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EXPERIMENTAL DETERMINATION OF
SYSTEM PARAMETERS FOR
THIN-WALLED CYLINDERS 4

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FOREWORD

This report was prepared by Republic Aviation Division of the Fairchild Hiller Corporation, Farmingdale, New York, under Contract No. NAS 8-11514, "Experimental Determination of System Parameters for Thin-Walled Cylinders."

The work was administered under the direction of the Structures Branch, Propulsion and Vehicle Engineering Laboratory, of the George C. Marshall Space Flight Center, of the National Aeronautics and Space Administration by Mr. J. H. Farrow and Mr. C. E. Lifer, principal and alternate technical representatives, respectively, of the Contracting Officer.

The Republic program manager was Dr. R. S. Levy, Project Engineer, Preliminary Design Section. The analytical work was performed by Dr. Marvin Forray. Laboratory work was performed by Messrs. Elbert Wilcox and Joseph Hoermann. Mr. Julius Port performed an invaluable service in reshaping the analysis to adapt it to computation, then programming and performing the numerical analysis. Mr. Sol Davis was responsible for the correlation of the computer results with test data and for the review of the analysis section (Section III) of this report.

This document is the Third Annual Summary Report and covers the period from 1 July 1965 through 31 August 1966.

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SECTION I - SUMMARY AND CONCLUSIONS

This report describes the third year's work on an investigation directed toward developing methods to predict the dynamic response of a stiffened, pressurized cylinder with attached concentrated-weight items. The work described herein was a coordinated program of analysis and test with emphasis on the problem of predicting the natural frequencies and mode shapes of weighted, pressurized cylinders without stringers.

During the second year of the program, a method of analysis was developed that gave satisfactory results when applied to cylinders with stringers. The method also gave good results for very small weights on cylinders without stringers, but gave poor correlation with test data on such cylinders with large weights. The objective of the current phase was to develop an improved method of analysis for this latter case, and to substantiate the analytical predictions by test data. The test program was conducted on a new cylinder No. 2 similar to the cylinder No. 2 used during the second year's work.

The specific accomplishments achieved are described below.

1) Prediction of Free Vibration Characteristics

A new digital computer program was developed and will be supplied to the George C. Marshall Space Flight Center. This program is documented in the User's Manual, Reference 9; the analytical basis of the program is contained in Section III of the present report. The program is presently capable of computing the resonant frequencies of pressurized cylinders with frames and an attached mass. Mode shapes should also be derivable; however, this part of the program is still in work.

The comparison of analytical predictions of fundamental resonant frequency with test results of the new cylinder No. 2 showed good agreement even for the heavy weights considered. Good correlation was obtained at both mid-bay and quarter-bay locations for a Fourier series order of truncation $N^* = 75$.

2) Improvement in Experimental Techniques

Mechanical impedance measurements were made with a new impedance head (Wilcoxon Model Z13) that had a weight downstream of the of the force gage of only 0.35 gram. The use of the light head reduced the influence of head weight on the test data. Understanding of the operation of the "mass cancellation" circuitry was greatly increased by a special experimental investigation.

The use of the MTI Fotonic Sensor Model KD45 to measure cylinder motion during acoustic noise excitation permitted measurement of the dynamic response of a truly unweighted cylinder for the first time. Shell modes higher than the fundamental were identified and measured for the first time on a weighted cylinder.

Cylinder excitation using the vibration shaker, acoustic horn, and the mallet "thump" gave consistent values of the fundamental resonant frequency.

SECTION II - EXPERIMENTAL PROGRAM

A. OBJECTIVES

The experimental program had four main objectives:

- Investigate the Spectral Dynamics mechanical impedance measuring system, with particular emphasis on the operation and effectiveness of the mass cancellation circuitry.
- Develop refined techniques for obtaining measurements of resonant frequencies of weighted and unweighted cylinders.
- Develop refined techniques for measuring mode shapes of weighted and unweighted cylinders.
- Measure the natural frequencies and mode shapes for a pressurized, ring-stiffened cylinder with concentrated weights of approximately 0.0, 0.1, 0.2, 0.3, and 0.4 pounds at mid-bay and quarter-bay locations.

B. TEST SPECIMEN AND FIXTURES

Tests were conducted upon a new test specimen made in accordance with Figure 1, consisting of an 0.020 gage aluminum cylinder, 36 inches in diameter, 36 inches long, with heavy frames spaced at 6 inches. This new cylinder was a duplicate of cylinder No. 2 which had been tested in the previous year. The new specimen was fabricated in order to eliminate inaccuracy or potential peril which might result from further testing upon the old specimen.

To facilitate testing, it was desirable to eliminate the steel armor that had surrounded the specimen for safety in the previous tests. It was decided that this could be done if a static test indicated adequate reserve strength. Therefore, a hydrostatic pressure test was performed on the old cylinder No. 2. In this test, deflections and permanent set were measured with dial indicators positioned at each of the six mid-bays. The cylinder was supported on one end with the cylinder axis vertical, and filled with water. Test pressure was supplied through the top bulkhead by using the shop air supply with a pressure regulator. Mid-bay deflections and permanent set were measured for test pressures up to 20 psig. After this test, the dial indicators were removed and a burst pressure test was con-

ducted. This test was terminated after the pressure had been slowly increased to 45 psig, at which point the cylinder had not yet burst. The mode of failure that did occur was opening of the splice with ejection of a sizeable stream of water; the most severe gap is shown in Figure 2. Since the hydrostatic test pressure was more than four times the working pressure, it was decided that the safety barricades were unnecessary.

The new cylinder No. 2 was installed in a jig duplicating those used previously (Figure 3). The ends were closed by steel plates, which were supported by the jig in a manner that permitted the cylinder to expand axially. Air pressure was supplied through the reducing valve shown at the right in Figure 3. Pressure was monitored by the gage shown at the left. This gage was later replaced by an inclined tube water manometer to improve sensitivity when it was discovered that the cylinder responses were affected by small changes in pressure.

Figure 4 shows the electrical equipment which is functionally arranged according to the block diagram of Figure 5. The Mosley X-Y plotter that produced the plots of output versus frequency is shown at the left of Figure 4. Next to it is the M-B vibration exciter control panel which supplied the input signal. On top of this is the Spectral Dynamics constant output level adapter which closed the feedback loop. The tall cabinet contained the Spectral Dynamics impedance measuring system, Model SD1002. The oscilloscope shown was used for troubleshooting and was not part of the data acquisition system.

A new piece of equipment which proved valuable in the program is the Fotonic Sensor, Model KD45, made by Mechanical Technology Incorporated, which is shown in Figure 6. This instrument employs fiber optics to obtain extremely sensitive measurements of motion by comparison of light transmitted through some of the fibers with that reflected from the specimen back through the other, initially un-illuminated, fibers. The sensor is shown in the left jaw of the calibration fixture which was used to compare its static measurement directly with that of a micrometer.

When used on test, the simple fixture shown in Figure 7 proved adequate to support the probe. A reference amplitude was picked up directly from the shaker armature as shown in Figure 8.

C. MASS CANCELLATION INVESTIGATION

Mechanical driving-point impedance is the ratio of force to velocity measured at the point of excitation; it is a function of frequency. The data are obtained simultaneously from an impedance head which comprises an accelerometer and a force gage. The accelerometer output can be integrated to produce the velocity signal. The force gage is in the end of the head attached to the specimen and should measure, directly, the force applied. However, there is always some mass between the specimen and the force-sensing element; this would influence the measurements. Spectral Dynamics has included a mass cancellation circuit to remove this error electronically.

During the previous year's work, this circuit was employed and, as discussed in Reference 1, ambiguous results were obtained for the unweighted cylinder case. This was attributed to the sensitivity of the dynamic characteristics of the cylinder to small added weights, and inexact functioning of the mass cancellation circuit. The first phase of the experimental program was directed toward improving the understanding of mass cancellation and toward refining test techniques.

A meeting was held with Mr. Kerfoot of Spectral Dynamics, and the following explanation was obtained:

The mass cancellation circuit employs an amplifier with variable gain producing a signal shifted 180 degrees from the input, as shown in Figure 9. This circuit is theoretically capable of reducing the total force measured ("F") by the amount required to accelerate the downstream weight. The resultant force ("F") should be proportional to the force going into the elastic system being excited. However, due to imperfections and nonlinearities, this cancellation cannot be equally effective over the entire frequency range.

The technique recommended by Mr. Kerfoot and employed during subsequent tests was to attach the impedance head to a shaker and set the mass cancellation at the frequency of interest, prior to attachment of the impedance head to the specimen. This setting is accomplished by varying the gain of the mass cancellation amplifier to produce the minimum force signal.

From a practical point of view, the difficulty associated with mass cancellation was largely circumvented on the cylinder test program by the use of a Wilcoxon Model Z-13 impedance head which has a weight of 0.35 gram below the force link. However, tests were run to explore the operation of the mass cancellation circuitry. Figure 10 shows the result of a test in which a weight of 0.10 pound was added downstream of the impedance head to check the function of the mass cancellation circuit. For this test, the weight was suspended from a pendulum and there was no spring in the system. The upper plot on the figure (Test Run 74) shows the ratio of force to acceleration versus frequency with mass cancellation "out" (circuit deactivated). The mass cancellation circuit was then activated and adjusted to give a minimum force signal at 200 cps. The frequency sweep was repeated and is shown as the lower curve (Test Run 75). It is evident that at 200 cps the F/A ratio is 30 db down. At other frequencies the curve is somewhat erratic but is always well below the reference curve. The ragged nature of the curve may be explained by the fact that the equipment is amplifying what is nearly a null signal and, therefore, the output is very sensitive to noise and nonlinearities.

Another interesting test was conducted employing two weights each of 0.10 pound connected in series, pendulum-mounted, downstream of the force gage. The upper curve of Figure 11 (Test Run 72) shows the ratio of force to acceleration versus frequency for the combined weight of 0.20 pound. The mass cancellation circuit was then activated at the setting for 0.10 pound at 300 cycles per second and the frequency sweep repeated. The lower curve on the figure (Test Run 73) shows the 6 db reduction (in force required for constant acceleration), which should be expected because the mass cancellation cancelled one half of the initial weight.

$$\begin{aligned}
 \text{db} &= 20 \log \left[\frac{(F/A)_{0.2 \text{ lb}}}{(F/A)_{0.1 \text{ lb}}} \right] \\
 &= 20 \log \left[\frac{(M)_{0.2 \text{ lb}}}{(M)_{0.1 \text{ lb}}} \right] \\
 &= 20 \log \left[\frac{0.2}{0.1} \right] = 20 \log 2 \\
 &= 6
 \end{aligned}$$

It is evident that the system functions much more smoothly in this configuration, when the force gage is putting out a reasonable signal level (compare this result with that shown previously in Figure 10 for Test Run 75).

Additional runs were made with and without mass cancellation when the shaker was attached to the pressurized cylinder. These runs were intended to observe the functioning of the mass cancellation circuit when the weight was working against a spring.

Test Run 65 plotted in Figure 12 shows a typical response for a weighted pressurized cylinder without mass cancellation of the attached 0.1-pound weight. A clearly defined resonance appears at about 255 cps. Test Run 64 of the same figure shows how effectively the mass cancellation circuit set up for 0.10-pound attached weight can mask this response. This curve presents convincing evidence that the mass cancellation circuit, properly employed, can achieve the task for which it was designed.

Figure 13 is a plot of the response of a lightly weighted cylinder. For this run the mass cancellation circuit was "Out", which introduced a weight of about 0.01 pounds because of the force link and the plastic button used for cylinder attachment. The sharply defined resonance at 790 cps agrees very well with the value of 800 cps calculated by the method of Reference 2. This is shown in Figure 14 in which the test frequency is superposed on the theoretical curve. Figure 15 is a rerun of the frequency sweep with mass cancellation "In". It may be noted that the response at 790 cps is greatly attenuated.

The initial calibration runs on the new impedance head were performed on the Calidyne Model C-6A 25 force-pound electrodynamic shaker that was used to excite the cylinder for the mid-bay and quarter-bay response tests. All other instrumentation equipment was identical to that used in the calibration of the first impedance head except that the standard accelerometer for the initial calculations was Endevco Model 2224C and the one used for performing the later calibrations was an Endevco Model 2219. The sensitivity of the Model 2219 is approximately thirty times greater than that of the 2224C.

The testing was conducted substantially according to the test plan set forth in Reference 3 with the detailed test procedures described in Reference 4. This plan had been predicated upon the use of a Wilcoxon Model Z602 impedance head

as had previously been used, unless a lighter head could be found. Such a head was discovered; the Wilcoxon Model Z13 has a weight downstream of the force gage equal to 0.35 gram (0.00077 pounds). The use of the light head made unnecessary certain of the preliminary tests aimed at reducing the influence of head weight upon the test data.

Another improvement in testing technique resulted from adaptation of the Fotonic Sensor (Mechanical Technology Incorporated vibration measuring unit, Model KD45). This device proved to be sufficiently sensitive to pick up the higher vibration modes of a weighted cylinder, which theory predicts but were beyond the resolution of other types of instrumentation previously employed. The sensor, which touches the cylinder with only a beam of light, in conjunction with acoustic excitation, permitted measurement of the dynamic response of a truly unweighted cylinder for the first time.

D. EXPERIMENTAL TECHNIQUES FOR DETERMINING NATURAL FREQUENCIES AND MODE SHAPES

On the basis of experience acquired in previous phases of the program and preliminary investigations conducted during the past year, the following techniques were developed for measurement of the resonant frequencies and mode shapes of the cylinder (details are given in Reference 4).

- 1) Fundamental resonant frequencies of weighted cylinders were measured by three independent methods which produced consistent results:
 - Shaker excitation of weight, and observation of dip in F/A curve.
 - Acoustic horn excitation of cylinder at location of weight, and observation of dip in db/A curve.
 - Thumping weight with mallet, and measuring frequency of output of accelerometer attached to weight.
- 2) The frequencies of resonances higher than the fundamental were recorded for weighted cylinders by measuring shell motions, utilizing the Fotonic Sensor during shaker excitation.
- 3) Resonant frequencies of unweighted cylinders were measured using acoustic excitation and measuring responses by the Fotonic Sensor.
- 4) Mode shape measurements were made at resonances by surveys employing the Fotonic Sensor with shaker excitation.

E. SUMMARY OF EXPERIMENTAL RESULTS

Since the voluminous experimental results have been submitted in previous quarterly reports (References 3, 5, and 6) they will not be repeated herein. A typical set of data curves is included for illustrative purposes corresponding to the test condition with a nominal weight of 0.10 pound attached at mid-bay.

Figure 16 is a reproduction of the plot produced by the impedance measuring equipment using shaker excitation at the weight. The F/A signal was employed rather than impedance (F/V) in order to eliminate a possible inaccuracy that might be introduced by integrating the "A" signal to obtain "V". A sharply defined resonance is apparent at 245 cps. The exact value of the frequency was not read from the plot; it was obtained from a digital readout device attached to the signal generator while the F/A signal was minimized by manual manipulation of the signal generator after the approximate location of the resonance had been established from the frequency sweep. This technique was employed for all frequency measurements using the impedance equipment.

Figure 17 is a reproduction of the plot produced by feeding the output of the Fotonic Sensor through the narrow band filters of the impedance equipment. For this run the sensor was located in the plane of the weight, 12 degrees above the weight. A constant acceleration was supplied to the weight, producing the expected linear decrease in the displacement amplitude plotted versus the logarithm of the frequency. When the frequency reached the neighborhood of 1100 cps, abrupt increases in amplitude were recorded. The shell amplitude at these frequencies exceeded the input amplitude at the weight, indicating that true higher resonances had been excited. This type of response with the weight at a node had been predicted by the theory of Reference 2. As mentioned above, the exact values of the resonant frequencies were obtained from digital readout.

Figure 18 presents the results of substituting a signal proportional to the sound pressure level of a four-inch horn for the shaker force signal used in Figure 16. This horn was centered over the weight and was held a small distance away from the cylinder while a sweep was made varying the frequency of the sinusoidal acoustic input. A sharply defined resonance was recorded at 298 cps using

the digital readout. The apparent discrepancy in frequency between this result and that of Figure 16 is attributable to the fact that the actual weight used for the acoustic test (0.077 pound) was considerably lighter than that used for the shaker test (0.092 pound), although they were both nominally 0.10 pound.

Figure 19 presents the output of the Fotonic Sensor located 24 degrees away from the weight. Again, the presence of cylinder resonances above 1000 cps was noted.

Figure 20 is similar to Figure 18 except that a six-inch horn opening was employed instead of the four-inch horn. The resonant frequency was read as 300 cps (which compares very well with the value of 298 cps for the four-inch horn).

Figure 21 differs from Figure 19 in two respects. A six-inch horn was substituted for the four-inch horn. The plot is db/displacement, instead of displacement versus frequency. Therefore, the appearance of shell modes above 1000 cps is evidenced by drops in the curve rather than by increases. Nevertheless, the presence of the resonances is clearly indicated. Again, the approximately linear decrease in the curve prior to resonance is noted, corresponding to the constant sound pressure level maintained during the run.

Figure 22 is a reproduction of a portion of the oscillograph trace produced from the output of an accelerometer built into the weight used for the thump test. This portion of the trace was produced after the weight had been struck a sharp blow by a rawhide mallet. Comparison of the accelerometer record with the superimposed timing trace permitted determination of the resonant frequency. Again, the apparent discrepancy between this frequency and those mentioned above is attributable to the fact that the weight used for this test (0.101 pound) was much heavier than that employed in the shaker and horn tests (although still nominally 0.10 pound).

Figure 23 presents the mode shape measurements for 0.1 pound at mid-bay corresponding to the 245 cps resonant condition established in Figure 16. For these measurements, the shaker was held at the resonant frequency of 245 cps while the relative amplitudes were measured with the Fotonic Sensor. The reference amplitude of 1.00 was picked up from the back end of the shaker (see Figure 8). This

amplitude was taken as the motion of the weight and is plotted as unity in the upper curve at the 180 degree position in mid-bay ($x = 3$ inches). It is also plotted in the central one of the three lower curves at the $x = 3$ position for $\theta = 180$ degrees. The upper curve gives the circumferential mode shape at mid-bay. The lower three curves show mode shapes measured from frame to frame at the weight ($\theta = 180$ degrees), 6 degrees below the weight ($\theta = 174$ degrees), and 6 degrees above the weight ($\theta = 186$ degrees). The sharply cusped nature of the mode shape in the vicinity of the weight is clearly evident.

Figure 24 presents a mode shape measured at one of the higher resonant frequencies for the cylinder with 0.1 pound mounted at mid-bay. No measurement was made at the weight; the curves were normalized by taking the largest measured amplitude as unity (at $x = 3$, $\theta = 204$ degrees). The sharply cusped shape is not present.

Table 1 describes the configurations and lists the frequencies for which mode shapes were measured. These mode shapes are presented in Reference 6.

Table 2 summarizes the fundamental frequencies measured for each weight configuration by each of the various methods used. It is to be noted that the actual values of the test weights as determined by laboratory balances differed, in some cases considerably, from the nominal values. Since the fundamental frequency is strongly dependent upon weight (particularly for small added weights), it was decided that the best presentation would be on a graph of resonant frequency versus actual weight. Such curves are presented as Figures 25 and 26. It may be seen that on this type of presentation the data appear to be quite consistent. It is therefore concluded that the experimentally determined natural frequencies are reliable. A discussion of the comparison with analysis is provided in Subsection III-D of this report.

SECTION III - ANALYSIS PROGRAM

A. ANALYTICAL FORMULATIONS

1. Introduction

The problem to be solved is to determine the natural frequencies and mode shapes of a ring-stiffened pressurized cylinder with a concentrated weight of arbitrary magnitude attached at any longitudinal location between rings. For purposes of analysis the rings are assumed to be sufficiently rigid so that no distortions occur at the ring locations.

The shell configuration, and its displacement components and coordinate systems are shown in Figure 27. It is the objective of the first part of the investigation to determine the equations of motion (small oscillations) as deviations from the pressurized state and the form of the boundary conditions.

While the exact equations of motion may be obtained most directly by integrating the three dimensional equations of motion, it is realized that the boundary conditions would present a problem. Therefore, it was decided to use a variational technique together with the principle of minimum potential energy to deduce the equations of motion and the form of the boundary conditions. To this end, the strain energy of the cylinder is formed

$$2U = \int_x \int_\phi \int_h \left[\sigma_x \epsilon_x + \sigma_\phi \epsilon_\phi + 2\sigma_{x\phi} \epsilon_{x\phi} \right] a d\bar{z} d\phi dx \quad (1)$$

where:

- U = strain energy
- σ_x, σ_ϕ = direct stress components in the x and ϕ directions, respectively
- $\epsilon_x, \epsilon_\phi$ = direct strain components in the x and ϕ directions, respectively
- $\sigma_{x\phi}$ = middle-surface shear stress
- $\epsilon_{x\phi}$ = middle-surface shear strain
- $\bar{z}$ = directed perpendicular distance from shell mid-surface

2. Stress Resultant-Displacement Relations

We employ for the strain-displacement relations Love's first approximation and Donnell type nonlinear terms

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x} - \bar{z} \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \epsilon_\phi &= \frac{\partial v}{a \partial \phi} + \frac{w}{a} - \frac{\bar{z}}{a^2} \left(\frac{\partial^2 w}{\partial \phi^2} - \frac{\partial v}{\partial \phi} \right) + \frac{1}{2a^2} \left(\frac{\partial w}{\partial \phi} \right)^2\end{aligned}\quad (2)$$

$$\epsilon_{x\phi} = \frac{1}{2} \left[\left(\frac{\partial u}{a \partial \phi} + \frac{\partial v}{\partial x} \right) - \frac{2\bar{z}}{a} \left(\frac{\partial^2 w}{\partial x \partial \phi} - \frac{\partial v}{\partial x} \right) + \frac{1}{a} \frac{\partial w}{\partial \phi} \cdot \frac{\partial w}{\partial x} \right]$$

For convenience, introduce the stress and moment resultants

$$\begin{aligned}N_x &= \int_h \sigma_x d\bar{z} = \frac{Eh}{1-\nu^2} (\epsilon_x^0 + \nu \epsilon_\phi^0) \\ N_\phi &= \int_h \sigma_\phi d\bar{z} = \frac{Eh}{1-\nu^2} (\epsilon_\phi^0 + \nu \epsilon_x^0) \\ N_{x\phi} &= \int_h \sigma_{x\phi} d\bar{z} = 2Gh \epsilon_{x\phi}^0 = \frac{Eh}{1+\nu} \epsilon_{x\phi}^0\end{aligned}\quad (3)$$

where ν = Poisson's ratio, E = Young's modulus and where the super zero refers to mid-surface strains, e.g., $\epsilon_x^0 = (\epsilon_x)_{\bar{z}=0}$.

Similarly,

$$\begin{aligned}M_x &= \int_h \sigma_x \bar{z} d\bar{z} = -D(\chi_x + \nu \chi_\phi) \\ M_\phi &= \int_h \sigma_\phi \bar{z} d\bar{z} = -D(\chi_\phi + \nu \chi_x) \\ M_{x\phi} &= \int_h \sigma_{x\phi} \bar{z} d\bar{z} = -D(1-\nu) \chi_{x\phi}\end{aligned}\quad (4)$$

where $D = Eh^3/12(1 - \nu^2)$ and

$$\chi_x = \frac{\partial^2 w}{\partial x^2}, \quad \chi_\phi = \left(\frac{\partial^2 w}{\partial \phi^2} - \frac{\partial v}{\partial \phi} \right) \frac{1}{a}, \quad \chi_{x\phi} = \left(\frac{\partial^2 w}{\partial x \partial \phi} - \frac{\partial v}{\partial x} \right) \frac{1}{a} \quad (5)$$

are the expressions for the curvatures and twist.

The strain energy may be expressed in terms of the resultants:

$$2U = \frac{1}{Eh} \int_x \int_\phi \left[N_x^2 + N_\phi^2 - 2\nu N_x N_\phi + 2(1+\nu) N_{x\phi}^2 \right] a d\phi dx \\ + \frac{1}{D(1-\nu^2)} \int_x \int_\phi \left[M_x^2 + M_\phi^2 - 2\nu M_x M_\phi + 2(1+\nu) M_{x\phi}^2 \right] a d\phi dx \quad (6)$$

Let

$$u = \bar{u} + u' \\ v = \bar{v} + v' \\ w = \bar{w} + w' \quad (7)$$

where

$$\bar{u}, \bar{v}, \bar{w} = \text{static displacement field due to pressure} \\ u', v', w' = \text{dynamic component of displacement field.}$$

We may write

$$N_x = \bar{N}_x + N_x' + \frac{Eh}{2(1-\nu^2)} \left[\left(\frac{\partial w'}{\partial x} \right)^2 + \frac{\nu}{a^2} \left(\frac{\partial w'}{\partial \phi} \right)^2 \right] \\ N_\phi = \bar{N}_\phi + N_\phi' + \frac{Eh}{2(1-\nu^2)} \left[\left(\frac{\partial w'}{\partial \phi} \right)^2 \frac{1}{a^2} + \nu \left(\frac{\partial w'}{\partial x} \right)^2 \right] \\ N_{x\phi} = \bar{N}_{x\phi} + N_{x\phi}' + \frac{Eh}{2(1+\nu)} \frac{\partial w'}{\partial x} \cdot \frac{\partial w'}{a \partial \phi} \quad (8)$$

where

$$\bar{N}_x = \frac{pa}{2}$$

$$\bar{N}_\phi = pa \quad \text{Assumed membrane resultants due to static displacement field}$$

$$\bar{N}_{x\phi} \equiv 0$$

$$N_x' = \frac{Eh}{1-\nu^2} \left[\frac{\partial u'}{\partial x} + \nu \left(\frac{\partial v'}{a \partial \phi} + \frac{w'}{a} \right) \right]$$

$$N_\phi' = \frac{Eh}{1-\nu^2} \left[\frac{1}{a} \left(\frac{\partial v'}{\partial \phi} + w' \right) + \nu \frac{\partial u'}{\partial x} \right]$$

$$N_{x\phi}' = \frac{Eh}{2(1+\nu)} \left[\frac{\partial u'}{a \partial \phi} + \frac{\partial v'}{\partial x} \right]$$

Similarly, since $\bar{M}_x = \bar{M}_\phi = \bar{M}_{x\phi} = 0$ (because of the assumed initial membrane state) it follows that

$$M_x = M_x', \quad M_\phi = M_\phi', \quad M_{x\phi} = M_{x\phi}' \quad (9)$$

The assumption of an initial membrane stress state throughout the cylinder is invalid in the neighborhood of the ring frames where local, rapidly attenuating shell bending stresses occur. Thus, if the mass location were to lie in this "disturbance" region, the stiffening effect of internal pressurization might not be adequately accounted for. It is shown in Reference 7 that shell bending effects are attenuated to less than 20% of their peak values at longitudinal distances from the frames greater than $2\sqrt{ah} / \sqrt[4]{3(1-\nu^2)}$ which, for the test cylinder under consideration, amounts to .9 inches. The analysis should therefore be applied to masses located inside of this region.

Substituting equations (8) and (9) into the expression for the strain energy (6) and neglecting terms higher than quadratic in the prime quantities and disregarding linear terms which will correspond to the membrane solution (which we know), there results

$$\begin{aligned}
2U = & \frac{1}{Eh} \iint_{\phi x} \left\{ (N_x')^2 + (N_\phi')^2 - 2\nu N_x' N_\phi' + 2(1+\nu) (N_{x\phi}')^2 \right. \\
& + Eh \left[\bar{N}_x \left(\frac{\partial w'}{\partial x} \right)^2 + \frac{\bar{N}_\phi}{a^2} \left(\frac{\partial w'}{\partial \phi} \right)^2 + \frac{2}{a} \left(\frac{\partial w'}{\partial x} \cdot \frac{\partial w'}{\partial \phi} \right) \bar{N}_{x\phi} \right] a dx d\phi \\
& \left. + \frac{1}{D(1-\nu^2)} \iint_{\phi x} \left[(M_x')^2 + (M_\phi')^2 - 2\nu M_x' M_\phi' + 2(1+\nu) (M_{x\phi}')^2 \right] a dx d\phi \right\} \quad (10)
\end{aligned}$$

Assuming workless external forces, set the first variation of the strain energy equal to zero; there result

- (i) Equations of equilibrium
- (ii) Boundary conditions

3. Equations of Equilibrium

The equilibrium equations of the present theory are :

$$\begin{aligned}
\frac{\partial N_x'}{\partial x} + \frac{\partial N_{x\phi}'}{a \partial \phi} &= 0 \\
\frac{\partial N_\phi'}{a \partial \phi} + \frac{\partial N_{x\phi}'}{\partial x} + \frac{1}{a^2} \frac{\partial M_\phi'}{\partial \phi} + \frac{2 \partial M_{x\phi}'}{a \partial x} &= 0 \\
\frac{\partial^2 M_x'}{\partial x^2} + \frac{2 \partial^2 M_{x\phi}'}{a \partial x \partial \phi} + \frac{\partial^2 M_\phi'}{a^2 \partial \phi^2} - \frac{N_\phi'}{a} + \bar{N}_x \frac{\partial^2 w'}{\partial x^2} + \bar{N}_\phi \frac{\partial^2 w'}{a^2 \partial \phi^2} &= 0
\end{aligned} \quad (11)$$

It is interesting and important to compare (11) with the exact equations of equilibrium found by integrating the three dimensional equations through the shell thickness (Reference 8):

$$\begin{aligned}
\frac{\partial N_x'}{\partial x} + \frac{\partial N_{x\phi}'}{a \partial \phi} - \frac{\bar{N}_\phi}{a} \left[\frac{\partial^2 v}{\partial x \partial \phi} + \frac{\partial w}{\partial x} \right] &= 0 \\
\frac{\partial N_\phi'}{a \partial \phi} + \frac{\partial N_{x\phi}'}{\partial x} + \frac{1}{a^2} \frac{\partial M_\phi'}{\partial \phi} + \frac{\partial M_{x\phi}'}{a \partial x} + \bar{N}_x \frac{\partial^2 v}{\partial x^2} &= 0 \\
\frac{\partial^2 M_x'}{\partial x^2} + \frac{2 \partial^2 M_{x\phi}'}{a \partial x \partial \phi} + \frac{\partial^2 M_\phi'}{a^2 \partial \phi^2} - \frac{N_\phi'}{a} + \bar{N}_x \frac{\partial^2 w'}{\partial x^2} + \bar{N}_\phi \left[-\frac{\partial v'}{a^2 \partial \phi} + \frac{1}{a^2} \frac{\partial^2 w'}{\partial \phi^2} \right] &= 0
\end{aligned} \quad (12)$$

Note that there are two kinds of differences when a comparison of (11) with (12) is made:

(a) Differences due to type of nonlinearity in the strain-displacement relations (for example there is a term $-\frac{\bar{N}}{2} \phi \frac{\partial v'}{\partial \phi}$ which appears in the last equation of (12) but does not appear in the corresponding equation of (11)). It is anticipated that the nonlinear terms of the Donnell type in the strain displacement relations (2) should yield frequencies and mode shapes with satisfactory accuracy.

(b) The corresponding term to $\frac{1}{a} \frac{\partial M'_{x\phi}}{\partial x}$ in (11) is $\frac{2}{a} \frac{\partial M'_{x\phi}}{\partial x}$ in (12). This is an inconsistency of the variational principle (when applied to the relations (2)) with the exact theory. Since our primary interest in the variational principle is to determine the form of the boundary conditions, we shall utilize the exact equation of equilibrium by using $\frac{1}{a} \frac{\partial M'_{x\phi}}{\partial x}$ instead of $\frac{2}{a} \frac{\partial M'_{x\phi}}{\partial x}$ and we shall also correspondingly modify the boundary conditions when this term appears.

4. Equations of Motion

With the addition of the inertia terms we have the equations of motion :

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{x\phi}}{a \partial \phi} - \rho h \ddot{u} &= 0^\dagger \\ \frac{1}{a} \frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} + \frac{1}{2} \frac{\partial M_\phi}{\partial \phi} + \frac{\partial M_{x\phi}}{a \partial x} - \rho h \ddot{v} &= 0 \\ \frac{\partial^2 M_x}{\partial x^2} + \frac{2}{a} \frac{\partial^2 M_{x\phi}}{\partial x \partial \phi} + \frac{\partial^2 M_\phi}{a^2 \partial \phi^2} - \frac{N_\phi}{a} + \bar{N}_x \frac{\partial^2 w}{\partial x^2} + \bar{N}_\phi \frac{\partial^2 w}{a^2 \partial \phi^2} - \rho h \ddot{w} &= 0 \end{aligned} \quad (13)$$

where $\ddot{} \equiv \frac{\partial^2}{\partial t^2}$ and ρ = mass density of the shell

Substitution of (2) and (4) into (13) results in the following system of equations.

[†] For simplicity of notation we drop the primes from the stress resultants, moment resultants and displacements.

$$\begin{aligned}
u_{xx} + \frac{1-\nu}{2a^2} u_{\phi\phi} + \frac{1+\nu}{2a} v_{x\phi} + \frac{\nu}{a} w_x - \rho \frac{(1-\nu^2)}{E} \ddot{u} &= 0 \\
\frac{1+\nu}{2} u_{x\phi} + \frac{(1-\nu)a}{2} v_{xx} + \frac{1}{a} (v_{\phi\phi} + w_{\phi}) + \frac{h^2}{12a} \left[\frac{1}{2} (v_{\phi\phi} - w_{\phi\phi\phi}) - w_{xx\phi} + (1-\nu) v_{xx} \right] \\
- \rho \frac{a(1-\nu^2)}{E} \ddot{v} &= 0
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{ah^2}{12} \nabla^4 w - \frac{(2-\nu)h^2}{12a} v_{\phi xx} - \frac{h^2}{12a^3} v_{\phi\phi\phi} \\
+ \left[\frac{1}{a} (v_{\phi} + w) + \nu u_x \right] - \frac{a(1-\nu^2)}{Eh} \bar{N}_x w_{xx} - \frac{(1-\nu^2)}{Eha} \bar{N}_{\phi} w_{\phi\phi} + \frac{a\rho(1-\nu^2)}{E} \ddot{w} &= 0
\end{aligned}$$

Let $u(x, \phi, t) = u(x, \phi) \sin pt$
 $v(x, \phi, t) = v(x, \phi) \sin pt$
 $w(x, \phi, t) = w(x, \phi) \sin pt$

(15)

be substituted into (14) and the displacement equations of motion become

$$\begin{aligned}
a u_{xx} + \frac{1-\nu}{2a} u_{\phi\phi} + \frac{1+\nu}{2} v_{x\phi} + \nu w_x + k u &= 0 \\
\frac{1+\nu}{2} u_{x\phi} + \frac{1-\nu}{2} a v_{xx} + \frac{1}{a} (v_{\phi\phi} + w_{\phi}) \\
+ \frac{h^2}{12a} \left[\frac{1}{2} (v_{\phi\phi} - w_{\phi\phi\phi}) - w_{xx\phi} + (1-\nu) v_{xx} \right] + k v &= 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{ah^2}{12} \nabla^4 w - \frac{(2-\nu)h^2}{12a} v_{\phi xx} - \frac{h^2}{12a^3} v_{\phi\phi\phi} + \left[\frac{1}{a} (v_{\phi} + w) + \nu u_x \right] \\
- \frac{a(1-\nu^2)}{Eh} \bar{N}_x w_{xx} - \frac{1-\nu^2}{Eha} \bar{N}_{\phi} w_{\phi\phi} - k w &= 0
\end{aligned}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2}{a^2} \frac{\partial^4}{\partial x^2 \partial \phi^2} + \frac{\partial^4}{a^4 \partial \phi^4} \tag{17}$$

and

$$k = \frac{\rho a(1-\nu^2)p^2}{E} \text{ (in.}^{-1}\text{)} \quad (18)$$

5. Boundary and Continuity Conditions

The system of equations (16) must be solved subject to boundary and jump conditions at the section through the mass.

The boundary conditions at the cylinder ends are

$$[u]_{x=L} = [v]_{x=L} = [w]_{x=L} = 0 \quad (19)$$

$$[\alpha_1 w_{xx} + \beta_1 w_x]_{x=L} = 0 \quad (20)$$

and

$$[u]_{x=0} = [v]_{x=0} = [w]_{x=0} = 0 \quad (21)$$

$$[\alpha_2 w_{xx} + \beta_2 w_x]_{x=0} = 0 \quad (22)$$

where

α_1 , β_1 , α_2 and β_2 are constants depending on the boundary conditions.

Equations (19) and (21) follow from the assumption that there are no displacements at the cylinder ends while (20) and (22) permit the prescription of rotations or moments there. These assumptions are incompatible with "beam-bending" type trial functions (functions corresponding to one circumferential wave, i.e., $n=1$), for which rigid frames would experience rigid body displacements and rotations. However, during the analytical and experimental investigations of Ref. 2., it was discovered that "beam-bending" effects contribute negligibly toward the fundamental modal response of a weighted cylinder. Since the weighted mode shape is highly localized in nature, the major contributions come from trial functions consisting of many circumferential waves (corresponding to a large circumferential wave number, n). Frame displacements associated with these trial functions, which would consist of non-rigid body distortions, are negligibly small because of the high frame rigidities as compared to the shell.

The jump or continuity conditions at the section $x = b$ containing the mass are as follows:

$$\begin{bmatrix} u \end{bmatrix}_{b_-}^{b_+} = 0, \quad \begin{bmatrix} v \end{bmatrix}_{b_-}^{b_+} = 0, \quad \begin{bmatrix} w \end{bmatrix}_{b_-}^{b_+} = 0 \quad (23)$$

$$\begin{bmatrix} \frac{\partial w}{\partial x} \end{bmatrix}_{b_-}^{b_+} = 0 \quad (24)$$

$$\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \end{bmatrix}_{b_-}^{b_+} = 0 \quad (25)$$

Conditions (23) state that the displacements u , v , and w are continuous at $x = b$; (24) states the requirement of continuity of slope, while (25) follows from Newton's 3rd law of action and reaction neglecting the rotatory inertia of the mass.

The equilibrium conditions at the mass location for the u , v , w directions may be expressed:

$$-D \left[w_{xxx} + \frac{2-\nu}{a^2} (w_{x\phi\phi} - v_{x\phi}) \right]_{b_-}^{b_+} + f(\phi) = 0 \quad (26)$$

$$\frac{Eh}{2(1+\nu)} \left[\frac{u}{a} + \left(1 + \frac{h^2}{6a^2}\right) v_x - \frac{h^2}{6a^2} w_{x\phi} \right]_{b_-}^{b_+} = 0 \quad (27)$$

$$\frac{Eh}{1-\nu} \left[u_x + \frac{\nu}{a} (v_\phi + w) \right]_{b_-}^{b_+} + h(\phi) = 0 \quad (28)$$

where

$f(\phi)$ = inertia force per unit of circumferential length in the w direction - lb/in
 $h(\phi)$ = inertia force per unit of circumferential length in the u direction - lb/in
 m = attached mass - lb-sec²/in

$$f(\phi) \begin{cases} = \frac{mp^2}{2a\phi_0} \begin{bmatrix} w \end{bmatrix}_{\phi=0}^{x=b} & |\phi| \leq \phi_0 \\ = 0 & \phi_0 < |\phi| < \pi \end{cases} \quad (29)$$

and

$$h(\phi) \begin{cases} = \frac{m p^2}{2a\phi_0} \left[u \right]_{x=b, \phi=0} & |\phi| \leq \phi_0 \\ = 0 & \phi_0 < |\phi| < \pi \end{cases} \quad (30)$$

Equations (26) through (28) are expressions of Newton's second law ($\vec{F} = m\vec{a}$) and the terms which enter into these relations are derived from the variational procedure. The mass is assumed to be spread uniformly in section $x = b$ over an arc subtending a central angle of $2\phi_0$ so that $\frac{m}{2a\phi_0}$ is the mass per unit of circumferential length, $-\phi_0 \leq \phi \leq +\phi_0$. It is further stipulated that the distributed mass acts like a rigid particle with displacement field $[u, v, w]_{x=b, \phi=0}$. For example, (28) and (30) are explained in the following manner.

Consider an element made up of sections $x = b_+$, $x = b_-$ and a unit hoop length. Therefore from Newton's second law, for $|\phi| \leq \phi_0$ we have

$$N_x \Big|_{b_+} - N_x \Big|_{b_-} \begin{cases} = \frac{m}{2a\phi_0} \ddot{u} & -\phi_0 \leq \phi \leq \phi_0 \\ = 0 & \phi_0 < |\phi| < \pi \end{cases}$$

Then from the expression for N_x in terms of the displacements, the results follow.

Equation (27) represents dynamic equilibrium in the v or hoop direction. There is no jump in the reduced shear (parallel to shell mid-surface) because $[v]_{x=b, \phi=0} = 0$ for symmetrical modes of vibration. Finally (26) is a statement of dynamic equilibrium in the direction of w .

Equations (26) through (28) are further simplified by noting that

$$[w_{x\phi\phi}]_{b_-}^{b_+} = [u_\phi]_{b_-}^{b_+} = [w_{x\phi}]_{b_-}^{b_+} = [v_\phi]_{b_-}^{b_+} = [w]_{b_-}^{b_+} = 0, \quad (31)$$

since, for example $[w_x]_{b_-}^{b_+} = 0$ implies that $[w_{x\phi}]_{b_-}^{b_+} = [w_{x\phi\phi}]_{b_-}^{b_+} = 0$.

But from (27) and $\left[\begin{smallmatrix} u \\ \phi \end{smallmatrix} \right]_{b_-}^{b_+} = \left[\begin{smallmatrix} w_{x\phi} \end{smallmatrix} \right]_{b_-}^{b_+} = 0$ we have $\left[\begin{smallmatrix} v_x \end{smallmatrix} \right]_{b_-}^{b_+} = 0$ which in turn implies that $\left[\begin{smallmatrix} v_{x\phi} \end{smallmatrix} \right]_{b_-}^{b_+} = 0$.

Therefore (26) through (28) may be replaced by the simpler jump conditions:

$$-D \left[\begin{smallmatrix} w_{xxx} \end{smallmatrix} \right]_{b_-}^{b_+} + f(\phi) = 0 \quad (26a)$$

$$\left[\begin{smallmatrix} v_x \end{smallmatrix} \right]_{b_-}^{b_+} = 0 \quad (27a)$$

$$\frac{Eh}{1-\nu^2} \left[\begin{smallmatrix} u_x \end{smallmatrix} \right]_{b_-}^{b_+} + h(\phi) = 0 \quad (28a)$$

6. Solution of the Boundary Value Problem

To solve the system of partial differential equations (16) subject to the boundary and jump conditions, let

$$\begin{aligned} u(x, \phi) &= \sum_{n=0}^{\infty} u_n(x) \cos n\phi \\ v(x, \phi) &= \sum_{n=0}^{\infty} v_n(x) \sin n\phi \\ w(x, \phi) &= \sum_{n=0}^{\infty} w_n(x) \cos n\phi \end{aligned} \quad (32)$$

be substituted into the system. The equations will be satisfied termwise if u_n , v_n , w_n satisfy the following homogeneous set of equations:

$$\begin{aligned}
& a u_n'' - \frac{n^2(1-\nu)}{2a} u_n + \frac{n(1+\nu)}{2} v_n' + \nu w_n' + k u_n = 0 \\
& - \frac{n(1+\nu)u_n'}{2} + \frac{(1-\nu)av_n''}{2} - \frac{n}{a} (n v_n + w_n) \\
& + \frac{h^2}{12a} \left[-\frac{n^2}{a^2} (v_n - n w_n) + n w_n'' + (1-\nu) v_n'' \right] + k v_n = 0 \\
& \frac{ah^2}{12} \left[w_n''' - \frac{2n^2}{a^2} w_n'' + \frac{n^4}{a^4} w_n \right] - \frac{(2-\nu)h^2 n v_n''}{12a} \\
& + \frac{h^2 n^3}{12a^3} v_n + \left[\frac{1}{a} (n v_n + w_n) + \nu u_n' \right] - \frac{a(1-\nu^2)}{Eh} \bar{N}_x w_n'' \\
& + \frac{(1-\nu^2)n^2 \bar{N}_\phi w_n}{Eha} - k w_n = 0
\end{aligned} \tag{33}$$

where $' = \frac{d}{dx}$, etc.

Assume a solution of the system (33) in the form

$$\begin{aligned}
u_n &= A_n e^{\lambda_n x} \\
v_n &= B_n e^{\lambda_n x} \\
w_n &= C_n e^{\lambda_n x}
\end{aligned} \tag{34}$$

Then direct substitution yields three linear homogeneous equations in three unknowns A_n , B_n , C_n . This system will possess a solution if and only if the following determinantal equation is satisfied:

$$\begin{aligned}
& a\lambda_n^2 + k - \frac{n^2(1-\nu)}{2a} & \frac{n(1+\nu)\lambda_n}{2} & \nu\lambda_n \\
& - \frac{n(1+\nu)\lambda_n}{2} & \frac{(1-\nu)a}{2} \left(1 + \frac{h^2}{6a^2}\right) \lambda_n^2 & \frac{nh^2}{12a} \lambda_n^2 - \frac{n}{a} \left(1 + \frac{h^2 n^2}{12a^2}\right) \\
& & - \frac{n^2}{a} \left(1 + \frac{h^2}{12a^2}\right) + k & \\
& \nu\lambda_n & - \frac{(2-\nu)h^2 n \lambda_n^2}{12a} + \frac{h^2 n^3}{12a^3} + \frac{n}{a} & \frac{ah^2}{12} \left[\lambda_n^4 - \frac{2n^2}{a^2} \lambda_n^2 + \frac{n^4}{a^4} \right] \\
& & & + \frac{1}{a} - \frac{a(1-\nu^2)}{Eh} \bar{N}_x \lambda_n^2 \\
& & & + \frac{(1-\nu^2)n^2}{Eha} \bar{N}_\phi - k
\end{aligned}
= 0 \quad (35)$$

The expansion of (35) yields a polynomial of degree 8 in λ_n which is a quartic in λ_n^2 . We denote the 8 roots by λ_{ni} ($i = 1, 2, \dots, 8$). Three important facts should be noted:

(i) Since the coefficients of the polynomial in λ_n must be real, the roots of (35) must be real or appear as conjugate complex pairs.

(ii) The frequency p must be specified in order to determine λ_{ni} numerically. In actuality, p is unknown and this suggests that an appropriate iterative scheme could be utilized starting with "guessed" frequencies.

(iii) The ratios of $\frac{A_n}{C_n}$ and $\frac{B_n}{C_n}$ are known quantities. We write

$$\begin{aligned}\frac{A_n}{C_n} &= \mu_n \\ \frac{B_n}{C_n} &= r_n\end{aligned}\tag{36}$$

and consequently

$$\begin{aligned}u_n &= \sum_{i=1}^8 \mu_{ni} C_{ni} e^{\lambda_{ni} x} \\ v_n &= \sum_{i=1}^8 r_{ni} C_{ni} e^{\lambda_{ni} x} \\ w_n &= \sum_{i=1}^8 C_{ni} e^{\lambda_{ni} x}\end{aligned}\quad 0 \leq x \leq b\tag{37}$$

while in the interval (b, L) ,

$$\begin{aligned}u_n &= \sum_{i=1}^8 \mu_{ni} C'_{ni} e^{\lambda_{ni} x} \\ v_n &= \sum_{i=1}^8 r_{ni} C'_{ni} e^{\lambda_{ni} x} \\ w_n &= \sum_{i=1}^8 C'_{ni} e^{\lambda_{ni} x}\end{aligned}\quad b \leq x \leq L\tag{38}$$

Therefore from (32)

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \sum_{i=1}^8 \mu_{ni} C_{ni} e^{\lambda_{ni} x} \cos n\phi \\ v &= \sum_{n=0}^{\infty} \sum_{i=1}^8 r_{ni} C_{ni} e^{\lambda_{ni} x} \sin n\phi \end{aligned} \quad 0 \leq x \leq b \quad (39)$$

$$w = \sum_{n=0}^{\infty} \sum_{i=1}^8 C_{ni} e^{\lambda_{ni} x} \cos n\phi$$

and

$$\begin{aligned} u &= \sum_{n=0}^{\infty} \sum_{i=1}^8 \mu_{ni} C'_{ni} e^{\lambda_{ni} x} \cos n\phi \\ v &= \sum_{n=0}^{\infty} \sum_{i=1}^8 r_{ni} C'_{ni} e^{\lambda_{ni} x} \sin n\phi \\ w &= \sum_{n=0}^{\infty} \sum_{i=1}^8 C'_{ni} e^{\lambda_{ni} x} \cos n\phi \end{aligned} \quad b \leq x \leq L \quad (40)$$

where the C_{ni} and C'_{ni} must be determined for $i=1, 2, \dots, 8$; $n=0, 1, 2, \dots$. To find C_{ni} and C'_{ni} the series appearing in (39) and (40) must be truncated. Let N denote the index or order of truncation i.e., $n=0, 1, 2, \dots, N$. Thus

$$\left. \begin{aligned} u_N &= \sum_{n=0}^N \sum_{i=1}^8 \mu_{ni} C_{ni} e^{\lambda_{ni} x} \cos n\phi \\ v_N &= \sum_{n=0}^N \sum_{i=1}^8 r_{ni} C_{ni} e^{\lambda_{ni} x} \sin n\phi \\ w_N &= \sum_{n=0}^N \sum_{i=1}^8 C_{ni} e^{\lambda_{ni} x} \cos n\phi \end{aligned} \right\} \quad 0 \leq x \leq b \quad (41)$$

and

$$\left. \begin{aligned} u_N &= \sum_{n=0}^N \sum_{i=1}^8 \mu_{ni} C'_{ni} e^{\lambda_{ni} x} \cos n\phi \\ v_N &= \sum_{n=0}^N \sum_{i=1}^8 r_{ni} C'_{ni} e^{\lambda_{ni} x} \sin n\phi \\ w_N &= \sum_{n=0}^N \sum_{i=1}^8 C'_{ni} e^{\lambda_{ni} x} \cos n\phi \end{aligned} \right\} b \leq x \leq L \quad (42)$$

Substitution of (41) and (42) into the boundary and jump conditions exclusive of the jump conditions (26a) and (28a) yields a system of 14 linear equations in 16 unknowns for each n :

$$\begin{aligned} \sum_{i=1}^8 \mu_{ni} C'_{ni} e^{\lambda_{ni} L} &= 0 \\ \sum_{i=1}^8 r_{ni} C'_{ni} e^{\lambda_{ni} L} &= 0 \\ \sum_{i=1}^8 C'_{ni} e^{\lambda_{ni} L} &= 0 \\ \sum_{i=1}^8 (\alpha_1 \lambda_{ni}^2 + \beta_1 \lambda_{ni}) C'_{ni} e^{\lambda_{ni} L} &= 0 \\ \sum_{i=1}^8 \mu_{ni} C_{ni} &= 0 \\ \sum_{i=1}^8 r_{ni} C_{ni} &= 0 \\ \sum_{i=1}^8 C_{ni} &= 0 \end{aligned} \quad (43)$$

$$\begin{aligned}
\sum_{i=1}^8 (\alpha_2 \lambda_{ni}^2 + \beta_2 \lambda_{ni}) C_{ni} &= 0 & \left([\alpha_2 w_{xx} + \beta_2 w_x]_{x=0} = 0 \right) \\
\sum_{i=1}^8 \mu_{ni} (C_{ni} - C'_{ni}) e^{\lambda_{ni} b} &= 0 & ([u]_{b-}^{b+} = 0) \\
\sum_{i=1}^8 r_{ni} (C_{ni} - C'_{ni}) e^{\lambda_{ni} b} &= 0 & ([v]_{b-}^{b+} = 0) \\
\sum_{i=1}^8 (C_{ni} - C'_{ni}) e^{\lambda_{ni} b} &= 0 & ([w]_{b-}^{b+} = 0) \\
\sum_{i=1}^8 \lambda_{ni} (C_{ni} - C'_{ni}) e^{\lambda_{ni} b} &= 0 & \left(\left[\frac{\partial w}{\partial x} \right]_{b-}^{b+} = 0 \right) \\
\sum_{i=1}^8 \lambda_{ni}^2 (C_{ni} - C'_{ni}) e^{\lambda_{ni} b} &= 0 & \left(\left[\frac{\partial^2 w}{\partial x^2} \right]_{b-}^{b+} = 0 \right) \\
\sum_{i=1}^8 \lambda_{ni} r_{ni} (C_{ni} - C'_{ni}) e^{\lambda_{ni} b} &= 0 & ([v_x]_{b-}^{b+} = 0)
\end{aligned} \tag{43}$$

cont'd

System (43) can be solved for C_{ni} and C'_{ni} ($i = 2, \dots, 8$) in terms of C_{n1} and C'_{n1} where $n = 0, 1, \dots, N$. Therefore, there are essentially 2 $(N + 1)$ unknowns C_{n1} and C'_{n1} , $n = 0, 1, \dots, N$.

The remaining jump conditions (26a) and (28a) with a Fourier cosine series expansion for $f(\phi)$ and $h(\phi)$ results in the following system of equations:

$$\left. \begin{aligned} \sum_{i=1}^8 -D \left[\lambda_{ni}^3 (C'_{ni} - C_{ni}) - \frac{2-\nu}{a^2} n \lambda_{ni} (C'_{ni} - C_{ni}) \right] e^{\lambda_{ni} b} + \gamma_n &= 0 \\ \frac{Eh}{1-\nu} \sum_{i=1}^8 \lambda_{ni} \mu_{ni} (C'_{ni} - C_{ni}) e^{\lambda_{ni} b} + \tau_n &= 0 \end{aligned} \right\} \quad (44)$$

where

$$\left. \begin{aligned} \gamma_0 &= \frac{mp^2}{2\pi a} \sum_{j=0}^N \sum_{i=1}^8 C_{ji} e^{\lambda_{ji} b} \quad \text{if } n=0 \\ \gamma_n &= \frac{mp^2 \sin n\phi_0}{\pi a n \phi_0} \sum_{j=0}^N \sum_{i=1}^8 C_{ji} e^{\lambda_{ji} b} \quad n=1, 2, \dots, N \end{aligned} \right\} \quad (45)$$

and

$$\left. \begin{aligned} \tau_0 &= \frac{mp^2}{2\pi a} \sum_{j=0}^N \sum_{i=1}^8 \mu_{ji} C_{ji} e^{\lambda_{ji} b} \quad \text{if } n=0 \\ \tau_n &= \frac{mp^2 \sin n\phi_0}{\pi a n \phi_0} \sum_{j=0}^N \sum_{i=1}^8 \mu_{ji} C_{ji} e^{\lambda_{ji} b} \quad n=1, 2, \dots, N \end{aligned} \right\} \quad (46)$$

for

$$-\pi \leq \phi \leq \pi$$

Since the C_{ni} and C'_{ni} can be expressed in terms of C_{n1} and C'_{n1} from equation (43), equation (44) can be converted to a system of $2(N+1)$ equations in the $2(N+1)$ unknowns, C_{n1} and C'_{n1} :

$$\begin{aligned} A_n C_{n1} + B_n C'_{n1} &= 0 \\ D_n C_{n1} + E_n C'_{n1} &= 0 \end{aligned} \quad n = 0, 1, 2, \dots, N \quad (47)$$

This system (47) of linear homogeneous equations in C_{n1} and C'_{n1} has a non-trivial solution if and only if the determinant, Δ , of the coefficients of the C_{n1} and C'_{n1} vanishes. The frequency, p , for which this determinant is zero is the approximate resonant frequency of the cylinder vibration problem for the index or order of truncation N . The index or order of truncation N^* will be used to define the number of terms of the Fourier Series used in the analysis. If all consecutive terms from $n = 0$ to $n = N$ are used, then $N^* = N + 1$.

The value of the determinant is a complex number of the form $a + ib$ where a and b are real. The value of the determinant is zero if and only if $a = b = 0$. The value of p that makes a and b equal to zero will make the modulus, $\sqrt{a^2 + b^2}$, equal to zero also.

B. COMPUTATIONAL PROGRAM

1. Analytical Investigations

a. Boundary Condition Equations

During the course of the "debugging" of the program it was found that the nonhomogeneous linear system of fourteen equations in fourteen unknowns (43) was singular for $n = 0$, the axisymmetric component of the displacement field. This occurs because four of the equations degenerate to $0 = 0$. Therefore, either a special program had to be written which would solve a reduced 10×10 system of linear equations, or the original 14×14 system would have to be modified to yield the same results. The latter course of action was adopted. The rationale is described in the following paragraph, utilizing a simplified example.

Suppose that we have five nonhomogeneous linear equations in five unknowns $x_1, x_2, \dots, x_5$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5 &= c_2 \\ \cdot &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 + a_{55}x_5 &= c_5 \end{aligned} \tag{48}$$

This system of equations possesses a unique solution if, and only if, the determinant of the coefficients is not equal to zero. It is anticipated that for the non-axisymmetric components of motion, the determinant will be nonzero. However, for the axisymmetric component the reduced system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3 \end{aligned} \tag{49}$$

should be employed. In order to avoid reprogramming we may add to the system (49) the additional relations

$$x_4 = 0$$

$$x_5 = 0$$

or in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \\ 0 \end{bmatrix} \quad (50)$$

Utilization of (50) instead of (49) will not alter the frequency or mode shapes of the cylinder and should result in a more efficient program. The program has been modified so that its internal logic identifies the extraneous unknowns for the axisymmetric modes and alters the equation in a manner analogous to that shown in equation (50).

b. Determinant Equation

Since the size of the determinant of the coefficients of equation (47) required that so many multiplications and additions be performed, it was feared that the general procedure for evaluating the determinant would yield inaccurate results. This was so especially when the determinant is near zero which is the region of most interest. However, the determinant appeared to have a sufficiently special form to permit solution by other than the general method, and this approach was investigated with good results as described below.

Starting with the equations for the determinant (44), we obtain equations (51) and (52).

$$-D \sum_{i=1}^8 \lambda_{ni}^3 (C'_{ni} - C_{ni}) e^{\lambda_{ni}b} + \gamma_n = 0 \quad (n = 0, 1, 2, \dots, N) \quad (51)$$

$$\frac{Eh}{1 + \nu^2} \sum_{i=1}^8 \mu_{ni} \lambda_{ni} (C'_{ni} - C_{ni}) e^{\lambda_{ni}b} + \tau_n = 0 \quad (n = 0, 1, 2, \dots, N) \quad (52)$$

where

$$\gamma_n = \frac{mp^2}{\pi a} \psi_n \sum_{j=0}^N \sum_{i=1}^8 C_{ji} e^{\lambda_{ji}b} \quad (53)$$

$$\tau_n = \frac{mp^2}{\pi a} \psi_n \sum_{j=0}^N \sum_{i=1}^8 \mu_{ji} C_{ji} e^{\lambda_{ji}b} \quad (54)$$

and

$$\psi_n \begin{cases} = \frac{\sin n\phi_0}{n\phi_0} & \text{for } n > 0 \\ = \frac{1}{2} & \text{for } n = 0 \end{cases} \quad (55)$$

Note: n is the number of the Fourier coefficient.

$N^* = N + 1$ is the number of consecutive modes used for analysis.

Let

$$f_n = -D / \left(\psi_n \frac{mp^2}{\pi a} \right) \quad (56)$$

$$g_n = \frac{Eh}{1 - \nu^2} / \left(\psi_n \frac{mp^2}{\pi a} \right) \quad (57)$$

Equations (51) and (52) can now be written

$$f_n \sum_{i=1}^8 \lambda_{ni}^3 (C'_{ni} - C_{ni}) e^{\lambda_{ni}b} + \sum_{j=0}^N \sum_{i=1}^8 C_{ji} e^{\lambda_{ji}b} = 0 \quad (58)$$

$$g_n \sum_{i=1}^8 \mu_{ni} \lambda_{ni} (C'_{ni} - C_{ni}) e^{\lambda_{ni}b} + \sum_{j=0}^N \sum_{i=1}^8 C_{ji} e^{\lambda_{ji}b} = 0 \quad (59)$$

Prior to this, each C_{ni} and C'_{ni} have been evaluated from equation (43) as a linear combination of C_{n1} and C'_{n1} for the same n ; i.e.:

$$C_{ni} = r_{ni} C_{n1} + s_{ni} C'_{n1} \quad (60)$$

$$C'_{ni} = r'_{ni} C_{n1} + s'_{ni} C'_{n1} \quad (61)$$

Therefore we can let

$$\alpha_n C_{n1} + \beta_n C'_{n1} = f_n \sum_{i=1}^8 \lambda_{ni}^3 (C'_{ni} - C_{ni}) e^{\lambda_{ni}b} \quad (62)$$

$$\xi_n C_{n1} + \zeta_n C'_{n1} = g_n \sum_{i=1}^8 \mu_{ni} \lambda_{ni} (C'_{ni} - C_{ni}) e^{\lambda_{ni}b} \quad (63)$$

$$A_j C_{j1} + F_j C'_{j1} = \sum_{i=1}^8 C_{ji} e^{\lambda_{ji}b} \quad (64)$$

$$G_j C_{j1} + H_j C'_{j1} = \sum_{i=1}^8 \mu_{ji} C_{ji} e^{\lambda_{ji}b} \quad (65)$$

Substituting (62) to (65) in (58) and (59) we get

$$\alpha_n C_{n1} + \beta_n C'_{n1} + \sum_{j=0}^N (A_j C_{j1} + F_j C'_{j1}) = 0 \quad (66)$$

$$\xi_n C_{n1} + \zeta_n C'_{n1} + \sum_{j=0}^N (G_j C_{j1} + H_j C'_{j1}) = 0 \quad (67)$$

Equations (66) and (67) can be put into matrix form. This is shown for four modes in equation (68).

$$\begin{bmatrix} A_1 + \alpha_1 & F_1 + \beta_1 & A_2 & F_2 & A_3 & F_3 & A_4 & F_4 \\ G_1 + \xi_1 & H_1 + \zeta_1 & G_2 & H_2 & G_3 & H_3 & G_4 & H_4 \\ A_1 & F_1 & A_2 + \alpha_2 & F_2 + \beta_2 & A_3 + \alpha_3 & F_3 & A_4 & F_4 \\ G_1 & H_1 & G_2 + \xi_2 & H_2 + \zeta_2 & G_3 + \xi_3 & H_3 & G_4 & H_4 \\ A_1 & F_1 & A_2 & F_2 & A_3 & F_3 + \beta_3 & A_4 & F_4 \\ G_1 & H_1 & G_2 & H_2 & G_3 & H_3 + \zeta_3 & G_4 & H_4 \\ A_1 & F_1 & A_2 & F_2 & A_3 & F_3 & A_4 + \alpha_4 & F_4 + \beta_4 \\ G_1 & H_1 & G_2 & H_2 & G_3 & H_3 & G_4 + \xi_4 & H_4 + \zeta_4 \end{bmatrix} \begin{bmatrix} C_{11} \\ C'_{11} \\ C_{21} \\ C'_{21} \\ C_{31} \\ C'_{31} \\ C_{41} \\ C'_{41} \end{bmatrix} = 0 \quad (68)$$

Subtracting row 1 from rows 3, 5, 7, etc., and subtracting row 2 from rows 4, 6, 8, etc., (68) becomes

$$\begin{bmatrix}
A_1 + \alpha_1 & F_1 + \beta_1 & A_2 & F_2 & A_3 & F_3 & A_4 & F_4 \\
G_1 + \xi_1 & H_1 + \zeta_1 & G_2 & H_2 & G_3 & H_3 & G_4 & H_4 \\
-\alpha_1 & -\beta_1 & \alpha_2 & \beta_2 & 0 & 0 & 0 & 0 \\
-\xi_1 & -\zeta_1 & \xi_2 & \zeta_2 & 0 & 0 & 0 & 0 \\
-\alpha_1 & -\beta_1 & 0 & 0 & \alpha_3 & \beta_3 & 0 & 0 \\
-\xi_1 & -\zeta_1 & 0 & 0 & \xi_3 & \zeta_3 & 0 & 0 \\
-\alpha_1 & -\beta_1 & 0 & 0 & 0 & 0 & \alpha_4 & \beta_4 \\
-\xi_1 & -\zeta_1 & 0 & 0 & 0 & 0 & \xi_4 & \zeta_4
\end{bmatrix}
\begin{bmatrix}
C_{11} \\
C'_{11} \\
C_{21} \\
C'_{21} \\
C_{31} \\
C'_{31} \\
C_{41} \\
C'_{41}
\end{bmatrix} = 0 \quad (69)$$

From here on we consider only the determinant Δ of the matrix. If the even columns of the determinant are multiplied by $-\frac{\xi_i}{\zeta_i}$ and then added to the corresponding odd columns, the value of the determinant is unchanged and we get

$$\Delta = \prod_{i=1}^4 \frac{1}{\zeta_i} \begin{vmatrix}
(A_1\zeta_1 - F_1\xi_1) + (\alpha_1\zeta_1 - \beta_1\xi_1) & F_1 + \beta_1 & A_2\zeta_2 - F_2\xi_2 & F_2 & A_3\zeta_3 - F_3\xi_3 & F_3 & A_4\zeta_4 - F_4\xi_4 & F_4 \\
G_1\zeta_1 - H_1\xi_1 & G_1 + \zeta_1 & G_2\zeta_2 - H_2\xi_2 & H_2 & G_3\zeta_3 - H_3\xi_3 & H_3 & G_4\zeta_4 - H_4\xi_4 & H_4 \\
-(\alpha_1\zeta_1 - \beta_1\xi_1) & -\beta_1 & \alpha_2\zeta_2 - \beta_2\xi_2 & \beta_2 & 0 & 0 & 0 & 0 \\
0 & -\zeta_1 & 0 & \zeta_2 & 0 & 0 & 0 & 0 \\
-(\alpha_1\zeta_1 - \beta_1\xi_1) & -\beta_1 & 0 & 0 & \alpha_3\zeta_3 - \beta_3\xi_3 & \beta_3 & 0 & 0 \\
0 & -\zeta_1 & 0 & 0 & 0 & \zeta_3 & 0 & 0 \\
-(\alpha_1\zeta_1 - \beta_1\xi_1) & -\beta_1 & 0 & 0 & 0 & 0 & \alpha_4\zeta_4 - \beta_4\xi_4 & \beta_4 \\
0 & -\zeta_1 & 0 & 0 & 0 & 0 & 0 & \zeta_4
\end{vmatrix} \quad (70)$$

where the symbol $\prod_{i=1}^4$ indicates a product of four terms.

Let

$$Q_i = \alpha_i \zeta_i - \beta_i \xi_i \quad (71)$$

$$W_i = A_i \zeta_i - F_i \xi_i \quad (72)$$

$$Y_i = G_i \zeta_i - H_i \xi_i \quad (73)$$

Substituting (71) to (73) into (70) we obtain

$$\Delta = \prod_{i=1}^4 \frac{Q_i}{\zeta_i} \begin{vmatrix} \frac{W_1}{Q_1} + 1 & F_1 + \beta_1 & \frac{W_2}{Q_2} & F_2 & \frac{W_3}{Q_3} & F_3 & \frac{W_4}{Q_4} & F_4 \\ \frac{Y_1}{Q_1} & H_1 + \zeta_1 & \frac{Y_2}{Q_2} & H_2 & \frac{Y_3}{Q_3} & H_3 & \frac{Y_4}{Q_4} & H_4 \\ -1 & -\beta_1 & 1 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & -\zeta_1 & 0 & \zeta_2 & 1 & 0 & 0 & 0 \\ -1 & -\beta_1 & 0 & 0 & 0 & \beta_3 & 0 & 0 \\ 0 & -\zeta_1 & 0 & 0 & 0 & \zeta_3 & 1 & 0 \\ -1 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & \beta_4 \end{vmatrix} \quad (74)$$

If the odd columns of (74) are multiplied by $-\beta_1$ and added to the corresponding even columns, we get

$$\Delta = \prod_{i=1}^4 \frac{Q_i}{\zeta_i} \zeta_i \begin{vmatrix} \frac{W_1}{Q_1} + 1 & \frac{F_1 \alpha_1 + A_1 \beta_1}{Q_1} & \frac{W_2}{Q_2} & \frac{F_2 \alpha_2 - A_2 \beta_2}{Q_2} & \frac{W_3}{Q_3} & \frac{F_3 \alpha_3 - A_3 \beta_3}{Q_3} & \frac{W_4}{Q_4} & \frac{F_4 \alpha_4 - A_4 \beta_4}{Q_4} \\ \frac{Y_1}{Q_1} & \frac{F_1 \alpha_1 - G_1 \beta_1}{Q_1} + 1 & \frac{Y_2}{Q_2} & \frac{H_2 \alpha_2 - G_2 \beta_2}{Q_2} & \frac{Y_3}{Q_3} & \frac{H_3 \alpha_3 - G_3 \beta_3}{Q_3} & \frac{Y_4}{Q_4} & \frac{H_4 \alpha_4 - G_4 \beta_4}{Q_4} \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad (75)$$

Let

$$X_i = F_i \alpha_i - A_i \beta_i$$

$$Z_i = H_i \alpha_i - G_i \beta_i$$

Then (75) can be written

$$\Delta = \prod_{i=1}^4 Q_i \begin{vmatrix} \frac{W_1}{Q_1} + 1 & \frac{X_1}{Q_1} & \frac{W_2}{Q_2} & \frac{X_2}{Q_2} & \frac{W_3}{Q_3} & \frac{X_3}{Q_3} & \frac{W_4}{Q_4} & \frac{X_4}{Q_4} \\ \frac{Y_1}{Q_1} & \frac{Z_1}{Q_1} + 1 & \frac{Y_2}{Q_2} & \frac{Z_2}{Q_2} & \frac{Y_3}{Q_3} & \frac{Z_3}{Q_3} & \frac{Y_4}{Q_4} & \frac{Z_4}{Q_4} \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad (76)$$

If in the general array corresponding to (76) we add columns $2i(i^\dagger = 2, 3, \dots, N^*)$ to column 2 and columns $2(i - 1)$ ($i = 2, 3, \dots, N^*$) to column 1 we obtain

$$\Delta = \prod_{i=1}^{N^*} Q_i \begin{vmatrix} 1 + \sum_{i=1}^{N^*} \frac{W_i}{Q_i} & \sum_{i=1}^{N^*} \frac{X_i}{Q_i} & \frac{W_2}{Q_2} & \frac{X_2}{Q_2} & \dots & \frac{W_{N^*}}{Q_{N^*}} & \frac{X_{N^*}}{Q_{N^*}} \\ \sum_{i=1}^{N^*} \frac{Y_i}{Q_i} & \sum_{i=1}^{N^*} \frac{Z_i}{Q_i} & \frac{Y_2}{Q_2} & \frac{Z_2}{Q_2} & \dots & \frac{Y_{N^*}}{Q_{N^*}} & \frac{Z_{N^*}}{Q_{N^*}} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} \quad (77)$$

From which the value of the determinant is

$$\Delta = \prod_{i=1}^{N^*} Q_i \begin{vmatrix} 1 + \sum_{i=1}^{N^*} \frac{W_i}{Q_i} & \sum_{i=1}^{N^*} \frac{X_i}{Q_i} \\ \sum_{i=1}^{N^*} \frac{Y_i}{Q_i} & \sum_{i=1}^{N^*} \frac{Z_i}{Q_i} \end{vmatrix} \quad (78)$$

The only immediately apparent difficulty is the possibility that one or more of the Q_i may be equal to or very near zero. Runs from the existing routine indicate that the value of Δ is a continuous function of frequency, so that if a Q_i is zero for some frequency the value of Δ has a removable singularity at the frequency. In that case the program will shift the frequency slightly and recompute.

[†] i represents the input order of the mode numbers, not the mode numbers themselves. For N consecutive modes starting with $n = 0$, i would run from $i = 1$ to $i = N^* = (N + 1)$.

c. Scaling of Determinant Elements

An examination of the revised method (equation (78)) of computing the determinant, Δ , indicated that a likely source of difficulty in numerical computing is the range of values that have to be treated. For example, equation (78) contains the term $\prod_{i=1}^{N^*} Q_i$ where N^* is the number of modes considered. The operational limits of the 7094 restrict the computer range of permissible numerical values from approximately 10^{-39} to 10^{+39} . Any number less than 10^{-39} results in an underflow in which all digits are lost and the computer indicates a "flat zero". Any number greater than 10^{+39} results in an overflow in which the most significant digits are lost, giving incorrect answers.

Since large values of N^* of 50, 75, and even 100 can be selected for the order of truncation, it does not require many multiplications of terms of the order of 10^{10} (or 10^{-10}) to exceed the machine range. To avoid this difficulty, an automatic scaling procedure was devised in which an index is associated with each number would have to be multiplied by 2^{50} (roughly 10^{15}) to fall within the range of 10^{-15} to 10^{15} . If the number is initially in this range, the index is zero. When numbers are multiplied, the indices are carried along and then, if necessary, the product is rectified in the same manner by adjusting the index. When performing additions of numbers with different indices, numbers with a lower index are discarded.

A power of 2 was chosen as the scaling factor because, in a binary computer, multiplication by powers of two introduces no error. When the determinant is found, it is converted to the usual description in terms of powers of 10.

d. Normalization of Determinant Value

Data from initial IBM machine runs indicated that the value of the determinant decreased rapidly with frequency by orders of magnitude, even in regions removed from the resonant frequency. The rate of change (slope) of the determinant value was so steep that it was difficult to observe the relative minimums expected near the resonant frequency.

In the value of the determinant is the factor $\prod_{i=1}^{N^*} Q_i$ where each Q_i has a frequency factor of $(\frac{1}{p^2})^2$. Since there are N^* terms, the value of the

determinant has a factor of $(\frac{1}{2})^{2N^*}$ where p is the trial frequency. Since the factor decreases rapidly with frequency, it was decided to normalize the determinant value by multiplying by $(p^4)^{N^*}$. The resulting rate of change (slope) of the determinant value allows the resonant condition to be better observed.

In an attempt to keep the exponents associated with the determinant value in a reasonable range, the above normalized determinant is multiplied by $(2^{50})^{-2N^*}$ to get the shifted determinant values. This shifting does not change the shape of the determinant value versus frequency curve, since the value of N^* is held constant for each run.

The complete normalization procedure, therefore, multiplies the original determinant value by the factor $(\frac{p^2}{2^{50}})^{2N^*}$.

e. Evaluation of C_{n1} and C'_{n1}

Starting with equation (69), which illustrates the procedure for $N^* = 4$,

Evaluate C_{41} , C'_{41} in terms of C_{11} , C'_{11}

Evaluate C_{31} , C'_{31} in terms of C_{11} , C'_{11}

Evaluate C_{21} , C'_{21} in terms of C_{11} , C'_{11}

Substitute in the equation implied by the 1st row to obtain an equation in terms of only C_{11} and C'_{11} .

Let $C_{11} = 1.0 + 0i$.

Solve for C'_{11} .

Then continue and obtain C_{21} and C'_{21} , etc., from expressions of C_{21} , C'_{21} , etc., in terms of C_{11} and C'_{11} .

2. Computational Procedure

The IBM 7094 Fortran IV Computer Program, 64E629, is used to predict the natural frequencies of a ring-stiffened pressurized cylinder with a concentrated attached weight based on the preceeding analysis of Dr. Forray. Program details are discussed in the User's Manual (Reference 9).

The general solution procedure to obtain the fundamental natural frequency is outlined below:

- (1) Establish the values of the geometrical and physical data (cylinder properties, location of mass, etc.).
- (2) Choose an order of truncation N^* and the mode numbers to be used.
- (3) Choose a trial frequency, p_1 , in cycles per second.
- (4) Solve the biquartic determinantal equation (35) for λ_{ni} , where $n=0,1,2,\dots,N$; $i=1,2,\dots,8$.
- (5) Solve the system (43) of fourteen linear equations and sixteen unknowns for C_{ni} and C'_{ni} ($i=1,2,\dots,8$) in terms of C_{n1} and C'_{n1} for $n=0,1,2,\dots,N$.
- (6) Establish the coefficients of C_{n1} and C'_{n1} of equation (47).
- (7) Evaluate the $2(N+1)$ determinant of the coefficients of the linear homogeneous equations (47). The value of this determinant is of the form $a + ib$ where a and b are real numbers. The modulus is $\sqrt{a^2 + b^2}$.
- (8) Repeat steps (3) through (7) for the next frequency, p_2 , etc.

In practice, a series of uniformly spaced frequencies of the form $p_i = p_1 + c_i \Delta p$ are specified for the first run. The initial frequency is p_1 , the constant frequency increment is Δp , and c_i takes on successive integer values until the last frequency is reached. Based on a visual or graphical examination of the absolute values (modulus) of the determinant for these values of p_i , the resonant frequency is determined to lie between two coarse values of frequency. (If Δp were 10 cps, then the resonant frequency might be determined to be between 270 and 280 cps.) A second run with a series of uniformly spaced frequencies p_i between the two coarse values will yield the resonant frequencies to the accuracy desired. (If p_1 is taken as 270 cps, and $\Delta p = 1$ cps, then the frequency will be determinable within 1 cps for the truncation value of N^* that has been used.)

The general solution procedure for determining the mode shape is outlined below, but has not been completed in the IBM program.

- (9) Once the value of resonant frequency, p_r , has been determined, set $C_{01} = 1$ and determine the remaining C_{n1} and C'_{n1} ($n = 0, 1, 2, \dots, N$) from the system of $2(N + 1)$ linear nonhomogeneous equations (47). This has been programmed.
- (10) Determine the C_{ni} and C'_{ni} ($i = 2, 3, \dots, 8$; $n = 0, 1, 2, \dots, N$) using the results of step (5).
- (11) Determine the mode shape corresponding to the value of p_r and the quantities C_{ni} and C'_{ni} using equations (41) and (42).

C. COMPUTATIONAL RESULTS

1. Cylinder No. 2, 0.1 Pound Weight at Mid-Bay

a. Geometric and Physical Data

- h - thickness of shell - 0.02 inch
- a - radius of shell - 18.0 inches
- ν - Poisson's ratio - 0.3 (dimensionless) for aluminum
- E - Young's modulus - 10^7 lb/in² for aluminum
- q - internal pressure - 10.0 lb/in²
- ρ - weight density of shell - 0.1 lb/in³ for aluminum
- α_1 } - boundary condition parameter for moment at frame
- α_2 } - 0.0 if clamped; 1.0 if simple support ($\alpha_1 = \alpha_2 = 1.0$)
- β_1 } - boundary condition parameter for slope at frame
- β_2 } - 1.0 if clamped; 0.0 if simple support ($\beta_1 = \beta_2 = 0.0$)
- b - distance of weight from frame - 3.0 inches
- L - length of cylinder between frames - 6.0 inches
- m - attached weight - 0.1 pound

b. Other Required Inputs

- ϕ_0 - 1/2 circumferential arc over which weight is assumed to be distributed - 0.0125 radians
- N* - number of consecutive modes starting with n = 0
- 30 (dimensionless)
- P_i - frequencies at which determinant is to be evaluated - cps

c. Value of Determinant Versus Frequency

The initial series of uniformly spaced trial frequencies, p_1 , started with 100 cycles per second and continued in increments of 50 cps to 1000 cps. The normalized modulus value of the determinant was plotted versus frequency as in Figure 28.

Because the modulus changes rapidly with trial frequency, the data is plotted on five-scale semilog paper with occasional shifts in the ordinate scale. Thus, at the trial frequency of 300 cps, the modulus value of $1.32 \times 10^{+34}$. The vertical arrow indicates a scale factor shift and the same point is plotted as 1.32×10^{-1} with a scale factor of 10^{+32} . The curve shown in Figure 28 is continuous at 300 cps, and also at the other frequencies where scale shifts are indicated (i.e., 600 cps, 750 cps, 850 cps, 950 cps, 1000 cps).

From the plot of the moduli for the 50 cps interval frequencies, it can be observed that there is a discontinuity in the plotted points in the frequency region of 250 to 350 cps. A second series of uniformly spaced frequencies starting with 250 cps, with 10 cps increments, was chosen. The plot of these additional points on the curve indicates that the resonant frequency lies between 310 and 320 cps because the relative minimum value of the modulus appears to lie in this range. Furthermore, an examination of the real and imaginary parts of the value of the determinant in the printout confirmed that they both changed sign (went through zero) within the frequency range of 310 to 320 cps. A final sweep, at 1 cps frequency intervals from 311 to 319 cps, yielded the resonant frequency of 313 cps within 1 cps. For this example, an additional sweep at intervals of 0.1 cps yielded the resonant frequency of 313.4 cps within 0.1 cps.

It is clear from this sample problem, that the two criteria for determining the fundamental resonant frequency are that the modulus have a relative minimum, and that the real and imaginary parts of the value of the determinant change sign. These practical criteria are used because the theoretical criterion of zero modulus would occur only if the trial frequency were exactly the resonant frequency and there was absolutely no numerical computation error.

d. Effect of Order of Truncation N^*

The number of terms of the infinite series (32) to be used is specified by the order of truncation, N^* . For consecutive terms of the series

starting with $n = 0$ to $n = N$, $N^* = (N + 1)$. For the previous problem, with $m = 0.1$ pound at mid-bay, the effect of N^* on the computed resonant frequency was investigated. In Figure 29, the modulus value of the determinant is plotted versus frequency for $N^* = 30, 50, 75, 100$, and 150 for trial frequencies near the resonant frequency.

For $N^* = 30$, each dot represents a trial frequency and its associated normalized modulus value. This curve is an enlargement of the data in Figure 28, in the region of 200-340 cps. The resonant frequency is indicated as between 313.3 and 313.4 cps. For an $N^* = 50$, the relative minimum of the + points indicates the resonant frequency between 276 and 278 cps. For $N^* = 75$, Δ points indicate a resonant frequency between 260 and 262 cps.

For $N^* = 100$, the resonant frequency is still slightly lower, between 255 and 256 cps; however, it is noted that the points no longer form a smooth curve as obtained for $N^* = 30$. The limited number of points for $N^* = 150$ also show erratic behavior. It is speculated that numerical round-off errors may limit the usable values of N^* to less than 100.

Since the theory indicates that the infinite series ($N^* = \infty$) will give the exact solution for the resonant frequency, an estimate of the resonant frequency for $N^* = \infty$ was obtained with the aid of Figure 30. This is a graph of the "resonant" frequency obtained for each N^* plotted against the reciprocal of N^* , the number of consecutive modes. The data suggest an extrapolated true resonant frequency value of 246 cps. The percent error for various values of N^* , based on the extrapolated value of 246 cps, is shown in Table 3.

Since the plotted data for $N^* = 100$ indicated potential problems with numerical round-off errors, and the accuracy with $N^* = 75$ is better than 10%, the decision was made that for the determinations of frequency in this report, calculations would be based on a value of $N^* = 75$.

2. Cylinder No. 2, Weights at Mid-Bay

The modulus versus frequency curves for weights of 0.2, 0.3, and 0.4 pound at mid-bay were plotted for an order of truncation $N^* = 75$.

The curves were smooth and similar to the $N^* = 75$ curve of Figure 29. The computed resonant frequency values are listed in Table 4:

3. Cylinder No. 2, 0.4 Pound at Quarter-Bay

a. Geometric and Physical Data

These data are the same as previously given, except that:

b - distance of weight from frame - 1.5 inches

m - attached weight - 0.4 pound

b. Other Required Input

These data are also the same as previously given, except that

N^* - number of consecutive modes starting with $n = 0$
= 75 (dimensionless)

c. Value of Determinant versus Frequency

The normalized value of the determinant plotted versus frequency is shown by the x points in Figure 31. The curve shows erratic behavior similar to that indicated in Figure 29, $N^* = 100$. The solution of the $2N^*$ by $2N^*$ determinant could not be evaluated by the machine in the frequency range of 136 to 140 cps because certain divisors in the computation were small enough to be called zero in single precision arithmetic.

In spite of the erratic behavior of the points, the existence of a relative minimum in the region of 135 to 140 cps is indicated by the low modulus value at 136 cps and the very high modulus value at 140 cps. Examination of the printout indicated a zero crossing between 135 and 136 cps; therefore, the predicted frequency was taken as 136⁻ cps.

4. Cylinder No. 2, Weights at Quarter-Bay

The modulus versus frequency curves for weights of 0.1, 0.2, and 0.3 pound at quarter-bay were plotted for an order of truncation $N^* = 75$. The curves showed erratic behavior similar to the quarter-bay data of Figure 31; however, the existence of a resonant frequency was confirmed by a zero crossing. These predicted frequencies are also listed in Table 4.

5. Cylinder No. 2, Weights at Three-Quarter-Bay

Because of symmetry of the boundary conditions and of the mass location, the fundamental resonant frequency of a cylinder with an attached mass at the three-quarter-bay point should be exactly the same as the frequency obtained for the quarter-bay point. However, the analysis equations (34) contain terms of the form $e^{\lambda_n x}$ which are not symmetric since $e^{\lambda_n b}$ does not equal $e^{\lambda_n(L-b)}$. It was therefore decided to check the three-quarter-bay computation to demonstrate that, even though the intermediate numerical results might be different from the quarter-bay case, the computed resonant frequency would be the same.

The normalized value of the determinant plotted versus frequency is shown by the points (•) of Figure 31. It is noted that the curve is smooth, and though the modulus values are orders of magnitude greater than the quarter-bay points, the three-quarter-bay predicted resonant frequency of 136 cps is the same as the previously predicted frequency for the quarter-bay. The predicted frequencies for 0.1 and 0.4 pound at three-quarter-bay are listed in Table 4.

D. COMPARISON OF FREQUENCY PREDICTIONS WITH TEST DATA

1. Cylinder No. 2, Weight at Mid-Bay

The solid curve of Figure 25 is the predicted curve of fundamental natural frequency for cylinder No. 2 with an attached weight at mid-bay, based on the new Fourier series method of analysis and using the new computer program 64E629 (Reference 9). The curve is based on the frequency predictions for 0.1, 0.2, 0.3, and 0.4 pound with an order of truncation $N^* = 75$.

Superposed on Figure 25 are the test data for cylinder No. 2 showing the frequencies determined from shaker, horn, and thump tests. It is seen that the present Fourier series analysis gives very good results, even for large weights.

For comparison, the dashed curve is the predicted curve of frequency based on the previous Lagrangian energy method of analysis and computer program 61F603 (Reference 2).

2. Cylinder No. 2, Weight at Quarter-Bay

In theory, the natural frequency with an attached weight at the quarter-bay point should be slightly higher than the frequency for the same weight at the mid-bay. This is based on the observation that the local stiffness of the cylinder at the quarter-bay point should be higher than the local stiffness at the mid-bay point because of the greater influence of the frame boundary. The computed values shown in Table 4 confirm that the frequencies for the quarter-bay location are about 4% higher than the mid-bay frequencies.

The solid curve of Figure 26 is the predicted curve of fundamental natural frequency for cylinder No. 2 with an attached weight at the quarter-bay point based on the Fourier series method of analysis. This curve is based on the frequency predictions for quarter-bay and three-quarter-bay listed in Table 4.

Superposed on Figure 26 is the comparable test data for cylinder No. 2 showing the frequencies determined from shaker, horn, and thump

tests. It is seen that the present Fourier series analysis gives very good results, even for the quarter-bay location.

For comparison, the dashed curve is the predicted curve of frequency based on the previous Lagrangian energy method of analysis.

SECTION IV - THREE-YEAR PROGRAM SUMMARY AND SUGGESTED ADDITIONAL INVESTIGATIONS

A. GENERAL

The subject program was directed toward development of methods to predict the dynamic responses of a pressurized stiffened cylinder to which concentrated mass items are attached, when the entire assembly is exposed to random vibrations. The end application would be a section of a Saturn vehicle to which the vibrations would be transmitted during boost from the engines either mechanically or acoustically.

To achieve this goal, a coordinated program of test and analysis has been conducted, proceeding in an orderly manner from simple cases to those of greater complexity. At each step, the analysis has been compared with test results, employing both mechanical and acoustic excitations. It seems appropriate, at this point, to summarize the accomplishments to date and suggest those additional steps which can be taken, based upon the foundation so far obtained.

1) A wealth of experimental data was obtained and organized, (Reference 10), establishing the relationships between the fundamental frequency and the magnitude of attached weight, stiffness of cylindrical shell, stiffness and arrangement of stringers, and location of stiff frames. This work established the limitations of validity of a single-degree-of-freedom idealization and suggested the need for better analytical procedures.

2) Two methods of analysis were developed for analysis of unstiffened and stringer-stiffened shells with a single attached point mass (Reference 2). The mass was idealized as being located in the surface of the cylinder.

3) The analysis of Reference 2 was checked by a test program (Reference 1) and found to give good results for stringer-stiffened cylinders. The agreement between analysis and test was not as satisfactory for unstiffened cylinders with a heavy, attached mass.

4) Acoustic tests were run (Reference 1), with a single-tone loudspeaker, and in a reverberant chamber; empirical methods were derived which correlated the results with analytical predictions.

5) Refined testing techniques were developed and applied to measuring fundamental and higher mode frequencies and mode shapes of unstiffened cylinders with single attached weights of various magnitudes. Shaker excitation, monotonically acoustic excitation, and thump testing were all utilized (Reference 5).

6) A more refined analysis was developed and programmed (Reference 9) to treat the problem of an unstiffened cylinder with a heavy attached mass. This program has produced numerical results which demonstrate good agreement with test results for resonant frequencies. It is expected that the continuation of the present work will also produce satisfactory calculated mode shapes.

It is now possible to eliminate two simplifying assumptions which have been implicit in all of the work accomplished to date:

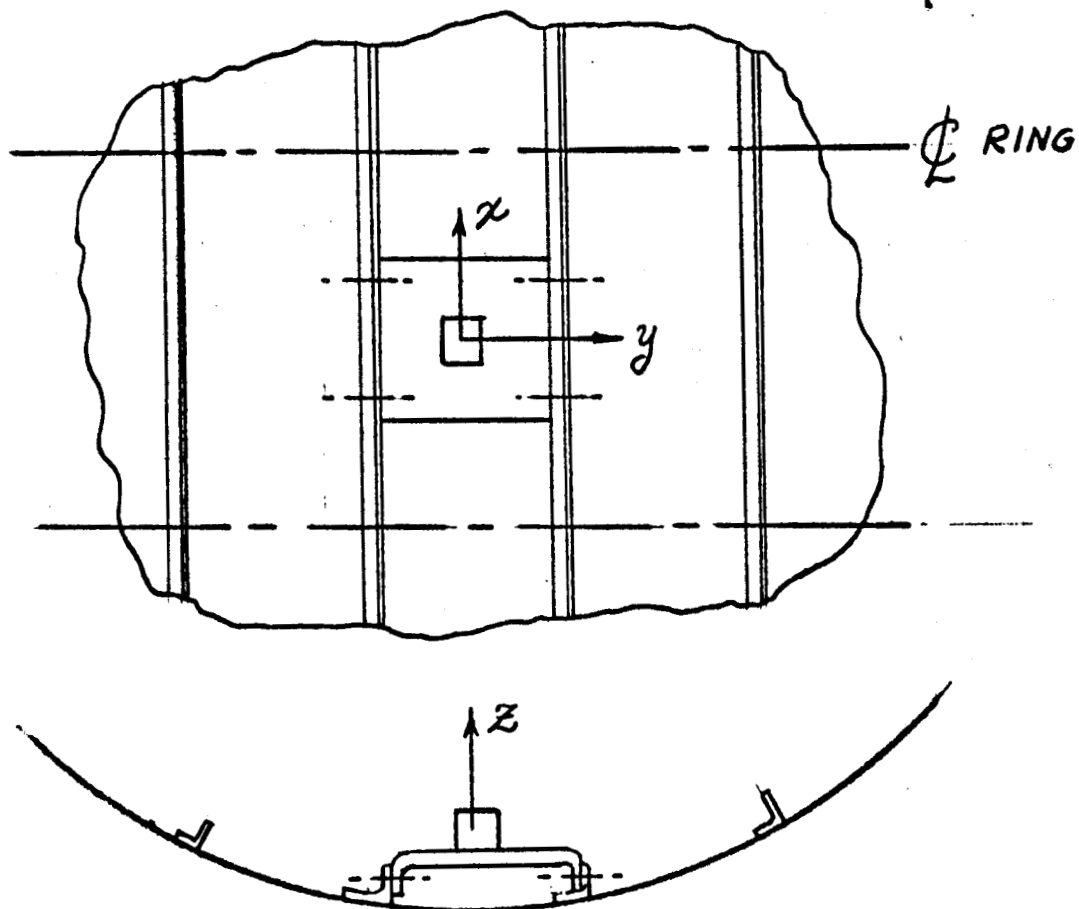
- 1) That the weights are point masses concentrated in the surface of the cylinder. Real objects are offset from the surface, have sizeable rotary inertias, and require support from several mounting points.
- 2) That one can obtain useful information treating the case of a single mass on the cylinder. In a real vehicle, many masses will be present in sufficiently close proximity that there is likely to be dynamic interaction.

A coordinated program of analysis and test has been developed to attack these two problems simultaneously. The analysis and test programs, described briefly below, would make maximum use of existing understanding of the problems, test specimens and jigs, testing techniques, and computer programs which have been developed in previous phases of the investigation.

B. SUGGESTED ANALYSIS PROGRAM

1. Cylinder Modes Including Internal Weight with Rotational Degrees of Freedom and Multiple Stringer Attachments

The problem to be solved is indicated schematically in the following sketch, which shows a weight attached to a rigid bracket which is in turn fastened to a pair of stringers.



Multiply Attached Weight

It is suggested that the discrete stringer-panel analysis (Reference 2) be extended to encompass natural mode calculations for the above configuration. This would require the inclusion of rotational degrees of freedom for the weight about the X and Y axes. The stringer-attachment fitting details must also be considered. To

simplify the analysis, the weight attachment fitting and ring frames could be assumed to be rigid and the fitting to have two attachments to each of two stringers.

The problem should be programmed for digital computer evaluation and should encompass variations of attached mass and moment of inertia, stringer stiffness and spacing, internal pressure, shell radius, wall thickness, and the geometric quantities describing the attachment points.

The following analysis procedure is suggested:

- 1) The rigid body motions of the weight should be expressed as functions of the displacements of the four attachment points.
- 2) The kinetic energy of the weight should be found in terms of these displacements and the appropriate mass properties, including rotary inertias.
- 3) The kinetic and potential energies of the stringers to which the weight is attached could be given special handling. The deflected shapes might be assumed to correspond to the form of the static deflection of a beam under concentrated loads at the attachment points. The values of these concentrated loads would correspond to the dynamic forces of the weight. The analysis would permit treatment of the case in which the loaded stringers may be made stiffer and more massive than the other stringers, as might well be done in practical design when heavy weights must be supported.
- 4) To complete the formulation of the problem, unweighted cylinder modes would be employed as reported in Reference 2, enforcing compatible deformations at the four weight attachment points. Terms describing modal shapes that are not symmetrical in the circumferential direction must be added.
- 5) Finally, eigenvalue solutions would be obtained as was done previously.

2. Natural Vibration Modes for Ring - and Stringer-Stiffened Cylinders with Multiple-Point Weights Attached to Stringers

It is suggested that both the orthotropic and discrete stringer-panel modal analyses be extended to multiple weights. This would require a generalization of the single-weight coupling technique which has been developed. The computational program would be extended to problems involving as many as three weights located in the same bay between frames. As in the current program, the

analysis would encompass variations with regard to stringer and frame spacing, cylinder pressure, stringer stiffness and spacing, shell stiffness, and weight sizes.

The following analysis procedure is suggested.

- 1) The coupling terms describing the kinetic energy of the attached mass would be generalized to include various masses at various locations.
- 2) Additional trial functions would be introduced into the family describing the uncoupled shell modes including antisymmetrical shapes, because symmetry in the circumferential direction will, in general, no longer be present.
- 3) The existing computer program would be revised according to the new formulation, and eigenvalue solutions found.

C. SUGGESTED TEST PROGRAM

1. Dynamic Response of Stringer-Stiffened Pressurized Cylinders with Multiply Attached Offset Mass

A test installation could be constructed as depicted in the sketch shown above (Multiply Attached Weight). The platform which supports the weight would have four attachments which pick up the quarter-span points of two stringers 15 degrees apart. This installation would pick up adjacent stringers of cylinder No. 3, and alternate stringers of cylinder No. 4. Various modes would be excited using a shaker and impedance head as in previous tests. In all cases, frequency responses would be obtained and mode shapes measured at resonances. These tests would be conducted with three different weights attached to the platform.

Current planning would have the unweighted platform include a three-axis accelerometer near the center of gravity, and piezoelectric accelerometers to measure motions normal to the shell at each attachment point. Additional instrumentation would include a Fotonic sensor to check the piezoelectric accelerometers. The Fotonic sensor would also be used to measure shell modes by picking up displacements at mid-stringer and mid-panel around the cylinder in the excited bay.

2. Dynamic Response of Stringer-Stiffened Pressurized Cylinders with Three Attached Concentrated Weights

These tests would be conducted on cylinders Nos. 3 and 4. A reference mass would be kept at the mid-span of one stringer. A second mass (not equal in size) including an accelerometer would be located at various other stringers in the same bay to obtain results to be compared with analysis. Exploratory investigations would also be made locating the roving weight in adjacent bays, along the same stringer. Tests would also be run with three weights (all different) located in the same bay in one quadrant of the cylinder.

Excitation would always be applied to the reference weight; frequency responses would be obtained at all weights. Mode shapes would be measured at resonances. For all runs, the modal responses of the weights would be defined by comparison of the amplitude and phase of the outputs of the accelerometers on the weights with the reference input at the shaker. The Fotonic sensor would be employed to further define shell modes as described above.

3. Horn Tests of Cylinders with Multiply Attached Offset Mass

After installation of the weight support platform, one might excite cylinder Nos. 3 and 4 with the horn, directing sound at a 6 inch by 6 inch area and repeat tests with various weights installed. The tests would employ incident (plane wave) input and comprise a frequency sweep to determine resonances by plots of db/A. The response of the weight to various excitation levels would be determined and mode shapes explored. The weight response under the horn could be established by the accelerometer outputs. The shell mode shapes could be measured by the Fotonic sensor.

4. Horn Tests of Cylinders with Three Attached Concentrated Weights

Several tests would be conducted on configurations similar to those described in Paragraph IV-C-2 above, except that all three weights would have accelerometers. Acoustic excitation would be supplied through a shroud directing the sound over the bay between frames, and the sector containing the weights. A frequency sweep would be made to establish resonant frequencies, and the mode shape established by comparing accelerometer outputs.

5. Reverberant Chamber Test, Cylinders with a Multiply Attached Offset Mass and with Two Attached Concentrated Weights

Two test cylinders could be put into a large reverberant chamber and subjected to a random acoustic noise field of about 130 decibels. One would be cylinder No. 3 with multiply connected offset weight; the other would be cylinder No. 4 with two attached concentrated weights. The configurations chosen would be based upon previous test results. Power spectra of ambient noise and response at the weights would be obtained.

SECTION V - REFERENCES

1. "Second Annual Summary Report - Experimental Determination of System Parameters for Thin-Walled Cylinders," Republic Aviation Corporation Report No. RAC 1117-14, June 1965.
2. "User's Manual for Free Vibration Analysis of Stiffened Cylinders with an Attached Weight," Fairchild Hiller, Republic Aviation Division Report No. FHR 1117-11 Rev. A, 1 July 1966.
3. "Ninth Quarterly Technical Progress Report - Experimental Determination of System Parameters for Thin-Walled Cylinders," Republic Aviation Corporation Report No. RAC 1117-15, 15 September 1965.
4. "Informal Report on Test Procedure for Experimental Determination of System Parameters for Thin-Walled Cylinders," Fairchild Hiller, Republic Aviation Division Report No. FHR 1117-21 (Previously FHR 1117-17), 30 December 1965.
5. "Tenth Quarterly Technical Progress Report - Experimental Determination of System Parameters for Thin-Walled Cylinders," Fairchild Hiller, Republic Aviation Division Report No. FHR 1117-16, 15 December 1965.
6. "Eleventh Quarterly Technical Progress Report - Experimental Determination of System Parameters for Thin-Walled Cylinders," Fairchild Hiller, Republic Aviation Division Report No. FHR 1117-17, 15 March 1966.
7. Timoshenko, S. and Woinowsky-Kreiger, S., "Theory of Plates and Shells," McGraw Hill, Second Edition, 1959, pp 468-474.
8. Fung, Y. C., Sechler, E. E., and Kaplan, A., "On The Vibration of Thin Cylindrical Sheels Under Internal Pressure," Journal of the Aeronautical Sciences, Volume 24, September 1957, pp 650-660.
9. "User's Manual for Natural Frequencies of Ring-Stiffened Cylinders with an Attached Weight," Fairchild Hiller, Republic Aviation Division Report No. FHR 1117-19, 1 August 1966.
10. "First Annual Summary Report for Thin-Walled Cylinders," Republic Aviation Corporation Report No. RAC 1117-6, 29 June 1964.

TABLE 1. SUMMARY OF MODE SHAPE MEASUREMENTS

Weight	Weight Location	Mode Shapes Measured at Frequency
Lb		CPS
0.0	Mid-Bay	760, 853, 1090, 1120
0.1	Mid-Bay	245, 309, 1120
0.2	Mid-Bay	193, 1120
0.3	Mid-Bay	136, 161, 1062
0.4	Mid-Bay	121
0.0	Quarter-Bay	1023, 1094, 1122, 1245
0.1	Quarter-Bay	254, 280, 337
0.2	Quarter-Bay	186
0.3	Quarter-Bay	143, 169
0.4	Quarter-Bay	132

TABLE 2. SUMMARY OF FUNDAMENTAL RESONANT FREQUENCIES AND TEST WEIGHTS

Weight, Nominal Value	Weight Location	Shaker Excitation			Acoustic Excitation						Thump Excitation		
		Test Weight			4-Inch Horn			6-Inch Horn			Test Weight		
					Test Weight		Freq.	Test Weight		Freq.			
Lb		Lb	Grams		Lb	Grams	Cps	Lb	Grams	Cps	Lb	Grams	Cps
0	Mid-Bay				0.0	0.0	1013						
0	Mid-Bay	0.010	5.0	790									
0.1	Mid-Bay	0.092	42.0	245	0.077	35.0	298	0.077	35.0	300	0.101	46.0	230
0.2	Mid-Bay	0.196	89.4	193	0.204	93.0	171				0.204	93.0	164
0.3	Mid-Bay	0.281	127.7	136	0.282	128.0	147				0.282	128.0	145
0.4	Mid-Bay	0.378	172.0	121	0.380	172.7	129	0.380	172.7	127	0.380	172.7	125
0.1	Quar-Bay	0.092	42.0	254	0.077	35.0	298	0.077	35.0	306	0.101	46.0	245
0.2	Quar-Bay	0.196	89.4	186	0.204	93.0	171				0.204	93.0	178
0.3	Quar-Bay	0.281	127.7	143	0.282	128.0	156				0.282	128.0	150
0.4	Quar-Bay	0.378	172.0	132	0.380	172.7	129	0.380	172.7	132	0.380	172.7	128

TABLE 3. PER CENT ERROR IN NATURAL FREQUENCY FOR VARIOUS N*

N*	"Resonant" Frequency - CPS	% Error
30	313.4	27.4
50	276	12.2
75	262	6.5
100	256	4.1
∞	246 (extrapolated)	-

TABLE 4. SUMMARY OF PREDICTED FUNDAMENTAL FREQUENCIES (N* = 75)

Attached Weight	Weight Location Cylinder No. 2	Predicted Resonant Frequency - cps
0.1	Mid-Bay	260 ⁺
0.2	Mid-Bay	186 ⁺
0.3	Mid-Bay	152 ⁺
0.4	Mid-Bay	132 ⁺
0.1	Quarter-Bay	270 ⁻
0.2	Quarter-Bay	192 ⁻
0.3	Quarter-Bay	157 ⁻
0.4	Quarter-Bay	136 ⁻
0.1	Three-Quarter Bay	270 ⁻
0.4	Three-Quarter Bay	136 ⁻

NOTE: Predicted fundamental frequency has been obtained with a $\Delta p = 1$ cps. The "+" indicates that the predicted frequency lies slightly above the listed frequency. The "-" indicates that the frequency is slightly less than the listed frequency.

Technical drawings of a bridge structure, including details of rivets, section D-D, and detail B.

DETAIL C
FULL SIZE

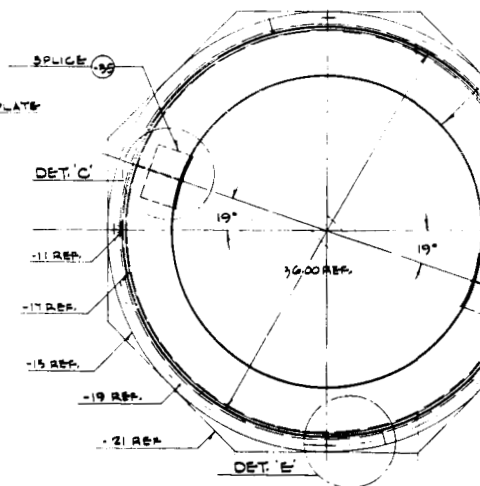
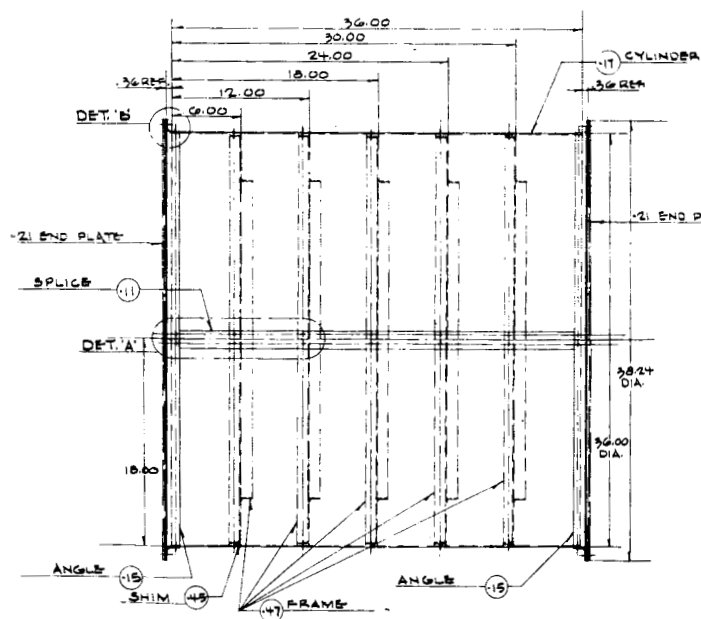
SECTION D-D
FULL SIZE

DETAIL B
FULL SIZE

DETAIL A
FULL SIZE

Key dimensions and notes include:

- 100 MAX.
- 38 TYP.
- 450
- 2 1/2 TYP.
- 38 TYP.
- 47 REF.
- 7 REF.
- 100
- 45 REF.
- 38
- 47 REF.
- 39 REF.
- 2.62
- 12 R TYP.
- 50
- 100
- 19°
- HORIZ. 22°
- MS 20470006 RIVETS
- 3000 DIA.
- 15AL PER AS-BUILT-2 BOTH ENDS
- 38
- 51 TYP.
- 32 TYP.
- 32 TYP.
- 32 TYP.
- 11 REF.
- 150
- 17 REF.
- 38
- 112
- 25 R.
- 5 R.A.R.
- 112
- 120
- 19 REF.
- 70
- 45 REF.
- 47 REF.
- MS 2047005 SEE NOTE 1
- 1100
- 45 REF.
- 47 REF.
- MS 2047005 SEE NOTE 1



⑤ CYLINDER ASSEMBLY
1/4 SCALE

67 (D)

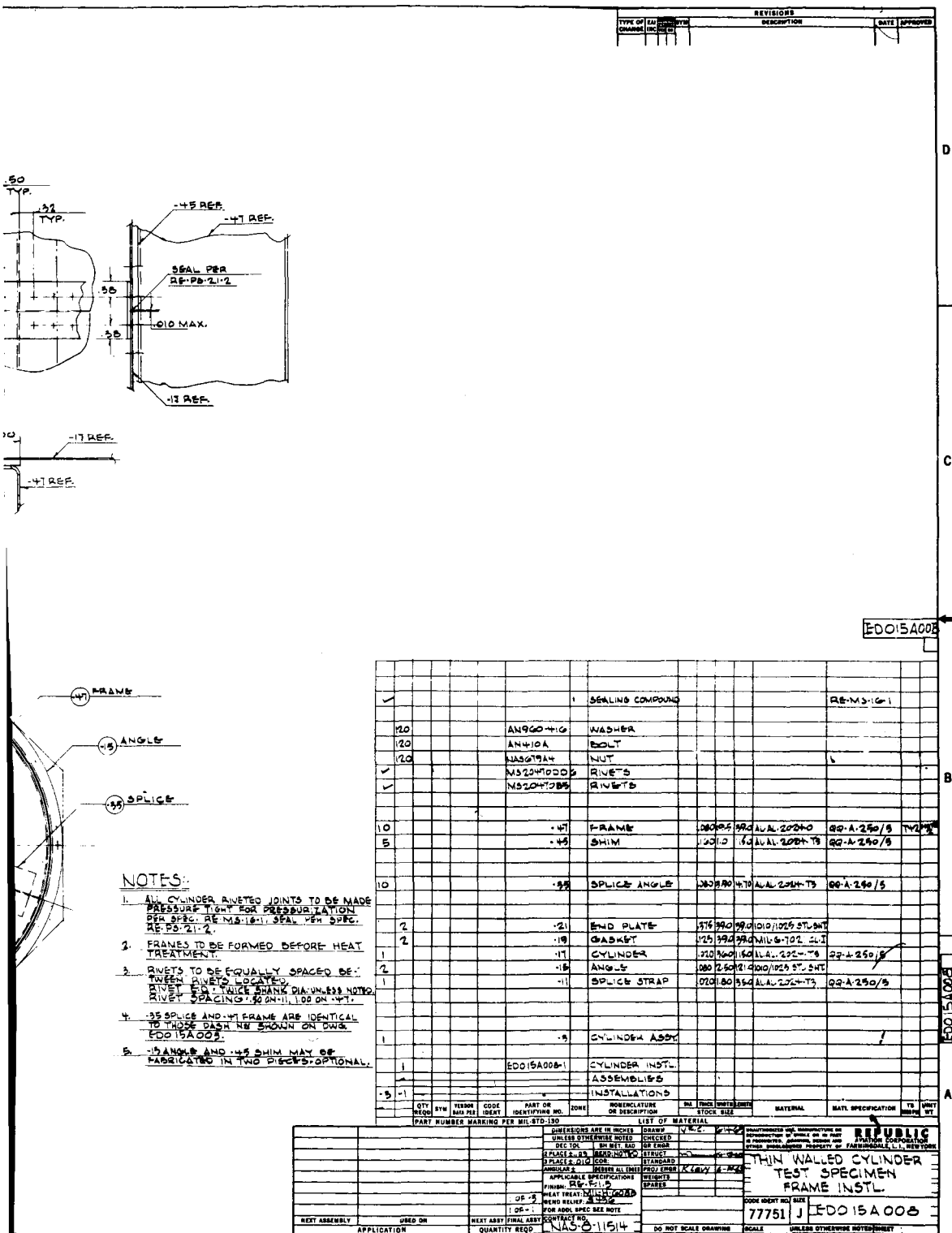
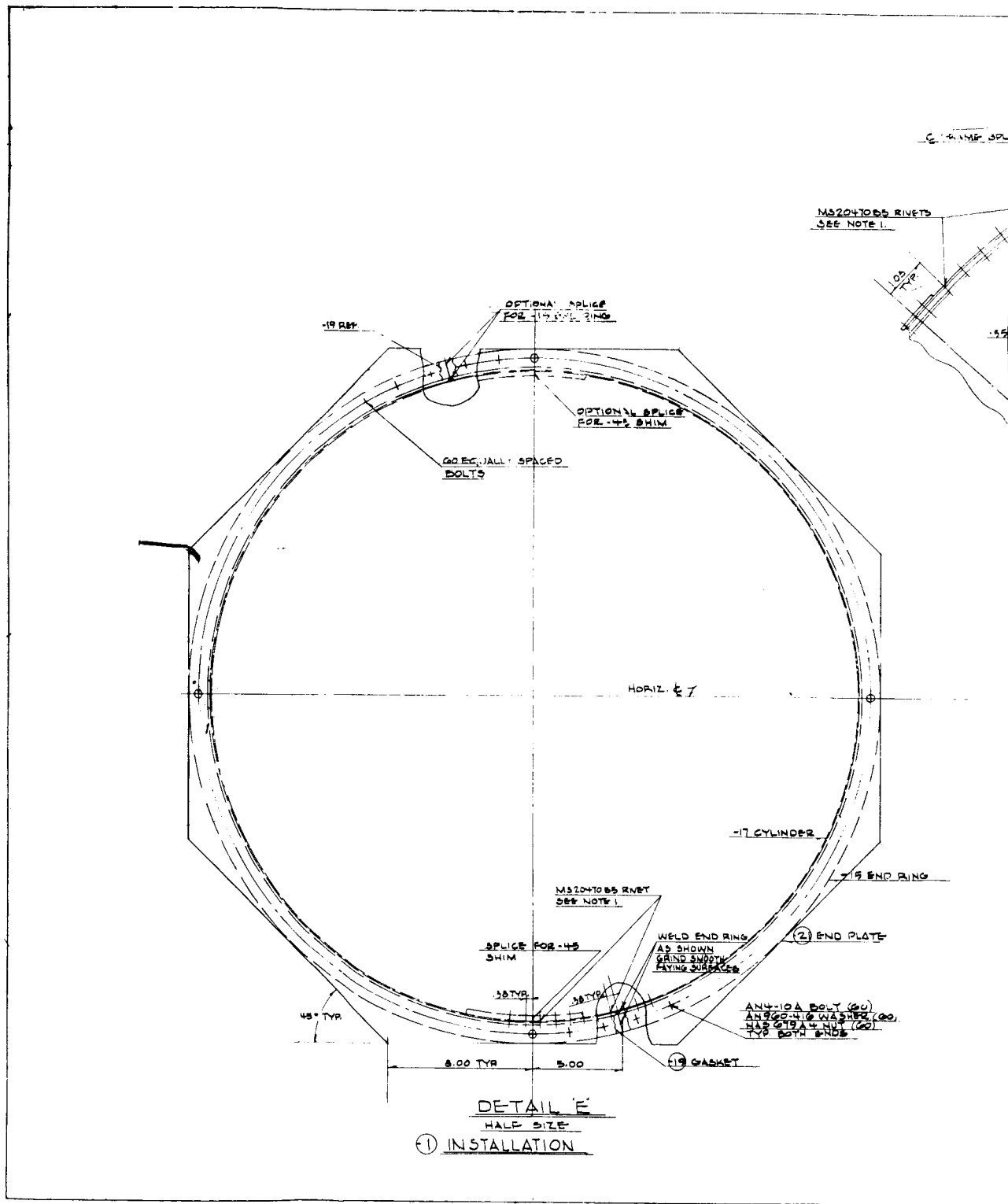
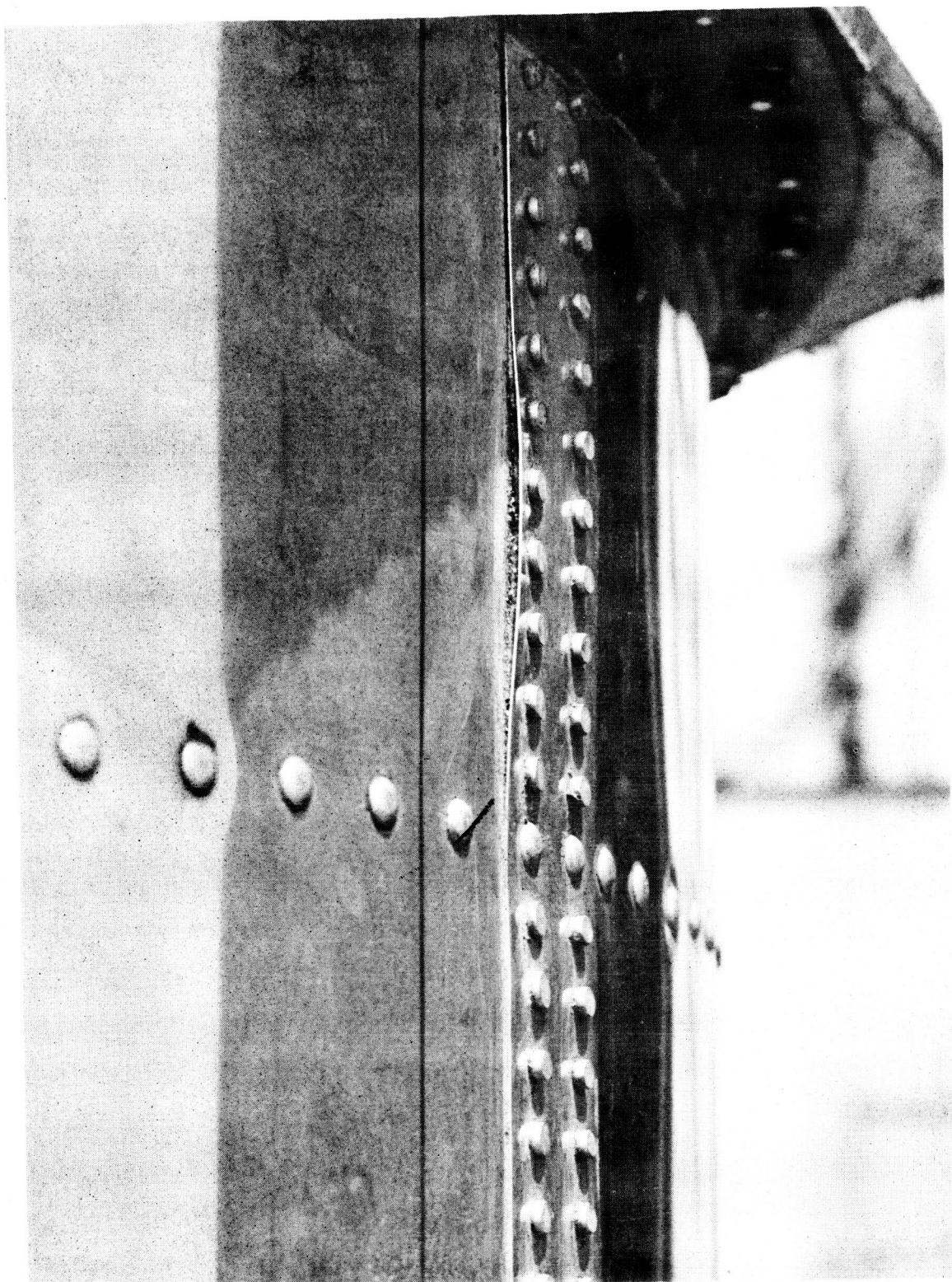


Figure 1. Thin-Walled Cylinder Test Specimen

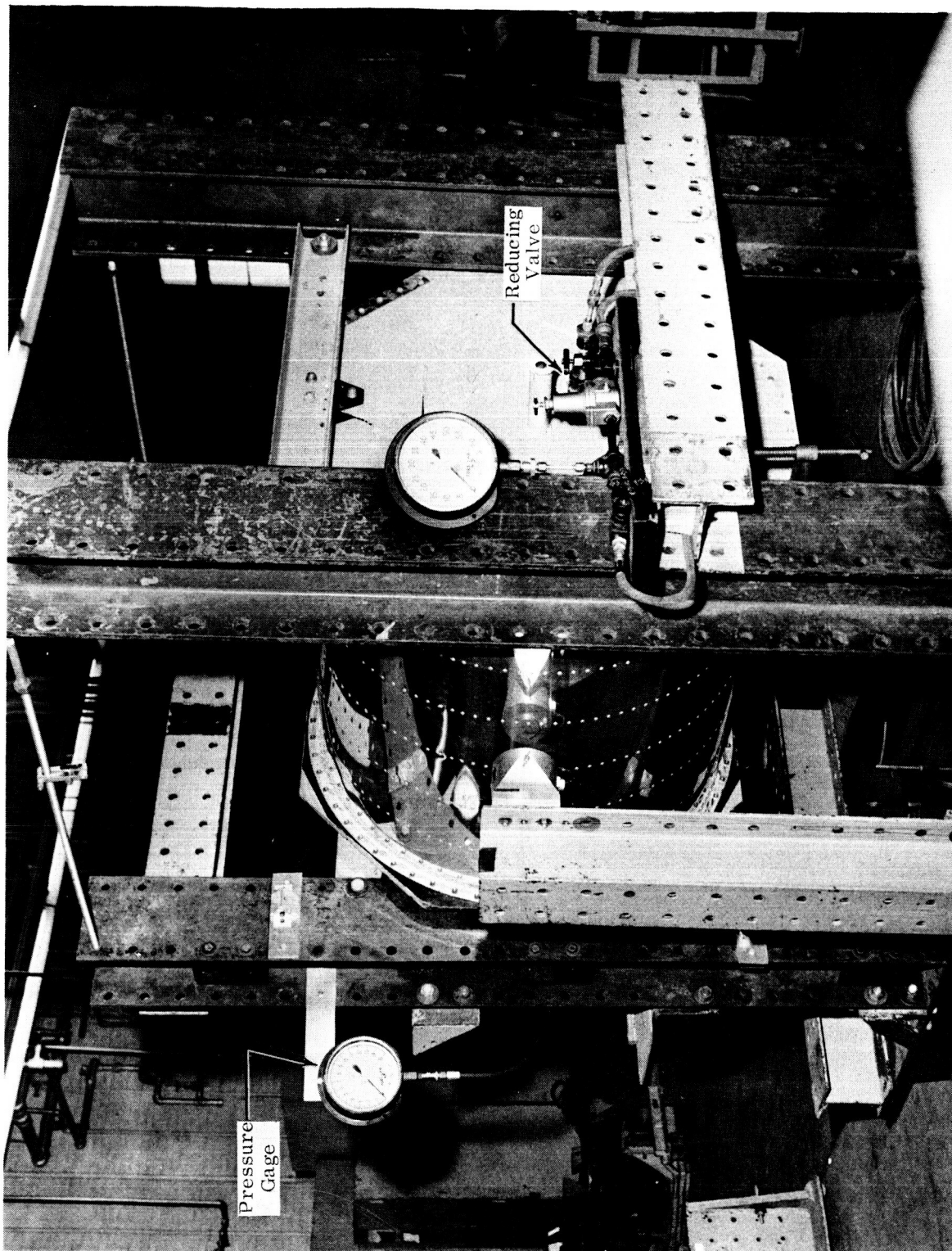


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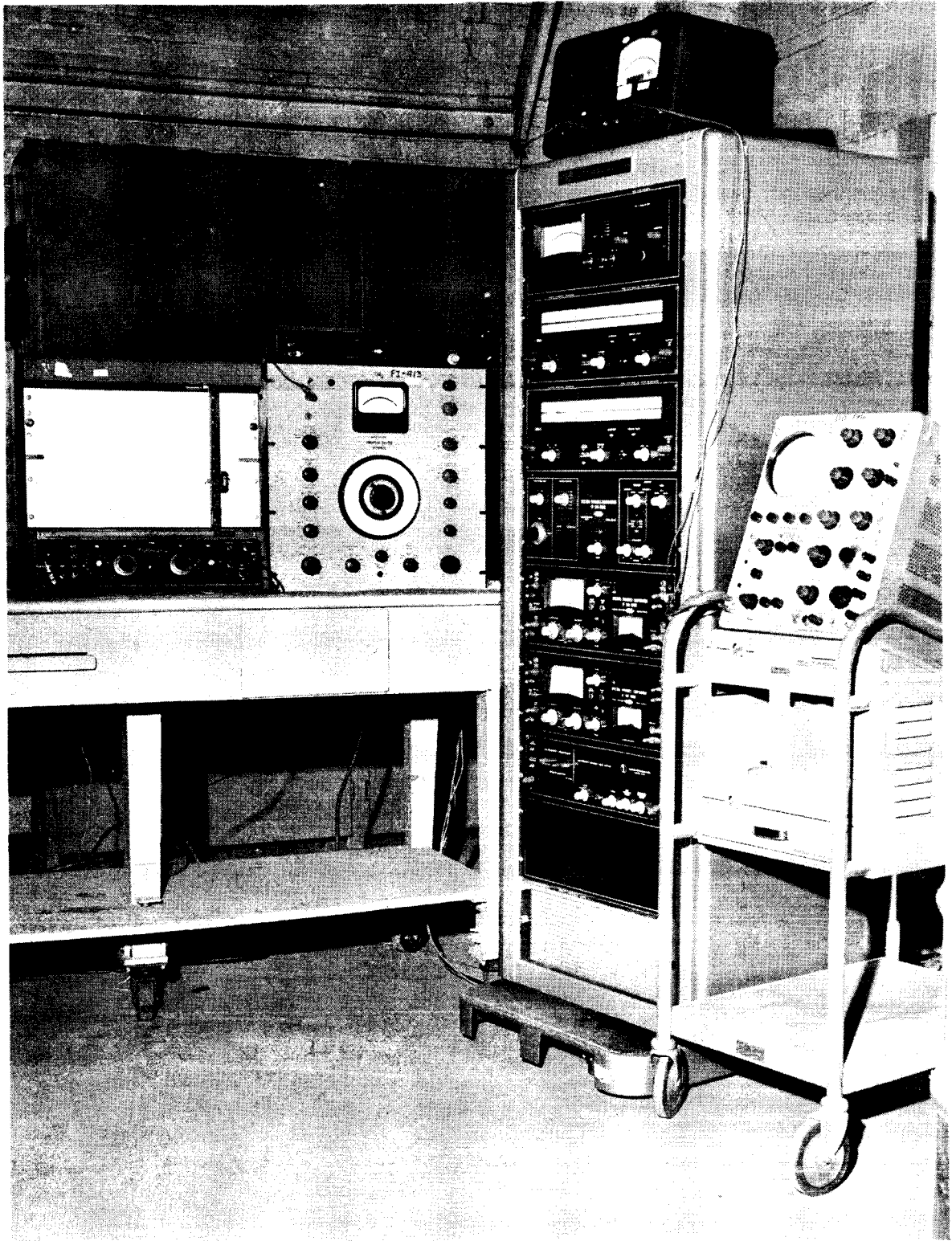
P-17522

Figure 2. Skin Splice After Test



P-17543

Figure 3. New Cylinder No. 2 in Test Fixture



P-17551

Figure 4. Photograph of Electrical Test Equipment

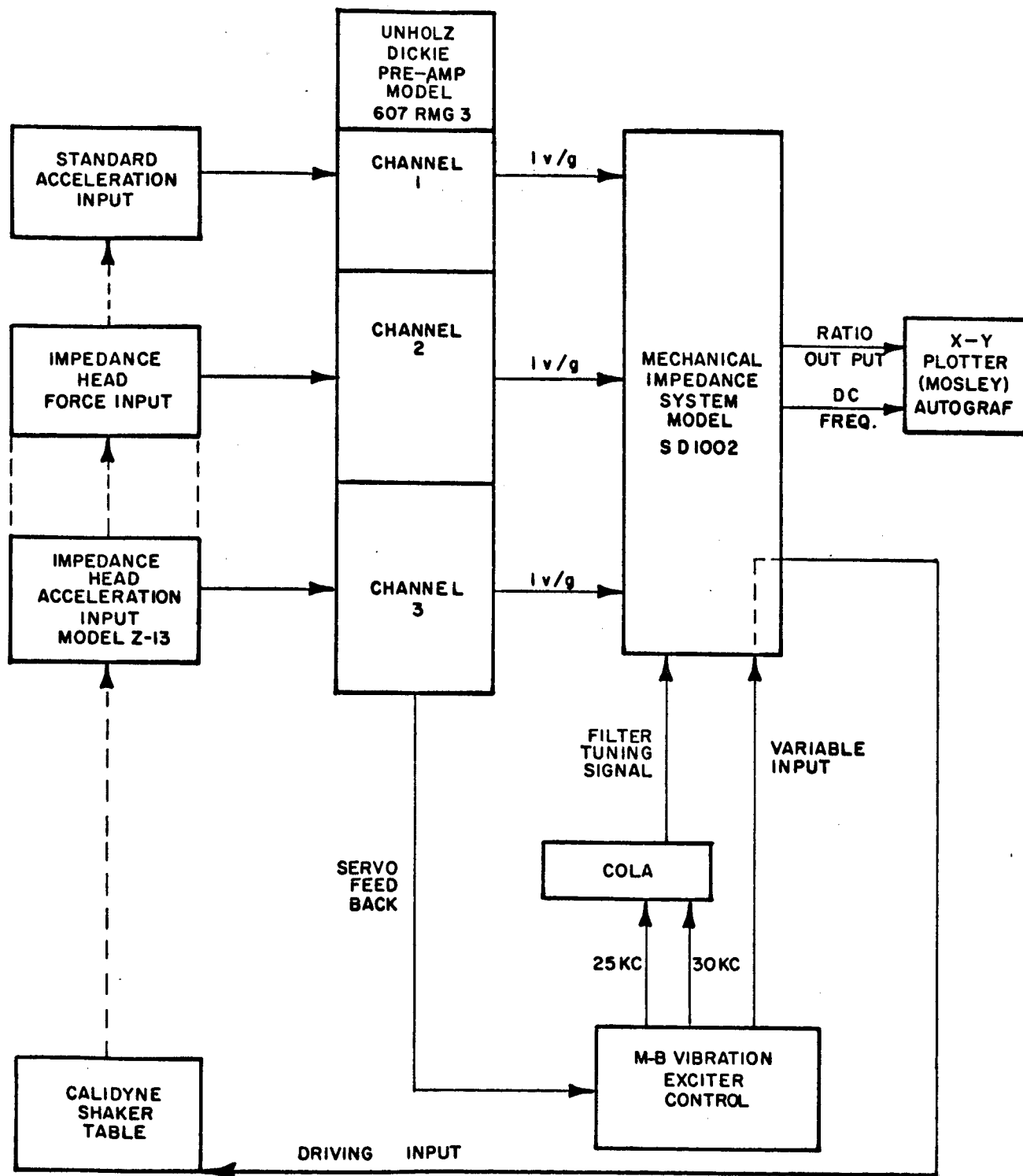


Figure 5. Block Diagram of Test Equipment

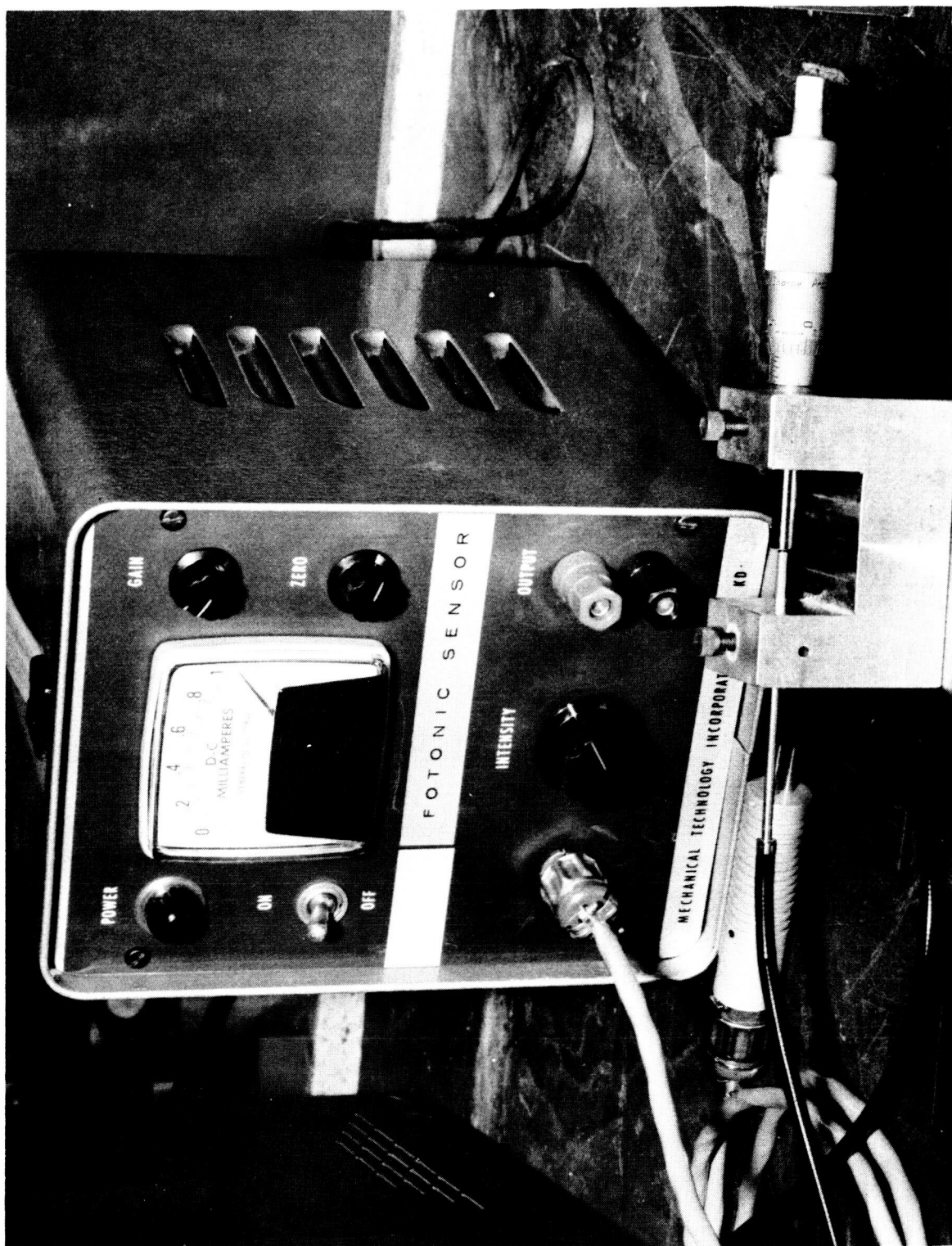
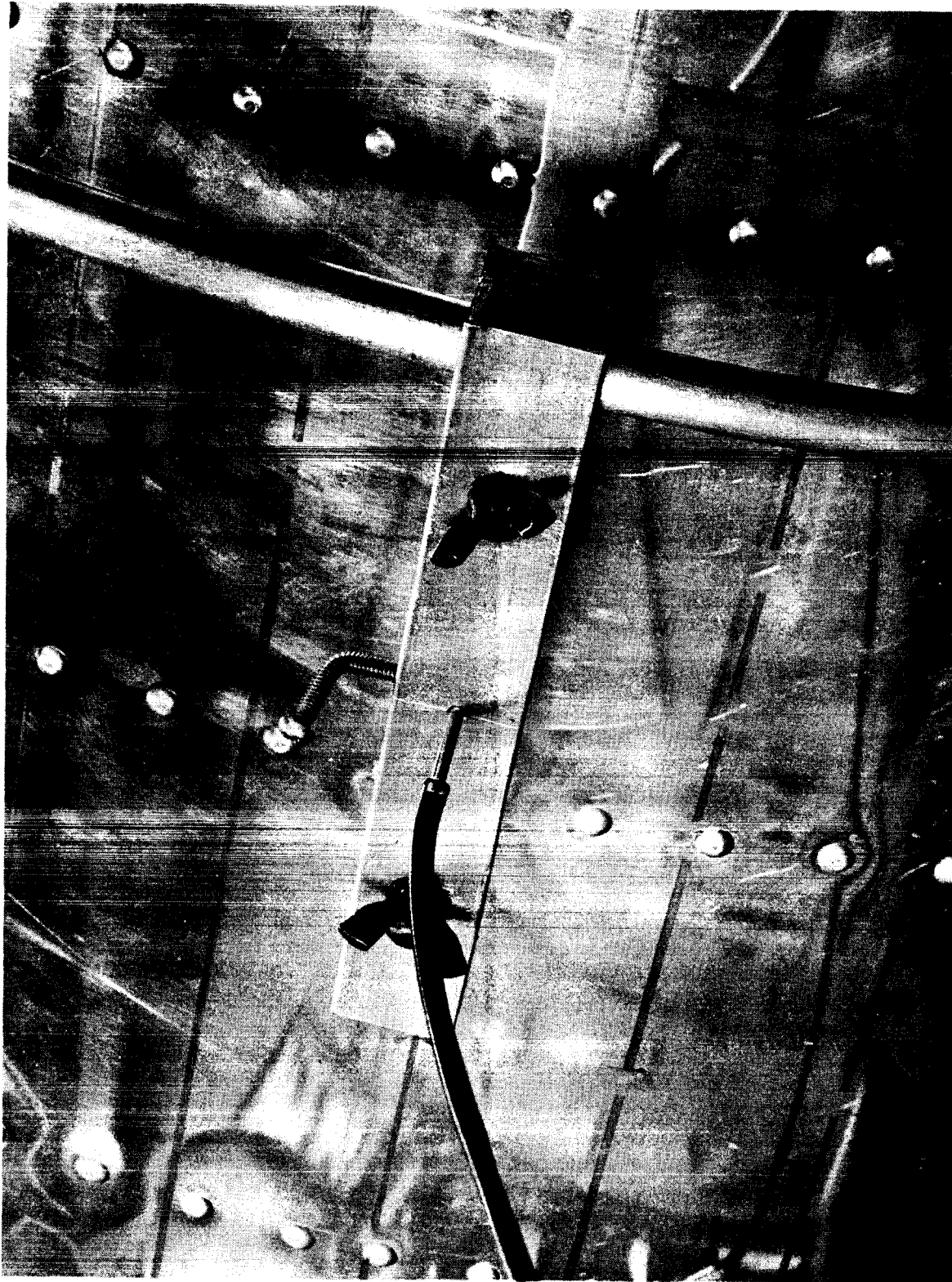


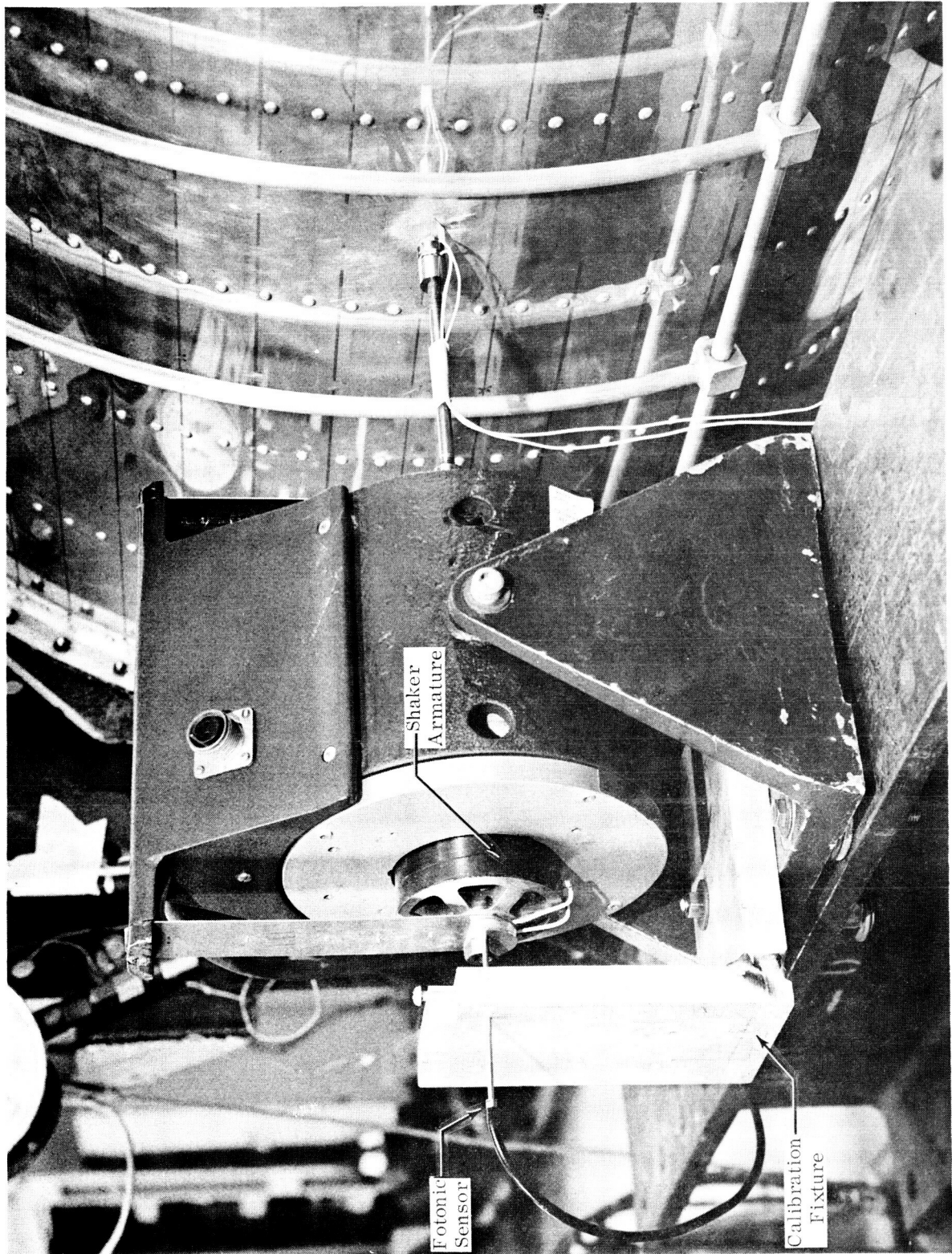
Figure 6. MTI Model KD45 - FOTONIC Sensor

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P-17561

Figure 7. Closeup of Fotonic Sensor Holding Fixture



P-17559

Figure 8. Closeup of Fotonic Sensor Calibration Fixture

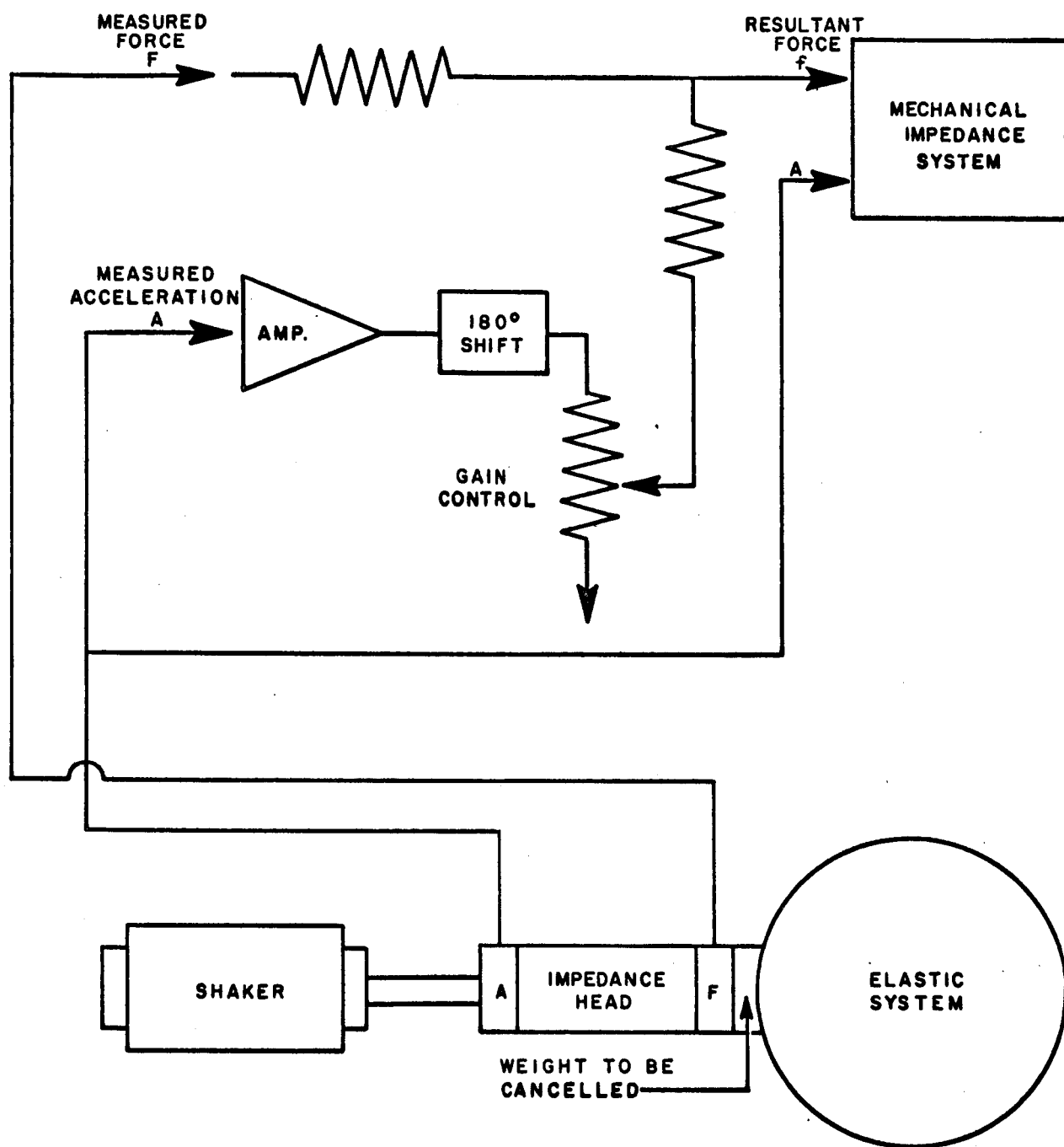


Figure 9. Schematic of Mechanical Impedance System with Mass Cancellation

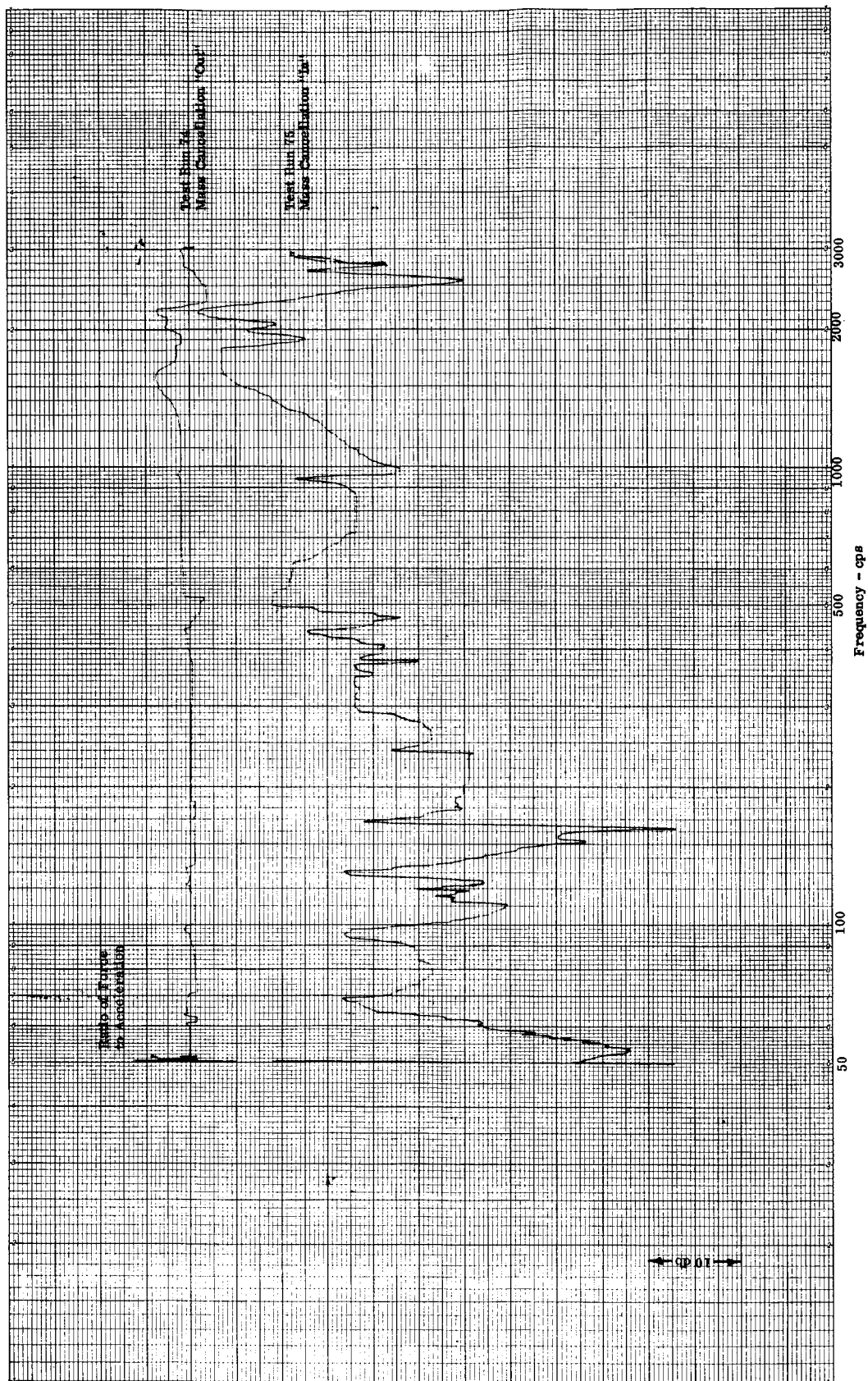


Figure 10. Effect of Mass Cancellation on Pendulum Weight of 0.1 Pound (Test Runs 74 and 75)

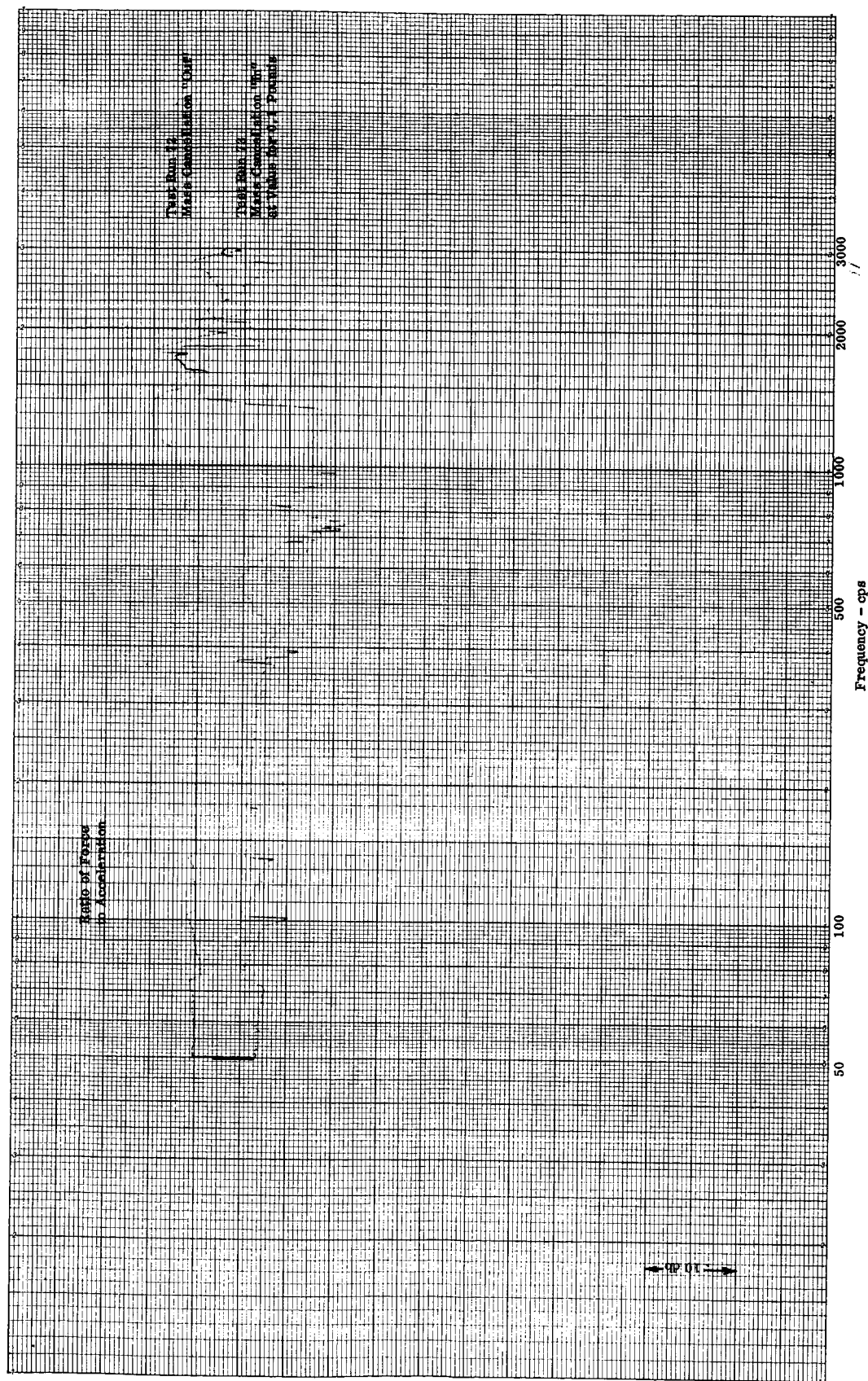


Figure 11. Effect of Mass Cancellation on Pendulum Weight of 0.2 Pound (Test Runs 72 and 73)

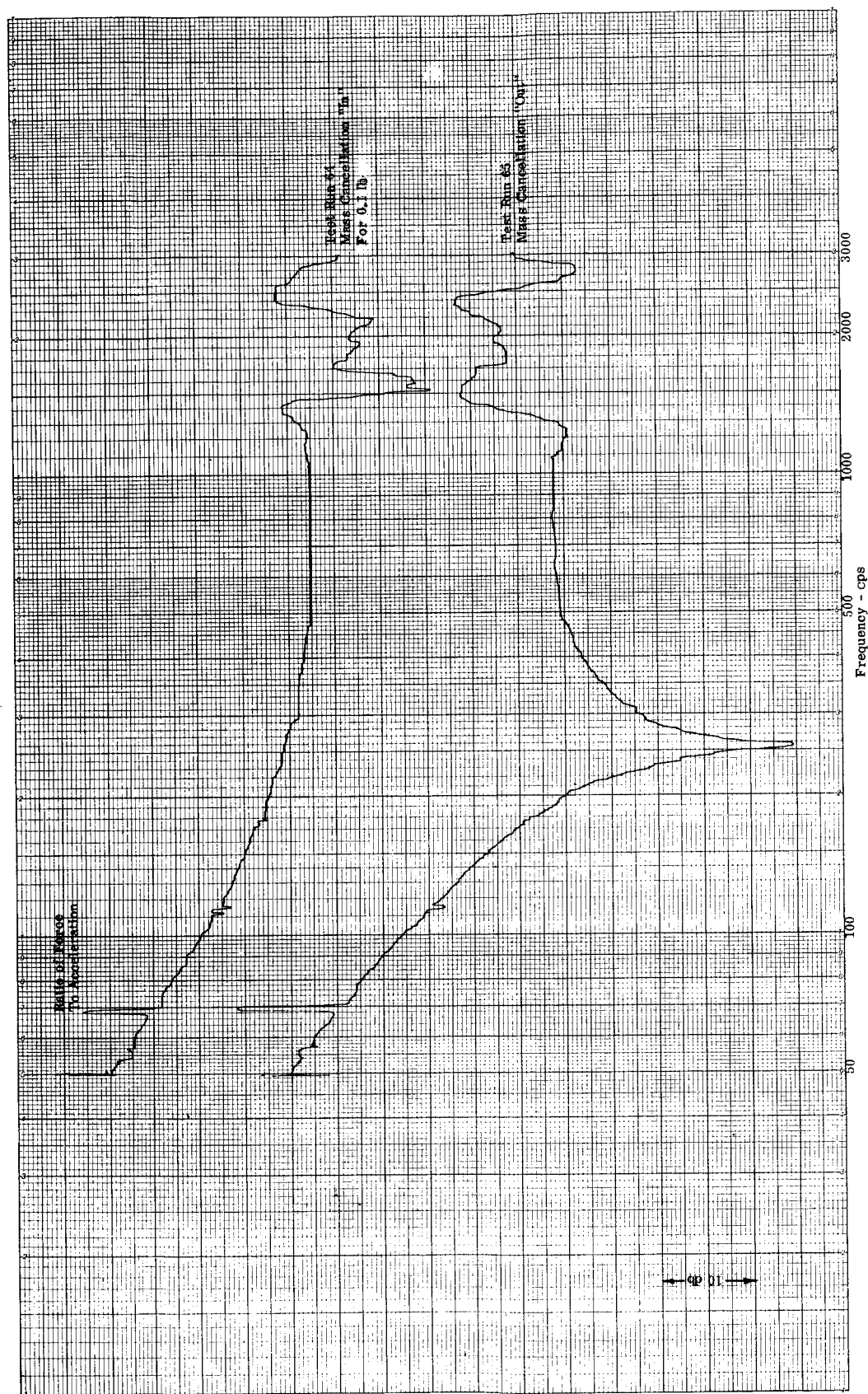


Figure 12. Effect of Mass Cancellation, 0.10-Pound Weight on Cylinder (Test Runs 64 and 65)

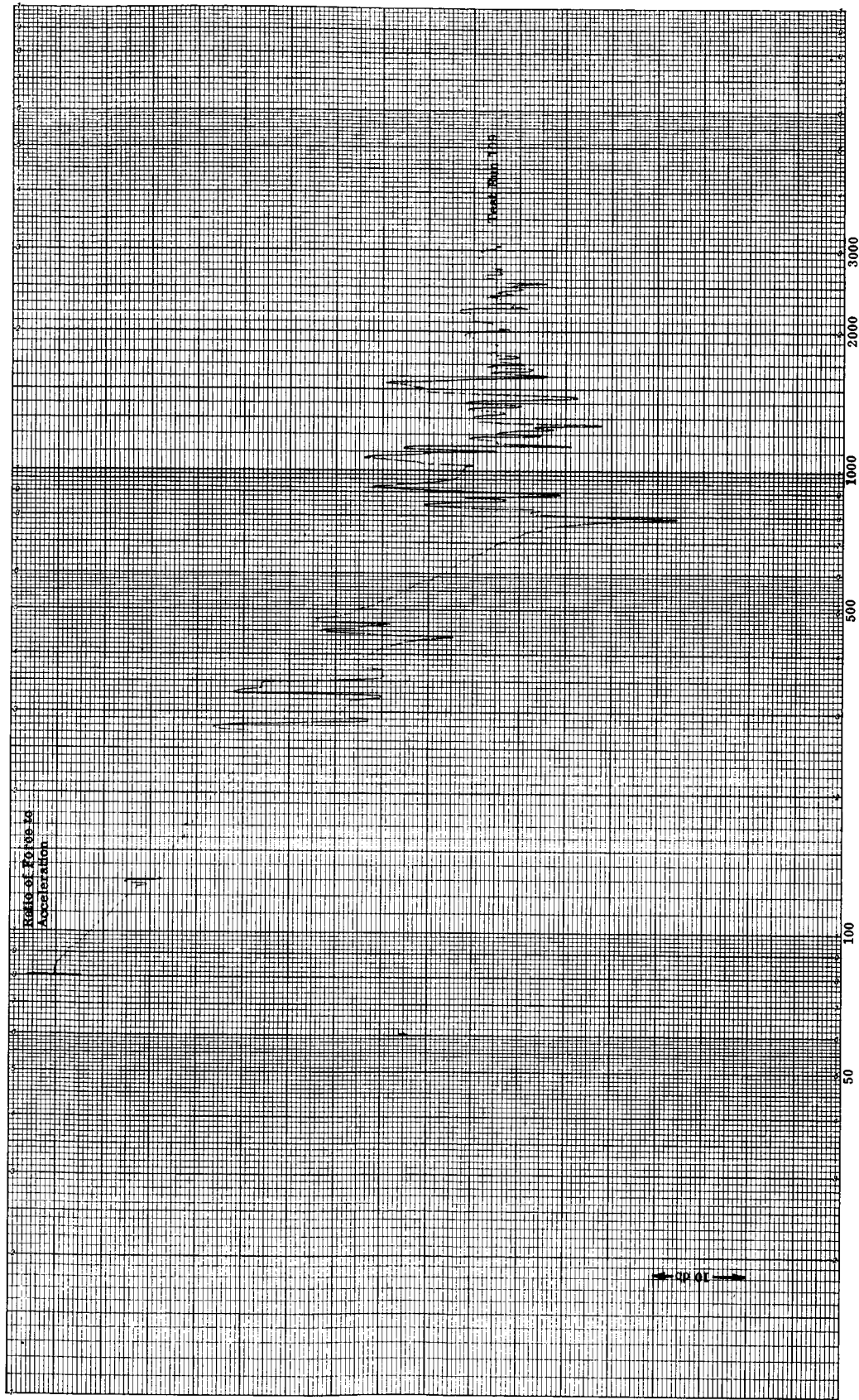


Figure 13. Response of Cylinder with 0.01 Pound Added, Mass Cancellation "Out" (Test Run 109)

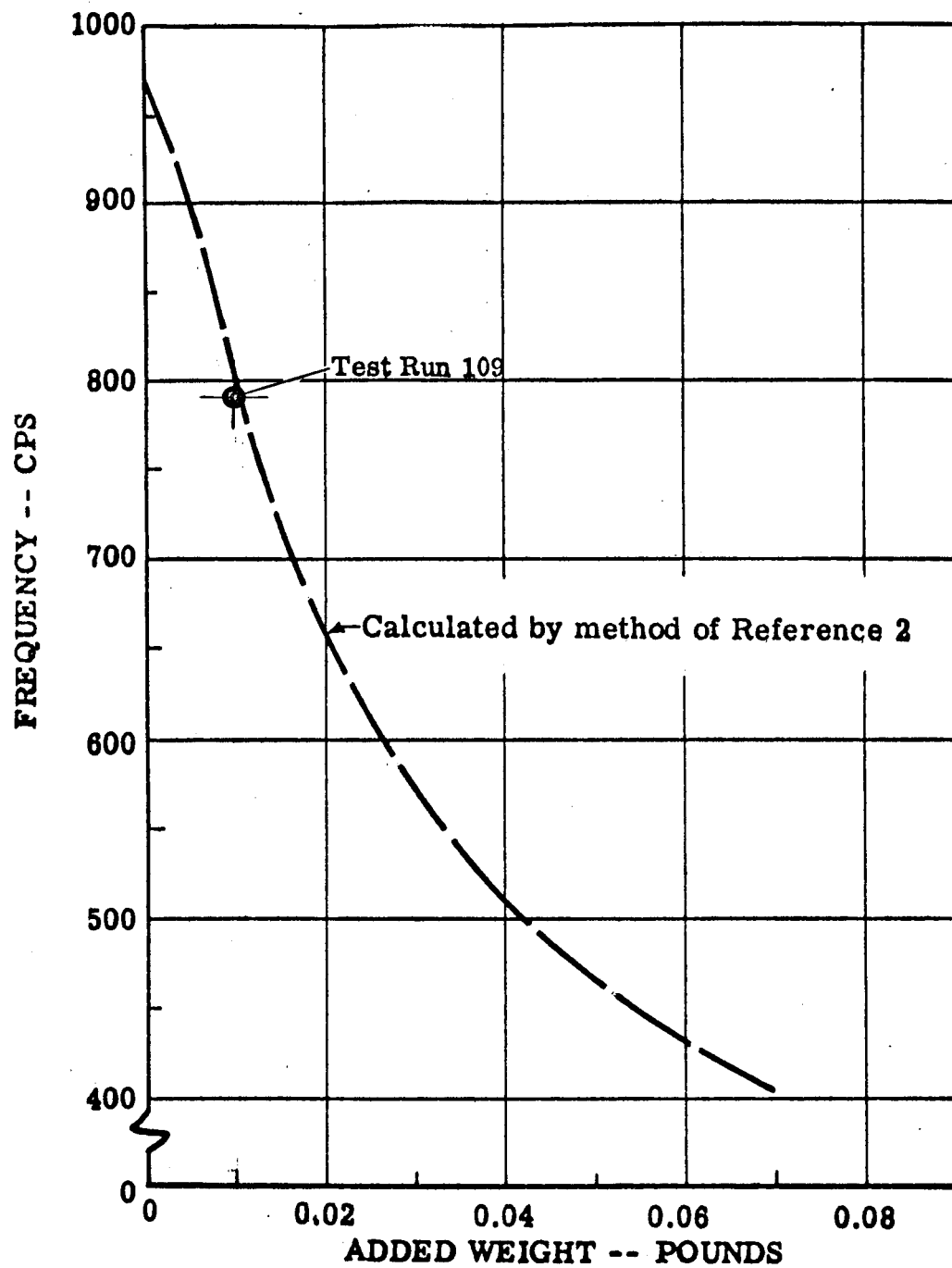


Figure 14. Effect of Small Added Weights Midway between Ring Stiffeners Upon Fundamental Frequency of Cylinder No. 2

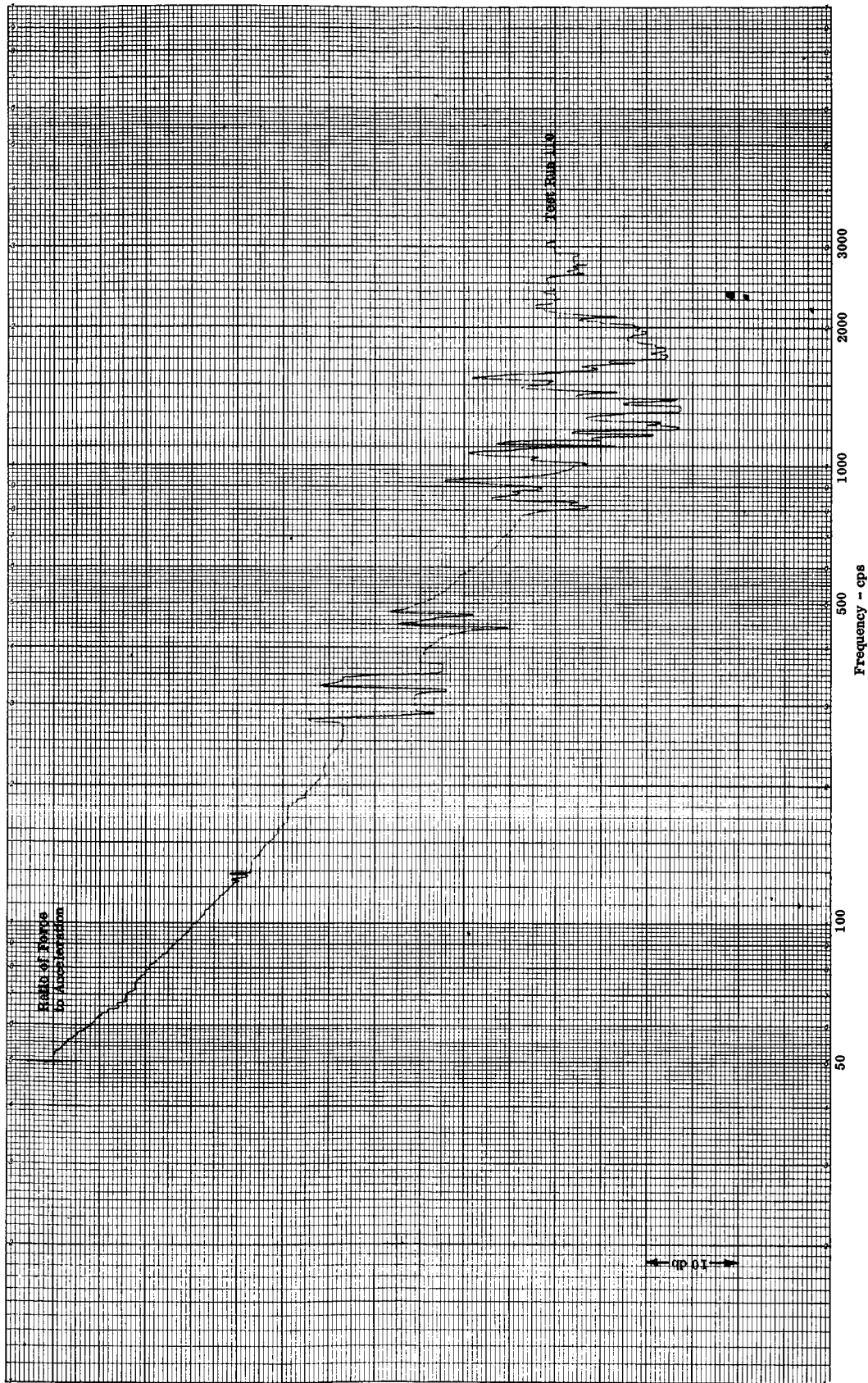


Figure 15. Response of Cylinder with 0.01 Pound Added, Mass Cancellation "In" (Test Run 110)

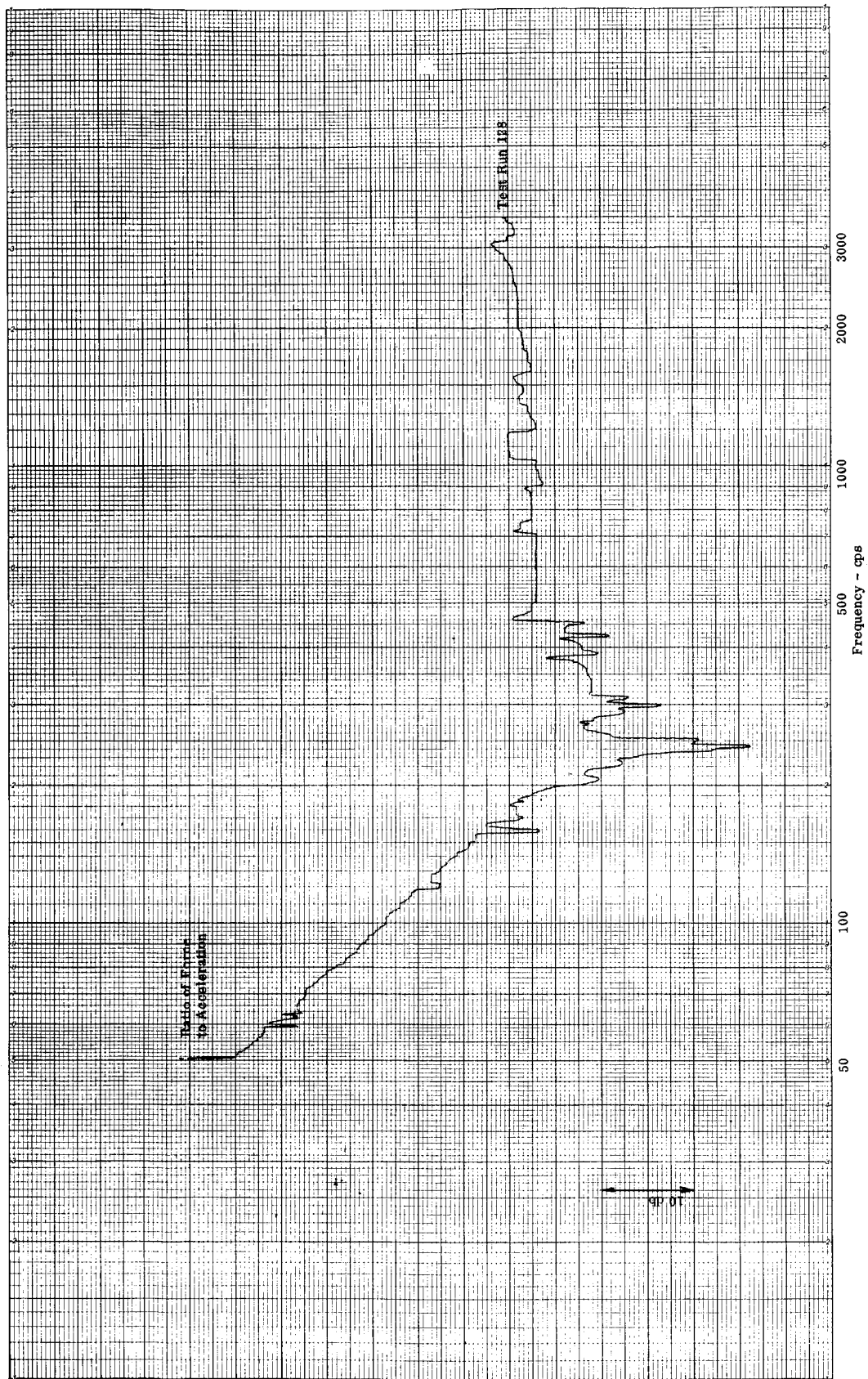


Figure 16. 0.1 Pound at Mid-Bay, Shaker Excitation, Plot of F/A (Test Run 128)

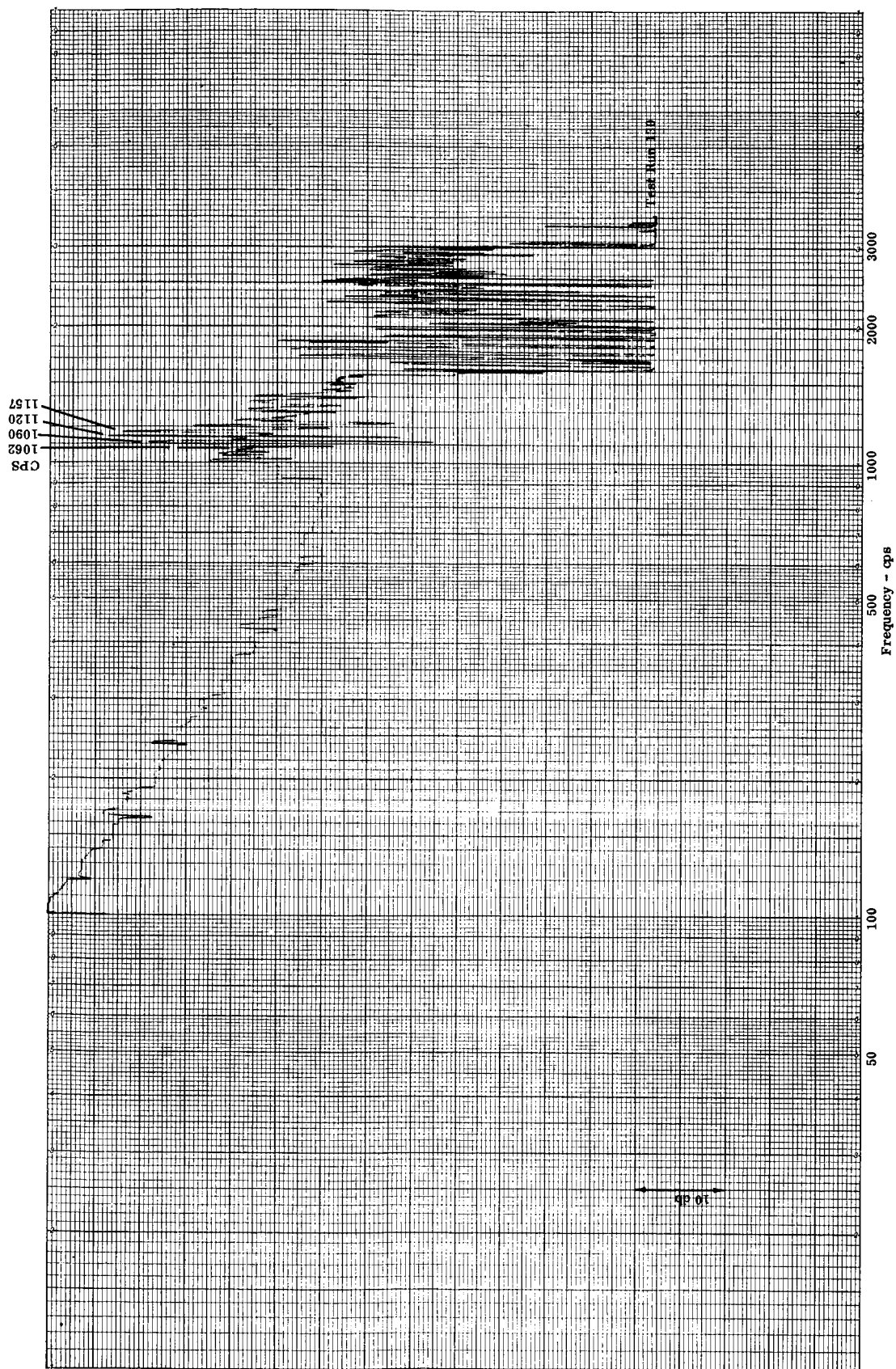


Figure 17. 0.1 Pound at Mid-Bay, Shaker Excitation, Plot of Displacement at 192° (Test Run 130)

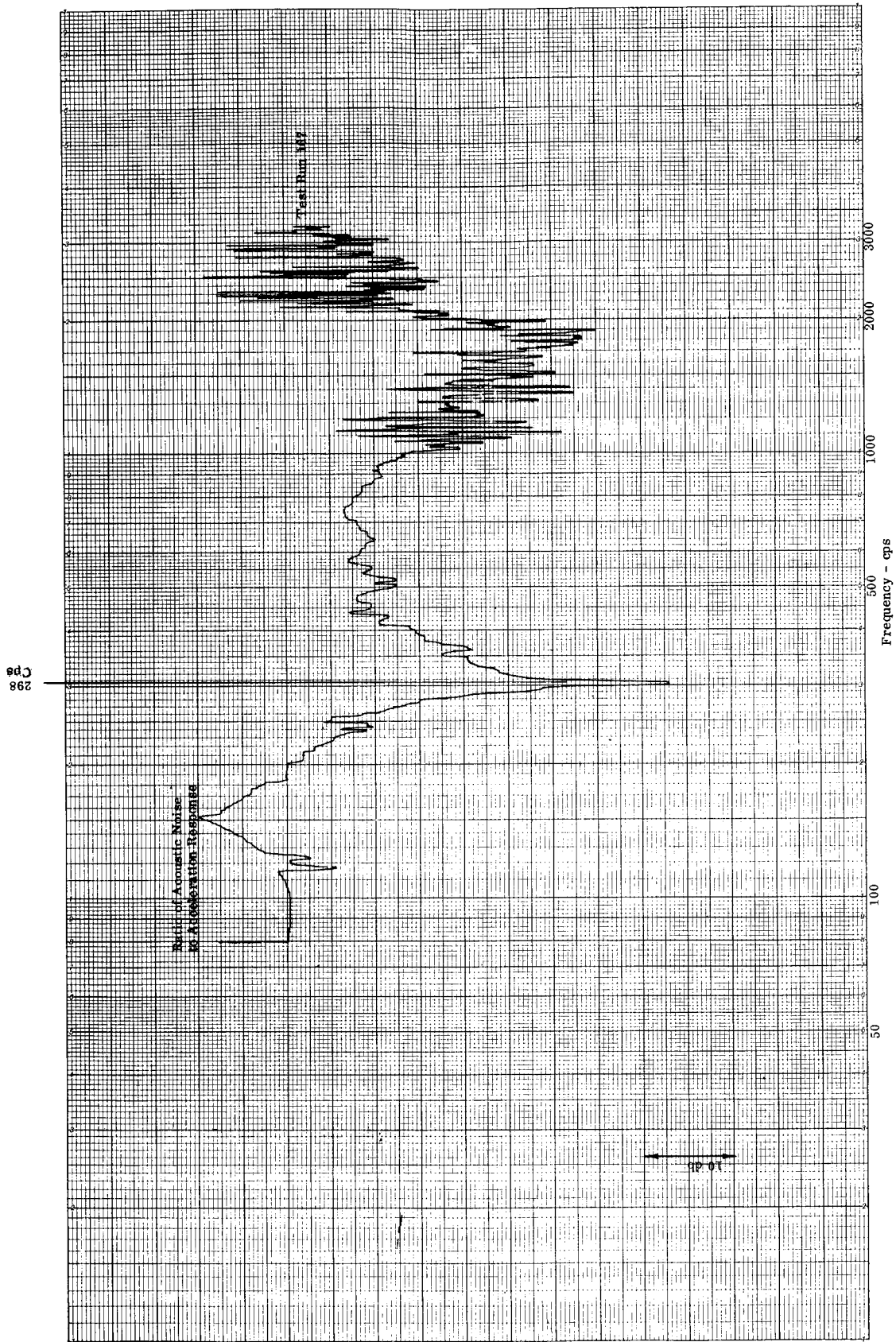


Figure 18. 0.1 Pound at Mid-Bay, Four-Inch Horn Excitation, Plot of db/Acceleration (Test Run 167)

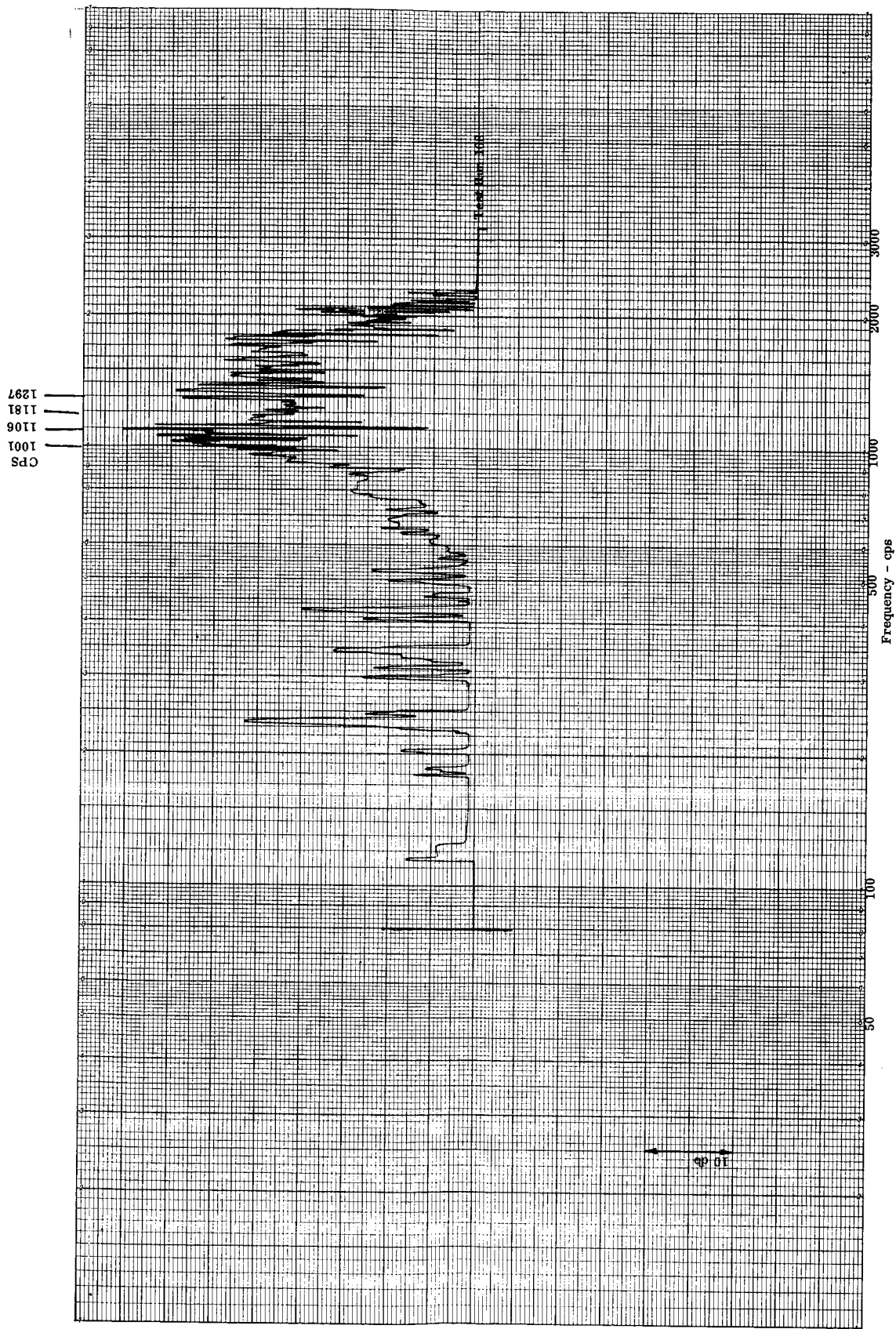


Figure 19. 0.1 Pound at Mid-Bay, Four-Inch Horn Excitation, Displacement at 204 Degrees (Test Run 168)

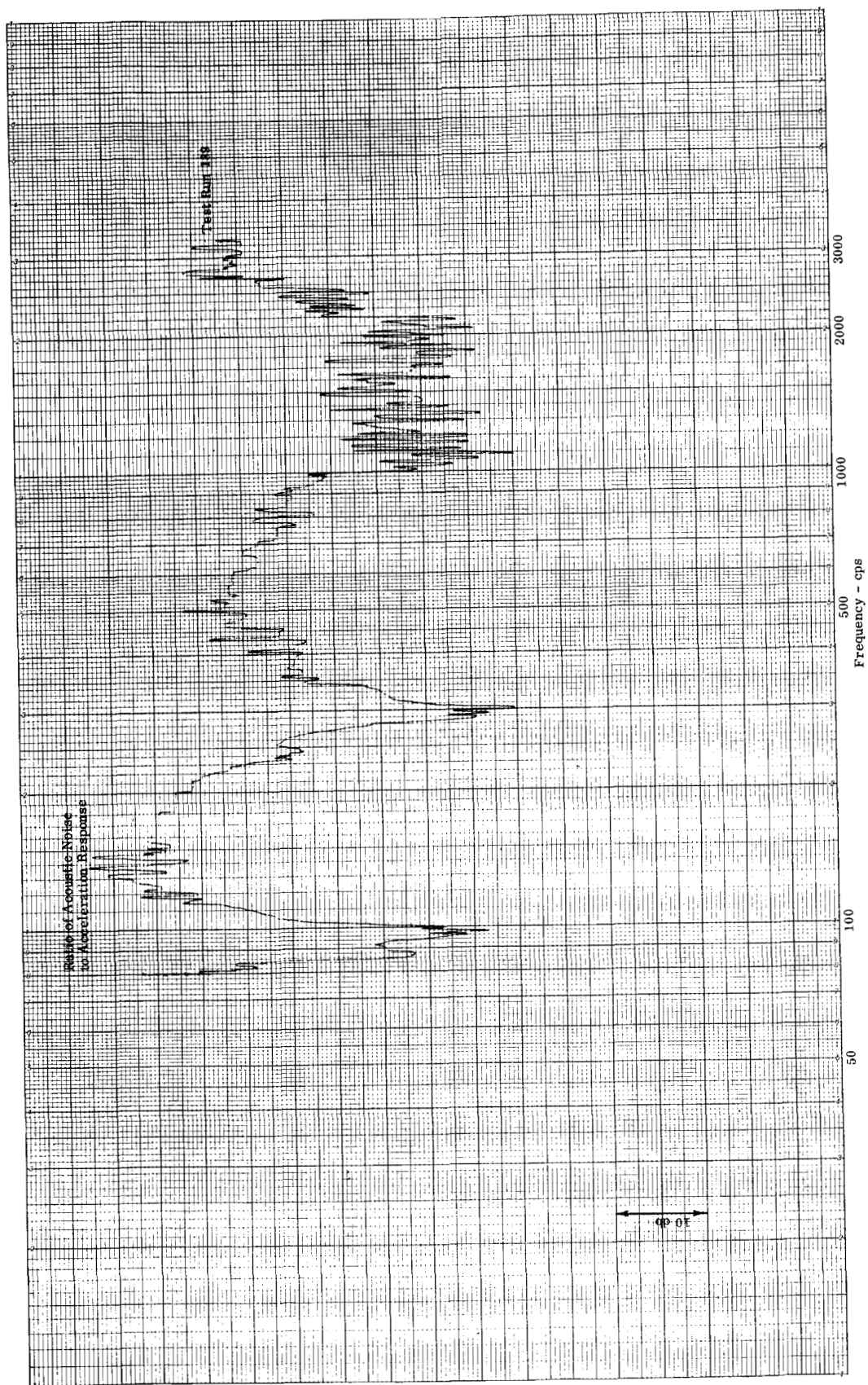


Figure 20. 0.1 Pound at Mid-Bay, Six-Inch Horn Excitation, Plot of db/Acceleration (Test Run 189)

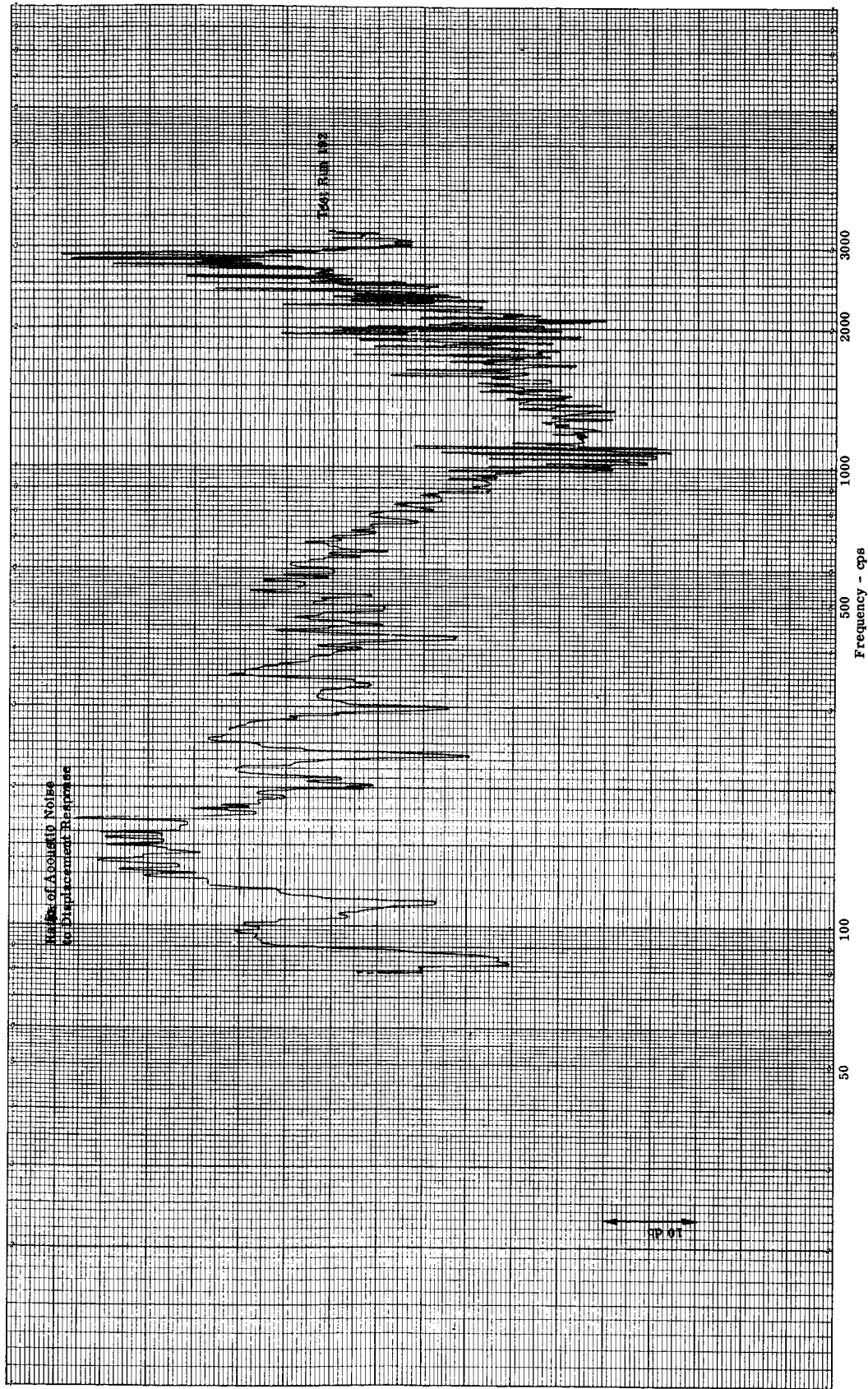


Figure 21. 0.1 Pound at Mid-Bay, Six-Inch Horn Excitation, Plot of db/Displacement, Sensor at 195 Degrees (Test Run 192)

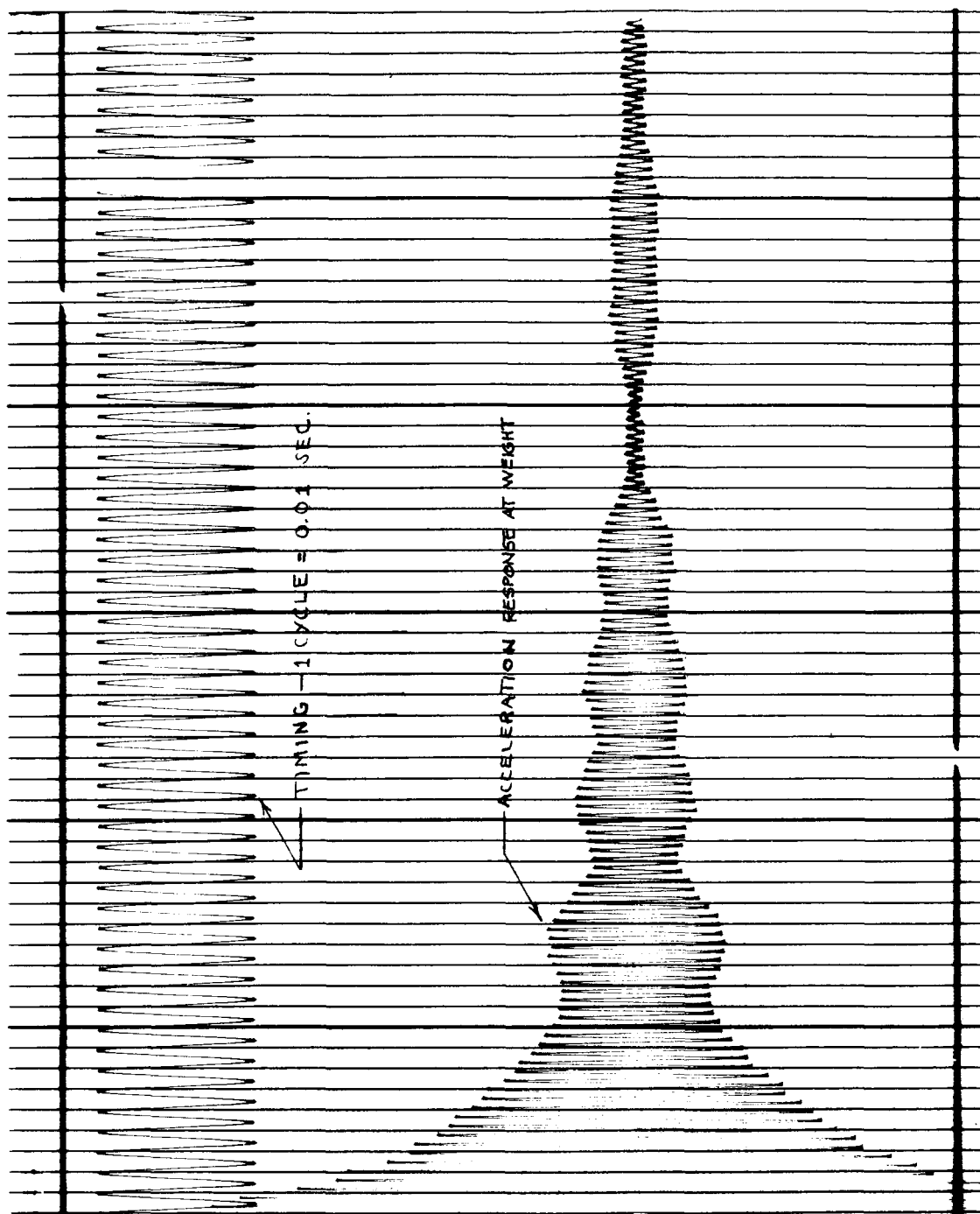


Figure 22. Thump Test, 0.1-Pound Weight Attached at Mid-Bay, Fundamental Resonant Frequency 230 cps

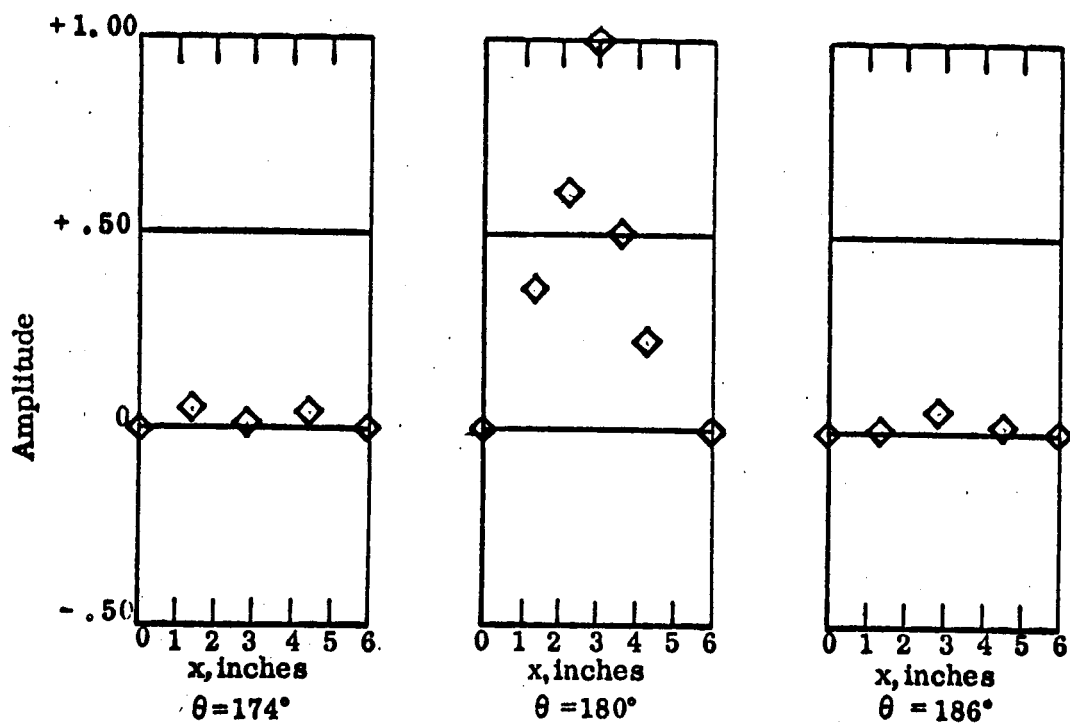
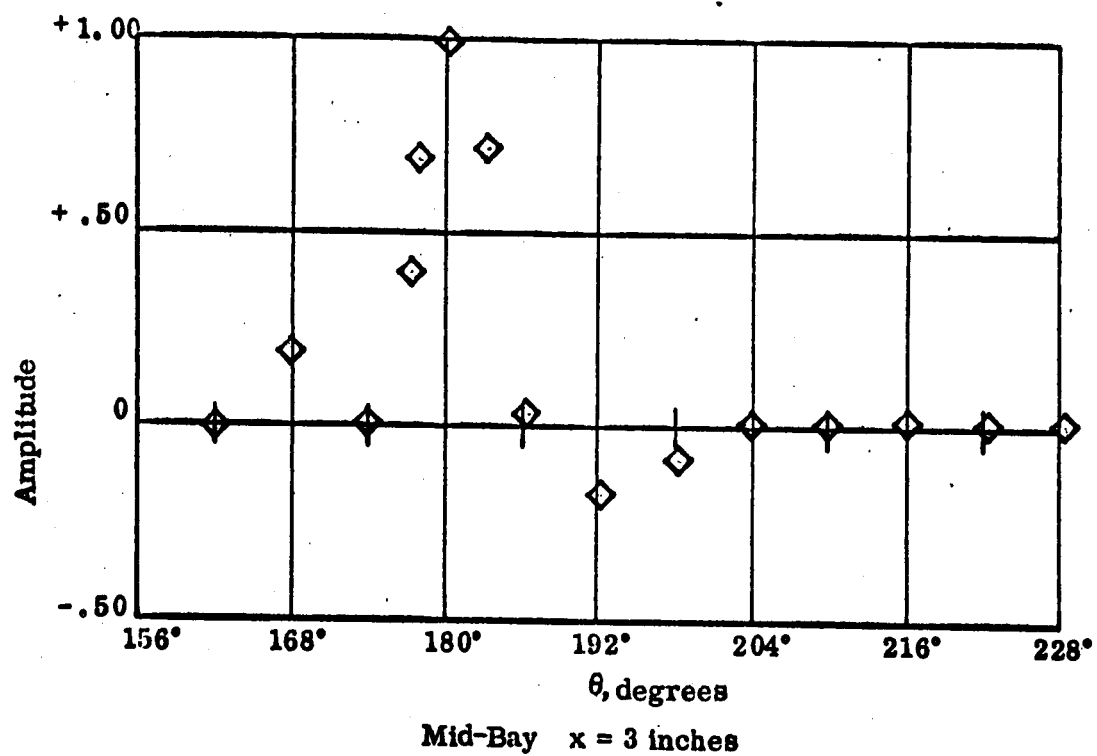


Figure 23. Fundamental Mode Shape at 245 cps, 0.1 Pound at Mid-Bay (Test Run 129A)

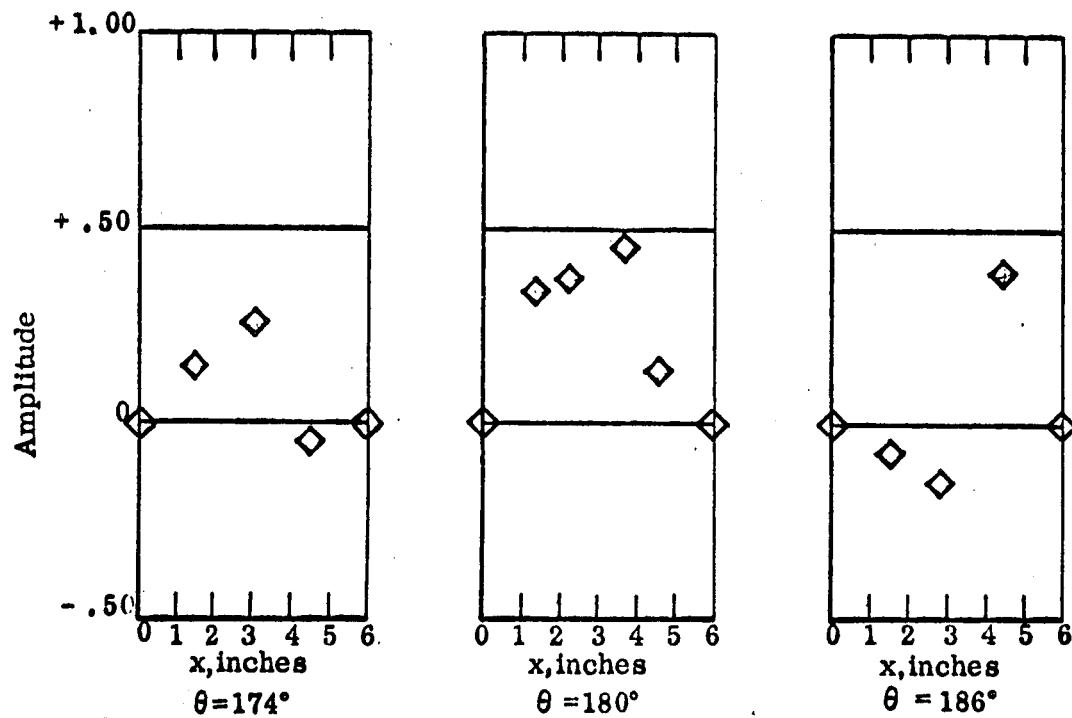
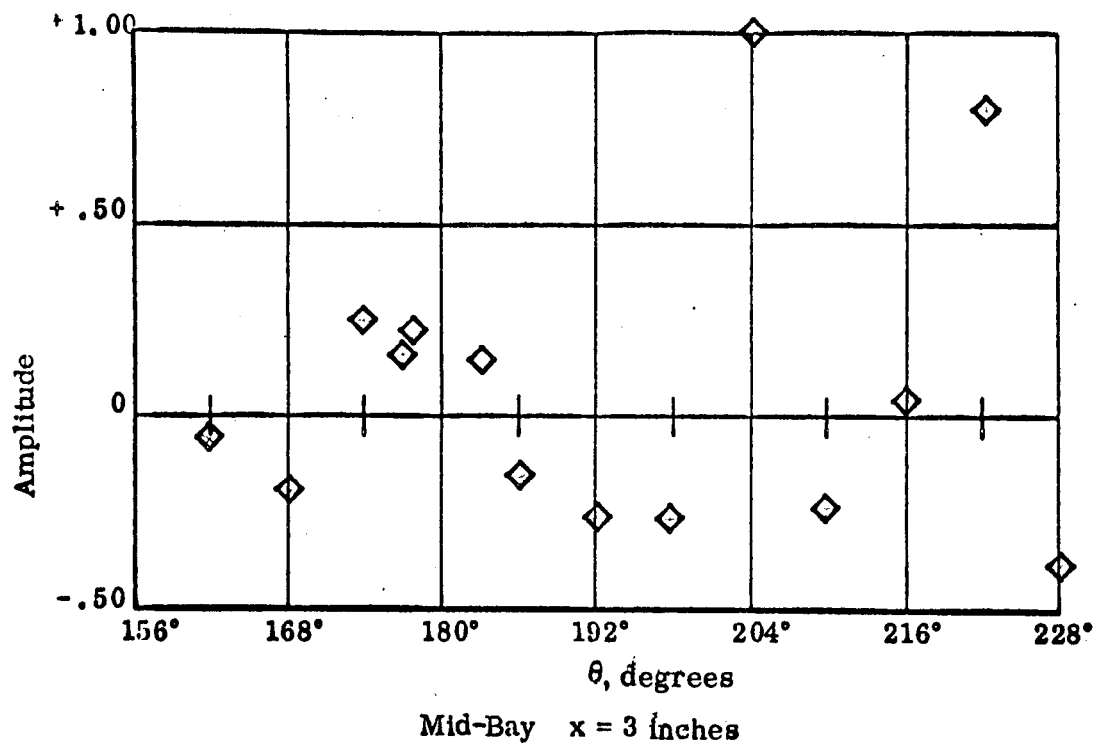


Figure 24. Mode Shape at 1120 cps, 0.1 Pound at Mid-Bay (Test Run 130C)

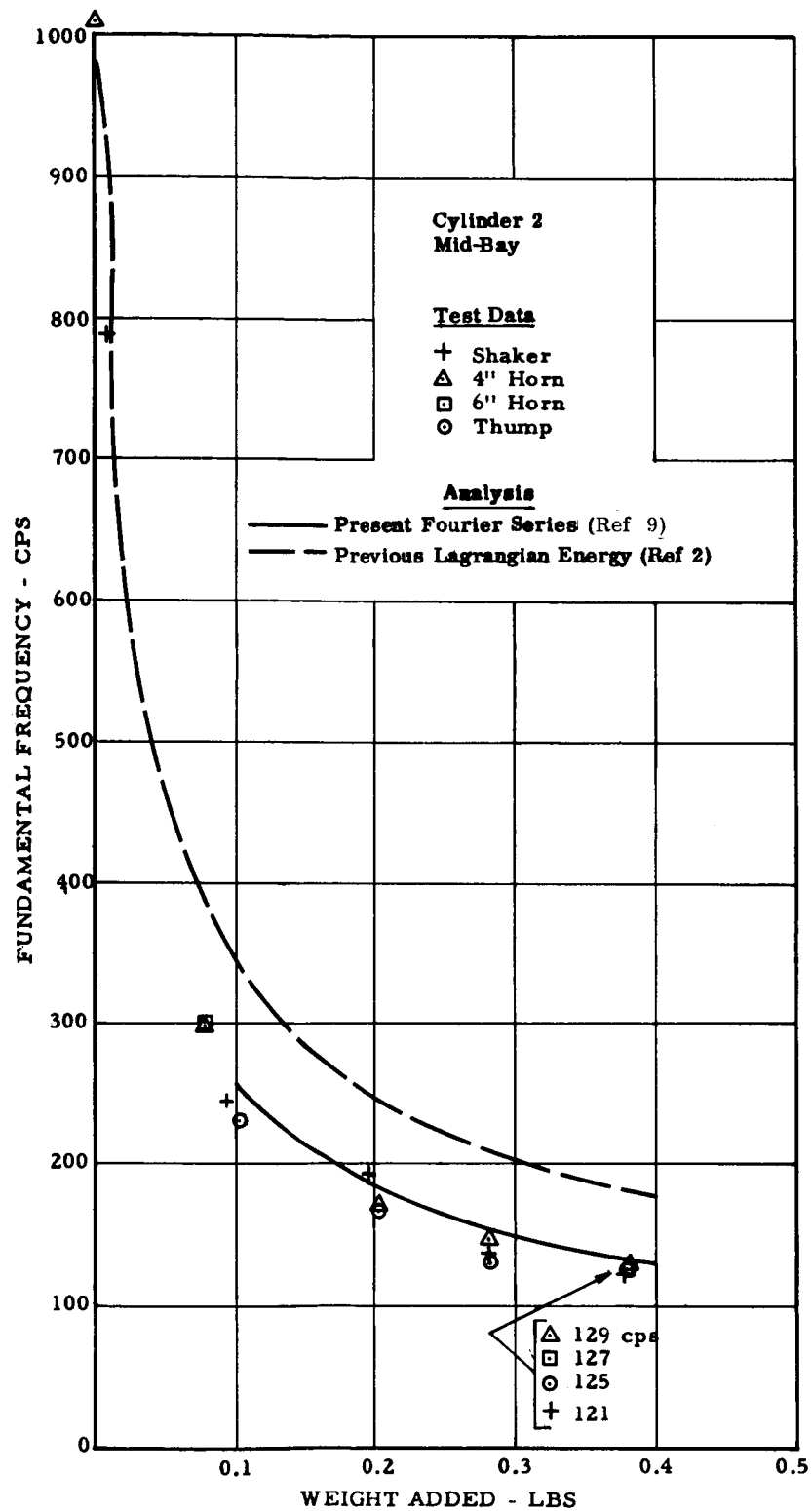


Figure 25. Fundamental Frequency versus Weight Added at Mid-Bay

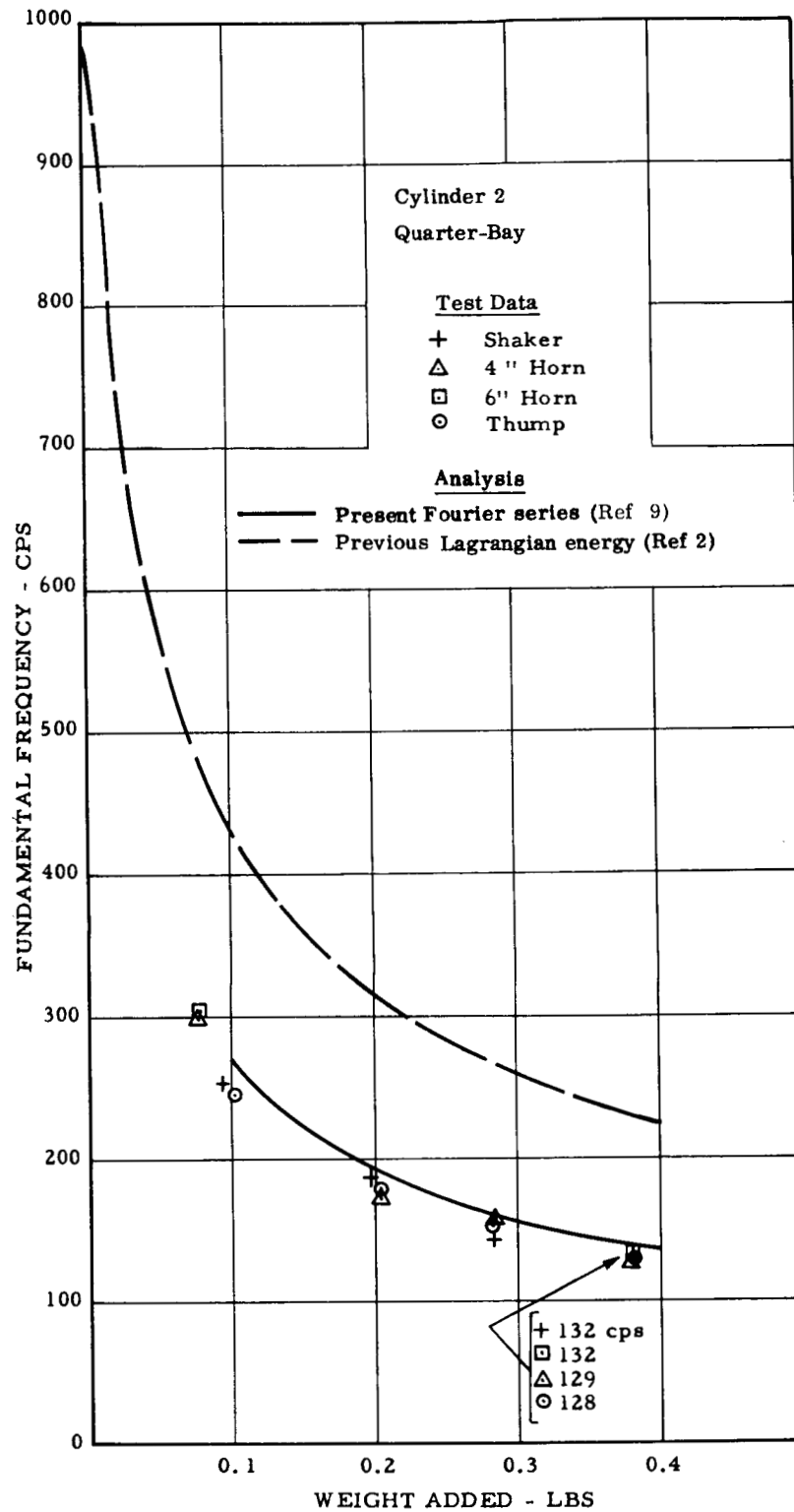


Figure 26. Fundamental Frequency versus Weight Added at Quarter-Bay

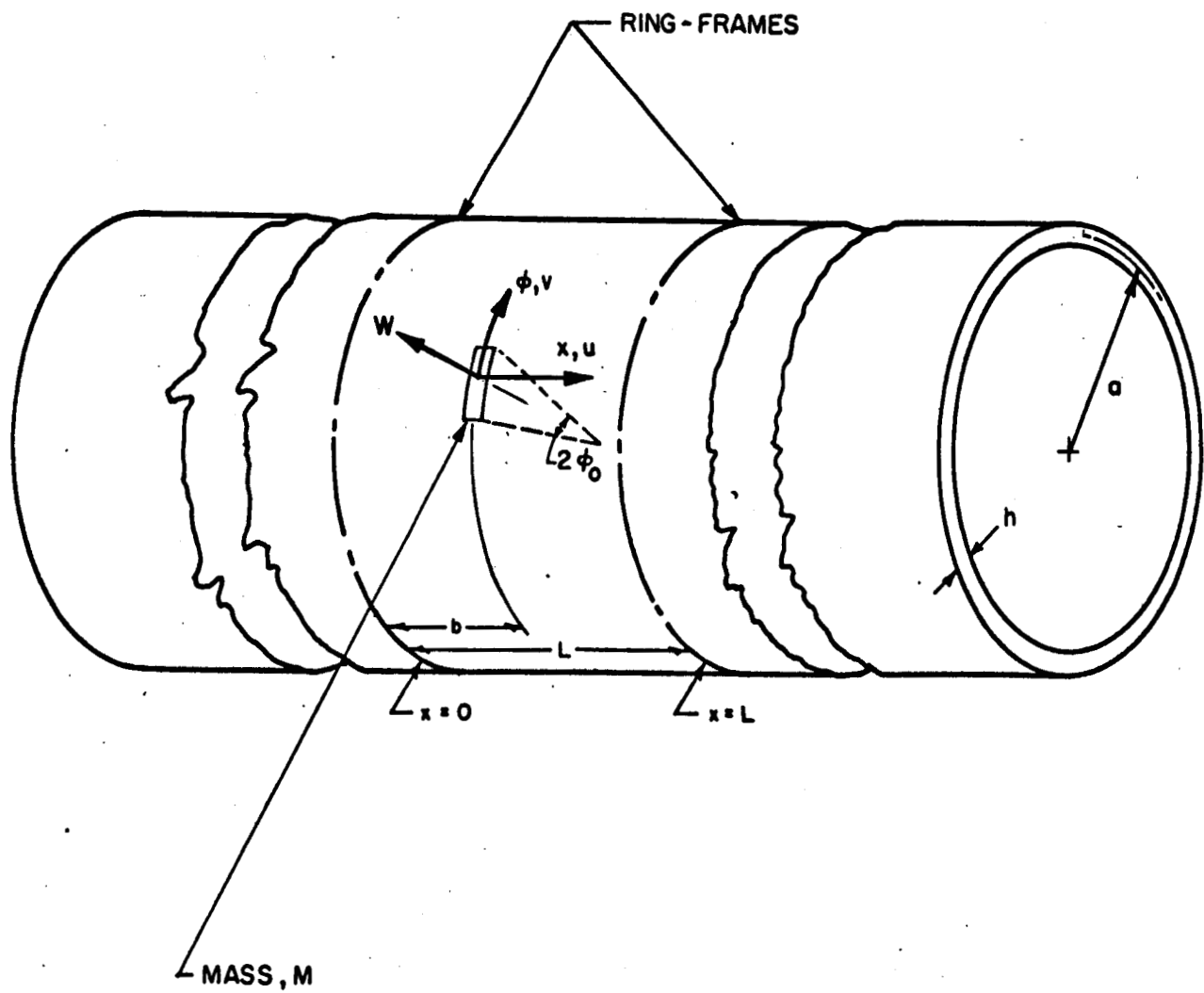


Figure 27. Cylinder Geometry and Displacement Field

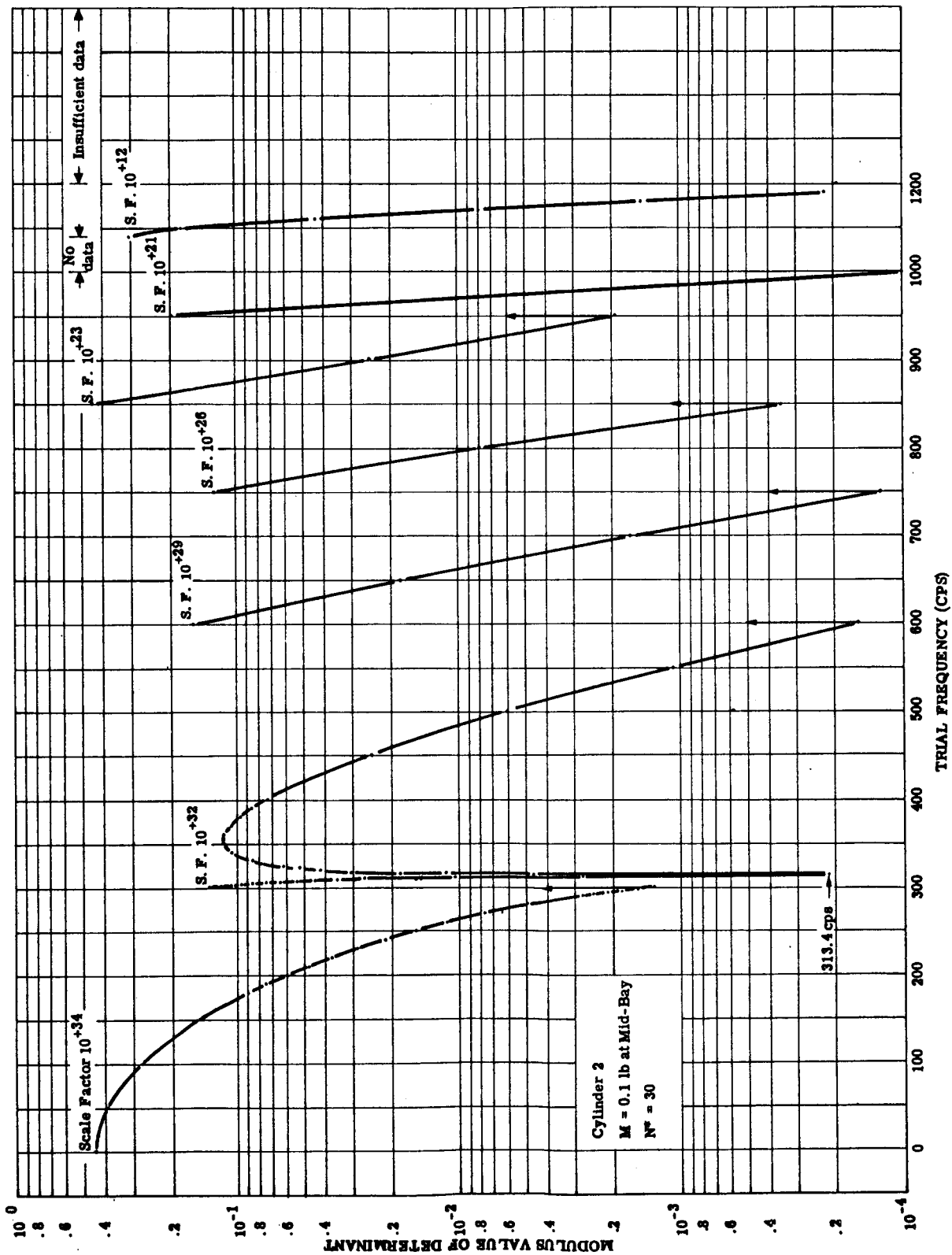


Figure 28. Modulus Value of Determinant versus Frequency

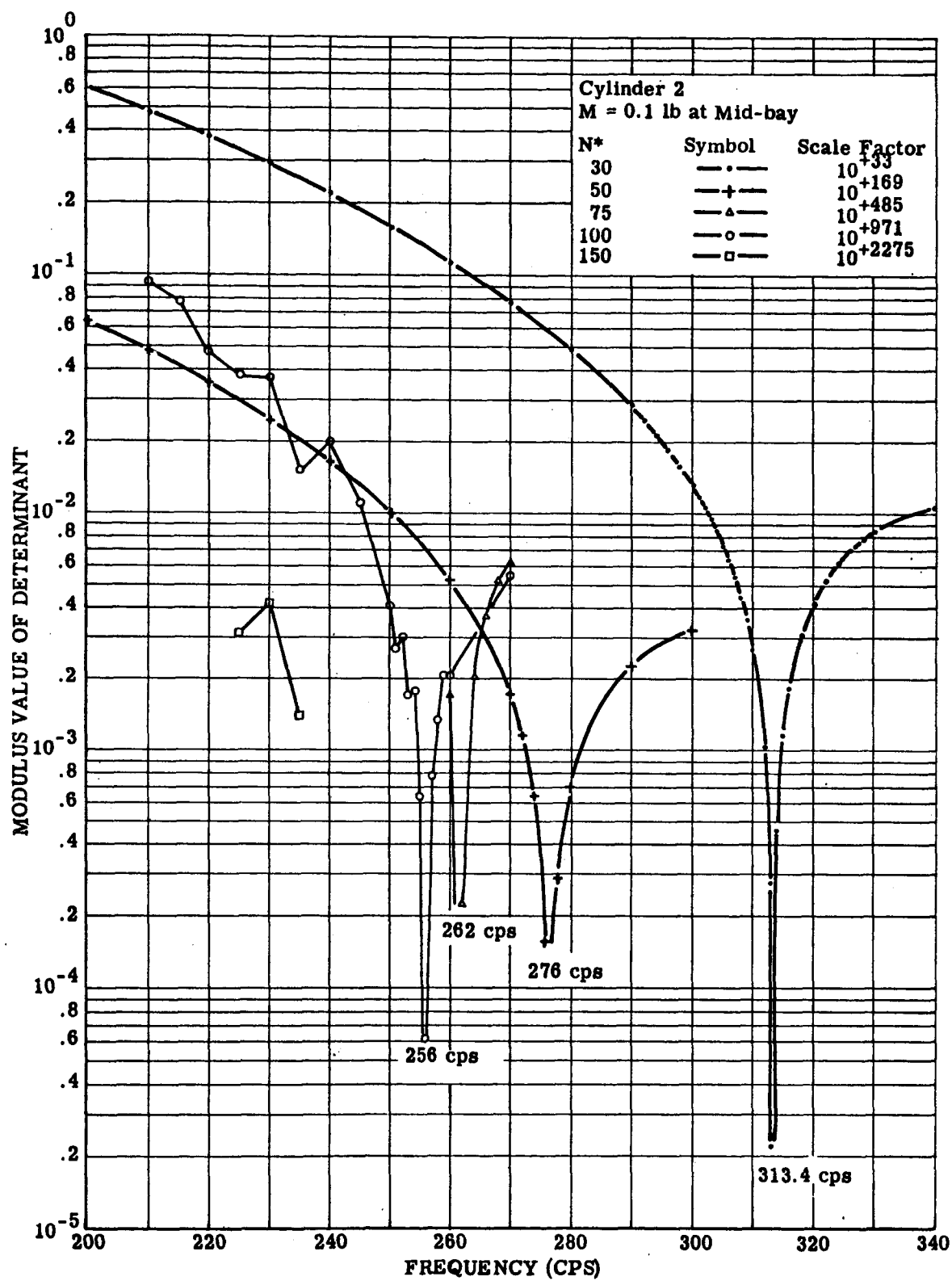


Figure 29. Effect of N* on Predicted Resonant Frequency

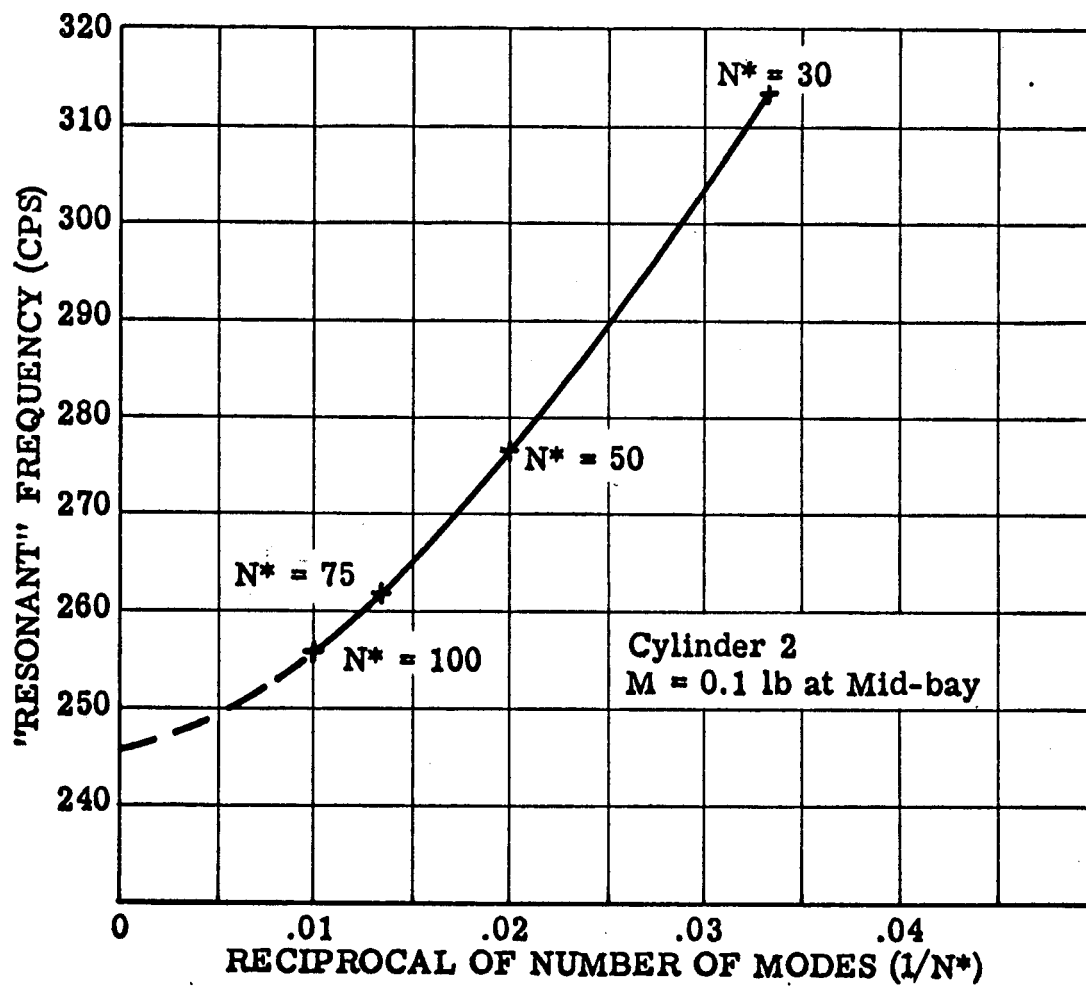


Figure 30. Extrapolated Resonant Frequency for $N^* = \infty$

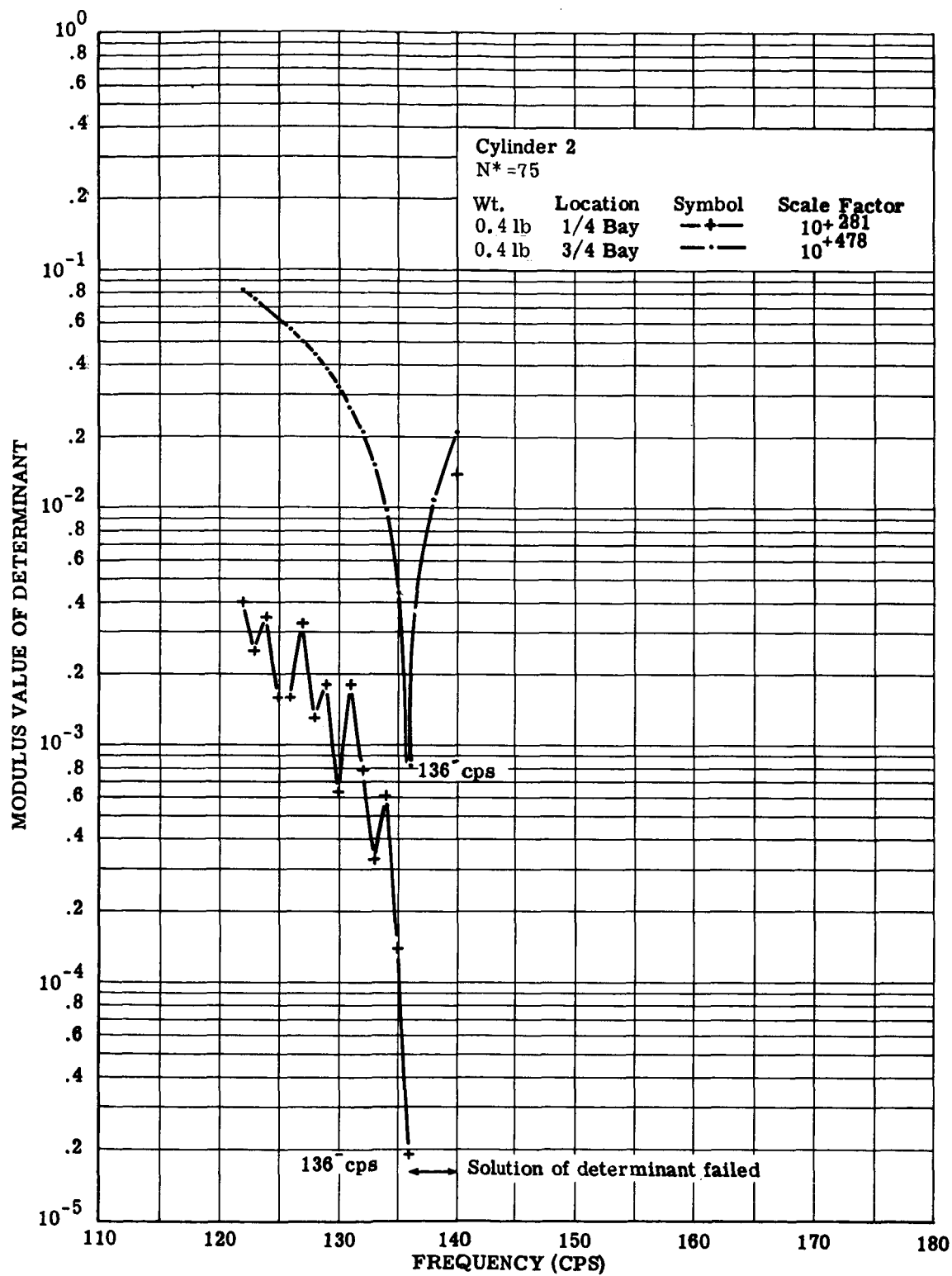


Figure 31. Modulus Value of Determinant versus Frequency