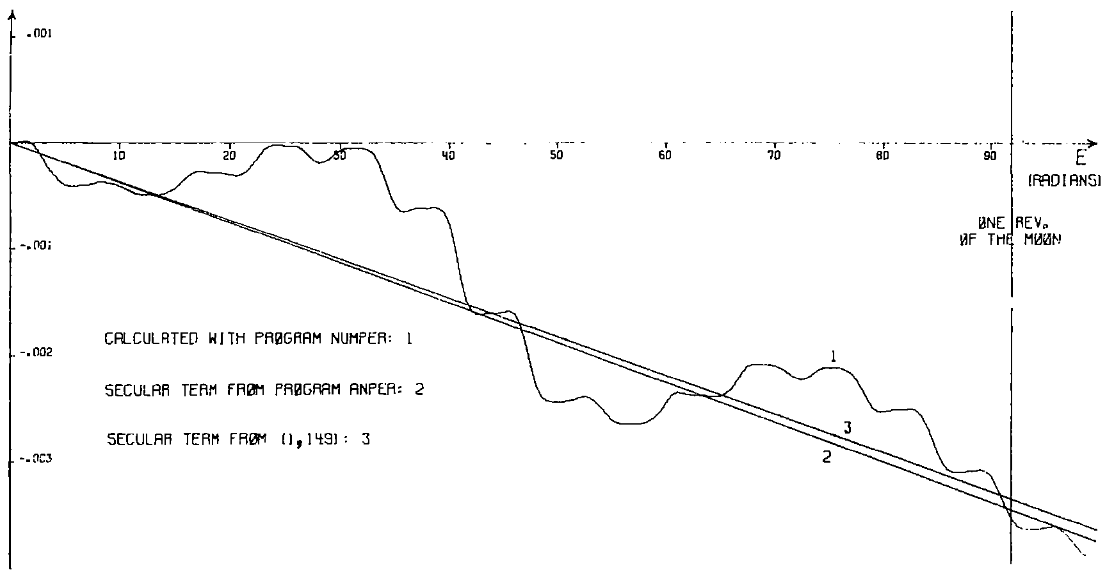


Fig. 2.5. Fourth example. Perturbation $\Delta\alpha_2$.

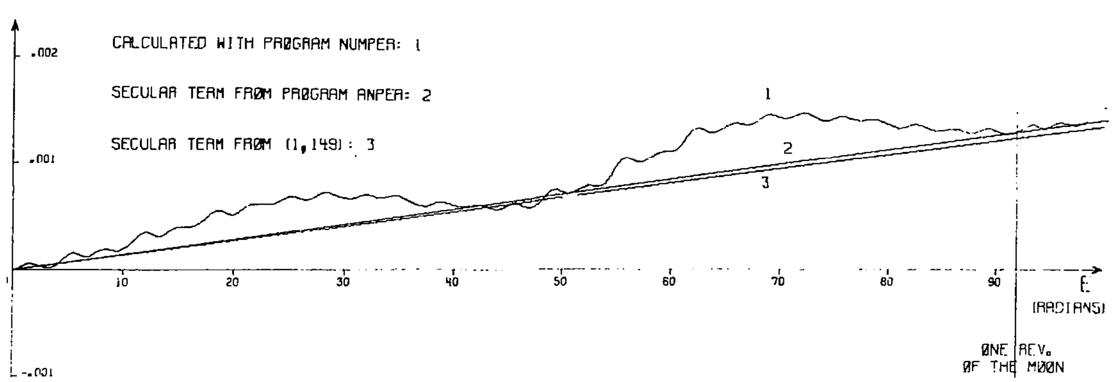


Fig. 2.6. Fourth example. Perturbation $\Delta\beta_4$.

2.2.5.4 Third method. Analytical first-order perturbations: (Appendix 2.4)
Program ANPER.
Input data: EO = 0, ESO = 0, JKMAX = 13, TF = TFT = 10^{14}, I = 1, E = 80.
The second and third methods are used to compare the numerical and first-order analytical perturbations. Again the results of the third method are listed in Table 2.3 (under the heading ANPER); the corresponding plot in Fig. 2.5 and Fig. 2.6 coincides practically with the plot of the second method. Furthermore the secular perturbations are listed in Table 2.4 under the heading ANPER, and the secular perturbations of $\Delta\alpha_2$ and $\Delta\beta_4$ are plotted in Fig. 2.5 and Fig. 2.6 as straight lines. Appendix 2.4 is a part of the results output by the Control Data 1604-A.

2.2.5.5 Fourth method. Secular perturbations according to the formulae (1,149):
The results are listed in Table 2.4 under the heading (1,149) and plotted (for $\Delta\alpha_2$ and $\Delta\beta_4$) in Fig. 2.5 and Fig. 2.6. Because the ratio of the major axes is rather small, the results of this rough computation have an acceptable accuracy; they coincide with the results of ANPER within a relative error of about 4%.

Table 2.4. Secular perturbations.

ANPER	(1,149)
$\Delta\alpha_1 = 0.27595\cdot10^{-5}$ E	$\Delta\alpha_1 = 0.19290\cdot10^{-5}$ E
$\Delta\alpha_2 = -3.77738\cdot10^{-5}$ E	$\Delta\alpha_2 = -3.66512\cdot10^{-5}$ E
$\Delta\alpha_3 = 2.00678\cdot10^{-5}$ E	$\Delta\alpha_3 = 1.91096\cdot10^{-5}$ E
$\Delta\alpha_4 = 0.60747\cdot10^{-5}$ E	$\Delta\alpha_4 = 0.68200\cdot10^{-5}$ E
$\Delta\beta_1 = -0.72616\cdot10^{-5}$ E	$\Delta\beta_1 = -0.68200\cdot10^{-5}$ E
$\Delta\beta_2 = 0.66424\cdot10^{-5}$ E	$\Delta\beta_2 = 0.68200\cdot10^{-5}$ E
$\Delta\beta_3 = 0.20442\cdot10^{-5}$ E	$\Delta\beta_3 = 0.19290\cdot10^{-5}$ E
$\Delta\beta_4 = 1.41098\cdot10^{-5}$ E	$\Delta\beta_4 = 1.35031\cdot10^{-5}$ E
$\Delta t = 3.81647\cdot10^{-4}$ E $+1.05457\cdot10^{-4}$ E cos E $+0.76276\cdot10^{-4}$ E sin E	

2.2.6 Fifth numerical example. Convergence of the Fourier expansion in the case of an ejection orbit.

2.2.6.1 Configuration: (Fig. 2.7)

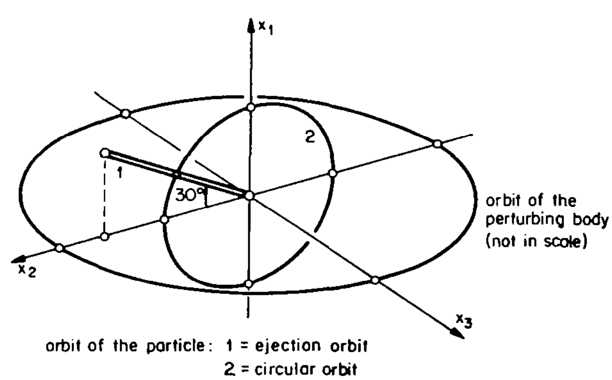


Fig. 2.7. Fifth and sixth example. Configuration.

As described in section 1.6.4. The numerical constants appearing in formula (1,157) were chosen as follows: $\frac{\bar{M}}{M} = 0.01$, $\frac{a}{\bar{a}} = \frac{1}{9}$. The same value $\varphi = 60^{\circ}$ was adopted.

Elements of the particle: $\alpha_j = 0$, $(j = 1,2,3,4)$, $\beta_1 = \frac{\sqrt{3}}{2}$, $\beta_2 = \frac{1}{2}$, $\beta_3 = 0$, $\beta_4 = 0$.

Elements of the perturbing body: $\bar{\alpha}_1 = \frac{3}{2}$, $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \frac{3}{2}$, $\bar{\alpha}_4 = 0$, $\bar{\beta}_1 = 0$, $\bar{\beta}_2 = \frac{3}{2}$, $\bar{\beta}_3 = 0$, $\bar{\beta}_4 = -\frac{3}{2}$. (Start on the positive x_3-axis).

2.2.6.2 Fourier-series:

Program ANPER.

Input data: EO = 0, ESO = 0, JKMAX = 13, TF = TFT = 10^{11}, I = 0.

In Table 2.5 the cosine-coefficients $a_{\nu n}$ (multiplied by TF) of the Fourier expansion

$$c(E-E_0) + \sum\sum\left[a_{\nu n}\cos(\nu E + nE_1) + b_{\nu n}\sin(\nu E + nE_1)\right]$$

of the perturbation $\Delta\alpha_1$ are listed. A row corresponds to running values of ν and a fixed value of n. This gives a picture of the convergence of such a series.

2.2.6.3 Secular perturbations:

They are, computed by ANPER,

$$\Delta\alpha_1 = 3.80195232\cdot10^{-6}\,E \quad , \quad \Delta\alpha_2 = -11.04974880\cdot10^{-6}\,E \quad ;$$

the remaining secular perturbations $\Delta\alpha_j$, $\Delta\beta_j$ vanish.

Table 2.5. Ejection orbit. Cosine-coefficients of $\Delta\alpha_1$.

n \ ν	0	1	2	3	4	5	6	7	8
-11		-1	0	0	0	0	0	0	0
-10		0	0	0	0	0	0	0	0
-9		15	-4	1	0	0	0	0	0
-8		0	0	0	0	0	0	0	0
-7		-376	94	-25	5	0	0	0	0
-6		0	0	0	0	0	0	0	0
-5		8 974	-1 966	339	-12	-8	1	0	0
-4		0	0	0	0	0	0	0	0
-3		-200 422	27 926	343	-724	0	-1	0	0
-2		0	0	0	0	0	0	0	0
-1		-263 426	61 044	-10 564	881	-7	0	0	0
0	11 090 045	0	0	0	0	0	0	0	0
1	-8 753 885	251 765	-62 904	11 774	-1 173	37	-1	0	0
2	0	0	0	0	0	0	0	0	0
3	-2 350 625	180 234	-34 856	2 042	613	-59	3	0	0
4	0	0	0	0	0	0	0	0	0
5	51 296	-7 006	2 186	-533	71	2	-2	0	0
6	0	0	0	0	0	0	0	0	0
7	-1 322	252	-97	34	-9	2	0	0	0
8	0	0	0	0	0	0	0	0	0
9	37	-8	4	-2	1	0	0	0	0
10	0	0	0	0	0	0	0	0	0
11	-1	0	0	0	0	0	0	0	0

Table 2.6. Circular orbit. Cosine-coefficients of $\Delta\alpha_1$

$n \backslash \nu$	0	1	2	3	4	5	6	7	8
-11		0	0	0	0	0	0	0	0
-10		0	0	0	0	0	0	0	0
-9		0	0	0	0	0	0	0	0
-8		0	0	0	0	0	0	0	0
-7		6	-2	1	0	0	0	0	0
-6		7	-12	4	-5	1	-1	0	0
-5		-442	170	-65	11	-8	-3	0	0
-4		-700	1 253	-317	311	-60	5	-1	0
-3		30 025	-9 591	3 182	1 085	37	15	0	0
-2		71 948	-134 254	23 169	-1 182	245	-11	2	0
-1		5 530	19 958	-9 002	-3 241	-73	-29	-1	0
0	-3 040 890	-468 685	256 216	-44 593	1 729	-360	15	-3	0
1	2 386 433	-5 133	-19 228	8 782	3 181	71	28	1	0
2	0	61 978	-124 618	22 047	-1 138	238	-11	2	0
3	3 983 381	-23 993	8 576	-2 953	-1 026	-36	-14	0	0
4	0	-519	1 079	-287	289	-61	4	-1	0
5	-2 709	304	-141	58	-10	8	3	0	0
6	0	5	-10	3	-4	1	-1	0	0
7	23	4	2	-1	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0

Comparison with the rough method of section 1.6.4:
From (1,156)(1,157) one obtains

$$\Delta\alpha_1 = 3.34114\cdot 10^{-6} E \quad , \quad \Delta\alpha_2 = -10.93105\cdot 10^{-6} E \quad .$$

2.2.7 Sixth numerical example. Convergence of the Fourier expansion in the case of a circular orbit.

2.2.7.1 Configuration: (Fig. 2.7)

The unperturbed orbit of the particle is a circle in the x_1, x_2 -plane; the perturbing body is as in the fifth example. $\frac{\bar{M}}{M} = 0.01$, $\frac{a}{\bar{a}} = \frac{1}{9}$ (as in the fifth example).

Elements of the particle: $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\beta_1 = \frac{1}{2}$, $\beta_2 = -\frac{1}{2}$, $\beta_3 = 0$, $\beta_4 = 0$ (coinciding with the reduced elements $(\alpha_j)_R$, $(\beta_j)_R$; therefore $E_0 = 0$).

2.2.7.2 Fourier-series:

In Table 2.6 the cosine-coefficients of $\Delta\alpha_1$ (computed by ANPER) are listed in the same arrangement as in the fifth example.

2.2.7.3 Conclusions:

As can be seen from the two Tables 2.5 and 2.6 the convergence of the series is not sensitive to the eccentricity. We have also carried out numerical experiments with the ratio $a:\bar{a} = 1:9$ in the more classical case where the orbit of the particle is in the plane of the orbit of the perturbing body. Also in this case the convergence behaviour of the Fourier-series was practically the same for an ejection orbit as for a circular orbit of the particle.

2.2.8 First-order perturbations of the orbit of the planetoid Vesta. The theory of the general perturbations of Vesta was established in 1880 by M.G. Leveau [10] according to Hansen's method. His results on the first-order perturbations by Jupiter have been compared with the results obtained by our program ANPER (cf. [5]). Since the set of elements used by Leveau is quite different from our regularized elements, it was only possible to compare the distance of the planetoid from the plane of its initial osculating Kepler orbit. The Fourier expansions of this distance as obtained by ANPER agreed perfectly with Leveau's results.

Appendix 2.1. Program NUMPER.

```
1: BEGIN
      REAL  T0,H,M,X1,X2,X3,V1,V2,V3,OM,C1,C2,C3,MP ;
      INTEGER  N,NTAB,NDEG,NFCT ;   BOOLEAN  NEARCENTRE ;
      ARRAY  AL0,BE0[1:4] ;
      FORMAT  INF := ''(22H START FAR FROM CENTRE//
         23H INITIAL POSITION  X1 =,E18.10,4X,4HX2 =,E18.10,4X,4HX3 =,
         E18.10/23H INITIAL VELOCITY  V1 =,E18.10,4X,4HV2 =,E18.10,4X,
         4HV3 =,E18.10//18H SEMI-MAJOR AXIS =,E18.10,4X,14HECCENTRICITY =,
         E18.10,4X,22HPERIOD OF REVOLUTION =,E18.10)'' ;
      FORMAT  INFNEARCENTRE := ''(18H START NEAR CENTRE//
         23H INITIAL POSITION  X1 =,E18.10,4X,4HX2 =,E18.10,4X,4HX3 =,
         E18.10/36H DIRECTION OF INITIAL VELOCITY  V1 =,E18.10,4X,4HV2 =,
         E18.10,4X,4HV3 =,E18.10//9H ENERGY =,E18.10//
         18H SEMI-MAJOR AXIS =,E18.10,4X,14HECCENTRICITY =,E18.10,4X,
         22HPERIOD OF REVOLUTION =,E18.10)'' ;

2:    PROCEDURE  REGEL(NEARCENTRE,M,X1,X2,X3,V1,V2,V3,OM,AL,BE,C1,C2,C3,
         L) ;
         VALUE  M,X1,X2,X3,V1,V2,V3 ;
         REAL  M,X1,X2,X3,V1,V2,V3,OM,C1,C2,C3 ;   ARRAY  AL,BE ;
         BOOLEAN  NEARCENTRE ;   LABEL  L ;
         BEGIN
            REAL  R,V ;   INTEGER  K ;
            R := SQRT(X1*X1+X2*X2+X3*X3) ;
            V := SQRT(V1*V1+V2*V2+V3*V3) ;
            IF  ¬ NEARCENTRE  THEN
            BEGIN
               OM := M/R/2-V*V/4 ;
               IF  OM≤0  THEN  GOTO  L ;
               OM := SQRT(OM) ;
            END  ;
            IF  R≠0 ∧ X1≥0  THEN
            BEGIN
               AL[1] := SQRT((R+X1)/2) ; AL[2] := X2*AL[1]/(R+X1) ;
               AL[3] := X3*AL[1]/(R+X1) ; AL[4] := 0 ;
            END
                              ELSE  IF  R≠0  THEN
            BEGIN
               AL[2] := SQRT((R-X1)/2) ; AL[1] := X2*AL[2]/(R-X1) ;
               AL[4] := X3*AL[2]/(R-X1) ; AL[3] := 0 ;
            END
                                                    ELSE
            AL[1] := AL[2] := AL[3] := AL[4] := 0 ;
            IF  R≠0  THEN
            BEGIN
               BE[1] := ( AL[1]*V1+AL[2]*V2+AL[3]*V3)/OM/2 ;
               BE[2] := (-AL[2]*V1+AL[1]*V2+AL[4]*V3)/OM/2 ;
               BE[3] := (-AL[3]*V1-AL[4]*V2+AL[1]*V3)/OM/2 ;
               BE[4] := ( AL[4]*V1-AL[3]*V2+AL[2]*V3)/OM/2 ;
               IF  NEARCENTRE  THEN
               BEGIN
                  REAL  VC ;
                  VC := 2*M/R-4*OM*OM ;
                  IF  VC<0  THEN  GOTO  L ;
                  VC := SQRT(VC) ;
                  FOR  K:=1  STEP  1  UNTIL  4  DO  BE[K] := BE[K]/V*VC ;
               END  ;
            END
                     ELSE  IF  V1≥0  THEN
            BEGIN
               BE[1] := SQRT((V+V1)*4/V)/OM/2 ; BE[2] := V2*BE[1]/(V+V1) ;
               BE[3] := V3*BE[1]/(V+V1) ; BE[4] := 0 ;
            END
                                       ELSE
```

```
         BEGIN
            BE[2] := SQRT((V-V1)*M/V)/OM/2 ; BE[1] := V2*BE[2]/(V-V1) ;
            BE[4] := V3*BE[2]/(V-V1) ; BE[3] := 0 ;
         END ;
         C1 := AL[1]*AL[1]+AL[2]*AL[2]+AL[3]*AL[3]+AL[4]*AL[4] ;
         C2 := BE[1]*BE[1]+BE[2]*BE[2]+BE[3]*BE[3]+BE[4]*BE[4] ;
         C1 := (C1+C2)/2 ;
         C2 := C1-C2 ;
         C3 := AL[1]*BE[1]+AL[2]*BE[2]+AL[3]*BE[3]+AL[4]*BE[4] ;
      END  REGEL ;

3:    READ(N,NEARCENTRE,T0) ;
      IF  N=10  THEN  OUTPUT(51,''(26H1REGULARIZATION  DT = R*DS////)'')
               ELSE  OUTPUT(51,''(15H1REGULARIZATION,2X,
         20HDT = SQRT(A/A0)*R*DS////)'') ;
      IF  NEARCENTRE  THEN
      BEGIN
         READ(H) ;
         IF  H>0  THEN  GOTO  INERROR ;
         OM := SQRT(-H/2) ;
      END ;
      OUTPUT(51,''(5H T0 =,E18.10////)'',T0) ;
      READ(M,X1,X2,X3,V1,V2,V3) ;
      OUTPUT(51,''(13H CENTRAL MASS//4H M =,E18.10////)'',M) ;
      REGEL(NEARCENTRE,M,X1,X2,X3,V1,V2,V3,OM,AL0,BE0,C1,C2,C3,INERROR) ;
      OUTPUT(51,''(10H SATELLITE/)'') ;
      IF  NEARCENTRE  THEN
      OUTPUT(51,INF,NEARCENTRE,X1,X2,X3,V1,V2,V3,H,C1,SQRT(C2↑2+C3↑2)/C1,
         3.1415926536*C1/OM)
                     ELSE
      OUTPUT(51,INF,X1,X2,X3,V1,V2,V3,C1,SQRT(C2↑2+C3↑2)/C1,
         3.1415926536*C1/OM) ;
      READ(MP,NTAB) ;
      OUTPUT(51,''(////16H PERTURBING MASS//4H M =,E18.10/)'',MP) ;
      IF  NTAB≠0  THEN  READ(NDEG)  ELSE  NDEG := 0 ;
      NFCT := 3 ;

4:    BEGIN
         REAL  XP1,XP2,XP3,VP1,VP2,VP3,OMP,CP1,CP2,CP3,TBEG,DTTAB,TFL,
            DS,TMAX,S,CS,SN,T,R,VF ;
         INTEGER  I,NOUT,NOUTI ;
         ARRAY  ALP,BEP[1:4],TAB[1:NFCT,0:NTAB],LAM[0:NDEG],DEL[1:N],
            FCT[1:NFCT],AL,BE,U,DUDS[1:4] ;

         PROCEDURE  LAINTAB(T,FCT) ;
            VALUE  T ;  REAL  T ;  ARRAY  FCT ;
            COMMENT  GLOBAL: NDEG,NFCT,NTAB,TBEG,DTTAB,LAM[0:NDEG],
               TAB[1:NFCT,0:NTAB],OUT ;
            BEGIN
               INTEGER  N,L,I,J ;  REAL  P,K,SS ;
               ARRAY  S[1:NFCT],MY[0:NDEG] ;
               P := (T-TBEG)/DTTAB ; N := P ; K := NDEG/2 ;
               L := N-K+(K-ENTIER(K))*SIGN(P-N) ;
               IF  L<0 ∨ L+NDEG>NTAB  THEN  GOTO  OUT ;
               IF  P=N  THEN
               FOR  I:=1  STEP  1  UNTIL  NFCT  DO  FCT[I] := TAB[I,N]
                     ELSE
               BEGIN
                  FOR  I:=1  STEP  1  UNTIL  NFCT  DO  S[I] := 0 ;
                  SS := 0 ;
                  FOR  J:=0  STEP  1  UNTIL  NDEG  DO
                  BEGIN
                     MY[J] := LAM[J]/(P-L-J) ;
                     FOR  I:=1  STEP  1  UNTIL  NFCT  DO
                     S[I] := S[I]+MY[J]*TAB[I,L+J] ;
                     SS := SS+MY[J] ;
                  END ;
                  FOR  I:=1  STEP  1  UNTIL  NFCT  DO  FCT[I] := S[I]/SS ;
               END ;
            END  LAINTAB ;
```

```
5:      PROCEDURE  RK1ST(X,Y,N,H,F) ;
           VALUE  N,H ;  REAL  X,H ;  INTEGER  N ;
           ARRAY  Y ;  PROCEDURE  F ;
           BEGIN
              REAL  XI ;  INTEGER  K,J ;  ARRAY  Y1,Y2,Z[1:N],A[1:5] ;
              A[1] := A[2] := A[5] := H/2 ; A[3] := A[4] := H ;
              XI := X ;
              FOR  K:=1  STEP  1  UNTIL  N  DO  Y1[K] := Y2[K] := Y[K] ;
              FOR  J:=1  STEP  1  UNTIL  4  DO
              BEGIN
                 F(XI,Y2,N,Z) ;
                 XI := X+A[J] ;
                 FOR  K:=1  STEP  1  UNTIL  N  DO
                 BEGIN
                    Y2[K] := Y[K]+A[J]*Z[K] ;
                    Y1[K] := Y1[K]+A[J+1]*Z[K]/3 ;
                 END  ;
              END  ;
              X := X+H ;
              FOR  K:=1  STEP  1  UNTIL  N  DO  Y[K] := Y1[K] ;
           END  RK1ST ;

6:      PROCEDURE  F(S,DEL,N,G) ;
           VALUE  S,N ;  REAL  S ;  INTEGER  N ;  ARRAY  DEL,G ;
           COMMENT  GLOBAL: T0,OM,C1,C2,C3,NTAB,CP1,CP2,CP3,ALP,BEP,
              AL0,BE0,LAINTAB ;
           BEGIN
              REAL  CS,SN,T,XP1,XP2,XP3,R,X1,X2,X3,DEN1,DEN2,P1,P2,P3,
                 SUM ;  INTEGER  I ;  ARRAY  AL,BE,U,DUDS,G[1:4] ;
              T := T0+C1*S+C2*SIN(2*OM*S)/OM/2+C3*(1-COS(2*OM*S))/CM/2
                 +DEL[9] ;
              IF  NTAB=0  THEN
7:            BEGIN
                 REAL  SP,SP1 ;  ARRAY  UP[1:4] ;
                 SP1 := (T-T0)/CP1-CP3/CP1/OMP/2 ;
                 LOOP: SP := SP1-(CP1*SP1+CP2*SIN(2*OMP*SP1)/OMP/2+CP3*
                    (1-COS(2*OMP*SP1))/OMP/2-T+T0)/(CP1+CP2*COS(2*CMP*SP1)
                    +CP3*SIN(2*OMP*SP1)) ;
                 IF  ABS(SP-SP1)>₁₀-9/OMP/2  THEN
                 BEGIN  SP1 := SP ;  GOTO  LOOP  END  ;
                 CS := COS(OMP*SP) ; SN := SIN(OMP*SP) ;
                 FOR  I:=1  STEP  1  UNTIL  4  DO
                 UP[I] := ALP[I]*CS+BEP[I]*SN ;
                 XP1 := UP[1]*UP[1]-UP[2]*UP[2]-UP[3]*UP[3]+UP[4]*UP[4] ;
                 XP2 := 2*(UP[1]*UP[2]-UP[3]*UP[4]) ;
                 XP3 := 2*(UP[1]*UP[3]+UP[2]*UP[4]) ;
              END
                           ELSE
8:            BEGIN
                 LAINTAB(T,FCT) ;
                 XP1 := FCT[1] ; XP2 := FCT[2] ; XP3 := FCT[3] ;
              END  ;
9:            CS := COS(OM*S) ; SN := SIN(OM*S) ;
              FOR  I:=1  STEP  1  UNTIL  4  DO
              BEGIN
                 AL[I] := AL0[I]+DEL[I] ; BE[I] := BE0[I]+DEL[I+4] ;
                 U[I] := AL[I]*CS+BE[I]*SN ;
                 DUDS[I] := OM*(-AL[I]*SN+BE[I]*CS) ;
              END  ;
              R := U[1]*U[1]+U[2]*U[2]+U[3]*U[3]+U[4]*U[4] ;
              X1 := U[1]*U[1]-U[2]*U[2]-U[3]*U[3]+U[4]*U[4] ;
              X2 := 2*(U[1]*U[2]-U[3]*U[4]) ;
              X3 := 2*(U[1]*U[3]+U[2]*U[4]) ;
              DEN1 := ((X1-XP1)*(X1-XP1)+(X2-XP2)*(X2-XP2)+(X3-XP3)
                 *(X3-XP3))↑1.5 ;
              DEN2 := (XP1*XP1+XP2*XP2+XP3*XP3)↑1.5 ;
              P1 := -MP*((X1-XP1)/DEN1+XP1/DEN2) ;
              P2 := -MP*((X2-XP2)/DEN1+XP2/DEN2) ;
              P3 := -MP*((X3-XP3)/DEN1+XP3/DEN2) ;
```

```
            Q[1] := 2*( U[1]*P1+U[2]*P2+U[3]*P3) ;
            Q[2] := 2*(-U[2]*P1+U[1]*P2+U[4]*P3) ;
            Q[3] := 2*(-U[3]*P1-U[4]*P2+U[1]*P3) ;
            Q[4] := 2*( U[4]*P1-U[3]*P2+U[2]*P3) ;
            SUM := Q[1]*DUDS[1]+Q[2]*DUDS[2]+Q[3]*DUDS[3]+Q[4]*DUDS[4] ;
            IF  N=10  THEN
10:         BEGIN
               REAL  DAL2,DBE2,DALBE,DR ;
               FOR  I:=1  STEP  1  UNTIL  4  DO
               BEGIN
                  G[I] := (R*Q[I]+2*DEL[10]*U[I])/OM/4 ;
                  G[I+4] := G[I]*CS ;
                  G[I] := -G[I]*SN ;
               END  ;
               DAL2 := (2*AL0[1]+DEL[1])*DEL[1]+(2*AL0[2]+DEL[2])*DEL[2]
                  +(2*AL0[3]+DEL[3])*DEL[3]+(2*AL0[4]+DEL[4])*DEL[4] ;
               DBE2 := (2*BE0[1]+DEL[5])*DEL[5]+(2*BE0[2]+DEL[6])*DEL[6]
                  +(2*BE0[3]+DEL[7])*DEL[7]+(2*BE0[4]+DEL[8])*DEL[8] ;
               DALBE := AL0[1]*DEL[5]+BE0[1]*DEL[1]+DEL[1]*DEL[5]
                  +AL0[2]*DEL[6]+BE0[2]*DEL[2]+DEL[2]*DEL[6]
                  +AL0[3]*DEL[7]+BE0[3]*DEL[3]+DEL[3]*DEL[7]
                  +AL0[4]*DEL[8]+BE0[4]*DEL[4]+DEL[4]*DEL[8] ;
               DR := (DAL2+DBE2)/2+(DAL2-DBE2)/2*COS(2*OM*S)
                  +DALBE*SIN(2*OM*S) ;
               G[9] := DR ;
               G[10] := SUM ;
            END
                     ELSE
11:         BEGIN
               REAL  A ;
               A := (AL[1]*AL[1]+AL[2]*AL[2]+AL[3]*AL[3]+AL[4]*AL[4]
                  +BE[1]*BE[1]+BE[2]*BE[2]+BE[3]*BE[3]+BE[4]*BE[4])/2 ;
               FOR  I:=1  STEP  1  UNTIL  4  DO
               BEGIN
                  G[I] := A/C1*(R*Q[I]+DUDS[I]*SUM/OM/OM)/OM/4 ;
                  G[I+4] := G[I]*CS ;
                  G[I] := -G[I]*SN ;
               END  ;
               G[9] := SQRT(A/C1)*R-(C1+C2*COS(2*OM*S)+C3*SIN(2*CM*S)) ;
            END  ;
         END  F ;

12:   IF  NTAB=0  THEN
      BEGIN
         READ(XP1,XP2,XP3,VP1,VP2,VP3) ;
         REGEL( FALSE ,M+MP,XP1,XP2,XP3,VP1,VP2,VP3,OMP,ALP,BEP,
            CP1,CP2,CP3,INERROR) ;
         OUTPUT(51,INF,XP1,XP2,XP3,VP1,VP2,VP3,CP1,SQRT(CP2↑2+CP3↑2)/
            CP1,3.1415926536*CP1/OMP) ;
      END
                     ELSE
      BEGIN
         READ(TBEG,DTTAB,TFL) ;
         FOR  I:=0  STEP  1  UNTIL  NTAB  DO
         READ(TAB[1,I],TAB[2,I],TAB[3,I]) ;
         OUTPUT(51,''(7H NDEG =,I3//10X,1HT,17X,2HX1,18X,2HX2,18X,
            2HX3/)'',NDEG) ;
         FOR  I:=0  STEP  1  UNTIL  NTAB  DO
         BEGIN
            TAB[1,I] := TFL*TAB[1,I] ;
            TAB[2,I] := TFL*TAB[2,I] ;
            TAB[3,I] := TFL*TAB[3,I] ;
            OUTPUT(51,''(1X,E18.8,3E20.10)'',TBEG+I*DTTAB,
               TAB[1,I],TAB[2,I],TAB[3,I]) ;
         END  ;
         LAM[0] := 1 ;
         FOR  I:=0  STEP  1  UNTIL  NDEG-1  DO
         LAM[I+1] := -LAM[I]*(NDEG-1)/(I+1) ;
      END  ;
```

```
        READ(DS,NOUT,TMAX) ;
        OUTPUT(51,''(////22H INTEGRATION STEP DS =,E18.10/7H NOUT =,I4/
           1H1,10X,1HT,16X,8HX1,X2,X3,11X,8HV1,V2,V3,14X,5HALPHA,15X,
           4HBETA/)'',DS,NOUT) ;
        S := 0 ;
        FOR  I:=1  STEP  1  UNTIL  N  DO  DEL[I] := 0 ;
13:     TR3:
        T := T0+C1*S+C2*SIN(2*OM*S)/OM/2+C3*(1-COS(2*OM*S))/OM/2+DEL[9] ;
        CS := COS(OM*S) ; SN := SIN(OM*S) ;
        FOR  I:=1  STEP  1  UNTIL  4  DO
        BEGIN
           AL[I] := AL0[I]+DEL[I] ; BE[I] := BE0[I]+DEL[I+4] ;
           U[I] := AL[I]*CS+BE[I]*SN ;
           DUDS[I] := OM*(-AL[I]*SN+BE[I]*CS) ;
        END  ;
        R := U[1]*U[1]+U[2]*U[2]+U[3]*U[3]+U[4]*U[4] ;
        X1 := U[1]*U[1]-U[2]*U[2]-U[3]*U[3]+U[4]*U[4] ;
        X2 := 2*(U[1]*U[2]-U[3]*U[4]) ;
        X3 := 2*(U[1]*U[3]+U[2]*U[4]) ;
        VF :=  IF  R=0  THEN  1  ELSE   IF  N=10  THEN  2/R  ELSE
           2/R/SQRT((AL[1]*AL[1]+AL[2]*AL[2]+AL[3]*AL[3]+AL[4]*AL[4]
           +BE[1]*BE[1]+BE[2]*BE[2]+BE[3]*BE[3]+BE[4]*BE[4])/2/C1) ;
        V1 := VF*(U[1]*DUDS[1]-U[2]*DUDS[2]-U[3]*DUDS[3]+U[4]*DUDS[4]) ;
        V2 := VF*(U[1]*DUDS[2]+U[2]*DUDS[1]-U[3]*DUDS[4]-U[4]*DUDS[3]) ;
        V3 := VF*(U[1]*DUDS[3]+U[2]*DUDS[4]+U[3]*DUDS[1]+U[4]*DUDS[2]) ;
        IF  N=9  THEN
        OUTPUT(51,''(/5E20.10/20X,4E20.10/20X,4E20.10/60X,2E20.10)'',
           T,X1,V1,AL[1],BE[1],X2,V2,AL[2],BE[2],X3,V3,AL[3],BE[3],
           AL[4],BE[4])
                ELSE
        OUTPUT(51,''(/6E20.10/20X,5E20.10/20X,4E20.10/60X,2E20.10)'',
           T,X1,V1,AL[1],BE[1],R*DEL[10],X2,V2,AL[2],BE[2],2*OM*OM*
           ((2*AL0[1]+DEL[1])*DEL[1]+(2*AL0[2]+DEL[2])*DEL[2]+(2*AL0[3]
           +DEL[3])*DEL[3]+(2*AL0[4]+DEL[4])*DEL[4]+(2*BE0[1]+DEL[5])*
           DEL[5]+(2*BE0[2]+DEL[6])*DEL[6]+(2*BE0[3]+DEL[7])*DEL[7]+(2*
           BE0[4]+DEL[8])*DEL[8]),X3,V3,AL[3],BE[3],AL[4],BE[4]) ;
        IF  R=0  THEN
        OUTPUT(51,''(
           57H (COLLISION,   V1,V2,V3 IS THE DIRECTION OF THE VELOCITY))
           '') ;
        NOUTI := 0 ;
14:     INT:
        RK1ST(S,DEL,N,DS,F) ;
        NOUTI := NOUTI+1 ;
        IF  T<TMAX  THEN
        BEGIN
           IF  NOUTI=NOUT  THEN  GOTO  TR3  ELSE  GOTO  INT ;
        END
                      ELSE  GOTO  ENDOFPR ;
     END  ;
15:  INERROR: OUTPUT(51,''(20H ERROR IN INPUT DATA)'') ;  GOTO  ENDOFPR ;
     CUT: OUTPUT(51,''(23H TABLE NOT LARGE ENOUGH)'') ;
  ENDOFPR: END  ;
```

Appendix 2.2. Output of program NUMPER. First example.

REGULARIZATION DT = R*DS

T0 = 0

CENTRAL MASS

M = 2.9656218330E 15

SATELLITE

START NEAR CENTRE

INITIAL POSITION X1 = 1.0000000000E 04 X2 = 0 X3 = 0
DIRECTION OF INITIAL VELOCITY V1 = 0 V2 = 0 V3 = 1.0000000000E 00

ENERGY = -1.0000000000E 10

SEMI-MAJOR AXIS = 1.4828109165E 05 ECCENTRICITY = 9.3256051803E-01 PERIOD OF REVOLUTION = 6.5879553214E 00

PERTURBING MASS

M = 3.6374608520E 13

NDEG = 6

T	X1	X2	X3
-3.00000000E 00	1.9809297145E 05	2.8849105617E 05	1.1473066829E 05
-2.50000000E 00	1.5674343871E 05	3.0521744845E 05	1.2652534907E 05
-2.00000000E 00	1.1298101015E 05	3.1731274704E 05	1.3640116991E 05
-1.50000000E 00	6.7444433387E 04	3.2450251803E 05	1.4416832961E 05
-1.00000000E 00	2.0828555217E 04	3.2659449588E 05	1.4966976950E 05
-5.00000000E-01	-2.6130266695E 04	3.2348921755E 05	1.5278764309E 05
0	-7.2672082464E 04	3.1518757452E 05	1.5344842921E 05
5.00000000E-01	-1.1803349729E 05	3.0179443599E 05	1.5162675269E 05
1.00000000E 00	-1.6146903501E 05	2.8351799938E 05	1.4734699536E 05
1.50000000E 00	-2.0227182665E 05	2.6064435234E 05	1.4068284392E 05
2.00000000E 00	-2.3979272619E 05	2.3362802952E 05	1.3175484477E 05
2.50000000E 00	-2.7345631254E 05	2.0287942360E 05	1.2072590846E 05
3.00000000E 00	-3.0277291166E 05	1.6894927374E 05	1.0779545139E 05
3.50000000E 00	-3.2734641613E 05	1.3241258714E 05	9.3192588785E 04
4.00000000E 00	-3.4687808020E 05	9.3872146407E 04	7.7168843714E 04
4.50000000E 00	-3.6116616965E 05	5.3943171774E 04	5.9990900672E 04
5.00000000E 00	-3.7010285430E 05	1.3239442018E 04	4.1934034149E 04
5.50000000E 00	-3.7366880827E 05	-2.7638494525E 04	2.3275868301E 04
6.00000000E 00	-3.7192605906E 05	-6.8113051746E 04	4.2913082915E 03
6.50000000E 00	-3.6530969047E 05	-1.0763706827E 05	-1.4751815204E 04
7.00000000E 00	-3.5312028419E 05	-1.4569969756E 05	-3.3596171797E 04
7.50000000E 00	-3.3651536801E 05	-1.8183034951E 05	-5.1997675864E 04
8.00000000E 00	-3.1550162814E 05	-2.1560147311E 05	-6.9727428678E 04
8.50000000E 00	-2.9042811491E 05	-2.4563056227E 05	-8.6573450416E 04
9.00000000E 00	-2.6167888178E 05	-2.7458065249E 05	-1.0234153980E 05
9.50000000E 00	-2.2966760460E 05	-2.9916130785E 05	-1.1685628018E 05
1.00000000E 01	-1.9483167242E 05	-3.2012832811E 05	-1.2996154260E 05
1.05000000E 01	-1.5762703606E 05	-3.3728356403E 05	-1.4152083600E 05
1.10000000E 01	-1.1852373167E 05	-3.5047467541E 05	-1.5141769563E 05
1.15000000E 01	-7.8001123993E 04	-3.5959425872E 05	-1.5955568948E 05
1.20000000E 01	-3.6543945742E 04	-3.6457952872E 05	-1.6585867290E 05
1.25000000E 01	5.3619751314E 03	-3.6541120406E 05	-1.7027071226E 05
1.30000000E 01	4.7233529675E 04	-3.6211233575E 05	-1.7275586842E 05

INTEGRATION STEP DS = 9.9999999998E-07
NOUT = 1

T	X1,X2,X3	V1,V2,V3	ALPHA	BETA	
0	1.0000000000E 04	0	1.0000000000E 02	0	0
	0	0	0	0	0
	0	7.5704977812E 05	0	5.3531503183E 02	
			0	0	
1.0460476110E-02	8.5196584167E 03	-2.5972015213E 05	1.0000000252E 02	-6.4986520425E-05	-1.7408365416E 08
	-1.1284299675E-04	-3.2927806659E-03	-7.7565357193E-08	-6.7277181532E-06	-1.7408236294E 08
	7.5452877411E 03	6.5857504854E 05	6.1478988576E-07	5.3531501509E 02	
			6.3601542619E-08	-4.1521899746E-07	
2.3672772422E-02	4.1081910175E 03	-3.7771411557E 05	1.0000001471E 02	-1.7635229444E-04	-4.3347483410E 08
	2.8376700312E-04	4.6518038437E-02	-7.6575866379E-06	5.8874266518E-05	-4.3347309170E 08
	1.4939919542E 04	4.6917968983E 05	3.5523731444E-06	5.3531498858E 02	
			4.8872925222E-06	-4.0992213733E-05	

T	X1,X2,X3	V1,V2,V3	ALPHA	BETA	
4.2333763423E-02	-3.1463198764E 03	-3.8780140977E 05	1.0000004747E 02	-3.5682443068E-04	-1.1440575004E 09
	4.3665715308E-03	3.7920544392E-01	-4.6251024858E-05	2.6812252992E-04	-1.1440464525E 09
	2.2036246450E 04	3.9994550439E 05	1.7036176582E-05	5.3531491610E 02	
			4.3478829120E-05	-2.4758871832E-04	
6.9031528642E-02	-1.3099023900E 04	-3.5635489788E 05	1.0000011761E 02	-6.3174787540E-04	-3.7106870800E 09
	2.2852822876E-02	1.0507253090E 00	-1.6853169315E-04	7.4335946018E-04	-3.7106368817E 09
	2.8692572818E 04	2.0262863885E 05	8.2878563439E-05	5.3531466327E 02	
			2.1309627434E-04	-9.0217576732E-04	
1.0619367229E-01	-2.5551188293E 04	-3.1568490122E 05	1.0000024524E 02	-1.0168919375E-03	-1.1924825078E 10
	8.5246819986E-02	2.3787313124E 00	-4.7057041134E-04	1.6495150844E-03	-1.1924677331E 10
	3.4775982740E 04	1.3336963780E 05	3.4897996243E-04	5.3531387220E 02	
			7.5449832756E-04	-2.5190359429E-03	
1.5603883600E-01	-4.0254148891E 04	-2.7668964238E 05	1.0000044338E 02	-1.4989517315E-03	-3.3616354644E 10
	2.5960288007E-01	4.7175635723E 00	-1.1005436888E-03	3.1741359177E-03	-3.3616021176E 10
	4.0164989972E 04	9.8009585040E 04	1.2071982095E-03	5.3531180910E 02	
			2.1525807783E-03	-5.8913847980E-03	
2.2053235967E-01	-5.6914240676E 04	-2.4213127869E 05	1.0000069721E 02	-2.0119120305E-03	-8.2158235635E 10
	6.8200705971E-01	3.5103003506E 00	-2.2569357878E-03	5.4962257587E-03	-8.2157604679E 10
	4.4751959511E 04	5.7374199254E 04	3.5006221159E-03	5.3530722754E 02	
			5.2433819838E-03	-1.2081754062E-02	
3.0134695368E-01	-7.5198597444E 04	-2.1214604545E 05	1.0000102714E 02	-2.5735511701E-03	-1.8112218995E 11
	1.5747478007E 00	1.3631388166E 01	-3.9870772898E-03	8.4376227981E-03	-1.8112101470E 11
	4.8445249896E 04	3.5999801978E 04	9.0077266332E-03	5.3529792174E 02	
			1.0703513389E-02	-2.1343558134E-02	
3.9982913035E-01	-9.4741705887E 04	-1.8614744640E 05	1.0000129460E 02	-2.9681271866E-03	-3.5406636604E 11
	3.2635977074E 00	2.0875882024E 01	-6.6103997526E-03	1.2248665309E-02	-3.5406467326E 11
	5.1171038486E 04	2.0637516361E 04	2.0196145814E-02	5.3528171614E 02	
			2.0388659841E-02	-3.5386882050E-02	
5.1697206880E-01	-1.1515261907E 05	-1.6343916614E 05	1.0000106760E 02	-2.6951511963E-03	-6.2735288631E 11
	6.2945416400E 00	3.1190631174E 01	-1.0505089451E-02	1.7129860661E-02	-6.2735062676E 11
	5.2874824586E 04	9.3099945430E 03	4.0577924500E-02	5.3525622661E 02	
			3.7053640874E-02	-5.6236494653E-02	
6.5[illegible]39544048E-01	-1.3602264791E 05	-1.4339438384E 05	9.9999478247E 01	-9.8658684501E-04	-1.0272798299E 12
	1.1504053072E 01	4.5599334987E 01	-1.6059686132E-02	2.3165194003E-02	-1.0272769997E 12
	5.3522553823E 04	7.7730614666E 02	7.4760431327E-02	5.3521915325E 02	
			6.4467256375E-02	-8.5972919220E-02	
8.0933258454E-01	-1.5693341501E 05	-1.2549288376E 05	9.9995587356E 01	2.6810181920E-03	-1.5770217530E 12
	2.0080443553E 01	6.4271549711E 01	-2.3304175793E-02	3.0015046450E-02	-1.5770183698E 12
	5.3101352681E 04	-5.7619952660E 03	1.2854542601E-01	5.3516849616E 02	
			1.0555167713E-01	-1.2475786584E-01	
9.8462527602E-01	-1.7746509534E 05	-1.0931446283E 05	9.9988803337E 01	8.1876080304E-03	-2.2923830399E 12
	3.3183233779E 01	4.6055629731E 01	-3.1712748842E-02	3.6891141876E-02	-2.2923792354E 12
	5.1619735748E 04	-1.0840814813E 04	2.0885365680E-01	5.3510287224E 02	
			1.6070235274E-01	-1.6977873084E-01	
1.1787262304E 00	-1.9720478604E 05	-9.4522093062E 04	9.9973443209E 01	1.8976410876E-02	-3.1710510798E 12
	5.2737316217E 01	1.1654705977E 02	-4.2958893015E-02	4.4834573182E-02	-3.1710470324E 12
	4.9107740127E 04	-1.4820120702E 04	3.2126768591E-01	5.3502352129E 02	
			2.4611983571E-01	-2.2999859748E-01	
1.3907094644E 00	-2.1575480909E 05	-9.0843882491E 04	9.9950228864E 01	3.2997994109E-02	-4.1915713409E 12
	8.1097468275E 01	1.5125282840E 02	-5.5178445805E-02	5.2249887837E-02	-4.1915670917E 12
	4.5616336252E 04	-1.7953323553E 04	4.7207897777E-01	5.3493277222E 02	
			3.5421124942E-01	-2.9544413715E-01	
1.6192881669E 00	-2.3274071063E 05	-6.8058246530E 04	9.9907049141E 01	5.5003431684E-02	-5.3329786050E 12
	1.2085091414E 02	1.9882556943E 02	-7.0407667946E-02	6.0054431765E-02	-5.3329744578E 12
	4.1216481975E 04	-2.0415542265E 04	6.6524830656E-01	5.3483325165E 02	
			5.1381689173E-01	-3.7703096748E-01	
1.8628400244E 00	-2.4781901390E 05	-5.5982760309E 04	9.9835071503E 01	8.5644337684E-02	-6.5474206777E 12
	1.7686454368E 02	2.6161973361E 02	-8.8270856650E-02	6.7697360074E-02	-6.5474164843E 12
	3.5998564953E 04	-2.2332740797E 04	9.0159280014E-01	5.3473245555E 02	
			7.3826583721E-01	-4.7277072845E-01	
2.1194394368E 00	-2.6068433511E 05	-4.4463955414E 04	9.9728694705E 01	1.2211562523E-01	-7.7780664540E 12
	2.5322334879E 02	3.3744388601E 02	-1.0678563413E-01	7.4091658292E-02	-7.7780626406E 12
	3.0069701010E 04	-2.3796393374E 04	1.1804138994E 00	5.3463609780E 02	
			1.0271049249E 00	-5.7208698839E-01	
2.3868959989E 00	-2.7107597282E 05	-3.3371126760E 04	9.9579603274E 01	1.6193110389E-01	-8.9593353383E 12
	3.5520476499E 02	4.2648019430E 02	-1.2464515260E-01	7.8909255597E-02	-8.9593304705E 12
	2.3552541774E 04	-2.4870947480E 04	1.4949881039E 00	5.3455168547E 02	
			1.3853431533E 00	-6.6803755015E-01	
2.6627987806E 00	-2.7878371585E 05	-2.2592578472E 04	9.9341852353E 01	2.0745179831E-01	-1.0086943114E 13
	4.8868399181E 02	5.4722992306E 02	-1.4247033049E-01	8.2378305516E-02	-1.0086938723E 13
	1.6582927969E 04	-2.5590993016E 04	1.8237366995E 00	5.3448774786E 02	
			1.8849901578E 00	-7.6410733820E-01	
2.9445659929E 00	-2.8365305941E 05	-1.2026150331E 04	9.9023740706E 01	2.4569013448E-01	-1.1106202739E 13
	6.6207953395E 02	6.9584664490E 02	-1.5573174270E-01	8.4033558958E-02	-1.1106193487E 13
	9.3084243888E 03	-2.5991029140E 04	2.1520847267E 00	5.3444764551E 02	
			2.4808447080E 00	-8.3618118425E-01	
3.2294980449E 00	-2.8558917160E 05	-1.5902886795E 03	9.8537255144E 01	2.6912255935E-01	-1.2157018249E 13
	8.8230081978E 02	8.4730544632E 02	-1.6257759906E-01	8.4431060711E-02	-1.2157009419E 13
	1.8841885892E 03	-2.6066854523E 04	2.4131820785E 00	5.3443381407E 02	
			3.2703485706E 00	-8.7490351314E-01	
3.5148355435E 00	-2.8456061709E 05	9.8050746534E 03	9.7921695726E 01	2.5582464727E-01	-1.3261880868E 13
	1.1573548191E 03	1.0623027307E 03	-1.5855979823E-01	8.4587523132E-02	-1.3261860089E 13
	-5.5282716005E 03	-2.5837224282E 04	2.5913714838E 00	5.3443671658E 02	
			4.1458067292E 00	-8.5679448189E-01	

etc.

Appendix 2.3. Program ANPER.

```
BEGIN

PROCEDURE  PF(M,MS,E,E1,ALO,BEO,AO,EXZO,ALS,BES,AS,EXZS,F) ;
   VALUE  M,MS,E,E1,AO,EXZO,AS,EXZS ;
   REAL  M,MS,E,E1,AO,EXZO,AS,EXZS ;
   ARRAY  ALO,BEO,ALS,BES,F ;
   BEGIN
      REAL  ES,ESA,XS,YS,ZS,R,X,Y,Z,DVDX,DVDY,DVDZ,SUM,H,HCOS,HSIN ;
      INTEGER  L ;
      ARRAY  U,US,DUDE,DVDU[1:4] ;
      H := E1-EXZO*SQRT((M+MS)/M)*(AO/AS)↑1.5*SIN(E) ;
      ESA := H ;
      LOOP: ES := ESA-(ESA-EXZS*SIN(ESA)-H)/(1-EXZS*COS(ESA)) ;
            IF  ABS(ES-ESA) > ⏨-9  THEN
            BEGIN  ESA := ES ;  GOTO  LOOP  END  ;
      HCOS := COS(ES/2) ; HSIN := SIN(ES/2) ;
      FOR  L:=1  STEP  1  UNTIL  4  DO
      US[L] := ALS[L]*HCOS+BES[L]*HSIN ;
      XS := US[1]*US[1]-US[2]*US[2]-US[3]*US[3]+US[4]*US[4] ;
      YS := 2*(US[1]*US[2]-US[3]*US[4]) ;
      ZS := 2*(US[1]*US[3]+US[2]*US[4]) ;
      HCOS := COS(E/2) ; HSIN := SIN(E/2) ;
      FOR  L:=1  STEP  1  UNTIL  4  DO
      BEGIN
         U[L] := ALO[L]*HCOS+BEO[L]*HSIN ;
         DUDE[L] := -ALO[L]/2*HSIN+BEO[L]/2*HCOS ;
      END  ;
      R := U[1]*U[1]+U[2]*U[2]+U[3]*U[3]+U[4]*U[4] ;
      X := U[1]*U[1]-U[2]*U[2]-U[3]*U[3]+U[4]*U[4] ;
      Y := 2*(U[1]*U[2]-U[3]*U[4]) ;
      Z := 2*(U[1]*U[3]+U[2]*U[4]) ;
      H := ((X-XS)*(X-XS)+(Y-YS)*(Y-YS)+(Z-ZS)*(Z-ZS))↑1.5 ;
      DVDX := (X-XS)/H ; DVDY := (Y-YS)/H ; DVDZ := (Z-ZS)/H ;
      H := (XS*XS+YS*YS+ZS*ZS)↑1.5 ;
      DVDX := -MS*(DVDX+XS/H) ;
      DVDY := -MS*(DVDY+YS/H) ;
      DVDZ := -MS*(DVDZ+ZS/H) ;
      DVDU[1] := 2*( U[1]*DVDX+U[2]*DVDY+U[3]*DVDZ) ;
      DVDU[2] := 2*(-U[2]*DVDX+U[1]*DVDY+U[4]*DVDZ) ;
      DVDU[3] := 2*(-U[3]*DVDX-U[4]*DVDY+U[1]*DVDZ) ;
      DVDU[4] := 2*( U[4]*DVDX-U[3]*DVDY+U[2]*DVDZ) ;
      SUM := DVDU[1]*DUDE[1]+DVDU[2]*DUDE[2]+DVDU[3]*DUDE[3]
         +DVDU[4]*DUDE[4] ;
      FOR  L:=1  STEP  1  UNTIL  4  DO
      BEGIN
         F[L] := AO/M/2*(R*DVDU[L]+4*DUDE[L]*SUM) ;
         F[L+4] := F[L]*HCOS ;
         F[L] := -F[L]*HSIN ;
      END  ;
   END  PF ;

PROCEDURE  PDF9DDELAN(M,ALO,BEO,AO,EXZO,I,DF9DDELAN,JKMAX) ;
   VALUE  M,AO,EXZO,I,JKMAX ;
   REAL  M,AO,EXZO ;  INTEGER  I,JKMAX ;
   ARRAY  ALO,BEO,DF9DDELAN ;
   BEGIN
      REAL  MSQ,AOSQ ;
      INTEGER  J,K ;
      MSQ := SQRT(M) ; AOSQ := SQRT(AO) ;
      FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      BEGIN  DF9DDELAN[J,K,1] := 0 ; DF9DDELAN[J,K,2] := 0  END  ;
      DF9DDELAN[0,0,1] := AOSQ/MSQ*( IF  I≤4  THEN  ALO[I]  ELSE
         BEO[I-4])+AOSQ/MSQ/2*( IF  I≤4  THEN  ALO[I]  ELSE  BEO[I-4]) ;
```

```
      DF9DDELAN[1,0,1] := (A0SQ/MSQ*( IF  I≤4  THEN  ALO[I] :ELSE
         -BE0[I-4])-EXZ0*A0SQ/MSQ/2*( IF  I≤4  THEN  ALO[I] :ELSE
         BE0[I-4]))/2 ;
      DF9DDELAN[1,0,2] := -(A0SQ/MSQ*( IF  I≤4  THEN  BE0[I]  ELSE
         ALO[I-4]))/2 ;
   END  PDF9DDELAN ;

PROCEDURE  DFOURAN(JKMAX,F,A) ;
   VALUE  JKMAX ;  INTEGER  JKMAX ;  ARRAY  F,A ;
   BEGIN
      PROCEDURE  SFOURAN(JMAX,F,A) ;
         VALUE  JMAX ;  INTEGER  JMAX ;  ARRAY  F,A ;
         BEGIN
            INTEGER  J,N,JNMOD2JMAX ;
            ARRAY  COSARRAY,SINARRAY[0:2*JMAX-1] ;
            FOR  J:=0  STEP  1  UNTIL  JMAX-1  DO
            BEGIN
               COSARRAY[J] := COS(3.1415926536/JMAX*J) ;
               SINARRAY[J] := SIN(3.1415926536/JMAX*J) ;
               COSARRAY[J+JMAX] := -COSARRAY[J] ;
               SINARRAY[J+JMAX] := -SINARRAY[J] ;
            END  ;
            A[0,1] := 0 ;
            FOR  N:=0  STEP  1  UNTIL  2*JMAX-1  DO
            A[0,1] := A[0,1]+F[N] ;
            A[0,2] := 0 ;
            FOR  J:=1  STEP  1  UNTIL  JMAX-1  DO
            BEGIN
               A[J,1] := 0 ; A[J,2] := 0 ;
               FOR  N:=0  STEP  1  UNTIL  2*JMAX-1  DO
               BEGIN
                  JNMOD2JMAX := J*N-ENTIER(J*N/(2*JMAX))*2*JMAX ;
                  A[J,1] := A[J,1]+F[N]*COSARRAY[JNMOD2JMAX] ;
                  A[J,2] := A[J,2]-F[N]*SINARRAY[JNMOD2JMAX] ;
               END  ;
            END  ;
            FOR  J:=0  STEP  1  UNTIL  JMAX-1  DO
            BEGIN
               A[J,1] := A[J,1]/2/JMAX ;
               A[J,2] := A[J,2]/2/JMAX ;
            END  ;
         END  SFOURAN ;
      INTEGER  J,K ;  ARRAY  AI[0:JKMAX-1,1:2],FI[0:2*JKMAX] ;
      FOR  K:=0  STEP  1  UNTIL  2*JKMAX  DO
      BEGIN
         FOR  J:=0  STEP  1  UNTIL  2*JKMAX  DO  FI[J] := F[J,K] ;
         SFOURAN(JKMAX,FI,AI) ;
         FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
         BEGIN
            F[J,K] := AI[J,1] ;
            F[J+JKMAX,K] := AI[J,2] ;
         END  ;
      END  ;
      FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
      BEGIN
         FOR  K:=0  STEP  1  UNTIL  2*JKMAX  DO  FI[K] := F[J,K] ;
         SFOURAN(JKMAX,FI,AI) ;
         FOR  K:=0  STEP  1  UNTIL  JKMAX-1  DO
         BEGIN
            F[J,K] := AI[K,1] ;
            F[J,K+JKMAX] := AI[K,2] ;
         END  ;
         FOR  K:=0  STEP  1  UNTIL  2*JKMAX  DO  FI[K] := F[J+JKMAX,K];
         SFOURAN(JKMAX,FI,AI) ;
         FOR  K:=0  STEP  1  UNTIL  JKMAX-1  DO
         BEGIN
            F[J+JKMAX,K] := AI[K,1] ;
            F[J+JKMAX,K+JKMAX] := AI[K,2] ;
         END  ;
      END  ;
```

```
      FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
      FOR  K:=0  STEP  1  UNTIL  JKMAX-1  DO
      BEGIN
         A[J,K,1] := F[J,K]-F[J+JKMAX,K+JKMAX] ;
         A[J,K,2] := F[J,K+JKMAX]+F[J+JKMAX,K] ;
         A[J,-K,1] := F[J,K]+F[J+JKMAX,K+JKMAX] ;
         A[J,-K,2] := -F[J,K+JKMAX]+F[J+JKMAX,K] ;
      END ;
   END  DFOURAN ;

PROCEDURE  DFOURINT(JKMAX,A,C,D,X0) ;
   VALUE  JKMAX,C,D,X0 ;  INTEGER  JKMAX ;  REAL  C,D ;  ARRAY  A ;
   BEGIN
      COMMENT   INT(SUM(A[J,K,1]+I*A[J,K,2])*E↑I(JX+K(DX+C))) =:
         A[0,0,2]*(X-X0)+SUM(A[J,K,1]+I*A[J,K,2])*E↑I(JX+K(DX+C))) ;
      INTEGER  J,K ;  REAL  H,HCOS,HSIN ;
      FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      IF  J≠0 ∨ K≠0  THEN
     BEGIN
         H := A[J,K,2]/(J+K*D) ;
         A[J,K,2] := -A[J,K,1]/(J+K*D) ;
         A[J,K,1] := H ;
      END
                   ELSE
      A[0,0,2] := A[0,0,1] ;
      A[0,0,1] := 0 ;
      HCOS := COS(J*X0+K*(D*X0+C)) ; HSIN := SIN(J*X0+K*(D*X0+C)) ;
      FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      A[0,0,1] := A[0,0,1]-( IF  J≠0  THEN  2  ELSE  IF  J=0∧K≠0  THEN
         1  ELSE  0)*(A[J,K,1]*HCOS-A[J,K,2]*HSIN) ;
   END  DFOURINT ;

PROCEDURE  DFOURPRODSP(JKMAX,A,B,C) ;
   INTEGER  JKMAX ;  ARRAY  A,B,C ;
   BEGIN  COMMENT  SUM(A[J,K,1]+I*A[J,K,2])*E↑I(JX+KY)
      *  ((B[-1,0,1]+I*B[-1,0,2])*E↑-IX + B[0,0,1] + (B[1,0,1]
      +I*B[1,0,2])*E↑IX)  =:  SUM(C[J,K,1]+I*C[J,K,2])*E↑I(JX+KY) ;
      INTEGER  J,K ;
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      BEGIN
         C[0,K,1] := A[1,K,1]*B[1,0,1]+A[1,K,2]*B[1,0,2]
            +A[0,K,1]*B[0,0,1]+A[1,-K,1]*B[1,0,1]+A[1,-K,2]*B[1,0,2] ;
         C[0,K,2] := -A[1,K,1]*B[1,0,2]+A[1,K,2]*B[1,0,1]
            +A[0,K,2]*B[0,0,1]+A[1,-K,1]*B[1,0,2]-A[1,-K,2]*B[1,0,1] ;
         FOR  J:=1  STEP  1  UNTIL  JKMAX-2  DO
         BEGIN
            C[J,K,1] := A[J+1,K,1]*B[1,0,1]+A[J+1,K,2]*B[1,0,2]
               +A[J,K,1]*B[0,0,1]+A[J-1,K,1]*B[1,0,1]
               -A[J-1,K,2]*B[1,0,2] ;
            C[J,K,2] := -A[J+1,K,1]*B[1,0,2]+A[J+1,K,2]*B[1,0,1]
              +A[J,K,2]*B[0,0,1]+A[J-1,K,1]*B[1,0,2]
              +A[J-1,K,2]*B[1,0,1] ;
         END ;
         C[JKMAX-1,K,1] := A[JKMAX-1,K,1]*B[0,0,1]+A[JKMAX-2,K,1]
            *B[1,0,1]-A[JKMAX-2,K,2]*B[1,0,2] ;
         C[JKMAX-1,K,2] := A[JKMAX-1,K,2]*B[0,0,1]+A[JKMAX-2,K,1]
            *B[1,0,2]+A[JKMAX-2,K,2]*B[1,0,1] ;
      END ;
   END  DFOURPRODSP ;
```

```
PROCEDURE  DFOUREV(X,X0,JKMAX,A,C,D,AEV) ;
   VALUE  X,X0,JKMAX,C,D ;
   REAL  X,X0,C,D,AEV ;  INTEGER  JKMAX ;  ARRAY  A ;
   BEGIN
      INTEGER  J,K ;
      AEV := A[0,0,1]+A[0,0,2]*(X-X0) ;
      FOR  K:=1  STEP  1  UNTIL  JKMAX-1  DO
      AEV := AEV+2*A[0,K,1]*COS(K*(D*X+C))-2*A[0,K,2]*SIN(K*(D*X+C)) ;
      FOR  J:=1  STEP  1  UNTIL  JKMAX-1  DO
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      AEV := AEV+2*A[J,K,1]*COS(J*X+K*(D*X+C))-2*A[J,K,2]*SIN(J*X
         +K*(D*X+C)) ;
   END  DFOUREV ;

REAL  M,MS,E0,ES0,A0,EXZ0,AS,EXZS,D,C,RES,TF,TFT ;
INTEGER  JKMAX,J,K,I ;
ARRAY  AL0,BE0,ALS,BES[1:4] ;
FORMAT  INF := ''(5H E0 =,F14.10/6H ALPHA,3X,4E20.10/5H BETA,4X,
   4E20.10/16H SEMI-MAJOR AXIS,E20.10/13H ECCENTRICITY,E20.10/4H T =,
   E18.10,13H * (E-E0)    +,E20.10,19H * (SIN(E)-SIN(E0)))'' ;
DATA IN:
   READ(M,E0,AL0[1],AL0[2],AL0[3],AL0[4],BE0[1],BE0[2],BE0[3],BE0[4]) ;
   OUTPUT(51,''(13H1CENTRAL MASS//4H M =,E18.10////
      22H SATELLITE UNPERTURBED/)'',M) ;
   A0 := AL0[1]*AL0[1]+AL0[2]*AL0[2]+AL0[3]*AL0[3]+AL0[4]*AL0[4] ;
   EXZ0 := BE0[1]*BE0[1]+BE0[2]*BE0[2]+BE0[3]*BE0[3]+BE0[4]*BE0[4] ;
   A0 := (A0+EXZ0)/2 ;
   EXZ0 := (-A0+EXZ0)/A0 ;
   OUTPUT(51,INF,E0,AL0[1],AL0[2],AL0[3],AL0[4],BE0[1],BE0[2],BE0[3],
      BE0[4],A0,EXZ0,A0↑1.5/SQRT(M),-EXZ0*A0↑1.5/SQRT(M)) ;
   READ(MS,ES0,ALS[1],ALS[2],ALS[3],ALS[4],BES[1],BES[2],BES[3],BES[4]);
   OUTPUT(51,''(///16H PERTURBING MASS//4H M =,E18.10)'',MS) ;
   AS := ALS[1]*ALS[1]+ALS[2]*ALS[2]+ALS[3]*ALS[3]+ALS[4]*ALS[4] ;
   EXZS := BES[1]*BES[1]+BES[2]*BES[2]+BES[3]*BES[3]+BES[4]*BES[4] ;
   AS := (AS+EXZS)/2 ;
   EXZS := (-AS+EXZS)/AS ;
   OUTPUT(51,INF,ES0,ALS[1],ALS[2],ALS[3],ALS[4],BES[1],BES[2],BES[3],
      BES[4],AS,EXZS,AS↑1.5/SQRT(M+MS),-EXZS*AS↑1.5/SQRT(M+MS)) ;
   READ(JKMAX) ;
   OUTPUT(51,''(///
      59H APPROXIMATION OF THE FOURIER SERIES BY FOURIER POLYNOMIALS//
   8H JKMAX =,I5)'',JKMAX) ;
   READ(TF,TFT) ;
RESONANCE ANALYSIS:
   D := SQRT(1+MS/M)*(A0/AS)↑1.5 ;
   C := ES0-EXZS*SIN(ES0)-D*(E0-EXZ0*SIN(E0)) ;
   OUTPUT(51,''(///19H RESONANCE ANALYSIS//5H E1 =,F13.10,6H * E +,
      F14.10/)'',D,C) ;
   RES := 1 ;
   FOR  J:=1  STEP  1  UNTIL  2*JKMAX  DO
   BEGIN
      I := ENTIER(J/D) ;
      FOR  K:=I-1,I,I+1,I+2  DO
      IF  ABS(J-K*D) ≤ RES  THEN
      BEGIN
         RES := ABS(J-K*D) ;
         OUTPUT(51,''(1X,I4,2H -,I4,2H *,F13.10,4H    =,F15.10)'',
            J,K,D,J-K*D) ;
      END  ;
   END  ;
```

```
FIRST ORDER PERTURBATIONS:
   OUTPUT(51,''(26H1FIRST ORDER PERTURBATIONS)'') ;
   REWIND(1) ; REWIND(2) ;
   FOR  J:=0  STEP  1  UNTIL  2*JKMAX  DO
   BEGIN
      ARRAY  F(1:8),FK(1:8,0:2*JKMAX) ;
      FOR  K:=0  STEP  1  UNTIL  2*JKMAX  DO
      BEGIN
         PF(M,MS,3.1415926536/JKMAX*J,3.1415926536/JKMAX*K,AL0,BE0,
            A0,EXZ0,ALS,BES,AS,EXZS,F) ;
         FOR  I:=1  STEP  1  UNTIL  8  DO  FK(I,K) := F(I) ;
      END  ;
      BINWRITE(2, FOR  I:=1  STEP  1  UNTIL  8  DO  ( FOR  K:=0  STEP
         1  UNTIL  2*JKMAX  DO  FK(I,K))) ;
   END  ;
   FOR  I:=1  STEP  1  UNTIL  8  DO
   BEGIN
      INTEGER  II ;
      ARRAY  FJK(0:2*JKMAX,0:2*JKMAX),DEL(0:JKMAX-1,-(JKMAX-1):JKMAX-1,
         1:2) ;
      REWIND(2) ;
      FOR  J:=0  STEP  1  UNTIL  2*JKMAX  DO
      BINREAD(2, FOR  II:=1  STEP  1  UNTIL  I  DO  ( FOR  K:=0  STEP
         1  UNTIL  2*JKMAX  DO  FJK(J,K))) ;
      DFOURAN(JKMAX,FJK,DEL) ;
      DFOURINT(JKMAX,DEL,C,D,E0) ;
      BINWRITE(1, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
         K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
         (DEL(J,K,1),DEL(J,K,2)))) ;
      IF  I≤4  THEN
      OUTPUT(51,''(////8H D ALPHA,I2,4H    *,E8.0//)'',I,TF)
               ELSE
      OUTPUT(51,''(////7H D BETA,I2,4H    *,E8.0//)'',I-4,TF) ;
      OUTPUT(51,''(15H SECULAR TERM =,F13.0,11H  *  (E-E0)//2X,1HE,2X,
         2HE1,10X,3HCOS,11X,3HSIN,8X,1HE,2X,2HE1,10X,3HCOS,11X,3HSIN,8X,
         1HE,2X,2HE1,10X,3HCOS,11X,3HSIN/)'',TF*DEL(0,0,2)) ;
      OUTPUT(51,''(I3,I4,F14.0)'',0,0,TF*DEL(0,0,1)) ;
      FOR  K:=1  STEP  1  UNTIL  JKMAX-1  DO
      OUTPUT(51,''(I3,I4,2F14.0)'',0,K,2*TF*DEL(0,K,1),
         -2*TF*DEL(0,K,2)) ;
      FOR  J:=1  STEP  3  UNTIL  JKMAX-3  DO
      BEGIN
         OUTPUT(51,''(1X)'') ;
         FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
         OUTPUT(51,''(I3,I4,2F14.0,4X,2I4,2F14.0,4X,2I4,2F14.0)'',J,K,
            2*TF*DEL(J,K,1),-2*TF*DEL(J,K,2),J+1,K,2*TF*DEL(J+1,K,1),
            -2*TF*DEL(J+1,K,2),J+2,K,2*TF*DEL(J+2,K,1),-2*TF
            *DEL(J+2,K,2)) ;
      END  ;
   END  ;
   REWIND(1) ; REWIND(2) ;
   FOR  I:=1  STEP  1  UNTIL  8  DO
   BEGIN
      REAL  DEL91I,DEL92I ;
      ARRAY  DEL,DF9DDELAN,DEL9I(0:JKMAX-1,-(JKMAX-1):JKMAX-1,1:2) ;
      BINREAD(1, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
         K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
         (DEL(J,K,1),DEL(J,K,2)))) ;
      DEL(0,0,1) := DEL(0,0,1)-DEL(0,0,2)*E0 ;
      PDF9DDELAN(M,AL0,BE0,A0,EXZ0,I,DF9DDELAN,JKMAX) ;
      DEL91I := DF9DDELAN(1,0,1)*DEL(0,0,2) ;
      DEL92I := DF9DDELAN(1,0,2)*DEL(0,0,2) ;
      DEL(0,0,2) := 0 ;
      DFOURPRODSP(JKMAX,DEL,DF9DDELAN,DEL9I) ;
      BINWRITE(2, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
         K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
         (DEL9I(J,K,1),DEL9I(J,K,2))),DEL91I,DEL92I) ;
   END  ;
```

```
REWIND(2) ;
BEGIN
   REAL  DEL91,DEL92,DEL91I,DEL92I ;
   ARRAY  DEL9,DEL9I[0:JKMAX-1,-(JKMAX-1):JKMAX-1,1:2] ;
   DEL91 := 0 ; DEL92 := 0 ;
   FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
   FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
   BEGIN  DEL9[J,K,1] := 0 ; DEL9[J,K,2] := 0  END  ;
   FOR  I:=1  STEP  1  UNTIL  8  DO
   BEGIN
      BINREAD(2, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
         K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
         (DEL9I[J,K,1],DEL9I[J,K,2])),DEL91I,DEL92I) ;
      DEL91 := DEL91+DEL91I ; DEL92 := DEL92+DEL92I ;
      FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      BEGIN
         DEL9[J,K,1] := DEL9[J,K,1]+DEL9I[J,K,1] ;
         DEL9[J,K,2] := DEL9[J,K,2]+DEL9I[J,K,2] ;
      END  ;
   END  ;
   DFOURINT(JKMAX,DEL9,C,D,E0) ;
   DEL9[0,0,1] := DEL9[0,0,1]-2*COS(E0)*DEL91+2*SIN(E0)*DEL92 ;
   DEL9[1,0,1] := DEL9[1,0,1]+DEL91 ;
   DEL9[1,0,2] := DEL9[1,0,2]+DEL92 ;
   BINWRITE(1, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
      K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      (DEL9[J,K,1],DEL9[J,K,2])),DEL91,DEL92) ;
   OUTPUT(51,''(////8H D T   *,E8.0///16H SECULAR TERMS =,F13.0,
      11H  *  (E-E0)/16X,F13.0,26H  *  (E*COS(E)-E0*COS(E0))/16X,
      F13.0,26H  *  (E*SIN(E)-E0*SIN(E0))//2X,1HE,2X,2HE1,10X,3HCOS,
      11X,3HSIN,8X,1HE,2X,2HE1,10X,3HCOS,11X,3HSIN,8X,1HE,2X,2HE1,
      10X,3HCOS,11X,3HSIN/)'',
      TFT,TFT*DEL9[0,0,2],2*TFT*DEL92,2*TFT*DEL91) ;
   OUTPUT(51,''(I3,I4,F14.0)'',0,0,TFT*DEL9[0,0,1]) ;
   FOR  K:=1  STEP  1  UNTIL  JKMAX-1  DO
   OUTPUT(51,''(I3,I4,2F14.0)'',0,K,2*TFT*DEL9[0,K,1],
      -2*TFT*DEL9[0,K,2]) ;
   FOR  J:=1  STEP  3  UNTIL  JKMAX-3  DO
   BEGIN
      OUTPUT(51,''(1X)'') ;
      FOR  K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
      OUTPUT(51,''(I3,I4,2F14.0,4X,2I4,2F14.0,4X,2I4,2F14.0)'',
         J,K,2*TFT*DEL9[J,K,1],-2*TFT*DEL9[J,K,2],
         J+1,K,2*TFT*DEL9[J+1,K,1],-2*TFT*DEL9[J+1,K,2],
         J+2,K,2*TFT*DEL9[J+2,K,1],-2*TFT*DEL9[J+2,K,2]) ;
   END  ;
END  ;
EVALUATION OF THE SERIES:
  BEGIN
   REAL  E,DELEV,DEL91,DEL92,T ;   INTEGER  I1,I2 ;
   ARRAY  DEL[0:JKMAX-1,-(JKMAX-1):JKMAX-1,1:2] ;
   READ(I) ;
   IF  I≠0  THEN  OUTPUT(51,''(25H1EVALUATION OF THE SERIES)'') ;
   FOR  I1:=1  STEP  1  UNTIL  I  DO
   BEGIN
      READ(E) ;
      OUTPUT(51,''(////4H E =,F14.10/26X,11HUNPERTURBED,9X,
         12HPERTURBATION,9X,9HPERTURBED/10X,5HALPHA)'',E) ;
      REWIND(1) ;
      FOR  I2:=1  STEP  1  UNTIL  4  DO
      BEGIN
         BINREAD(1, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
            K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
            (DEL[J,K,1],DEL[J,K,2]))) ;
         DFOUREV(E,E0,JKMAX,DEL,C,D,DELEV) ;
         OUTPUT(51,''(20X,3E20.10)'',AL0[I2],DELEV,AL0[I2]+DELEV) ;
      END  ;
```

```
        OUTPUT(51,''(10X,4HBETA)'') ;
        FOR  I2:=1  STEP  1  UNTIL  4  DO
        BEGIN
           BINREAD(1, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
              K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
              (DEL[J,K,1],DEL[J,K,2]))) ;
           DFOUREV(E,E0,JKMAX,DEL,C,D,DELEV) ;
           OUTPUT(51,''(20X,3E20.10)'',BE0[I2],DELEV,BE0[I2]+DELEV) ;
        END  ;
        BINREAD(1, FOR  J:=0  STEP  1  UNTIL  JKMAX-1  DO  ( FOR
           K:=-(JKMAX-1)  STEP  1  UNTIL  JKMAX-1  DO
           (DEL[J,K,1],DEL[J,K,2])),DEL91,DEL92) ;
        DFOUREV(E,E0,JKMAX,DEL,C,D,DELEV) ;
        DELEV := 2*DEL92*(E*COS(E)-E0*COS(E0))+2*DEL91*(E*SIN(E)
           -E0*SIN(E0))+DELEV ;
        T := A0↑1.5/SQRT(M)*((E-E0)-EXZ0*(SIN(E)-SIN(E0))) ;
        OUTPUT(51,''(/10X,1HT,9X,3E20.10)'',T,DELEV,T+DELEV) ;
     END  ;
  END  ;
  CUTPUT(51,''(//////11H END OUTPUT)'') ;

END  ;
```

Appendix 2.4. Output of program ANPER. Fourth example.

```
CENTRAL MASS

M := 1.0000000000E 00


SATELLITE UNPERTURBED

E0 =             0
ALPHA         7.0710678118E-01                        0    1.0000000000E 00              0
BETA          1.0000000000E 00   1.0000000000E 00   -7.0710678118E-01   -1.4142135624E 00
SEMI-MAJOR AXIS    3.0000000000E 00
ECCENTRICITY     4.9999999999E-01
T =  5.1961524227E 00 * (E-E0)    +    -2.5980762114E 00 * (SIN(E)-SIN(E0))


PERTURBING MASS

M =  1.0000000000E-02
E0 =             0
ALPHA         3.0000000000E 00                        0    3.0000000000E 00              0
BETA                         0   3.0000000000E 00                   0   -3.0000000000E 00
SEMI-MAJOR AXIS    1.8000000000E 01
ECCENTRICITY                  0
T =  7.5988534829E 01 * (E-E0)    +                  -0 * (SIN(E)-SIN(E0))


APPROXIMATION OF THE FOURIER SERIES BY FOURIER POLYNOMIALS

JKMAX =    13


RESONANCE ANALYSIS

E1 =  .0683807424 * E +             0

   1 -  13 *  .0683807424   =    .1110503494
   1 -  14 *  .0683807424   =    .0426696070
   1 -  15 *  .0683807424   =   -.0257111353
   2 -  29 *  .0683807424   =    .0169584717
   3 -  44 *  .0683807424   =   -.0087526636
   5 -  73 *  .0683807424   =    .0082058080
   8 - 117 *  .0683807424   =   -.0005468554


FIRST ORDER PERTURBATIONS


D ALPHA 1    *    1E 14

SECULAR TERM =     275947504   *  (E-E0)
```

E	E1	COS	SIN	E	E1	COS	SIN	E	E1	COS	SIN
0	0	34081166103									
0	1	-3015179143	-3860292829								
0	2	-28107172082	-30626281948								
0	3	3212603509	4732426530								
0	4	-378689109	-778492691								
0	5	41934877	132542819								
0	6	-3689407	-22885097								
0	7	50795	3968004								
0	8	92879	-686471								
0	9	-34153	117906								
0	10	9166	-20013								
0	11	-2168	3342								
0	12	458	-560								
1	-12	2303	6428	2	-12	17	-1887	3	-12	-568	1527
1	-11	-5004	-24228	2	-11	-1360	8666	3	-11	3636	-6612
1	-10	5380	96623	2	-10	12173	-38840	3	-10	-20956	27570
1	-9	36772	-394696	2	-9	-82204	169319	3	-9	112202	-109589
1	-8	-400077	1624170	2	-8	488758	-713869	3	-8	-564932	408581
1	-7	2738195	-6656752	2	-7	-2680999	2883938	3	-7	2677481	-1387379
1	-6	-16094494	26903298	2	-6	13791445	-10989819	3	-6	-11835369	4032664
1	-5	87186253	-106000613	2	-5	-66568032	38374159	3	-5	47627154	-8371713
1	-4	-444646559	400609482	2	-4	297081710	-115108193	3	-4	-164362673	1208256
1	-3	2136046561	-1402294290	2	-3	-1166002161	252292767	3	-3	407645682	100957562
1	-2	-9359996203	4369492102	2	-2	3291249051	77515843	3	-2	-448054896	-466821730
1	-1	532758988	582558535	2	-1	-401825744	-266533904	3	-1	122022511	106595160
1	0	-2273519972	-430785509	2	0	846486879	243758879	3	0	-51729011	-118971655
1	1	464804596	25783603	2	1	-142150513	-23841479	3	1	24761486	20136318
1	2	469844040	1649072456	2	2	53071443	-138028738	3	2	-6962909	-501412
1	3	-57473137	-352537867	2	3	-15741467	42480499	3	3	3726154	-2076612
1	4	2219937	71631240	2	4	4684735	-9881782	3	4	-1217364	770296
1	5	1475683	-13998751	2	5	-1282004	2061932	3	5	337303	-187072
1	6	-664042	2657102	2	6	324861	-400715	3	6	-85157	37155
1	7	200030	-491072	2	7	-77629	73479	3	7	20147	-6212
1	8	-51772	88334	2	8	17716	-12710	3	8	-4539	812
1	9	12329	-15422	2	9	-3896	2048	3	9	983	-48
1	10	-2780	2598	2	10	830	-298	3	10	-205	-16
1	11	598	-421	2	11	-169	41	3	11	41	1
1	12	-111	84	2	12	26	-33	3	12	-11	37

E	E1	COS	SIN	E	E1	COS	SIN	E	E1	COS	SIN
4	-12	993	-1196	5	-12	-1215	744	6	-12	1155	-269
4	-11	-5184	4657	5	-11	5565	-2387	6	-11	-4655	396
4	-10	25539	-16954	5	-10	-24064	6404	6	-10	17437	1070
4	-9	-118848	55932	5	-9	97246	-11100	6	-9	-59334	-12794
4	-8	519383	-156548	5	-8	-360344	-13115	6	-8	176122	69351
4	-7	-2099780	302636	5	-7	1181498	233959	6	-7	-420377	-268333
4	-6	7616931	100434	5	-6	-3185935	-1271424	6	-6	659237	751797
4	-5	-23194927	-4432082	5	-5	5899852	4499646	6	-5	-299118	-1271126
4	-4	49895948	24117595	5	-4	-4242510	-9432795	6	-4	-266453	250861
4	-3	-46408442	-67849068	5	-3	-804785	1865628	6	-3	279197	-741716
4	-2	2778991	12831040	5	-2	840251	-4791539	6	-2	-113554	130374
4	-1	-1403017	-27175508	5	-1	-258537	1150298	6	-1	173370	-143360
4	0	1793137	7116746	5	0	532360	-906866	6	0	-54622	51667
4	1	11602	-3940146	5	1	-111434	338371	6	1	36220	-21557
4	2	509590	1057683	5	2	52185	-129988	6	2	-11887	10300
4	3	-271227	-184391	5	3	-709	45992	6	3	3992	-3915
4	4	144293	26541	5	4	-5729	-13328	6	4	-916	1663
4	5	-50458	-4856	5	5	3688	4226	6	5	110	-623
4	6	14371	1828	5	6	-1396	-1315	6	6	15	227
4	7	-3639	-749	5	7	410	394	6	7	-12	-75
4	8	850	265	5	8	-104	-112	6	8	4	23
4	9	-187	-82	5	9	24	30	6	9	-1	-5
4	10	39	21	5	10	-6	-6	6	10	2	-0
4	11	-10	4	5	11	7	-8	6	11	-9	7
4	12	20	-42	5	12	-34	38	6	12	44	-25
7	-12	-860	-75	8	-12	489	209	9	-12	-199	-179
7	-11	2989	711	8	-11	-1401	-878	9	-11	425	549
7	-10	-9342	-3801	8	-10	3381	3039	9	-10	-632	-1361
7	-9	25075	15499	8	-9	-6114	-8461	9	-9	264	2474
7	-8	-52602	-49603	8	-8	5681	17411	9	-8	1179	-2572
7	-7	67093	117348	8	-7	4205	-21091	9	-7	-1758	285
7	-6	-4060	-166367	8	-6	-10421	3392	9	-6	1966	-1714
7	-5	-56884	31041	8	-5	11520	-13993	9	-5	-748	-510
7	-4	61481	-105640	8	-4	-4686	-1005	9	-4	1310	-31
7	-3	-25743	7958	8	-3	7557	-1609	9	-3	-315	-197
7	-2	39328	-17846	8	-2	-2235	-219	9	-2	365	222
7	-1	-12761	4408	8	-1	2086	554	9	-1	-126	-99
7	0	10080	-900	8	0	-810	-225	9	0	69	82
7	1	-3917	660	8	1	418	205	9	1	-30	-35
7	2	1787	-103	8	2	-178	-75	9	2	12	13
7	3	-668	132	8	3	72	33	9	3	-5	-7
7	4	225	-74	8	4	-28	-9	9	4	2	3
7	5	-69	43	8	5	10	2	9	5	-1	-1
7	6	19	-20	8	6	-4	0	9	6	0	0
7	7	-5	8	8	7	1	-0	9	7	-0	-0
7	8	1	-3	8	8	-0	0	9	8	-0	0
7	9	-0	1	8	9	-0	0	9	9	0	-0
7	10	-2	1	8	10	2	-0	9	10	-2	-0
7	11	10	-4	8	11	-9	0	9	11	6	2
7	12	-44	9	8	12	34	4	9	12	-21	-9
10	-12	48	93	11	-12	-1	-31	12	-12	-3	6
10	-11	-51	-209	11	-11	-15	44	12	-11	7	-3
10	-10	-37	337	11	-10	41	-32	12	-10	-6	-2
10	-9	236	-298	11	-9	-43	-5	12	-9	7	-1
10	-8	-281	5	11	-8	47	-18	12	-8	-2	-5
10	-7	314	-189	11	-7	-14	-27	12	-7	5	2
10	-6	-107	-130	11	-6	32	10	12	-6	1	-2
10	-5	211	31	11	-5	-1	-12	12	-5	1	2
10	-4	-33	-56	11	-4	8	12	12	-4	0	-1
10	-3	56	57	11	-3	-1	-5	12	-3	-0	1
10	-2	-14	-25	11	-2	0	4	12	-2	0	-0
10	-1	8	21	11	-1	0	-2	12	-1	-0	0
10	0	-3	-9	11	0	-0	1	12	0	0	-0
10	1	1	5	11	1	0	-0	12	1	-0	0
10	2	-0	-2	11	2	-0	0	12	2	0	-0
10	3	0	1	11	3	0	-0	12	3	-0	0
10	4	-0	-0	11	4	-0	0	12	4	-0	0
10	5	-0	0	11	5	0	0	12	5	0	-0
10	6	-0	-0	11	6	-0	0	12	6	-0	-0
10	7	-0	0	11	7	0	-0	12	7	-0	0
10	8	0	0	11	8	-0	-0	12	8	-0	-0
10	9	-0	-0	11	9	0	0	12	9	-0	-0
10	10	1	1	11	10	-1	-0	12	10	0	0
10	11	-3	-2	11	11	1	2	12	11	-0	-1
10	12	9	9	11	12	-3	-5	12	12	0	2

etc.

D T • 1E 14

```
SECULAR TERMS =  38164546416  *  (E-E0)
                 10545733673  *  (E*COS(E)-E0*COS(E0))
                  7627611101  *  (E*SIN(E)-E0*SIN(E0))
```

E	E1	COS	SIN	E	E1	COS	SIN	E	E1	COS	SIN
0	0	-11505645088									
0	1	21365564621	-28685093497								
0	2	-33987863205	160175750664								
0	3	8212315157	-24035014525								
0	4	-1902924203	3743615524								
0	5	420928879	-598881638								
0	6	-91195562	96617531								
0	7	19609821	-15515659								
0	8	-4220800	2438500								
0	9	916806	-364061								
0	10	-203010	47919								
0	11	46504	-3953								
0	12	-10951	-848								
1	-12	-77876	-21987	2	-12	-22605	-15658	3	-12	896	1213
1	-11	240445	107817	2	-11	64266	59899	3	-11	-2826	-5266
1	-10	-830906	-528871	2	-10	-189739	-235769	3	-10	7850	21645
1	-9	3044459	2626021	2	-9	555290	935566	3	-9	-16946	-83083
1	-8	-11462752	-13205628	2	-8	-1539407	-3693860	3	-8	13979	288848
1	-7	43318093	67342349	2	-7	3752486	14353117	3	-7	106393	-841133
1	-6	-160077712	-349286079	2	-6	-6470129	-54184411	3	-6	-679995	1498152
1	-5	552632476	1853243006	2	-5	-2824908	194805948	3	-5	1472965	3687268
1	-4	-1572113587	-10174502822	2	-4	92137645	-640086722	3	-4	8611642	-57577793
1	-3	1432287429	58862293100	2	-3	-477346760	1738563395	3	-3	-93720362	365696143
1	-2	40329233934	-380255199016	2	-2	762108197	-2142913678	3	-2	245666128	-10912653[illegible]
1	-1	-22734762425	24607948841	2	-1	-429056181	153986597	3	-1	-126849690	52652386
1	0	-11796390900	185699678144	2	0	331172366	32410592	3	0	252154856	-147578658
1	1	30547905560	-18691211525	2	1	-103323835	14354473	3	1	-40169326	7354417
1	2	-22441936241	37714880868	2	2	-26939283	99482891	3	2	34555371	-11196271
1	3	3283205066	-4037451117	2	3	-15623625	2424991	3	3	-6335954	1342825
1	4	-516533297	481442251	2	4	6126680	-1762946	3	4	965692	-67224
1	5	84511566	-58798306	2	5	-1603778	319984	3	5	-126243	-8623
1	6	-14086889	7048744	2	6	362191	-27736	3	6	13297	2752
1	7	2365967	-781824	2	7	-75506	-4265	3	7	-747	-236
1	8	-398017	69844	2	8	14915	2943	3	8	-129	-84
1	9	66783	-1861	2	9	-2825	-987	3	9	62	50
1	10	-11136	-1325	2	10	515	268	3	10	-16	-17
1	11	1827	502	2	11	-87	-65	3	11	3	4
1	12	-245	-132	2	12	1	10	3	12	2	1

etc.

EVALUATION OF THE SERIES

E = 80.0000000000

	UNPERTURBED	PERTURBATION	PERTURBED
ALPHA			
	7.0710678118E-01	7.3163459883E-04	7.0783841579E-01
	0	-2.5493975552E-03	-2.5493975552E-03
	1.0000000000E 00	1.6946540171E-03	1.0016946540E 00
	0	3.1952645720E-04	3.1952645720E-04
BETA			
	1.0000000000E 00	-5.5083833139E-04	9.9944916165E-01
	1.0000000000E 00	4.9473402216E-04	1.0004947340E 00
	-7.0710678118E-01	4.9363897422E-04	-7.0665314221E-01
	-1.4142135624E 00	1.3682831436E-03	-1.4128452792E 00
T	4.1827439227E 02	2.1180003860E-02	4.1829557228E 02

END OUTPUT

3. THE RESTRICTED ELLIPTIC THREE - BODY PROBLEM

by J. Waldvogel

3.1 Theory

In sections 1.1.2 and 1.2.2 the restricted circular three-body problem has been considered (computation of a particle's orbit in the force field of two attracting centers - referred to as earth and moon - on the assumption that the moon's orbit about the earth is a circle). In the 3-dimensional case the simultaneous regularization at both attracting centers could be carried out by the use of the B_3-transformation.

In the sequel we develop the regularization of the more general restricted elliptic three-body problem, but we content ourselves with the important points of the methods and proofs. A detailed analysis is contained in [4].

In the restricted elliptic three-body problem we again consider a particle of negligible mass moving in the force field of the earth and the moon, but the moon is allowed to move on a Kepler ellipse. The fact that the particle has negligible mass is the only assumption distinguishing the restricted elliptic problem from the general problem of the three bodies.

By means of a transformation to a suitable coordinate system the differential equations governing the motion of the particle in the restricted elliptic problem may be transformed to equations which are very similar to those governing the motion of the particle in the restricted circular problem. Consequently, the simultaneous regularization of the restricted elliptic problem at both attracting centers may also be carried out using the B_3-transformation.

3.1.1 Equations of motion. Let m be a particle of negligible mass moving in 3-dimensional physical space. The forces acting on the particle are the Newtonian attractions of two attracting centers - referred to as earth and moon - having the masses m_1 and m_2 respectively. As these point masses are not influenced by the particle, they move about their center of gravity O on Kepler orbits. Only the elliptic case of this Kepler motion is considered here.

We introduce a rectangular coordinate system η_1, η_2, η_3 with origin O, rotating about its η_3-axis with angular velocity ω in such a way that the earth and the moon always lie on the η_1-axis. Thus the η_1, η_2-plane is the orbital plane of m_1 and m_2. The varying distance between the earth and the moon is denoted by ρ. Let v be the true anomaly of the Kepler motion; this may be defined as the angle between the direction from the center of gravity to the pericenter of the moon's orbit and the positive η_1-axis (Fig. 3.1). The orbit of the moon with respect to a rectangular coordinate system centered at the earth (with axes of constant direction) is referred to as the relative Kepler orbit.

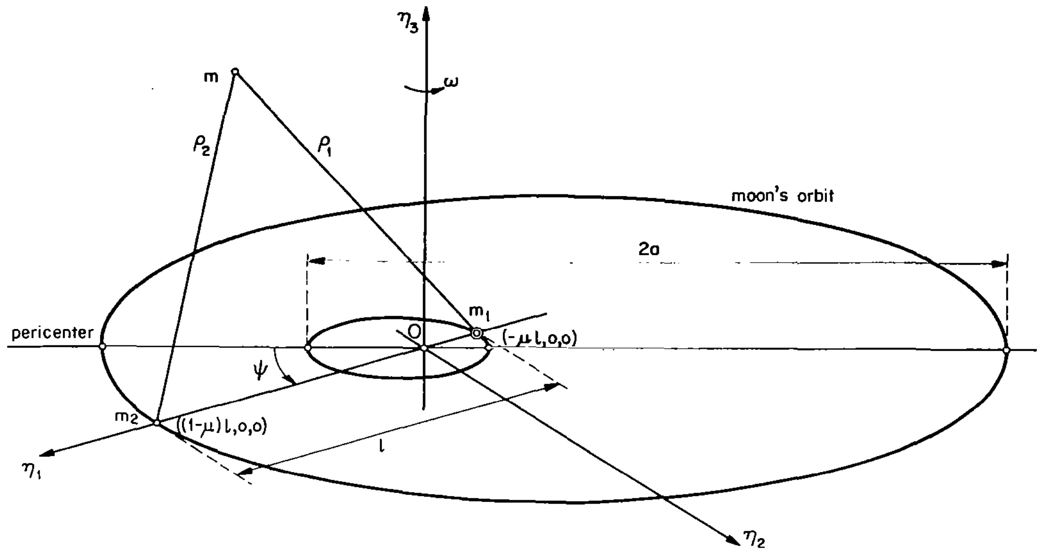


Fig. 3.1. The restricted elliptic three-body problem.

From the theory of Kepler motion [6] we recall the relations

$$\rho(\psi) = \frac{p}{1 + e \cos \psi} , \tag{3,1}$$

$$\frac{d\psi}{dt} = \omega = \frac{K\sqrt{p}}{\rho^2} . \tag{3,2}$$

Here

$$K^2 = \gamma (m_1 + m_2) \tag{3,3}$$

is the gravitational parameter, γ the gravitational constant, and p and e are respectively the semilatus rectum and eccentricity of the relative Kepler ellipse. In order to state the relationship between the true anomaly ψ and the physical time t we also introduce the eccentric anomaly E of the relative Kepler ellipse, defined by

$$tg \frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\; tg \frac{\psi}{2} , \qquad |\psi - E| < \pi . \tag{3,4}$$

Then, introducing the semi-major axis a of the relative Kepler ellipse given by

$$a = \frac{p}{1 - e^2} ,$$

Kepler's equation

$$Kt = a^{3/2}(E - e\sin E) \tag{3,5}$$

enables us to compute t from a given value of E.

Finally, in terms of the mass ratio μ, defined by either of

$$m_2 = \mu(m_1 + m_2), \qquad m_1 = (1-\mu)(m_1 + m_2), \tag{3,6}$$

the coordinates of the earth and the moon are

$$(-\mu\ell, 0, 0), \qquad ((1-\mu)\ell, 0, 0) \tag{3,7}$$

respectively.

In order to establish the differential equations of the particle's motion in the coordinate system η_1, η_2, η_3, we list the forces acting on the particle per unit of mass (denoting differentiation with respect to physical time t by a dot):

centrifugal force	$(\omega^2\eta_1, \ \omega^2\eta_2, \ 0)$
Coriolis force	$(2\omega\dot{\eta}_2, \ -2\omega\dot{\eta}_1, \ 0)$
force caused by the angular acceleration	$(\dot{\omega}\eta_2, \ -\dot{\omega}\eta_1, \ 0)$
gravitation	$\left(-K^2\frac{\partial\phi}{\partial\eta_1}, \ -K^2\frac{\partial\phi}{\partial\eta_2}, \ -K^2\frac{\partial\phi}{\partial\eta_3}\right).$

Here ϕ is the gravitational potential

(3,3)(3,6)
$$\phi = -\frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}, \tag{3,8}$$

and ρ_1 and ρ_2 are the distances of the particle from earth and moon respectively, given by

$$\rho_1 = \sqrt{(\eta_1 + \mu\ell)^2 + \eta_2^2 + \eta_3^2}, \qquad \rho_2 = \sqrt{(\eta_1 + \mu\ell - \ell)^2 + \eta_2^2 + \eta_3^2}. \tag{3,9}$$

The equations of motion of the particle are

$$\begin{aligned}
\ddot{\eta}_1 - 2\omega\dot{\eta}_2 - \omega^2\eta_1 - \dot{\omega}\eta_2 &= -K^2\frac{\partial\phi}{\partial\eta_1} \\
\ddot{\eta}_2 + 2\omega\dot{\eta}_1 - \omega^2\eta_2 + \dot{\omega}\eta_1 &= -K^2\frac{\partial\phi}{\partial\eta_2} \\
\ddot{\eta}_3 &= -K^2\frac{\partial\phi}{\partial\eta_3}.
\end{aligned} \tag{3,10}$$

As <u>Scheibner</u> [11] suggested in 1866 it is possible to reduce the restricted elliptic three-body problem to the restricted circular problem by simple substitutions of the variables. For this purpose we introduce into (3,10) the true anomaly ψ instead of the time as independent variable:

(3,2) $$\frac{d}{dt} = \omega \frac{d}{d\psi} . \qquad (3,11)$$

Denoting differentiation with respect to ψ by an accent, we obtain

$$\dot{\eta}_i = \omega \eta_i' , \qquad \ddot{\eta}_i = \omega^2 \eta_i'' + \omega\omega' \eta_i' , \qquad i = 1, 2, 3 .$$

Using these relations in (3,10) we find

$$\begin{aligned}
\omega^2 \eta_1'' + \omega\omega' \eta_1' - 2\omega^2 \eta_2' - \omega^2 \eta_1 - \omega\omega' \eta_2 &= - K^2 \frac{\partial \phi}{\partial \eta_1} \\
\omega^2 \eta_2'' + \omega\omega' \eta_2' + 2\omega^2 \eta_1' - \omega^2 \eta_2 + \omega\omega' \eta_1 &= - K^2 \frac{\partial \phi}{\partial \eta_2} \\
\omega^2 \eta_3'' + \omega\omega' \eta_3' &= - K^2 \frac{\partial \phi}{\partial \eta_3} .
\end{aligned} \qquad (3,12)$$

Taking into account

(3,2) $$\frac{\omega'}{\omega} = -2 \frac{\ell'}{\ell} ,$$

(3,12) can be written as

$$\begin{aligned}
\eta_1'' - 2\frac{\ell'}{\ell} \eta_1' - 2\eta_2' - \eta_1 + 2\frac{\ell'}{\ell} \eta_2 &= - \frac{K^2}{\omega^2} \frac{\partial \phi}{\partial \eta_1} \\
\eta_2'' - 2\frac{\ell'}{\ell} \eta_2' + 2\eta_1' - \eta_2 - 2\frac{\ell'}{\ell} \eta_1 &= - \frac{K^2}{\omega^2} \frac{\partial \phi}{\partial \eta_2} \\
\eta_3'' - 2\frac{\ell'}{\ell} \eta_3' &= - \frac{K^2}{\omega^2} \frac{\partial \phi}{\partial \eta_3} .
\end{aligned} \qquad (3,13)$$

Following Scheibner's proposal we further introduce the dimensionless variables y_i defined by

$$\eta_i = \ell y_i , \qquad i = 1, 2, 3 \qquad (3,14)$$

and restate some of the preceding results (in particular the differential equations (3,13)) in terms of these dimensionless variables. In the y_i-system the earth and the moon occupy the fixed points

(3,7)(3,14) $$(-\mu, 0, 0) , \qquad (1-\mu, 0, 0) \qquad (3,15)$$

respectively. It is convenient to introduce also the dimensionless distances

$$r_1 = \frac{\rho_1}{\ell} , \qquad r_2 = \frac{\rho_2}{\ell} ,$$

which are the distances of the point (y_1, y_2, y_3) from the points (3,15):

(3,9)(3,14) $$r_1 = \sqrt{(y_1+\mu)^2 + y_2^2 + y_3^2} , \qquad r_2 = \sqrt{(y_1+\mu-1)^2 + y_2^2 + y_3^2} . \qquad (3,16)$$

Remembering that ℓ is a function of ψ, we obtain for the derivatives of η_i the expressions

$$(3,14)\qquad \eta_i' = \rho y_i' + \rho' y_i \,, \qquad \eta_i'' = \rho y_i'' + 2\rho' y_i' + \rho'' y_i \,, \qquad i = 1,2,3.$$

Inserting these into (3,13) yields

$$\begin{aligned} y_1'' - 2y_2' + \left[\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2 - 1\right] y_1 &= -\frac{K^2}{\rho\omega^2}\frac{\partial\phi}{\partial\eta_1} \\ y_2'' + 2y_1' + \left[\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2 - 1\right] y_2 &= -\frac{K^2}{\rho\omega^2}\frac{\partial\phi}{\partial\eta_2} \\ y_3'' \qquad\quad + \left[\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2 \right] y_3 &= -\frac{K^2}{\rho\omega^2}\frac{\partial\phi}{\partial\eta_3} \,. \end{aligned} \qquad (3,17)$$

By using the differential equation

$$(3,1)\qquad \left(\frac{1}{\rho}\right)'' + \frac{1}{\rho} = \frac{1}{p}$$

satisfied by $\frac{1}{\rho}$, the expression occurring twice on the left-hand side of (3,17) is reduced to

$$\frac{\rho''}{\rho} - 2\left(\frac{\rho'}{\rho}\right)^2 - 1 = -\frac{\rho}{p} \,.$$

Furthermore the common factor on the right-hand side of (3,17) may be written as

$$(3,2)\qquad \frac{K^2}{\rho\omega^2} = \frac{\rho^3}{p} \,.$$

By finally substituting the dimensionless variables into ϕ and its partial derivatives,

$$\begin{aligned} \phi &= -\frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} = \frac{1}{\rho}\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right), \\ \frac{\partial\phi}{\partial\eta_i} &= \frac{1}{\rho}\frac{\partial\phi}{\partial y_i} = \frac{1}{\rho^2}\frac{\partial}{\partial y_i}\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right), \qquad i = 1,2,3, \end{aligned}$$

the differential equations (3,17) of the restricted elliptic problem are transformed into

$$\begin{aligned} y_1'' - 2y_2' &= -\frac{\rho}{p}\left[\frac{\partial}{\partial y_1}\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right) - y_1\right] \\ y_2'' + 2y_1' &= -\frac{\rho}{p}\left[\frac{\partial}{\partial y_2}\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right) - y_2\right] \\ y_3'' \qquad\quad &= -\frac{\rho}{p}\left[\frac{\partial}{\partial y_3}\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right) - y_3\right] - y_3 \,. \end{aligned} \qquad (3,18)$$

In the restricted circular problem ($e = 0$) ρ is constant ($= p$); therefore the factor ρ/p is the only correction to be made in order to generalize the circular to the elliptic case. In the circular case also ω is constant; thus the transformations (3,11) and (3,14) are merely magnifications of the time and space variables. These transformations then do no more than to introduce the special units defined at the beginning of section 1.1.2.

In the next section the differential equations (3,18) will be regularized. According to the methods given in section 1.1, we require quantities which correspond to the potential function $\mathcal{U}$ and the perturbing forces p_i of the table following (1,16). Thus, our intention is to find functions $\mathcal{U}(y_1, y_2, y_3, \psi)$ and $p_i(y_1', y_2', y_3')$ $(i = 1, 2, 3)$ so that (3,18) may be expressed in the form

$$y_i'' = -\frac{\partial \mathcal{U}}{\partial y_i} + p_i \, , \quad i = 1,2,3 \, . \tag{3,19}$$

This may be accomplished as follows. We notice that the expressions in the square brackets of (3,18) are the partial derivatives of the function

$$\mathcal{U}^* = -(1-\mu)\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) - \mu\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right) \tag{3,20}$$

with respect to y_1, y_2 and y_3 respectively. As the factor

(3,1) $$\frac{\varrho}{p} = \frac{1}{1 + e \cos \psi}$$

does not depend explicitly on y_i, the potential $\mathcal{U}$ is the function

$$\begin{aligned} \mathcal{U} &= \frac{\mathcal{U}^*}{1 + e \cos \psi} + \frac{1}{2} y_3^2 \\ \mathcal{U} &= \frac{-(1-\mu)\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) - \mu\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right)}{1 + e \cos \psi} + \frac{1}{2} y_3^2 \, . \end{aligned} \tag{3,21}$$

For the special case of the restricted circular problem ($e = 0$) it coincides with the potential (1,62). It should be stressed however that $\mathcal{U}$ depends explicitly on the independent variable ψ.

The perturbing forces p_i acting in the restricted elliptic problem are, according to the equations of motion (3,18) and (3,19),

$$p_1 = 2y_2' \, , \quad p_2 = -2y_1' \, , \quad p_3 = 0 \, . \tag{3,22}$$

This force may be regarded as a modified Coriolis force; the formulae (3,22) are similar to (1,30).

3.1.2 Regularization. The potential (3,21) occurring in the restricted elliptic three-body problem is singular at the two attracting centers (3,15). Because it depends explicitly on ψ, the theory of section 1.1 (in particular equations (1,20), (1,23), (1,24)) must be slightly generalized. But the method being used in section 1.1.2 in order to regularize the 3-dimensional restricted circular problem at both attracting centers can still be applied here. Thus we again introduce the four generalized coordinates v_j by formula (1,64):

$$y_1 = \frac{1}{2} - \mu + \frac{1}{2}\left[v_1 + \frac{v_1(v_4^2 + \frac{1}{4})}{v_1^2 + v_2^2 + v_3^2}\right]$$
$$y_2 = \frac{1}{2}\left[v_2 + \frac{v_2(v_4^2 - \frac{1}{4}) - v_3 v_4}{v_1^2 + v_2^2 + v_3^2}\right] \qquad (3,23)$$
$$y_3 = \frac{1}{2}\left[v_3 + \frac{v_3(v_4^2 - \frac{1}{4}) + v_2 v_4}{v_1^2 + v_2^2 + v_3^2}\right] .$$

The functional determinant D of this B_3-transformation is given by

(1,65) $$D = \frac{r_1 r_2}{v_1^2 + v_2^2 + v_3^2} , \qquad (3,24)$$

where the distances r_1, r_2 must be written in terms of the v_j:

$$r_1 = \frac{1}{2}\,\frac{(v_1+\frac{1}{2})^2 + v_2^2 + v_3^2 + v_4^2}{\sqrt{v_1^2 + v_2^2 + v_3^2}} , \qquad r_2 = \frac{1}{2}\,\frac{(v_1-\frac{1}{2})^2 + v_2^2 + v_3^2 + v_4^2}{\sqrt{v_1^2 + v_2^2 + v_3^2}} . \qquad (3,25)$$

For our elliptic problem the regularizing independent variable s plays the role of a "fictitious true anomaly" and is defined by

(1,18) $$d\psi = \lambda D \cdot ds . \qquad (3,26)$$

As in section 1.1, $\lambda = \lambda(v_1, v_2, v_3, v_4)$ is a scaling factor to be specified in the sequel. Now, the equations (1,22), adapted to our notations, have the form

(1,23) $$\frac{1}{\lambda}\frac{d}{ds}\left(\frac{1}{\lambda}\frac{dv_j}{ds}\right) - \frac{v^2}{2}\frac{\partial D}{\partial v_j} + D\frac{\partial \mathcal{U}}{\partial v_j} = D q_j , \quad j = 1,2,3,4 , \qquad (3,27)$$

where $\mathcal{U}$ is the function (3,21), v^2 is the squared "velocity", that is

$$v^2 = \sum_{i=1}^{3}\left(\frac{dy_i}{d\psi}\right)^2 ,$$

and the q_j are the components of the perturbing force (Coriolis force) in the parametric space. The rules (1,67) for computing these forces still hold true, but the scaling factor λ must be taken into account; this yields

$$q_j = \frac{2}{\lambda D}\sum_{k=1}^{4}(b_{1j}b_{2k} - b_{2j}b_{1k})\frac{dv_k}{ds} , \qquad b_{ik} = \frac{\partial y_i}{\partial v_k} . \qquad (3,28)$$

As in section 1.1, the final step of regularization is to eliminate the velocity v from (3,27) by the use of an energy equation. But we should remember that our potential $\mathcal{U}$ is not conservative, and therefore a vis viva integral like (1,11) (Jacobi integral) is not available.

In order to bypass this difficulty we propose the following method. Multiplying the i-th equation of (3,18) by y_i', summing over i and taking into account (3,20) yields

$$\frac{d}{d\psi}\left(\frac{v^2}{2}\right) + \frac{d}{d\psi}\left(\frac{y_3^2}{2}\right) + \frac{1}{1+e\cos\psi}\,\frac{d\mathcal{U}^*}{d\psi} = 0 . \tag{3,29}$$

Writing the last term on the left-hand side as

$$\frac{1}{1+e\cos\psi}\,\frac{d\mathcal{U}^*}{d\psi} = \frac{d}{d\psi}\left(\frac{\mathcal{U}^*}{1+e\cos\psi}\right) - \mathcal{U}^*\frac{d}{d\psi}\left(\frac{1}{1+e\cos\psi}\right),$$

equation (3,29) becomes

$$\frac{d}{d\psi}\left(\frac{v^2}{2}\right) + \frac{d}{d\psi}\left(\frac{\mathcal{U}^*}{1+e\cos\psi} + \frac{y_3^2}{2}\right) = \mathcal{U}^*\frac{d}{d\psi}\left(\frac{1}{1+e\cos\psi}\right). \tag{3,30}$$

By integrating from the initial value ψ_0 of the true anomaly to a general value ψ, equation (3,30) may be brought to a form similar to (1,11):

$$\frac{v^2}{2} + \mathcal{U} = h + W^*, \tag{3,31}$$

where

$$W^* = \int_{\psi_0}^{\psi} \mathcal{U}^*\,\frac{e\sin\psi}{(1+e\cos\psi)^2}\,d\psi \tag{3,32}$$

is an integral replacing the work W of section 1.1. The quantity h is an energy constant and may be computed from the initial velocity v_0 and the initial potential $\mathcal{U}_0$ at instant ψ_0 by

(3,31)(3,32) $$h = \frac{v_0^2}{2} + \mathcal{U}_0 . \tag{3,33}$$

Although $\mathcal{U}^*$ is infinite at collisions, the integral W^* exists for every finite value of ψ. This can be shown by substituting the fictitious anomaly s in the integral (3,32):

(3,25) $$W^* = \int \lambda\cdot D\,\mathcal{U}^*\,\frac{e\sin\psi}{(1+e\cos\psi)^2}\,ds . \tag{3,34}$$

Here the expression

$$\lambda\cdot D\,\mathcal{U}^* = \frac{\lambda}{V_1^2+V_2^2+V_3^2}\left[-(1-\mu)(r_2+\tfrac{1}{2}r_1^3 r_2) - \mu(r_1+\tfrac{1}{2}r_1 r_2^3)\right] \tag{3,35}$$

no longer has singularities at the attracting centers, provided that λ remains finite. The denominator $V_1^2+V_2^2+V_3^2$ is in general non-zero: it vanishes only if the particle is infinitely remote. This proves our statement.

The above mentioned final step of regularization is now carried out by eliminating v^2 between the equations (3,27) and (3,31). The result is (replacing the q_j by (3,28))

$$\begin{aligned} &\frac{1}{\lambda}\frac{d}{ds}\left(\frac{1}{\lambda}\frac{dv_j}{ds}\right) + \frac{\partial}{\partial v_j}\left[D(\mathcal{U}-h)\right] \\ &= \frac{2}{\lambda}\sum_{k=1}^{4}\left(b_{1j}b_{2k} - b_{2j}b_{1k}\right)\frac{dv_k}{ds} + W^*\frac{\partial D}{\partial v_j}, \quad j = 1,2,3,4 \end{aligned} \tag{3,36}$$

(cf. (1,68)). This system of differential equations must be integrated numerically, and, in order to do this, the values of ψ and W^* must be known at every step in the integration. Therefore we add the following two regular differential equations to the system (3,36):

$$\text{(3,34)} \qquad \frac{dW^*}{ds} = \lambda \cdot D\,\mathcal{U}^* \frac{e \sin\psi}{(1+e\cos\psi)^2} , \qquad (3,37)$$

$$\text{(3,26)} \qquad \frac{d\psi}{ds} = \lambda D . \qquad (3,38)$$

This terminates the regularization procedure. Equations (3,36),(3,37) and (3,38) form, in all, a simultaneous system of 10 regular first order differential equations for the unknowns v_j, dv_j/ds $(j=1,2,3,4)$, W^*, ψ as functions of s.

By using Birkhoff's transformation, the regularization of the 2-dimensional restricted elliptic three-body problem has already been performed by Szebehely and Giacaglia [12] in 1964. The result of these authors was a system of integro-differential equations.

According to section 1.1.2 the scaling factor $\lambda(v_1, v_2, v_3, v_4)$ might be chosen as $\lambda = 1$. In this case the equations (3,36) become very similar to the equations (1,68) governing the restricted circular three-body problem. Equation (3,38) then becomes

$$d\psi = \frac{r_1 r_2}{v_1^2 + v_2^2 + v_3^2}\, ds . \qquad (3,39)$$

In order to integrate the system (3,36),(3,37),(3,38) of differential equations numerically, the independent variable s is chosen to have a constant increment. As (3,39) shows, the corresponding increments in ψ become small whenever one of the distances r_1, r_2 becomes small (i.e. whenever the particle comes close to the earth or to the moon). This is the most important advantage produced by regularization.

On the other hand, however, any variation in the denominator $v_1^2 + v_2^2 + v_3^2$ modifies the step length of ψ. Since the v_4-axis, whose equation is $v_1^2 + v_2^2 + v_3^2 = 0$, corresponds to infinity in the physical space (cf. (3,23)), the denominator of (3,39) approaches zero if the particle escapes to infinity. From a numerical point of view a small denominator should be avoided. Our numerical experiments show that $v_1^2 + v_2^2 + v_3^2$ may approach zero even if the particle is not extremely far away in physical space. In such a case the increment in ψ becomes very large without any physical reason, and sometimes the numerical integration breaks down.

In order to avoid extremely large steps of ψ, in what follows we define the scaling factor λ as

$$\lambda = v_1^2 + v_2^2 + v_3^2 \qquad (3,40)$$

(in the sequel λ is used as an abbreviation for $v_1^2 + v_2^2 + v_3^2$). By this choice equation (3,38) may be written in the form

$$d\psi = r_1 r_2 \, ds$$

which avoids the difficulties associated with (3,39).

In section 3.1.3 we describe a method that may additionally be used in order to avoid extremely small values of the denominator $v_1^2 + v_2^2 + v_3^2$.

For our choice of λ (cf. (3,40)) we now proceed to establish an explicit form of the system (3,36),(3,37),(3,38).

We multiply equations (3,36) by λ^2; the first term becomes

$$\lambda \frac{d}{ds}\left(\frac{1}{\lambda}\frac{dv_j}{ds}\right) = \frac{d^2 v_j}{ds^2} - \frac{dv_j}{ds}\cdot\frac{1}{\lambda}\sum_{k=1}^{4}\lambda_k \frac{dv_k}{ds} \; . \tag{3,41}$$

where

$$\lambda_k = \frac{\partial\lambda}{\partial v_k}\,, \qquad k = 1,2,3,4\,,$$

$$\lambda_1 = 2v_1\,, \quad \lambda_2 = 2v_2\,, \quad \lambda_3 = 2v_3\,, \quad \lambda_4 = 0 \; . \tag{3,42}$$

The second term is transformed as follows:

$$\lambda^2\frac{\partial}{\partial v_j}\left[D(U-h)\right] = \frac{1}{1+e\cos\psi}\,\lambda^2\frac{\partial}{\partial v_j}\left(\frac{Q}{\lambda}\right) + \lambda^2\frac{\partial}{\partial v_j}\left[D\left(\frac{v_3^2}{2}-h\right)\right], \tag{3,43}$$

where Q is an abbreviation for the expression on the right-hand side of (3,35):

$$Q = \lambda D U^* = -(1-\mu)(r_2 + \tfrac{1}{2} r_1^3 r_2) - \mu(r_1 + \tfrac{1}{2} r_1 r_2^3) \; . \tag{3,44}$$

In order to carry out the partial differentiations required on the right-hand side of (3,43), we introduce the quantities

$$r_{1j} = \lambda\frac{\partial r_1}{\partial v_j}\,, \qquad\qquad r_{2j} = \lambda\frac{\partial r_2}{\partial v_j}\,,$$

(3,25)

$$\begin{array}{llll} r_{11} = \tfrac{1}{2}\sqrt{\lambda} + (\sqrt{\lambda}-r_1)v_1 & & r_{21} = -\tfrac{1}{2}\sqrt{\lambda} + (\sqrt{\lambda}-r_2)v_1 \\ r_{12} = (\sqrt{\lambda}-r_1)v_2 & & r_{22} = (\sqrt{\lambda}-r_2)v_2 \\ r_{13} = (\sqrt{\lambda}-r_1)v_3 & & r_{23} = (\sqrt{\lambda}-r_2)v_3 \\ r_{14} = \sqrt{\lambda}\cdot v_4 & = & r_{24} = \sqrt{\lambda}\cdot v_4 \end{array} \tag{3,45}$$

as well as

$$Q_j = \lambda\frac{\partial Q}{\partial v_j}\,,$$

$$Q_j = -\left[(1-\mu)\cdot\tfrac{3}{2}r_1^2 r_2 + \mu\left(1+\frac{r_2^3}{2}\right)\right]r_{1j} - \left[(1-\mu)\left(1+\frac{r_1^3}{2}\right) + \mu\cdot\tfrac{3}{2}r_1 r_2^2\right]r_{2j} \tag{3,46}$$

and

$$D_j = \lambda^2\frac{\partial D}{\partial v_j} = r_2 r_{1j} + r_1 r_{2j} - r_1 r_2 \lambda_j \; . \tag{3,47}$$

Then we obtain

$$\lambda^2 \frac{\partial}{\partial v_j}\left(\frac{Q}{\lambda}\right) = Q_j - Q\lambda_j ,$$
$$\lambda^2 \frac{\partial}{\partial v_j}\left[D\left(\frac{y_3^2}{2} - h\right)\right] = D_j\left(\frac{y_3^2}{2} - h\right) + r_1 r_2 y_3 \cdot \lambda b_{3j} . \qquad (3,48)$$

By using the symbol b_{jk} defined in (1,67) as well as the abbreviations (3,40), (3,42),(3,44) - (3,47), the regularized system of differential equations then becomes

$$(3,36) \quad \frac{d^2 v_j}{ds^2} = \frac{2}{\lambda}\sum_{k=1}^{4}\left\{\lambda b_{1j}\cdot\lambda b_{2k} - \lambda b_{2j}\cdot\lambda b_{1k} + \frac{1}{2}\sum_{\ell=1}^{4}\lambda_\ell \frac{dv_\ell}{ds}\cdot\delta_{jk}\right\}\frac{dv_k}{ds}$$
$$- \frac{1}{1+e\cos\psi}\left(Q_j - Q\lambda_j\right) - r_1 r_2 y_3\cdot\lambda b_{3j} + \left(h - \frac{y_3^2}{2} + W^*\right)D_j$$

(δ_{jk} = Kronecker's symbol) (3,49)

$$(3,37) \quad \frac{dW^*}{ds} = Q\,\frac{e\sin\psi}{(1+e\cos\psi)^2}$$

$$(3,38) \quad \frac{d\psi}{ds} = r_1 r_2 .$$

Finally, in order to evaluate the derivatives b_{jk} , we notice that by introducing the quantities

$$\beta_2 = v_2 v_4 - \frac{1}{2}v_3 , \qquad \beta_3 = v_3 v_4 + \frac{1}{2}v_2 \qquad (3,50)$$

and

$$\alpha_1 = \frac{v_1(v_4^2+\frac{1}{4})}{\lambda} , \qquad \alpha_2 = \frac{v_4\beta_2 - \frac{1}{2}\beta_3}{\lambda} , \qquad \alpha_3 = \frac{v_4\beta_3 + \frac{1}{2}\beta_2}{\lambda} \qquad (3,51)$$

the B_3-transformation (3,23) may be written as

$$y_1 = \frac{1}{2} - \mu + \frac{1}{2}(v_1+\alpha_1), \quad y_2 = \frac{1}{2}(v_2+\alpha_2), \quad y_3 = \frac{1}{2}(v_3+\alpha_3). \qquad (3,52)$$

Differentiating these equations while taking into account (3,50) and (3,51) yields

$$\begin{aligned}
\lambda b_{11} &= \tfrac{1}{2}(v_4^2+\tfrac{1}{4}+\lambda) - \alpha_1 v_1 & \qquad \lambda b_{21} &= -\alpha_2 v_1\\
\lambda b_{12} &= -\alpha_1 v_2 & \lambda b_{22} &= \tfrac{1}{2}(v_4^2-\tfrac{1}{4}+\lambda) - \alpha_2 v_2\\
\lambda b_{13} &= -\alpha_1 v_3 & \lambda b_{23} &= -\tfrac{1}{2}v_4 - \alpha_2 v_3\\
\lambda b_{14} &= v_1 v_4 & \lambda b_{24} &= \beta_2
\end{aligned}$$

(3,53)

$$\begin{aligned}
\lambda b_{31} &= -\alpha_3 v_1\\
\lambda b_{32} &= \tfrac{1}{2}v_4 - \alpha_3 v_2\\
\lambda b_{33} &= \tfrac{1}{2}(v_4^2-\tfrac{1}{4}+\lambda) - \alpha_3 v_3\\
\lambda b_{34} &= \beta_3
\end{aligned}$$

In the sequel the points $(\frac{1}{2}, 0, 0, 0)$ and $(-\frac{1}{2}, 0, 0, 0)$ are called <u>centers</u> (in the parametric space). By the B_3-transformation (3,23) each of them is mapped onto one of the attracting centers in the physical space.

If the point v_j is near one of the centers, the λb_{jj} ($j = 1, 2, 3$) appear in (3,53) as differences of two almost equal quantities. The following manner of computation avoids the loss of significant figures.

$$\beta_4 = \frac{1}{2}\left[-(v_1+\tfrac{1}{2})(v_1-\tfrac{1}{2}) - v_2^2 - v_3^2 + v_4^2\right], \tag{3,50a}$$

$$\begin{aligned} \lambda b_{11} &= \frac{1}{\lambda}(v_2^2+v_3^2)(v_4^2+\tfrac{1}{4}) - \beta_4 \\ \lambda b_{22} &= v_4^2 - \beta_4 - \alpha_2 v_2 \\ \lambda b_{33} &= v_4^2 - \beta_4 - \alpha_3 v_3 \ . \end{aligned} \tag{3,53a}$$

Assuming that the regularized variables v_j, dv_j/ds, W^*, ψ are given for a general point of the particle's orbit, the physical coordinates, velocity and time can be computed as follows:

a) The B_3-transformation yields the dimensionless coordinates y_i, and with e determined by (3,1), the physical coordinates η_i are obtained from (3,14).

b) The derivatives of y_i with respect to ψ are obtained from (1,19),

$$y_i' = \frac{dy_i}{d\psi} = \frac{1}{r_1 r_2}\sum_{j=1}^{4} b_{ij}\frac{dv_j}{ds}, \qquad i = 1, 2, 3, \tag{3,54}$$

where the b_{ij} are given by (3,53). A formula for the computation of the velocity $d\eta_i/dt$ in the physical space can be established by differentiating (3,14), taking into account (3,11),(3,2) and (3,1):

$$\dot\eta_i = \omega(e y_i' + e' y_i) = K\sqrt{p}\left[\frac{1}{e}y_i' - \left(\frac{1}{e}\right)' y_i\right],$$

$$\dot\eta_i = K\left(\frac{\sqrt{p}}{e}y_i' + \frac{e}{\sqrt{p}}\sin\psi\cdot y_i\right), \qquad i = 1, 2, 3. \tag{3,55}$$

c) The physical time t may be computed from ψ without integrating a differential equation by using the formulae (3,4) and (3,5).

We now add a few remarks concerning initial conditions. From a given initial anomaly ψ_0, initial position η_i and initial velocity $\dot\eta_i$ the initial dimensionless coordinates y_i may be computed by (3,1) and (3,14). The formulae for the computation of the initial derivatives $y_i' = dy_i/d\psi$ are obtained by solving (3,55) for y_i' and using (3,14):

$$y_i' = \frac{e}{K\sqrt{p}}\dot\eta_i - \frac{e}{p}\sin\psi_0\cdot\eta_i, \qquad i = 1, 2, 3. \tag{3,56}$$

The regularized coordinates v_j may be computed as described in the sentence following formula (1,69). Then, according to (1,19), the initial "velocity" dv_j/ds is given by

$$\frac{dv_j}{ds} = \lambda\sum_{i=1}^{3} b_{ij}\,y_i', \qquad j = 1, 2, 3, 4. \tag{3,57}$$

It may be verified that the components of the velocity computed by (3,57) satisfy the relation

$$v_1 v_4 \frac{dv_1}{ds} + \beta_2 \frac{dv_2}{ds} + \beta_3 \frac{dv_3}{ds} + \beta_4 \frac{dv_4}{ds} = 0 , \tag{3,58}$$

where β_2 , β_3 are given by (3,50), and β_4 may be defined by (3,50a) or by

$$\beta_4 = \frac{1}{2}\left(v_4^2 + \frac{1}{4} - \lambda\right) .$$

Equation (3,58) is the above mentioned <u>non-holonomic</u> <u>condition</u> belonging to the B_3-transformation. If this condition is satisfied by the initial values of the particle's motion, it is satisfied by the regularized variables of this motion at any time (i.e. for any s). The proof of this statement is contained in [4].

Finally we collect the formulae of this chapter in order to establish a set of guiding rules called

<u>Fifth procedure</u>

(Solution of the restricted elliptic three-body problem from given initial values by numerical integration of the regularized differential equations.)

<u>Data</u>

<u>Universal constant</u>:

γ gravitational constant.

<u>Constants characterizing the earth-moon system</u>:

m_1 , m_2 masses of earth and moon respectively.

Compute:

$$K = \sqrt{\gamma(m_1 + m_2)} \quad \text{(gravitational parameter)},$$

$$\mu = \frac{m_2}{m_1 + m_2} \quad \text{(mass ratio)}.$$

p, e semilatus rectum and eccentricity of the moon's relative Kepler ellipse about the earth ($p > 0$, $0 \leq e < 1$).

Compute:

$$a = \frac{p}{1 - e^2} \quad \text{(semi-major axis)}.$$

<u>Initial data of the particle's orbit</u>:

ψ_0 initial true anomaly of the moon in its relative Kepler ellipse.

η_1 , η_2 , η_3 initial position, $\dot\eta_1 , \dot\eta_2 , \dot\eta_3$ initial velocity — of the particle in the rotating coordinate system described in section 3.1.1. (3,59)

<u>Initial values for the regularized system</u>

Compute successively the following quantities (which are all evaluated at the instant ψ_0):

<u>Initial distance earth-moon</u>:

$$\ell = \frac{p}{1 + e \cos \psi_0}$$

Initial values y_i and derivatives y_i' of the dimensionless coordinates:

$$y_i = \frac{\eta_i}{\ell} , \quad y_i' = \frac{\ell}{K\sqrt{p}} \dot{\eta}_i - \frac{e}{p} \sin \psi_0 \cdot \eta_i , \quad i = 1, 2, 3 . \tag{3,60}$$

Initial distances particle-attracting centers:

$$r_1 = \sqrt{(y_1+\mu)^2 + y_2^2 + y_3^2} , \quad r_2 = \sqrt{(y_1+\mu-1)^2 + y_2^2 + y_3^2} .$$

Initial potential $\mathcal{U}$ and constant of energy h:

$$\begin{aligned} \mathcal{U} &= \frac{\ell}{p}\left[-(1-\mu)\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) - \mu\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right)\right] + \frac{y_3^2}{2} , \\ v^2 &= y_1'^2 + y_2'^2 + y_3'^2 , \qquad h = \frac{v^2}{2} + \mathcal{U} . \end{aligned} \tag{3,61}$$

The initial values v_j of the regularized coordinates
are computed by the following set of formulae (obtained by reading table (1,31) from bottom to top):

$$x_1 = 1 + \frac{y_1 + \mu - 1}{r_2^2} ; \quad x_i = \frac{y_i}{r_2^2} , \quad i = 2, 3 . \tag{3,62}$$

Inverse KS-transformation (cf. 2nd procedure):

$$\begin{aligned} u_1^2 + u_4^2 &= \tfrac{1}{2}(r + x_1) , & u_2^2 + u_3^2 &= \tfrac{1}{2}(r - x_1) , \\ u_2 &= \frac{x_2 u_1 + x_3 u_4}{r + x_1} , & u_1 &= \frac{x_2 u_2 + x_3 u_3}{r - x_1} , \qquad r = \sqrt{\textstyle\sum x_i^2} . \\ u_3 &= \frac{x_3 u_1 - x_2 u_4}{r + x_1} , & u_4 &= \frac{x_3 u_2 - x_2 u_3}{r - x_1} , \end{aligned}$$

Take the left- (right-) hand set if $x_1 \geq 0$ ($x_1 < 0$) and choose u_4 (u_3) arbitrarily. Finally the regularized coordinates are

$$v_1 = \frac{1}{2} + \frac{u_1 - 1}{(u_1-1)^2 + u_2^2 + u_3^2 + u_4^2} ; \quad v_j = \frac{u_j}{(u_1-1)^2 + u_2^2 + u_3^2 + u_4^2} , \quad j = 2, 3, 4 . \tag{3,63}$$

Initial derivatives dv_j/ds:
By applying the formulae (3,50),(3,51) and (3,53) with

$$\lambda = v_1^2 + v_2^2 + v_3^2$$

the values of the coefficients (λb_{ij}) at instant ψ_0 are obtained. The initial derivatives dv_j/ds are then given by

$$\frac{dv_j}{ds} = \sum_{i=1}^{3} (\lambda b_{ij})\, y_i' . \tag{3,64}$$

The initial values of W^* and ψ are 0 and ψ_0 respectively. At instant ψ_0 the independent variable s may be chosen as $s = 0$.

The regularized differential equations
for the restricted elliptic three-body problem are given by equations (3,49). In order to compute all the auxiliary variables occurring on the right-hand sides of these equations, formulae (3,40),(3,42),(3,50),(3,51),(3,52),(3,53),(3,25),(3,44),(3,45),(3,46),(3,47) must be applied in this order.

Motion in physical space
Whenever information about the motion of the particle is wanted, the results obtained in the parametric space must be transformed into the physical space. In order to do so, the quantities $r_1, r_2, \lambda, \beta_2, \beta_3, \beta_4, \alpha_i, (\lambda b_{ij})$ are first computed from the actual values of v_j, dv_j/ds, W^*, ψ by use of (3,25),(3,40),(3,50),(3,51),(3,53).

The values y_i and derivatives y_i' of the dimensionless coordinates are then given by

$$y_1 = \tfrac{1}{2} - \mu + \tfrac{1}{2}(v_1 + \alpha_1)\,, \quad y_2 = \tfrac{1}{2}(v_2 + \alpha_2)\,, \quad y_3 = \tfrac{1}{2}(v_3 + \alpha_3)\,,$$

$$y_i' = \frac{1}{\lambda r_1 r_2} \sum_{j=1}^{4} (\lambda b_{ij}) \frac{dv_j}{ds}\,, \qquad i = 1, 2, 3\,. \tag{3,65}$$

With

$$\varrho = \frac{p}{1 + e\cos\psi}$$

the position η_i and velocity $\dot\eta_i$ of the particle are given by

$$\eta_i = \varrho y_i\,, \quad \dot\eta_i = K\left(\frac{\sqrt{p}}{\varrho} y_i' + \frac{e}{\sqrt{p}} \sin\psi \cdot y_i\right), \quad i = 1, 2, 3. \tag{3,66}$$

In order to determine the physical time t, at which the particle attains this position η_i, first compute the eccentric anomaly E from

$$\operatorname{tg}\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\, \operatorname{tg}\frac{\psi}{2}\,, \qquad |\psi - E| < \pi\,,$$

and then t by Kepler's equation

$$t = \frac{a^{3/2}}{K}(E - e\sin E)\,.$$

Checks
Together with the transformation into physical space, two checks may easily be carried out:

a) The non-holonomic condition (3,58) must always be satisfied:

$$v_1 v_4 \frac{dv_1}{ds} + \beta_2 \frac{dv_2}{ds} + \beta_3 \frac{dv_3}{ds} + \beta_4 \frac{dv_4}{ds} = 0\,. \tag{3,67}$$

b) The equation

$$\frac{1}{2\lambda} \sum_{j=1}^{4} \left(\frac{dv_j}{ds}\right)^2 + \frac{\varrho}{p} Q + r_1 r_2 \left(\frac{y_3^2}{2} - W^* - h\right) = 0 \tag{3,68}$$

(which follows from the energy equation (3,31) by taking into account (1,20) and (3,21)) has to be satisfied at any time. The quantity Q is given by (3,44).

3.1.3 Remarks. If a solution of the differential equations (3,49) passes through one of the two centers $(\pm\frac{1}{2}, 0, 0, 0)$, the corresponding orbit in the physical space passes through one of the attracting centers. In this case the particle collides with the earth or the moon. As a consequence of regularization the derivatives dv_j/ds have finite limits even at collisions (in the physical space the the components of the particle's velocity generally tend to infinity if the particle collides with one of the attracting centers).

In order to discuss the two types of collisions together, we introduce the sign σ which takes the value $+1$ or -1, according as the particle collides with the moon or the earth. The attracting center with mass

$$[\tfrac{1}{2} + \sigma(\mu - \tfrac{1}{2})](m_1 + m_2) \tag{3,69}$$

then has coordinates

$$\eta_1 = (\tfrac{\sigma+1}{2} - \mu)\,\ell\,, \qquad \eta_2 = \eta_3 = 0 \tag{3,70}$$

in the physical space. The corresponding point in the parametric space is

$$v_1 = \frac{\sigma}{2}\,, \qquad v_2 = v_3 = v_4 = 0\,. \tag{3,71}$$

We now consider a collision of the particle with the attracting center indicated by σ. Then

$$v_1 \to \frac{\sigma}{2}\,, \quad v_2 \to 0\,, \quad v_3 \to 0\,, \quad v_4 \to 0\,.$$

According to (3,25),(3,40),(3,44) the following limiting values are obtained

$$\lambda \to \frac{1}{4}\,, \quad r_1 \to \frac{1+\sigma}{2}\,, \quad r_2 \to \frac{1-\sigma}{2}\,, \quad Q \to -\left(\tfrac{1}{2} + \sigma(\mu - \tfrac{1}{2})\right)\,.$$

Substituting these in the energy equation (3,68) gives

$$\sum_{j=1}^{4}\left(\frac{dv_j}{ds}\right)^2 \longrightarrow \frac{\frac{1}{2} + \sigma(\mu - \frac{1}{2})}{2(1 + e\cos\psi_c)}\,, \tag{3,72}$$

where ψ_c is the value of the true anomaly at the instant of the collision under consideration. Thus at a collision the limit of the squared velocity in the parametric space is finite and does not depend on the direction of the collision.

In the case of a collision, the velocity can no longer be transformed by using equations (3,64) and (3,65), because the physical velocity becomes infinite and all the b_{ij} vanish. However instead of mapping velocity vectors at one of the centers in the parametric space, we may establish a correspondence between the directions of vectors at these centers.

We add to the position vector $(\frac{\sigma}{2}, 0, 0, 0)$ of the center given by σ the small increment

$$(\sigma\bar{v}_1\,,\ \bar{v}_2\,,\ \bar{v}_3\,,\ \bar{v}_4)\,, \tag{3,73}$$

which is parallel to the velocity vector $\left(\frac{dv_1}{ds}, \frac{dv_2}{ds}, \frac{dv_3}{ds}, \frac{dv_4}{ds}\right)$ at this center. In order to obtain the corresponding increment which is denoted by

$$(\sigma \bar{y}_1 , \bar{y}_2 , \bar{y}_3) , \tag{3,74}$$

the point

$$\left(\sigma(\tfrac{1}{2}+\bar{v}_1), \bar{v}_2 , \bar{v}_3 , \bar{v}_4\right) \tag{3,75}$$

is mapped into the physical space by the B_3-transformation. Substituting (3,75) in (3,23) and expanding the results in power series at the point $(\frac{\sigma}{2}, 0, 0, 0)$ yields

$$\left.\begin{aligned} y_1 &= \frac{1+\sigma}{2} - \mu + \sigma(\bar{v}_1^{\,2} - \bar{v}_2^{\,2} - \bar{v}_3^{\,2} + \bar{v}_4^{\,2}) + \cdots \\ y_2 &= 2(\bar{v}_1 \bar{v}_2 - \bar{v}_3 \bar{v}_4) + \cdots \\ y_3 &= 2(\bar{v}_1 \bar{v}_3 + \bar{v}_2 \bar{v}_4) + \cdots \end{aligned}\right\} \text{terms of 3rd and higher order}$$

By keeping the direction of the increment (3,73) fixed, but allowing its length to tend to zero, it follows that the desired increment in the physical space is in fact given by (3,74) with

$$\begin{aligned} \bar{y}_1 &= \bar{v}_1^{\,2} - \bar{v}_2^{\,2} - \bar{v}_3^{\,2} + \bar{v}_4^{\,2} \\ \bar{y}_2 &= 2(\bar{v}_1 \bar{v}_2 - \bar{v}_3 \bar{v}_4) \\ \bar{y}_3 &= 2(\bar{v}_1 \bar{v}_3 + \bar{v}_2 \bar{v}_4) \ . \end{aligned} \tag{3,76}$$

This is exactly the KS-transformation (1,44). Since the $\bar{y}_i$ are homogeneous functions of the $\bar{v}_j$ (all having the same degree), the transformation (3,76) is a mapping of the increments' <u>direction</u>. For that reason the length of the increment (3,73) may now be chosen arbitrarily; for example, simply

$$\bar{v}_1 = \sigma \frac{dv_1}{ds} \ ; \quad \bar{v}_j = \frac{dv_j}{ds} \ , \quad j = 2, 3, 4 \ . \tag{3,77}$$

The vector (3,74) (with $\bar{y}_i$ given by (3,76)) then indicates the direction of the collision under consideration.

If the motion of the particle is started exactly at a collision (with the attracting center given by σ), one is concerned with the problem of finding an initial velocity vector $\left(\frac{dv_1}{ds}, \frac{dv_2}{ds}, \frac{dv_3}{ds}, \frac{dv_4}{ds}\right)$ corresponding to the given direction $(\sigma\bar{y}_1 , \bar{y}_2 , \bar{y}_3)$ of the collision in the physical space. This may be done by applying the inverse KS-transformation (1,47) to the vector $(\bar{y}_1 , \bar{y}_2 , \bar{y}_3)$. If for simplicity this vector is assumed to have unit length,

$$\bar{y}_1^{\,2} + \bar{y}_2^{\,2} + \bar{y}_3^{\,2} = 1 , \tag{3,78}$$

the following formulae are obtained:

$$\begin{aligned} \bar{v}_1^{\,2} + \bar{v}_4^{\,2} &= \tfrac{1}{2}(1+\bar{y}_1) \\ \bar{v}_2 &= \frac{\bar{y}_2 \bar{v}_1 + \bar{y}_3 \bar{v}_4}{1+\bar{y}_1} \\ \bar{v}_3 &= \frac{\bar{y}_3 \bar{v}_1 - \bar{y}_2 \bar{v}_4}{1+\bar{y}_1} \end{aligned} \quad \text{or} \quad \begin{aligned} \bar{v}_2^{\,2} + \bar{v}_3^{\,2} &= \tfrac{1}{2}(1-\bar{y}_1) \\ \bar{v}_1 &= \frac{\bar{y}_2 \bar{v}_2 + \bar{y}_3 \bar{v}_3}{1-\bar{y}_1} \\ \bar{v}_4 &= \frac{\bar{y}_3 \bar{v}_2 - \bar{y}_2 \bar{v}_3}{1-\bar{y}_1} \ . \end{aligned} \tag{3,79}$$

The vector $(\sigma\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)$ is parallel to the initial velocity vector in the parametric space and also has unit length. Thus, according to (3,72), the initial velocity vector is

$$\left(\frac{dv_1}{ds}, \frac{dv_2}{ds}, \frac{dv_3}{ds}, \frac{dv_4}{ds}\right) = \sqrt{\frac{\frac{1}{2}+\sigma(\mu-\frac{1}{2})}{2(1+e\cos\psi_0)}}\,(\sigma\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)\,. \tag{3,80}$$

Modifications of the fifth procedure for the case of an ejection
(the initial position of the particle is one of the attracting centers).

Only these parts of the 5th procedure, which must be modified in the case of an ejection, are recorded here. The subtitles are the same as in the 5th procedure.

Data

Initial data:

The initial position of the particle may now be indicated by the sign σ:

$$\sigma = \begin{Bmatrix} -1 \\ +1 \end{Bmatrix} : \begin{matrix}\text{motion}\\ \text{starts at}\end{matrix} \begin{cases} m_1 & \text{(earth)} \\ m_2 & \text{(moon)} \end{cases} \tag{3,59a}$$

$(\dot\eta_1, \dot\eta_2, \dot\eta_3)$ indicates the initial direction of the particle's orbit.

Furthermore the energy constant h must be given (cf. (3,61a)).

Initial values for the regularized system

Initial values y_i and derivatives y_i' of the dimensionless coordinates:

$$y_1 = \frac{\sigma+1}{2} - \mu\,, \quad y_2 = y_3 = 0\,.$$

By y_i' we now mean the components of the unit vector indicating the initial direction

$$y_i' = \frac{\dot\eta_i}{\sqrt{\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2}}\,, \quad i = 1, 2, 3\,. \tag{3,60a}$$

Initial potential $\mathcal{U}$ and energy constant h:

The formulae (3,61) can not be used. The energy constant h is given by the initial data. (3,61a)

Initial values of the regularized coordinates v_i:

$$v_1 = \frac{\sigma}{2}\,, \quad v_2 = v_3 = v_4 = 0\,. \tag{3,63a}$$

It is not necessary to apply (3,62) and the inverse KS-transformation (1,47).

Initial derivatives dv_i/ds:

The coefficients (λb_{ij}) cannot be used because they all vanish.

With

$$\bar{y}_1 = \sigma y_1'\,, \quad \bar{y}_2 = y_2'\,, \quad \bar{y}_3 = y_3'$$

compute the $\bar{v}_i$ from (3,79). Use the left-hand or the right-hand equations of (3,79), according as $\bar{y}_1$ is positive or negative. The initial derivatives are given by (3,64a)

$$\left(\frac{dv_1}{ds}, \frac{dv_2}{ds}, \frac{dv_3}{ds}, \frac{dv_4}{ds}\right) = \sqrt{\frac{\frac{1}{2} + \sigma(\mu - \frac{1}{2})}{2(1 + e\cos\psi_0)}}\,(\sigma\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4),$$

where ψ_0 is the initial true anomaly.

Referring to the remark following formula (3,40) in section 3.1.2 we now give a few words on avoiding small values of the denominator $v_1^2 + v_2^2 + v_3^2$ during numerical integration of the regularized differential equations (3,49).

As it is mentioned in the fifth procedure, there are generally many points in the 4-dimensional parametric space which are mapped onto the same point of the 3-dimensional physical space by the B_3-transformation. The set of points (w_1, w_2, w_3, w_4) having the same image as the fixed point (v_1, v_2, v_3, v_4) is called the fibre passing through the point (v_1, v_2, v_3, v_4) and is given by (cf. [4], page 26)

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \frac{\frac{1}{2}}{(\rho^2 + \frac{1}{4}) - (\rho^2 - \frac{1}{4})\cos\varphi + v_4 \sin\varphi} \begin{pmatrix} v_1 \\ v_2\cos\varphi + v_3\sin\varphi \\ -v_2\sin\varphi + v_3\cos\varphi \\ v_4\cos\varphi + (\rho^2 - \frac{1}{4})\sin\varphi \end{pmatrix}, \qquad (3,81)$$

where ρ^2 is the expression

$$\rho^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 .$$

In order to obtain all points of the fibre passing through the point v_j, the parameter φ must take all values in the interval $0 \le \varphi < 2\pi$. In general the fibres are circles, the only exceptions being the v_4-axis $v_1^2 + v_2^2 + v_3^2 = 0$ and the two centers $(\pm\frac{1}{2}, 0, 0, 0)$.

For the following discussion, on the fibre circle passing through the point v_j we introduce the points N and F. They have the property that, of all the points belonging to the considered fibre circle, their distances from the v_4-axis, d_N and d_F, are the least and the greatest respectively (nearest and farthest point). There are also fibre circles, where all the points have the same distance from the v_4-axis, but this case is not important here. The relation

$$d_N \cdot d_F = \frac{1}{4} \qquad (3,82)$$

holds true for every fibre circle.

Let us now consider for a point v_j lying on an orbit in the parametric space the fibre passing through this point. If for v_j the denominator

$$d^2 = v_1^2 + v_2^2 + v_3^2$$

is small compared with $\frac{1}{4}$, it follows from (3,82) that the point v_j lies near the point N of its fibre. In order to avoid a close approach of v_j to N we propose the following method.

If the denominator d^2 at a point (v_1, v_2, v_3, v_4) of the orbit becomes smaller than a certain limit $d_0^2 \ll \frac{1}{4}$, the motion in the parametric space is stopped and restarted at another point (w_1, w_2, w_3, w_4) of the fibre passing through v_j. The coordinates w_j are given by (3,81) with a suitable value of φ, and a formula for computing the derivatives $\frac{dw_j}{ds}$ from $\frac{dv_j}{ds}$ may be obtained by differentiation of (3,81) with respect to s.

The consequence of this procedure is not recognizable in the physical space because the B_3-transformation maps all the points of a fibre onto the same point.

A suitable choice of φ may be obtained from the following statement, [4]: We consider equation (3,81) as a transformation (depending on φ) of the parametric space onto itself keeping fixed the fibres. The special transformation that maps the farthest point F of a fibre onto N is given by (3,81) with $\varphi = \varphi_N = \pi$. On the other hand the transformation mapping F onto a general point (v_1, v_2, v_3, v_4) lying on the fibre of F is given by (3,81) with

$$\varphi = \arg\left[v_1^2 + v_2^2 + v_3^2 + \left(v_4 + \frac{i}{2}\right)^2\right] . \tag{3,83}$$

This information about the position of the point v_j on its fibre may be used to choose the angle φ occurring in (3,81) in such a way that the transformed coordinates w_j satisfy the inequation

$$w_1^2 + w_2^2 + w_3^2 > d_0^2 .$$

Although this procedure may sometimes help to avoid extremely small denominators during the numerical integration, the singularity occurring when the particle escapes to infinity is still present. But in practice the particle's orbit is of very little interest at a great distance from the earth and the moon.

3.2 Examples

The fifth procedure is very useful for computing orbits in the restricted elliptic problem whenever the particle comes close to one of the attracting centers. In order to illustrate this we give here some results of numerical experiments. All the computations were carried out on the Control Data 1604-A computer of the Swiss Federal Institute of Technology.

A computational program (referred to as SIMREG = simultaneous regularization) for the calculation of trajectories in restricted three-body problems was written in ALGOL. In its essential parts the program is a replica of the fifth procedure, but the transformation to an inertial coordinate system is added. The numerical integration of the regularized differential equations (3,49) is always performed by the Runge-Kutta method (single step method of error order 4).

The orbits resulting from the computations are displayed in two coordinate systems; we refer to them as

a) the inertial coordinate system,

b) the dimensionless rotating system.

The inertial coordinate system $\eta_1^*, \eta_2^*, \eta_3^*$ has its origin at the center of gravity of the two attracting bodies (earth m_1 and moon m_2). The η_1^*-axis initially (at time $t = 0$) passes through the attracting centers and is directed from earth to moon. The η_2^*-axis is obtained by rotating the η_1^*-axis through the angle $\pi/2$ in the moon's orbital plane (in the sense of the moon's revolution). The η_3^*-axis is then chosen to form a right-handed rectangular system together with the two previous axes η_1^*, η_2^*.

The dimensionless rotating system is the coordinate system y_1, y_2, y_3 introduced in (3,14). The origin is again the center of gravity, and the y_3-axis coincides with the η_3^*-axis. The system rotates about this axis and "pulsates" in such a way that the earth and the moon occupy fixed positions on the y_1-axis.

3.2.1 Transfer of a vehicle from earth to moon. In this first example the computation of a realistic orbit from earth to moon is described. In order to compute the vehicle's trajectory by the program SIMREG, the motion of the moon had to be approximated by a pure Kepler orbit which yields values for the orbital elements of the moon. This was performed by approximating a given exact ephemeris of the moon. We are indebted to Mr. B. Stanek for this auxiliary computation. Only perturbations by the moon have been taken into account. The resulting orbital elements of the moon are:

semi-major axis	a = 382 100 km
time of revolution	T = 648.61321 926 hrs
eccentricity	e = .05
initial true anomaly	ψ_0 = .3 rad
mass ratio	μ = .01211 68060 .

In all our examples we use "standard" units adapted to the earth-moon system under consideration:

unit of length: a (semi-major axis)
unit of time : $T/2\pi$ (3,84)
unit of mass : $m_1 + m_2$ (total mass)

By the laws of Kepler motion it follows that

$$p = 1 - e^2 \qquad \text{(semilatus rectum)}$$
$$T = 2\pi$$
$$m_1 + m_2 = 1$$
$$\gamma = 1 \qquad \text{(gravitational constant).}$$

In standard units the adopted initial conditions for the vehicle (in the rotating coordinate system described in section 3.1.1) are

$$\eta_1 = -.02182\,35477 \qquad \dot\eta_1 = 5.25062\,2867$$
$$\eta_2 = -.01299\,03502 \qquad \dot\eta_2 = -2.01747\,1424$$
$$\eta_3 = .00542\,30458 \qquad \dot\eta_3 = 8.94355\,4806.$$

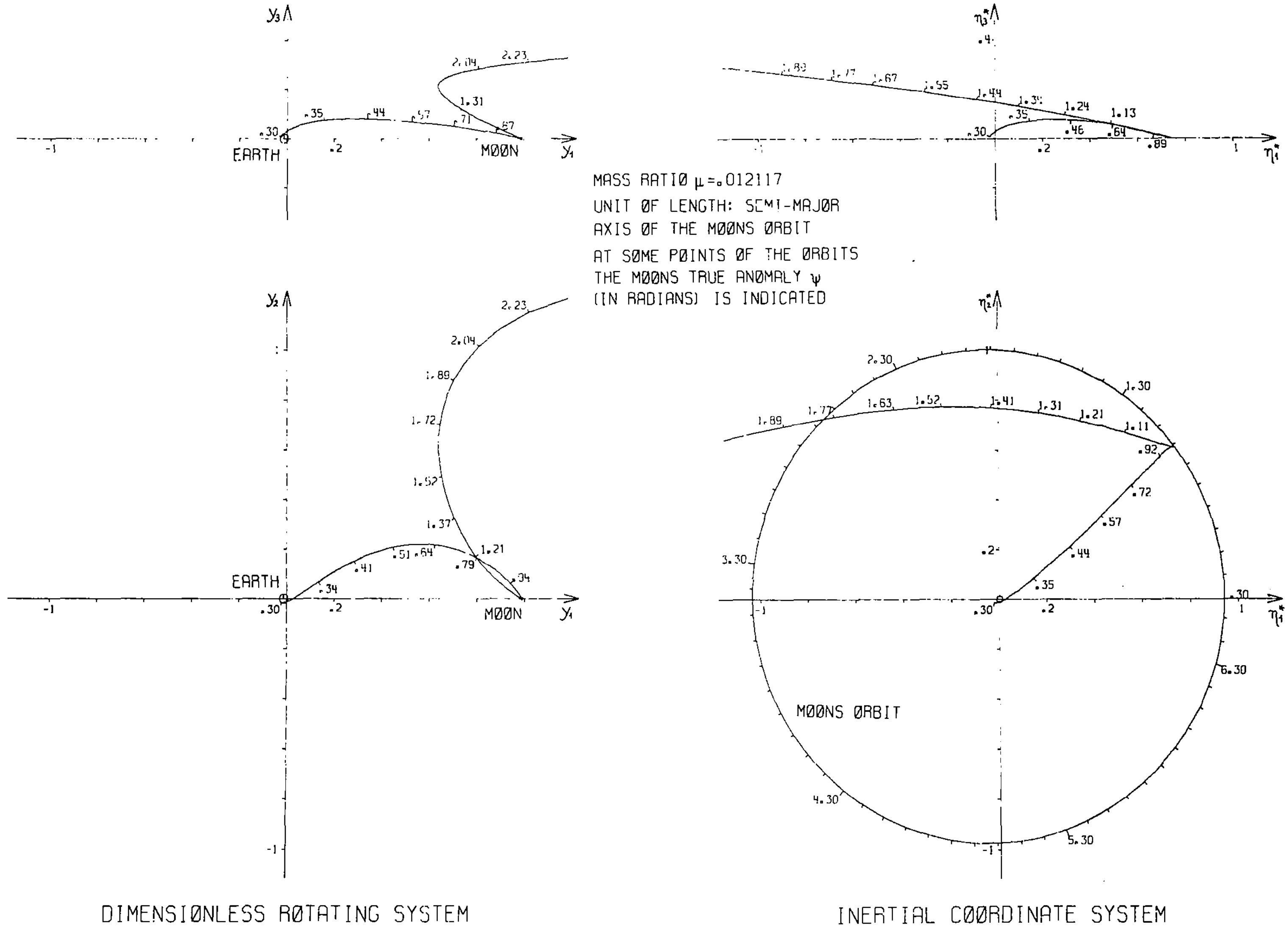
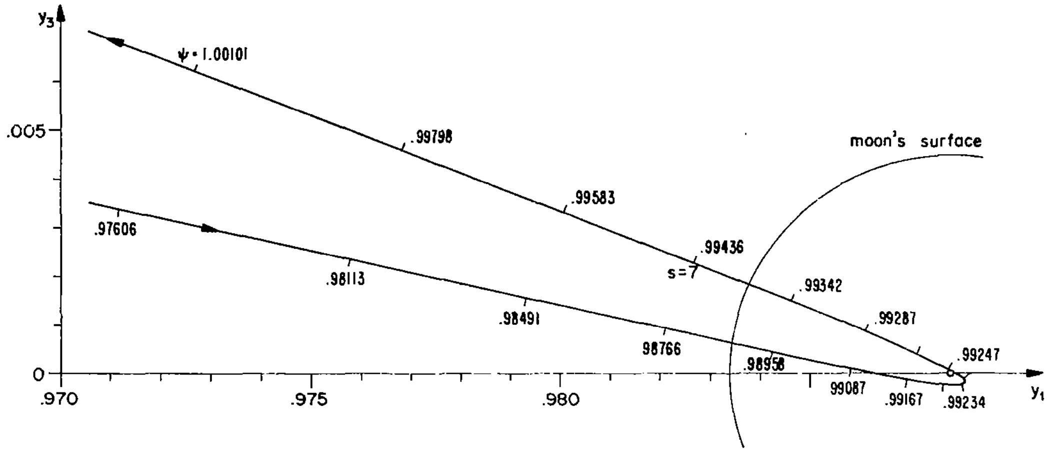


Fig. 3.2. Transfer orbit from the earth to the moon.



Unit of length = 382 100km (semi-major axis of the moon's orbit).

The points of the orbit with marks correspond to equal increments Δs = .2 of the fictitious anomaly s.
At each of these points the moon's true anomaly ψ (in radians) is indicated.

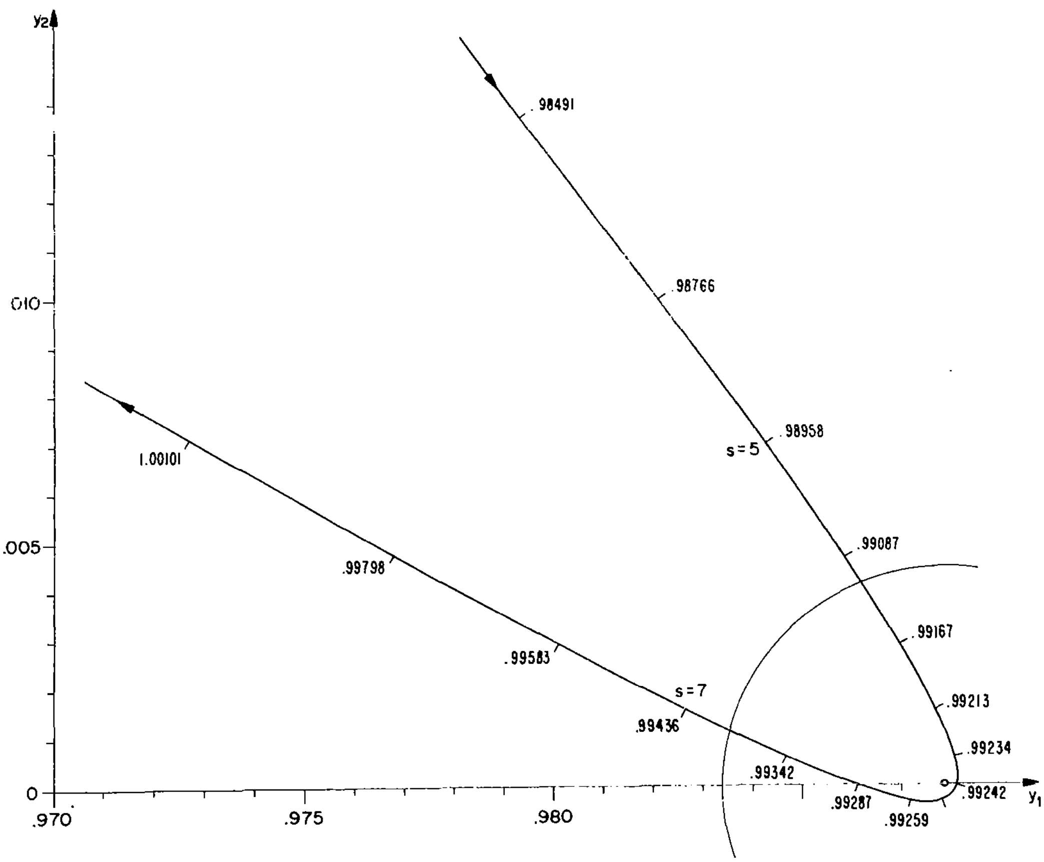


Fig. 3.3. Detail of Fig. 3.2: vicinity of the moon.

In Fig. 3.2 and Fig. 3.3 we show the transfer orbit from the earth to the moon resulting from the constants and initial values listed above. The trajectory starts about 285 km above the earth's surface and collides with the moon's surface. If this body is assumed to be a mass point, the orbit may be continued into the interior of the moon and further into deep space. The minimum of the vehicle's distance from the center of the moon is about $1/20$ of its radius. After this near-collision the vehicle escapes with high velocity from the earth-moon system.

For the numerical computation of this orbit a constant step $\Delta s = .02$ of the fictitious anomaly s has been chosen. Due to the influence of regularization the corresponding step $\Delta\psi$ of the true anomaly increased from $4\cdot10^{-4}$ up to its maximum $6\cdot10^{-3}$ between the earth and the moon and was finally reduced to $5\cdot10^{-6}$ at the closest approach to the moon. 143 Runge-Kutta integration steps were needed for reaching the moon's activity sphere (radius = 57 500 km), and 160 more steps were needed for the leg of the journey to the closest approach. No numerical instabilities are generated by this close approach.

In Fig. 3.4 the true anomaly ψ is plotted as a function of the fictitious anomaly s. $\psi(s)$ is monotonically increasing, but it increases very slowly in the neighbourhood of the points $s = 0$ and $s = 6.06$ corresponding to the earth and the moon.

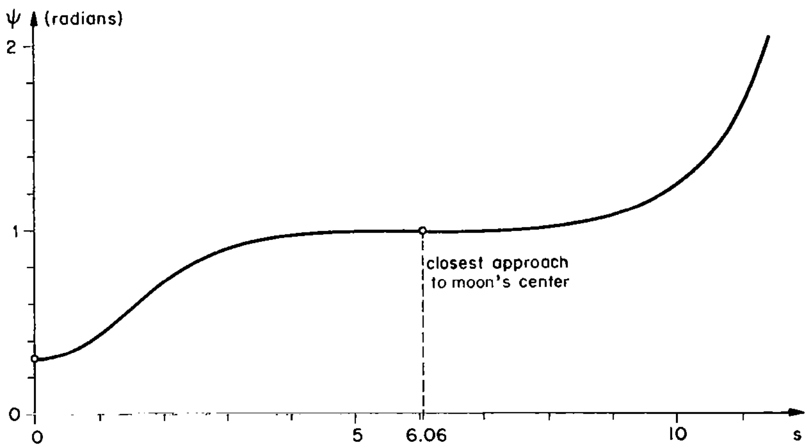


Fig. 3.4. The true anomaly ψ as a function of the fictitious anomaly s in the case of Fig. 3.2.

The values of the left-hand sides of the checks (3,67) and (3,68) did not exceed $1.7\cdot10^{-9}$ and $6.5\cdot10^{-9}$ respectively after 500 steps. In order to obtain information about the exactness of the numerical integration the same orbit was computed with a new step length $\Delta s = .04$, and two corresponding sets of coordinates y_i describing the arrival on the moon were compared. The maximum difference was $1.2\cdot10^{-6}$. Thus the orbit computed with $\Delta s = .02$ is exact to at least 6 decimal places.

Because the eccentricity of the moon's orbit, in this example, is very small, we carried out corresponding experiments with a fictitious moon m_2 moving in an orbit of high eccentricity. The following input data were chosen (standard units):

$$\mu = m_2 = .1 \,, \qquad \rho = .36 \,, \qquad e = .8 \,, \qquad \psi_0 = -.5 \,,$$

$$\eta_1 = 0 \qquad \dot{\eta}_1 = 0$$

$$\eta_2 = 0 \qquad \dot{\eta}_2 = -.5$$

$$\eta_3 = 0 \qquad \dot{\eta}_3 = 9.15 \,.$$

The resulting trajectory is displayed in Fig. 3.5. It is remarkable that the vehicle reaches the moon m_2, altough the initial velocity is almost perpendicular to the orbital plane of m_2.

The computation proceeds in the same way as in the preceding example. No difficulties occur because of the large eccentricity of the orbit of m_2.

3.2.2 A 3-dimensional periodic orbit in the restricted circular three-body problem. Recently, R.F. Arenstorf [13] has computed families of plane periodic orbits passing near both attracting centers of the restricted circular problem. On the other hand C.L. Goudas [14] constructed many 3-dimensional periodic orbits without close approach to both masses. In order to make a first step in synthesizing the methods of the two authors, we present in Fig. 3.6 an example of a 3-dimensional periodic orbit of a particle ejected from the first attracting center (earth) and approaching very close to the second center (moon). About 100 preliminary orbits have been computed by Mr. E. Sturzenegger in order to achieve periodicity. Up to the present we have not been able to construct a 3-dimensional periodic orbit colliding with both attracting centers.

The system of the attracting centers is characterized by the values (standard units)

$$\mu = m_2 = .1 \,, \qquad \rho = 1 \,, \qquad e = 0 \,, \qquad \psi_0 = 0.$$

The direction of the ejection needed for periodicity was found to be

$$(-1 \,,\ 0 \,,\ .06874\,215)$$

in the dimensionless rotating system, while a value

$$h = -.82448\,546$$

had to be taken for the energy constant. The half period $\tau/2$ thus became

$$\tau/2 = 7.77403\,9$$

($2\pi = 6.283...$ corresponds to one revolution of the moon).

The orbit resulting from these input data is symmetric with respect to the y_1, y_3-plane. This is a consequence of the facts that the initial position and the direction of ejection are in this plane, and that the orbit intersects it perpendicularly at the time $\tau/2$. Therefore only half the orbit is plotted in Fig. 3.6 (the projection to the y_1, y_3-plane is a curve being covered twice).

A final remark to this periodic orbit is added. At ejection the velocity component perpendicular to the y_1, y_2-plane is small, but later, after the close approach to the moon, it is very large. This fact raises some doubts about the stability of the many classical plane periodic orbits if perturbations perpendicular to the moon's orbital plane are allowed.

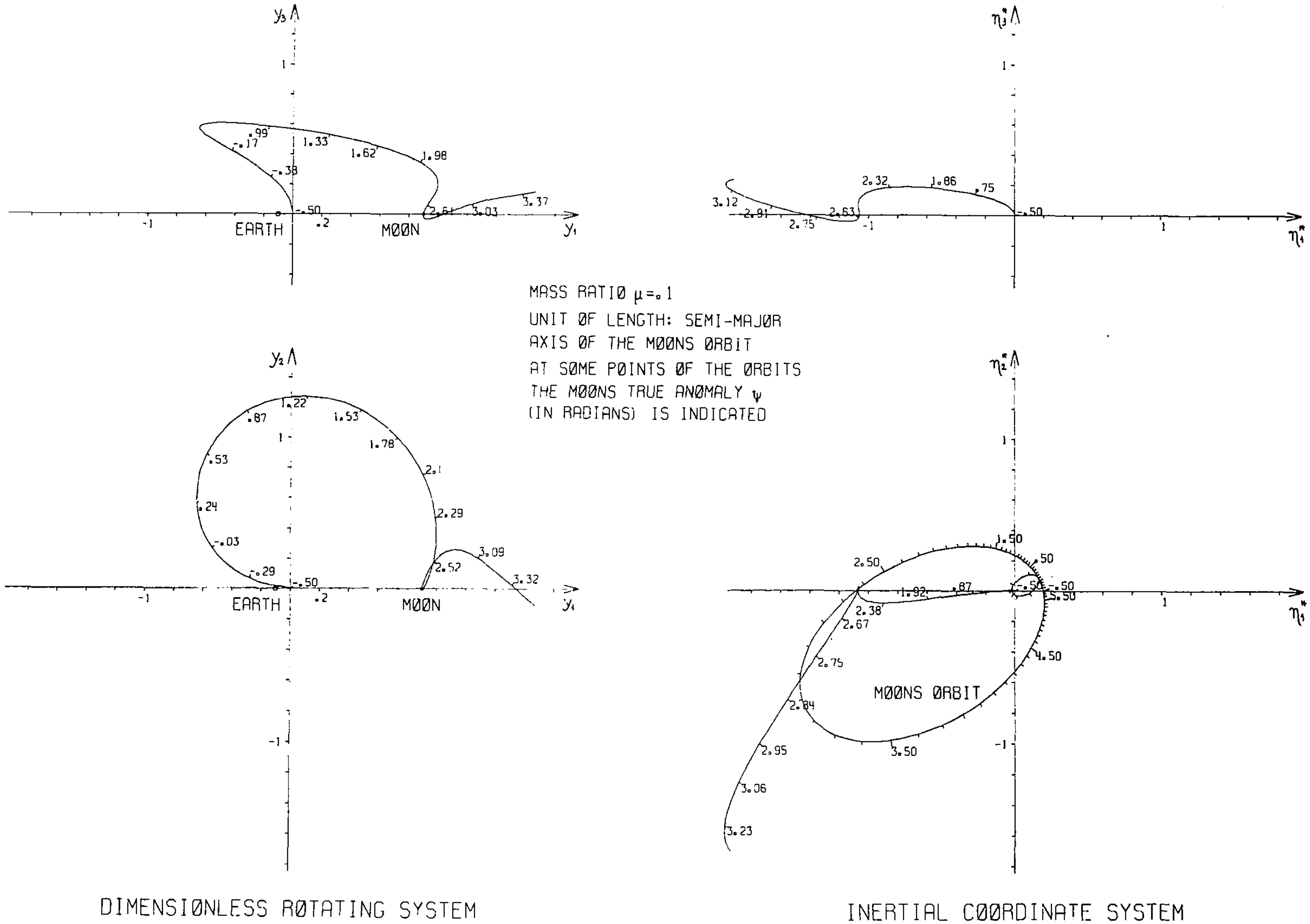


Fig. 3.5. Orbit of a particle in a fictitious earth-moon system with a lunar orbit of large eccentricity.

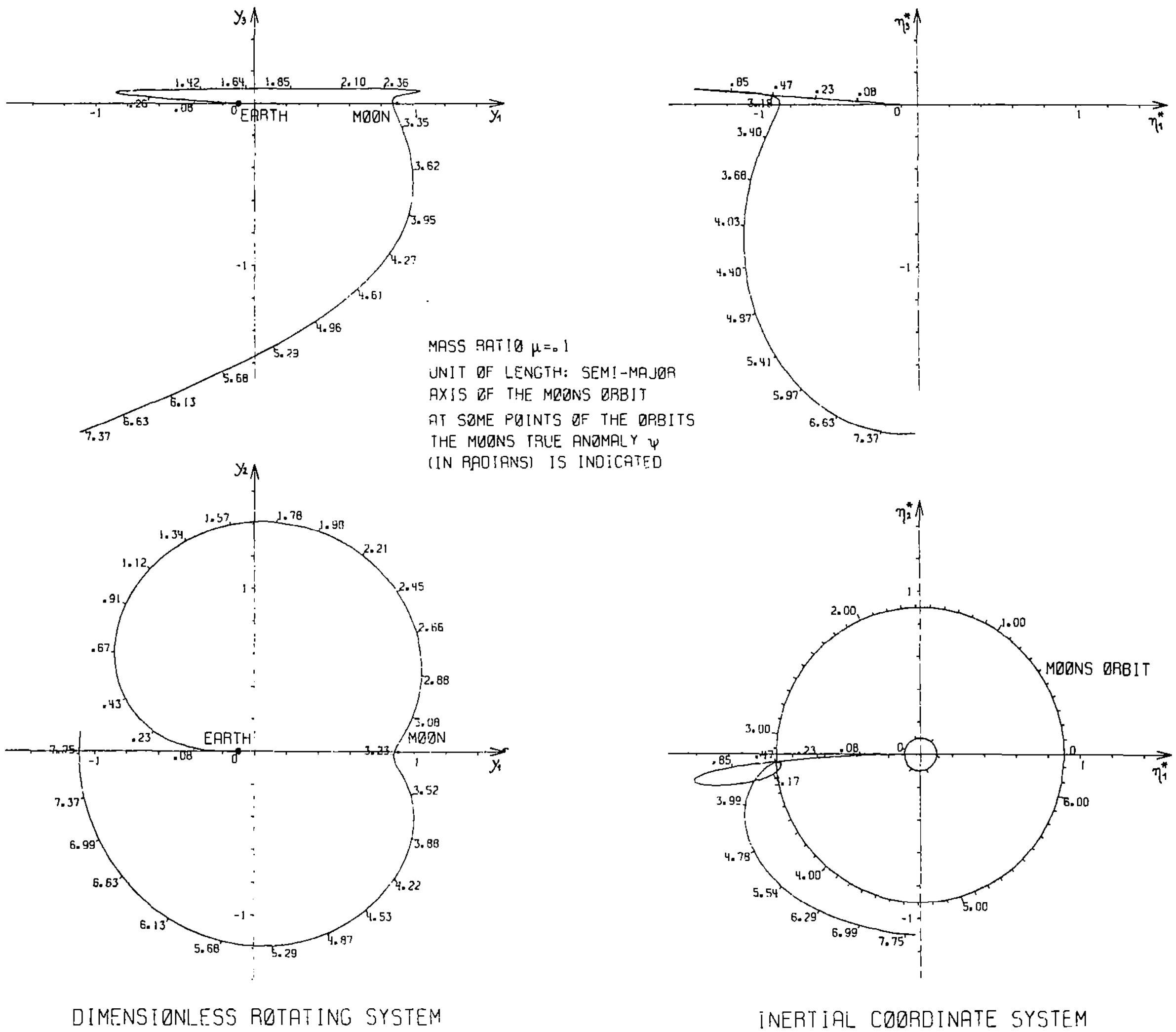


Fig. 3.6. A 3-dimensional periodic orbit in the restricted circular problem. Only half a revolution is plotted.

3.2.3 Conclusions.

- From a theoretical point of view the B_3-regularization of the elliptic restricted problem is very well suited to the qualitative discussion of trajectories and to obtaining information on the general behaviour of a three-body system.

- It may also be well suited to feasibility studies on transfer orbits from one celestial body to another, as for instance in problems of capture.

- For the exact numerical computation of transfers it is a disadvantage that the two attracting centers are assumed to move on exact Kepler orbits. If this assumption is not satisfied, one could use, at the beginning of the trip, KS-regularization centered at the earth and switch at a convenient instant to KS-regularization centered at the moon. We have no experience about the numerical behaviour of such a method.

- It should be mentioned in this connection that A. Deprit and R. A. Broucke [15] have suggested this idea in the special case of the 2-dimensional restricted circular problem by using Levi-Civita's transformation. They have developed a simple set of formulae containing a switching parameter. The generalization of such a procedure to 3-dimensional motion and to KS-transformation is obvious.

4. EXPERIMENTS CONCERNING NUMERICAL ERRORS

by C.A. Burdet

4.1 Configuration of the reference orbit

(with which all numerical experiments have been performed)

At the time this paper is being written, complete results of numerical experiments are only available for unperturbed and circular Kepler orbits (eccentricity = 0) and we shall therefore restrict our presentation to this special case.

In order to put the 3-dimensional KS-regularization (cf. 1.2.1) into operation and to investigate its numerical behaviour, we choose a circular trajectory with orbital plane in general position.

The gravitational parameter M was set equal to 1 in (1,51) and the radius of the orbit is 1 .

Exact initial conditions:

$$\begin{array}{llll} x_1 = .36235\,77544\,9 , & \dot{x}_1 = -.50358\,28673\,1 , \\ x_2 = .93203\,90859\,7 , & \dot{x}_2 = .19578\,27303\,0 , \\ x_3 = 0 , & \dot{x}_3 = .84147\,09848\,0 . \end{array} \tag{4,1}$$

The corresponding circular orbit has an inclination with respect to the x_1, x_2 - plane measuring roughly 57°.

From the above conditions, we derive the following formulae for the motion of our particle:

$$\begin{array}{ll} x_1 = .36235\,77544\,9 * \cos t & - .50358\,28673\,1 * \sin t , \\ x_2 = .93203\,90859\,7 * \cos t & + .19578\,27303\,0 * \sin t , \\ x_3 = & .84147\,09848\,0 * \sin t . \end{array} \tag{4,2}$$

t is the physical time.

Furthermore, we have for the radial distance r and the true anomaly φ the following exact expressions:

$$r \equiv 1 , \tag{4,3}$$

$$\varphi \equiv t . \tag{4,4}$$

4.2 Numerical integration of the equations of motion

A) Classical equations of Kepler motion. Our system of differential equations is composed of 6 first order equations which read

$$\begin{aligned} \dot{x}_i &= y_i \;, \\ \dot{y}_i &= -\frac{x_i}{r^3} \;, \end{aligned} \qquad (i = 1,2,3) \qquad (4,5)$$

where

$$r = \sqrt{\sum_{i=1}^{3} x_i^2} \;. \qquad (4,6)$$

We denote the solution of (4,5) obtained with numerical integration by:

$${}_{d}x_i \;, \qquad (i = 1,2,3)$$

initial conditions are given in (4,1).

<u>B) Regularized equations of motion</u>. The four parametric coordinates u_1, u_2, u_3, u_4 and the physical time t are computed from a system of 9 first order differential equations which read

(1,74)(1,83)

$$\begin{aligned} u_j' &= v_j \;, \\ v_j' &= -\frac{1}{4} u_j \;, \end{aligned} \qquad (j = 1,2,3,4) \qquad (4,7)$$

(1,57)(1,45)

$$t' = \sum_{j=1}^{4} u_j^2 \;. \qquad (4,8)$$

Here the independent variable is the fictitious time s ; after numerical integration the physical coordinates are obtained from

(1,44)

$$\begin{aligned} x_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \;, \\ x_2 &= 2\,(u_1 u_2 - u_3 u_4) \;, \\ x_3 &= 2\,(u_1 u_3 + u_2 u_4) \;, \end{aligned} \qquad (4,9)$$

and the velocities from

(1,98)

$$\begin{aligned} \dot{x}_1 &= \frac{2}{r}\,(u_1 u_1' - u_2 u_2' - u_3 u_3' + u_4 u_4') \;, \\ \dot{x}_2 &= \frac{2}{r}\,(u_1 u_2' + u_2 u_1' - u_3 u_4' - u_4 u_3') \;, \\ \dot{x}_3 &= \frac{2}{r}\,(u_1 u_3' + u_3 u_1' + u_2 u_4' + u_4 u_2') \;, \end{aligned} \qquad (4,10)$$

with

(1,45)

$$r = \sum_{j=1}^{4} u_j^2 \;. \qquad (4,11)$$

We denote the numerical value of the above coordinates obtained by numerical integration by:

$${}_{reg}x_i \;,$$

for the velocities:

$${}_{reg}\dot{x}_i$$

and for the physical time:

$${}_{reg}t \;.$$

It should be emphasized that throughout integration we constantly make use of the exact initial value $\omega^2 = \frac{1}{4}$ in the equations of motion (1,74).

The initial conditions for the parametric coordinates and velocities are taken from the left-hand version of (1,47) by choosing $u_4 = 0$ and from (1,48).

Thus, we have at our disposal the numerical values of

- the solution ${}_{cl}x_i$ for the classical case,
- the solution ${}_{reg}x_i$, ${}_{reg}t$ for the regularized case,
- and the solution ${}_{ex}x_i$ which denotes values of coordinates of the exact analytical solution (4,2).

Comparison of numerical solutions with the exact analytical solution was established for the distance r and true anomaly φ, in both classical and regularized cases. We computed r and φ from the Cartesian coordinates ${}_{cl}x_i$ and ${}_{reg}x_i$ respectively by projecting the point x_i onto the orbital plane of the exact solution. The results are denoted in the sequel by

${}_{cl}r$, ${}_{cl}\varphi$; ${}_{reg}r$, ${}_{reg}\varphi$ respectively.

Furthermore ${}_{ex}r$, ${}_{ex}\varphi$ denote the exact values (4,3),(4,4).

Numerical errors can now be defined as follows:

<u>for the classical solution:</u>

$$ {}_{cl}\Delta r(t) = {}_{cl}r(t) - {}_{ex}r(t) , \tag{4,12} $$

$$ {}_{cl}\Delta \varphi(t) = {}_{cl}\varphi(t) - {}_{ex}\varphi(t) , \tag{4,13} $$

<u>for the solution of the regularized system:</u>

$$ {}_{reg}\Delta r({}_{reg}t) = {}_{reg}r(s) - {}_{ex}r({}_{reg}t) , \tag{4,14} $$

$$ {}_{reg}\Delta \varphi({}_{reg}t) = {}_{reg}\varphi(s) - {}_{ex}\varphi({}_{reg}t) , \tag{4,15} $$

i.e. regularized coordinates ${}_{reg}r$ and ${}_{reg}\varphi$ are opposed to values ${}_{ex}r$ and ${}_{ex}\varphi$ of the exact solution taken at the computed time ${}_{reg}t$.

We also determined the influence of numerical errors on the most important of all elements of the orbit, namely the semi-major axis a; values of ${}_{cl}a$, ${}_{reg}a$ yield the following errors:

$$ {}_{cl}\Delta a(t) = {}_{cl}a(t) - 1 , \tag{4,16} $$

$$ {}_{reg}\Delta a({}_{reg}t) = {}_{reg}a({}_{reg}t) - 1 . \tag{4,17} $$

They were computed, during integration, for various values of time, from the corresponding values of the physical coordinates x_i and velocities $\dot{x}_i$.

All experiments were performed on a Control Data 1604-A computer using floating point arithmetic with ~ 11 decimal places and symmetric rounding.

The differential equations were integrated with the standard Runge-Kutta method of order 4.

4.3 Description and results of the numerical experiments

In the following figures, the unit on the time axis corresponds to one period of revolution of the exact Kepler orbit, i.e. $2\pi \approx 6.28$ units of t.

We describe two experiments:

A) Long term experiment. For both the classical and the regularized case, we choose a step size such that integration of one whole revolution is accomplished in $10 * 2\pi \approx 63$ integration steps.

This relatively large value of the step size (= 0.1) clearly brings truncation errors to the foreground so that round-off errors are imperceptible.

Fig. 4.1 represents the error behaviour of r and a; the scale factor imposed by the errors in the classical case is such that in the regularized case the error curve for r can hardly be distinguished from the error curve belonging to a.

Fig. 4.2 shows errors of the true anomaly.

B) Short term experiment. In contrast to experiment A), experiment B) is primarily designed for throwing some light on the behaviour of round-off errors.

This was done by choosing a smaller mesh which corresponds to $50 * 2\pi \approx 314$ steps per revolution (step size = 0.02).

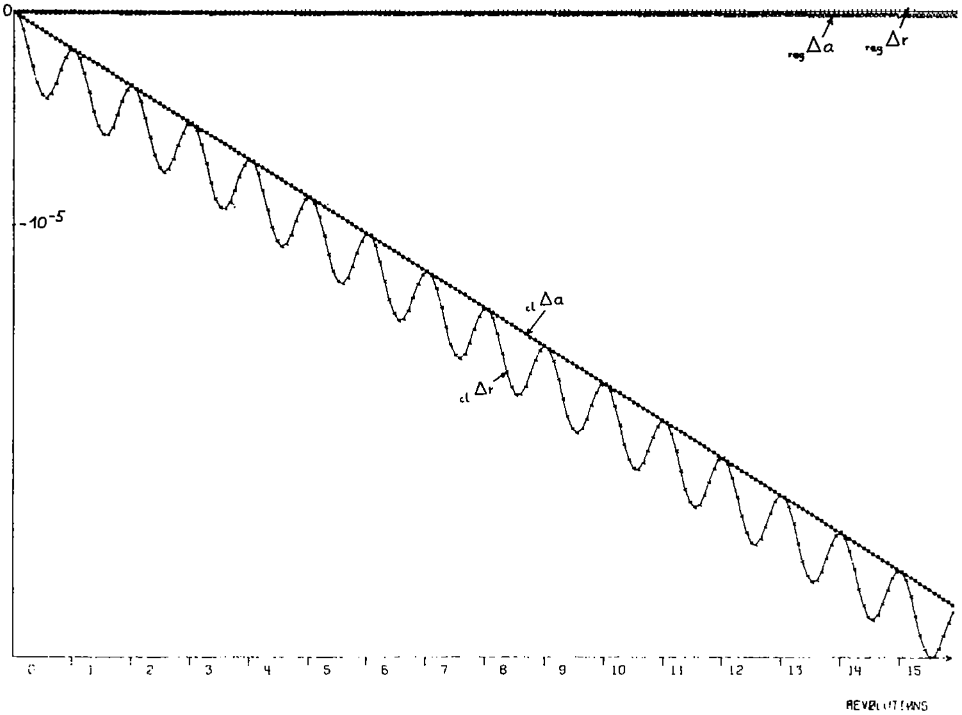
Here again results have been plotted in Fig. 4.3 and Fig. 4.4.

In Fig. 4.4, the curve ${}_{reg}\Delta\varphi$ requires some explanations; the main component of this error is due to the propagation of round-off errors in the integration of the physical time in equation (4,8).

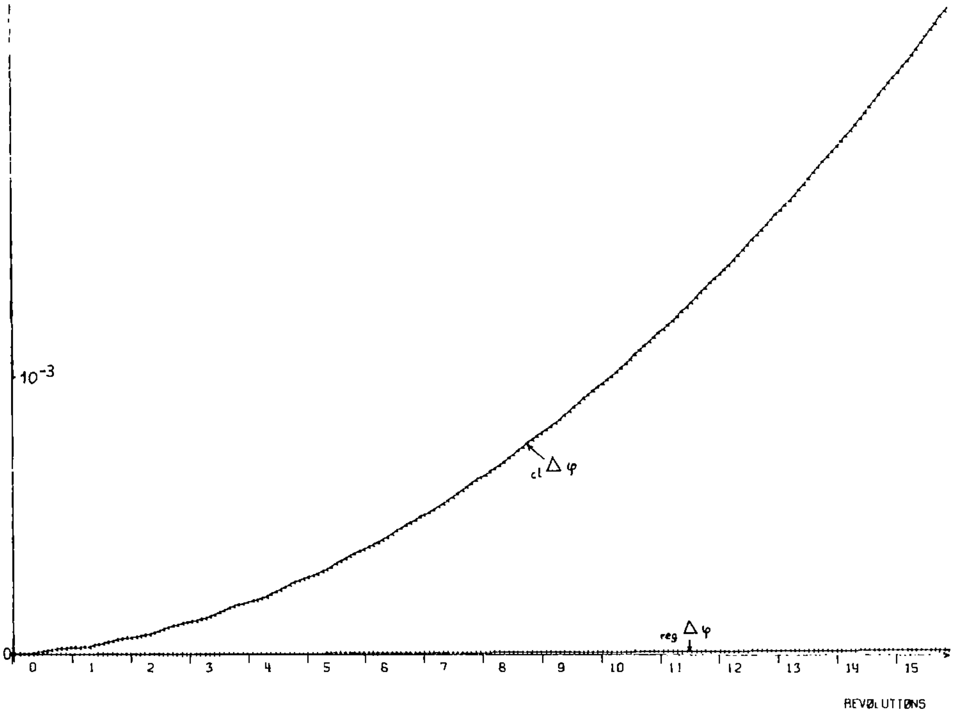
Integration of formula (4,8) with the above mentioned Runge-Kutta method is equivalent to Simpson's rule; for two consecutive values t_n and t_{n+1}, we have a relation of the type

$$ {}_{reg}t_{n+1} = {}_{reg}t_n + h \cdot F(s) \quad , \tag{4,18} $$

where $F(s)$ is a function determined by the numerical method of integration. Looking at the right-hand sides of (4,8) and (4,11) we see that, on account of orbital stability of Kepler motion, the value of $F(s)$ remains very close to 1 and is a smooth function of s. At each integration step the addition at the right-hand side of (4,18) is rounded thus creating a cumulative propagation of round-off errors and thereby erroneous values of ${}_{reg}t$.



<u>Fig</u>. <u>4.1</u>. Long term experiment: Total error in distance and semi-major axis.



<u>Fig</u>. <u>4.2</u>. Long term experiment: Total error in true anomaly.

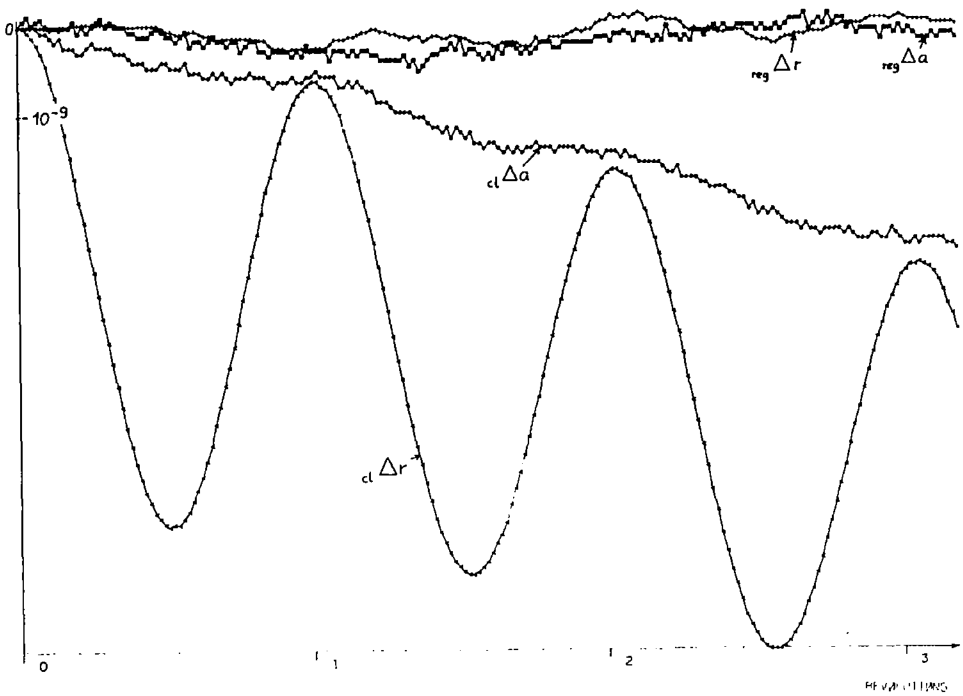


Fig. 4.3. Short term experiment: Total error in distance and semi-major axis.

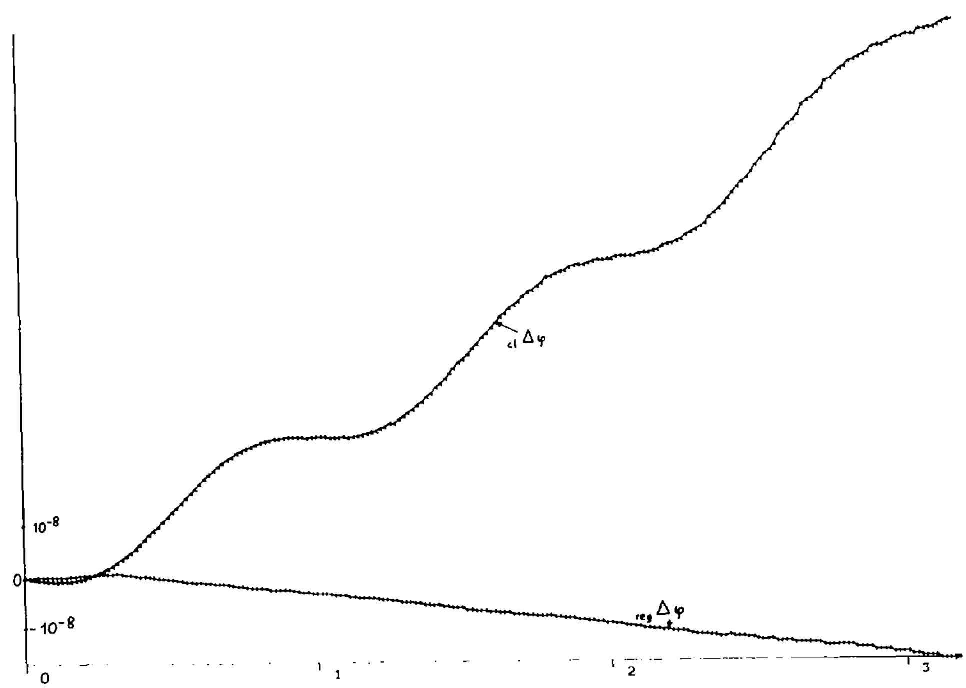


Fig. 4.4. Short term experiment: Total error in true anomaly.

However it should be emphasized that such a propagation of round-off errors while integrating a perturbed Kepler motion, is expected only if the physical time is integrated with formula (1,94) of the second procedure (cf. 1.3.2). This propagation no longer exists if physical time is integrated with the companion procedure of section 1.3.3, since only the perturbation of time is numerically integrated.

4.4 Conclusions

- The above experiments present numerical integrations of the coordinates x_i and consequently do not test the methods developed in chapter 1 and chapter 2 which only require integration of the perturbations of elements α_j , β_j .

- However it has become evident that regularized methods are significantly more stable than classical ones, during numerical integration; experiments have corroborated the theoretical considerations of section 1.7.1 and they show that the advantage of regularization outlined there is more pronounced than expected.

- Further studies (not published here) concerning elliptical orbits show that this behaviour also occurs in such cases; for higher values of the eccentricity, this beneficent tendency becomes even more significant.

- Theoretical investigations on such error behaviours are subject of a forthcoming thesis in which separation of truncation and round-off errors, as well as perturbed motion will be discussed.

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