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# ANALYSIS AND DESIGN OF SPACE VEHICLE FLIGHT CONTROL SYSTEMS

## VOLUME I - SHORT PERIOD DYNAMICS

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# FOREWORD

This report was prepared under NASA Contract NAS 8-11494 and is one of a series intended to illustrate methods used for the design and analysis of space vehicle flight control systems. Below is a complete list of the reports in the series:

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| Volume I | Short Period Dynamics |
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## NOMENCLATURE

$\bar{a}$ = acceleration vector; defined by Eq. (4)

A = transformation matrix; defined by Eq. (52)

$A_1, A_2, \cdots, A_6$ = reference areas; see Eqs. (110) through (115)

$(C_A, C_Y, C_N)$ = (axial, side, normal) force coefficient

$(C_l, C_m, C_n)$ = (roll, pitch, yaw) moment coefficient

D = drag

$\bar{F}$ = force vector; see Eq. (7)

$\bar{F}_i$ = force vector acting on element of mass, $m_i$

$(\Sigma F_x, \Sigma F_y, \Sigma F_z)$ = total perturbation force acting parallel to vehicle body axes

$[f_p(\ell,t), f_y(\ell,t), f_r(\ell,t)]$ = force (moment) causing bending (torsion) in (pitch, yaw, roll) plane

$(F_{XA}, F_{YA}, F_{ZA})$ = aerodynamic (axial, side, normal) force; see Eqs. (110) through (112)

g = gravity acceleration

$\bar{G}_o$ = moment of external forces about origin of inertial coordinates; see Eq. (16)

$\bar{G}_{S'}$ = moment of external forces about origin of body axis system

$\bar{H}_o$ = angular momentum defined by Eq. (12)

$\bar{H}_{S'}$ = angular momentum defined by Eq. (13)

$(H_x, H_y, H_z)$ = components of $\bar{H}_{S'}$ in body axis system

$(I_{xx}, I_{yy}, I_{zz})$ = moment of inertia of reduced vehicle (i.e., excluding sloshing masses) about vehicle body axes

$(I_{xy}, I_{xz}, I_{yz})$ = product of inertia of reduced vehicle about vehicle body axes

$I_o$ = moment of inertia of rocket engine about its c.g.

$I_R$ = moment of inertia of rocket engine about swivel point; $= I_o + M_R \ell_R^2$

$I$ = unit matrix

$I_r(\ell)$ = moment of inertia per unit length of reduced vehicle about longitudinal axis of vehicle

$\bar{i}, \bar{j}, \bar{k}$ = unit vector triad in body axis system

$K_a^{(p)}, K_a^{(y)}$ = accelerometer gain in (pitch, yaw) plane

$(K_{Ap}, K_{Ay}, K_{Ar})$ = servoamplifier gain in (pitch, yaw, roll) channel

$(K_{bp}, K_{by})$ = load torque feedback gain in (pitch, yaw) plane

$(K_{cp}, K_{cy})$ = engine servo gain in (pitch, yaw) plane

$(K_{Ip}, K_{Iy}, K_{Ir})$ = integrator gain in (pitch, yaw, roll) channel

$(K_{Rp}, K_{Ry}, K_{Rr})$ = rate gyro gain in (pitch, yaw, roll) plane

$(K_\alpha, K_\beta)$ = angle of attack meter gain in (pitch, yaw) plane

$\ell$ = length parameter along vehicle longitudinal axis; positive in aft direction

$(\ell_1, \ell_2, \ell_3)$ = reference lengths; see Eqs. (113) through (115)

$\ell_a$ = distance from nose of vehicle to origin of body axis system; see Figure 8

$\ell_A$ = distance from c.g. of vehicle to accelerometer location; positive forward

$\ell_c$ = distance from origin of body axis system to engine swivel point; see Figure 8

$\ell_m$ = distance from c.g. of vehicle to angle of attack sensor; positive forward

$\ell_R$ = distance from c.g. of rocket engine to engine swivel point; see Figure 7

$\ell_\alpha$ = distance from center of pressure in pitch plane to origin of body axis system; see Fig. 8

$\ell_\beta$ = distance from center of pressure in yaw plane to origin of body axis system; see Fig. 10

$\ell_{pi}$ = distance from hinge point of $i^{th}$ pendulum to origin of body axis system; see Fig. 5

$(L_A, M_A, N_A)$ = aerodynamic (roll, pitch, yaw) moment

$L_{pi}$ = length of $i^{th}$ pendulum; see Fig. 5

$L_\alpha$ = aerodynamic load in pitch plane; defined by Eq. (159)

$L_\beta$ = aerodynamic load in yaw plane

$m_i$ = element of mass

$m(\ell)$ = reduced mass per unit length along vehicle longitudinal axis; $\int_0^L m(\ell)\, d\ell = m_o$

$m_o$ = reduced mass of vehicle; $= M_t - \sum_i m_{pi}$

$m_{pi}$ = mass of $i^{th}$ pendulum

$M_t$ = total mass of vehicle

$M_R$ = mass of rocket engine

$(M_p^{(i)}, M_y^{(i)}, M_r^{(i)})$ = generalized mass of $i^{th}$ bending mode in (pitch, yaw, roll) plane

$(\Sigma M_x, \Sigma M_y, \Sigma M_z)$ = total perturbation moment along vehicle body axes

(P, Q, R) = angular velocities defined by Fig. 1 and Eq. (3)

$(P_o, Q_o, R_o)$ = steady-state values of (P, Q, R)

(p, q, r) = perturbation values of (P, Q, R)

$(q_p^{(i)}, q_y^{(i)}, q_r^{(i)})$ = generalized coordinate of $i^{th}$ bending mode in (pitch, yaw, roll) plane

$(Q_p^{(i)}, Q_y^{(i)}, Q_r^{(i)})$ = generalized force (moment) of $i^{th}$ bending mode in (pitch, yaw, roll) plane

s = Laplace operator

t = time

$T_c$ = control (gimballed) thrust

$T_s$ = ungimballed thrust

$T_r$ = control thrust moment in roll

$(T_{Ep}, T_{Ey})$ = engine servo torque in (pitch, yaw) plane

$(T_{fp}, T_{fy})$ = gimbal pivot friction torque in (pitch, yaw) plane

$(T_{Lp}, T_{Ly})$ = load feedback torque in (pitch, yaw) plane

$T'$ = kinetic energy of pendulum

$T''$ = kinetic energy of rocket engine

$(U, V, W)$ = components of velocity vector of origin of body axis system; see Eq. (1)

$(U_o, V_o, W_o)$ = steady-state values of (U, V, W)

$(U_w, V_w, W_w)$ = components of wind velocity vector in body axis system

$(u, v, w)$ = perturbation values of (U, V, W)

$(x, y, z)$ = coordinates specifying location of element of mass in body axis system

$(x_{cg}, y_{cg}, z_{cg})$ = coordinates specifying location of c.g. of reduced vehicle relative to body axis system

$(X_c, Y_c, Z_c)$ = coordinates specifying location of origin of body axis system relative to inertial space

$(X_e, Y_e, Z_e)$ = coordinates specifying location of engine c.g. relative to inertial space

$(X_G, Y_G, Z_G)$ = coordinates specifying location of engine gimbal point relative to inertial space

$\alpha$ = perturbation angle of attack in pitch plane

$\alpha_T$ = forward acceleration of vehicle; defined by Eq. (131)

$\beta$ = perturbation angle of attack in yaw plane

$\gamma$ = perturbation flight path angle; $= \alpha - \theta$

$(\Gamma_{pi}, \Gamma_{yi})$ = pendulum angle in (pitch, yaw) plane

$(\delta_p, \delta_y)$ = rocket engine deflection angle in (pitch, yaw) plane

$(\delta_{pc}, \delta_{yc})$ = command signal to rocket engine in (pitch, yaw) plane

$\zeta_a$ = relative damping factor for accelerometer

$(\zeta_p^{(i)}, \zeta_y^{(i)}, \zeta_r^{(i)})$ = relative damping ratio for $i^{th}$ bending mode in (pitch, yaw, roll) plane

$(\zeta_{ep}, \zeta_{ey})$ = relative damping ratio for engine servo controller in (pitch, yaw) plane

$(\zeta_{Rp}, \zeta_{Ry}, \zeta_{Rr})$ = relative damping coefficient for rate gyro in (pitch, yaw, roll) plane

$\zeta_\alpha$ = relative damping factor for angle of attack sensor

$(\theta, \psi, \varphi)$ = perturbation attitude angle in (pitch, yaw, roll)

$(\theta_c, \psi_c, \varphi_c)$ = command signal in (pitch, yaw, roll) channel

$(\theta_a, \psi_a)$ = accelerometer signal in (pitch, yaw) plane

$(\theta_\alpha, \psi_\beta)$ = angle of attack signal in (pitch, yaw) plane

$(\theta_F, \psi_F, \varphi_F)$ = feedback signal in (pitch, yaw, roll) channel

$(\theta_{PG}, \psi_{PG}, \varphi_{PG})$ = position gyro signal in (pitch, yaw, roll) channel

$(\theta_{RG}, \psi_{RG}, \varphi_{RG})$ = rate gyro signal in (pitch, yaw, roll) channel

$\bar{\lambda}$ = radius vector from origin of inertial reference to origin of body axis system

$\bar{\mu}$ = velocity vector from origin of inertial reference to origin of body axis system

$(\bar{\mu}_{pi}, \bar{\mu}_{yi})$ = velocity vector of $i^{th}$ pendulum in (pitch, yaw) plane relative to inertial reference

$\mu_c$ = control moment coefficient; defined by Eq. (164)

$\mu_\alpha$ = aerodynamic moment coefficient; defined by Eq. (165)

$[\xi_p(\ell, t), \xi_y(\ell, t), \xi_r(\ell, t)]$ = bending (torsion) deflection in (pitch, yaw, roll) plane

$\rho$ = atmospheric density

$\bar{\rho}$ = radius vector from element of mass to origin of body axis system

$\bar{\rho}_c$ = radius vector from c. g. of reduced mass of vehicle to origin of body axis system

$(\bar{\rho}_{pi}, \bar{\rho}_{yi})$ = radius vector from pendulum to origin of body axis system

$(\sigma_p^{(i)}, \sigma_y^{(i)}, \sigma_r^{(i)})$ = negative slope of $i^{th}$ bending mode in (pitch, yaw, roll) plane;

$$= \left( \frac{-\partial \varphi_p^{(i)}}{\partial \ell}, \frac{-\partial \varphi_y^{(i)}}{\partial \ell}, \frac{-\partial \varphi_r^{(i)}}{\partial \ell} \right)$$

$(\tau_p, \tau_y, \tau_r)$ = time constant of position gyro in (pitch, yaw, roll) channel

$(\varphi_p^{(i)}, \varphi_y^{(i)}, \varphi_r^{(i)})$ = normalized mode shape function for the $i^{th}$ bending (torsion) mode in the (pitch, yaw, roll) plane

$\omega_a$ = undamped natural frequency for accelerometer

$\omega_\alpha$ = undamped natural frequency for angle of attack meter

$(\omega_p^{(i)}, \omega_y^{(i)}, \omega_r^{(i)})$ = undamped natural frequency of the $i^{th}$ bending mode in the (pitch, yaw, roll) plane

$(\omega_{ep}, \omega_{ey})$ = undamped natural frequency of the engine servo controller in the (pitch, yaw) plane

$(\omega_{Rp}, \omega_{Ry}, \omega_{Rr})$ = undamped natural frequency of the rate gyro in the (pitch, yaw, roll) plane

$(\omega_{pi}, \omega_{yi})$ = undamped natural frequency of the $i^{th}$ pendulum in the (pitch, yaw) plane

$\bar{\omega}$ = velocity vector of body axis coordinate frame relative to inertial space

$(\bar{\ })$ = a vector

$(\dot{\ })$ = derivative with respect to time in local coordinate frame

$(\ )_o$ = steady-state value

## 1. INTRODUCTION

The performance quality of a space launch vehicle during the launch phase of flight is generally studied in two distinct, though related, phases. The first deals with the trajectory of the vehicle with reference to some specified inertial frame and is concerned with such factors as payload capacity, dispersions from nominal, and orbit capability. Dispersion of the actual from nominal trajectory due to such factors as parameter uncertainty and random loads is generally referred to as "long period dynamics." In this context, the vehicle is usually assumed to be a point mass and the oscillations about the nominal trajectory have a "long" period. However, the action of the control system in orienting the vehicle is not instantaneous. Oscillations about the vehicle center of mass are induced and these must be damped out if the mission is to be successful. These oscillations have a comparatively short period and the study of these motions constitutes the subject matter of the vehicle "short period dynamics."

As is well known, a meaningful investigation of the stability and performance quality of the vehicle autopilot requires that one take account of a multitude of factors which are known to have a significant influence. To effectively analyze the complex system which this approach entails, one adopts the classical method of studying the perturbations from nominal; this results in a linear system for which powerful analytical tools are available.

In this monograph, these so-called perturbation equations will be derived taking account of such factors as c.g. (center of gravity) eccentricity, bending, propellant sloshing, and engine inertia. The derivation will proceed from first principles, so that by suitable modifications in the development, special equations emphasizing certain facets of the problem may be obtained. Also, various effects may be either included or eliminated depending on particular study requirements.

A brief mention should be made of the coordinate axes adopted. Analysis of missile systems conventionally employs body axes rather than the "stability axes" which are commonly used in aircraft. Among the reasons for this preference is that the aerodynamic stability derivatives for missiles and space launch vehicles are obtained both experimentally and analytically via body axes. Furthermore, a space launch vehicle is not a lift body in the usual sense of the word. The steady-state angle of attack is generally zero; hence there is little to be gained by using stability axes.

It should also be emphasized that the equations developed here are valid only for short time periods -- on the order of a few seconds. Vehicle properties such as mass, c.g. location, and moment of inertia are assumed constant in this interval. Since these quantities vary slowly compared with control system time constants, this approximation is generally valid.

## 2. STATE OF THE ART

The present methods of stability and control analysis of aerospace vehicles are an extension of the techniques first used by Lanchester [3] and Bryan [4] shortly after the turn of the century. Although these investigators were concerned mainly with aircraft and gliders, their general approach, which was essentially the classical method of perturbations about a reference condition, is still used widely today. This is of course due to the fact that a general description of the motion of an airplane (or an aerospace vehicle) is highly complex, containing many nonlinear terms and intractable functions, such that closed-form solutions are impossible.

In applying this general approach to launch vehicles, further complications are introduced because of the fact that the vehicle mass varies with time. Also, since the control system for the vehicle is closed loop (whereas a pilot flying an airplane is essentially open loop), various other factors (such as influence of flexibility on the control system) must be considered.

In spite of this, the general approach is still essentially the same; the stability and control problem is analyzed via perturbation techniques. However, the period during which the resulting equations are valid is short, since constant mass and inertial properties are assumed during this interval. This raises some subtle questions on the legitimacy of the scheme since there is no strict mathematical assurance that stability in the "time-slice" sense is equivalent to stability in the time-varying case. Making use of some recent results in nonlinear theory, a partial answer to this question is given in the monograph "Nonlinear Systems", which constitutes part 2 of Vol. II in the present series of monographs.

While further developments in nonlinear theory may add fresh insight to the problem, the techniques discussed in this monograph are basic and yield results which are of fundamental importance in the design of control systems for aerospace vehicles.

# 3. RECOMMENDED PROCEDURES

The derivation of the equations of motion of a space launch vehicle -- including the effects of elasticity, fuel sloshing, and engine inertia -- may be approached in two ways. In one sense, it might appear simpler in principle to apply Newton's laws directly to the entire vehicle, incorporating all the necessary degrees of freedom, thereby obtaining a set of equations in which all the necessary elastic and inertial coupling terms appear automatically. This approach has two drawbacks. First, a degree of physical insight is lost in the mathematical manipulations. Second, and more important, the elastic modes would have to be computed with the sloshing propellants and engine masses included. This is a more difficult task than computing the free-free body modes alone.

Accordingly, we will take an alternate approach which involves the derivation of artificially uncoupled motions in which the coupling terms will appear via appropriately introduced mathematical constraints.

## 3.1 RIGID BODY EQUATIONS

The rigid body equations of motion will be derived to take account of the situation where the geometric center and the mass center do not coincide. A right-hand coordinate system, X' Y' Z', is fixed to the vehicle such that the origin is located at the vehicle geometric center (Fig. 1). The motion will be described with reference to an inertial coordinate system, XYZ. For brevity, we shall refer to the inertial and body coordinate systems as S and S' respectively.

For present purposes, it is not necessary to define the inertial frame explicitly. It is usually convenient to think of the inertial reference as earth-centered with the axes suitably oriented with respect to some stellar reference. However, the inertial frame may also be taken with the origin at the launch site with one axis directed along the local vertical of the launch site. This selection is sufficiently accurate since the flight time associated with the launch phase is on the order of a few minutes, and the earth's rotation is negligible for this interval. Since we are not at the moment concerned with the position and velocity of the vehicle with respect to a specific reference, the precise orientation of the inertial frame may be left undefined. However, for purposes of trajectory analysis, the appropriate reference frames must be specified precisely. This is done in the monograph on "Trajectory Equations," which constitutes part 3 of Vol. I in the present series.

We may now express the velocity of an element of mass by

$$\bar{\mu} + \frac{d\bar{\rho}}{dt} = \bar{\mu} + \bar{\omega} \times \bar{\rho}$$

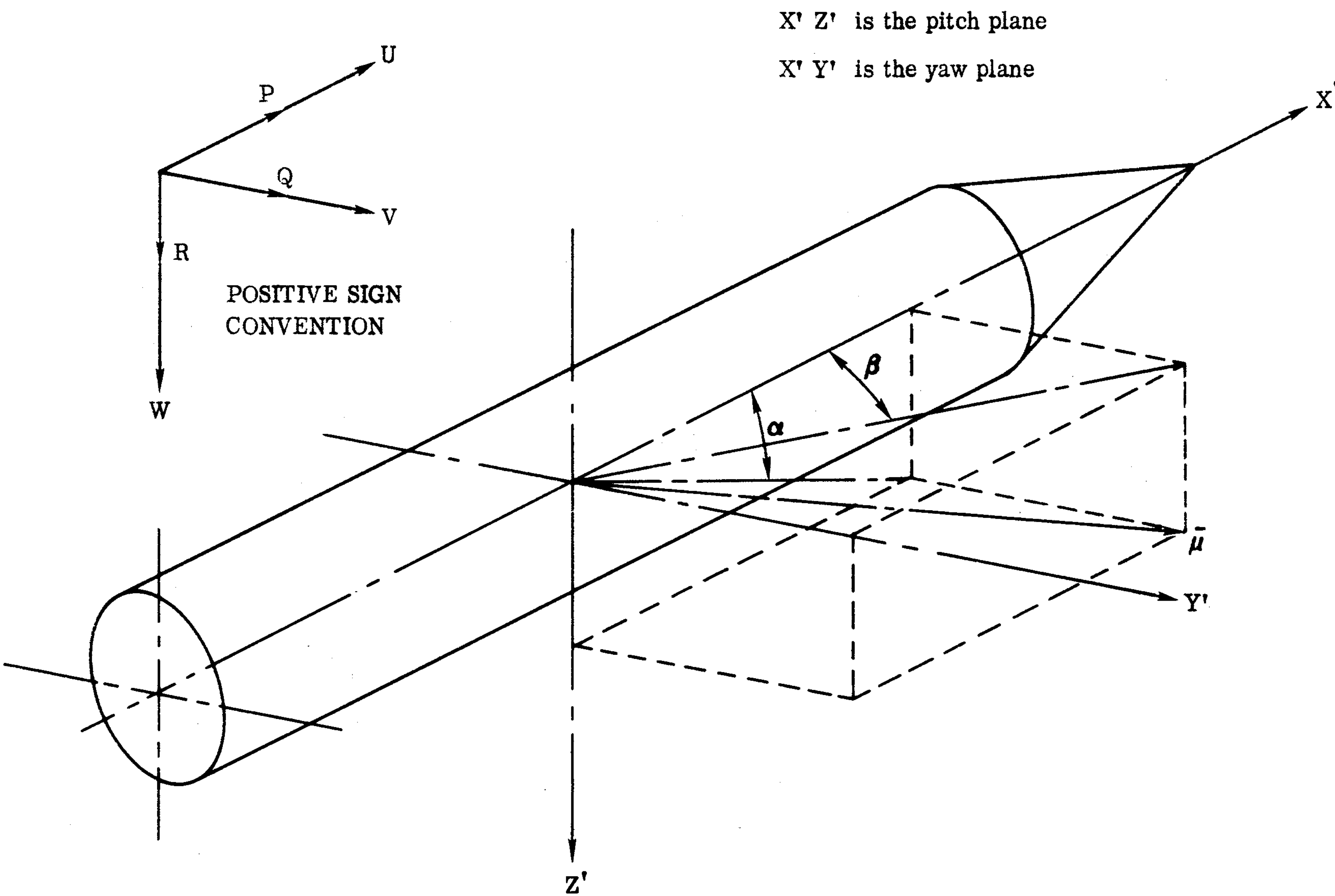


Figure 1. Vehicle Coordinate System

where $\bar{\mu}$ is the velocity of the origin of S′; $\bar{\rho}$ is the radius vector from the element of mass to the origin of S′; and $\bar{\omega}$ is the angular velocity of S′. We have*

$$\bar{\mu} = U\bar{i} + V\bar{j} + W\bar{k} \tag{1}$$

$$\bar{\rho} = x\bar{i} + y\bar{j} + z\bar{k} \tag{2}$$

$$\bar{\omega} = P\bar{i} + Q\bar{j} + R\bar{k} \tag{3}$$

The acceleration of the element of mass is therefore

$$\bar{a} = \frac{d\bar{\mu}}{dt} + \bar{\omega}\times\frac{d\bar{\rho}}{dt} + \frac{d\bar{\omega}}{dt}\times\bar{\rho} = \frac{\delta\bar{\mu}}{\delta t} + \bar{\omega}\times\bar{\mu} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + \frac{\delta\bar{\omega}}{\delta t}\times\bar{\rho} \tag{4}$$

where**

$$\frac{\delta\bar{\mu}}{\delta t} = \dot{U}\bar{i} + \dot{V}\bar{j} + \dot{W}\bar{k} \tag{5}$$

$$\frac{\delta\bar{\omega}}{\delta t} = \dot{P}\bar{i} + \dot{Q}\bar{j} + \dot{R}\bar{k} \tag{6}$$

The equation of motion of the vehicle is

$$\begin{aligned}\bar{F} &= \int \bar{a}\,dm = \int\left[\frac{\delta\bar{\mu}}{\delta t} + \bar{\omega}\times\bar{\mu} + \frac{\delta\bar{\omega}}{\delta t}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho})\right]dm \\ &= m_o\left[\frac{\delta\bar{\mu}}{\delta t} + \bar{\omega}\times\bar{\mu}\right] + \frac{\delta\bar{\omega}}{\delta t}\times\left[\int\bar{\rho}dm\right] + \bar{\omega}\times\left[\bar{\omega}\times\int\bar{\rho}dm\right]\end{aligned} \tag{7}$$

Now

$$\int\bar{\rho}dm = \bar{\rho}_c m_o \tag{8}$$

where $\bar{\rho}_c$ is the radius vector from the mass center to the origin of S′. If the mass center coincides with this origin, $\bar{\rho}_c = 0$, and Eq. (7) reduces to

$$\bar{F} = m_o\left[\frac{\delta\bar{\mu}}{\delta t} + \bar{\omega}\times\bar{\mu}\right] \tag{9}$$

---

* $\bar{i}$, $\bar{j}$, $\bar{k}$ denotes a unit vector triad in the S′ frame.

** $\frac{\delta}{\delta t}(\;)$ denotes time derivative with respect to body frame.

Noting that

$$\bar{\rho}_c = x_{cg}\,\bar{i} + y_{cg}\,\bar{j} + z_{cg}\,\bar{k} \tag{10}$$

we have on integrating Eq. 7 over the entire vehicle

$$\begin{aligned}
\bar{F} = &\left[ m_o\,(\dot{U} + QW - RV) - m_o x_{cg}\,(Q^2 + R^2) - m_o y_{cg}\,(\dot{R} - PQ) + m_o z_{cg}\,(\dot{Q} + PR) \right] \bar{i} \\
&+ \left[ m_o\,(\dot{V} + RU - PW) + m_o x_{cg}\,(\dot{R} + PQ) - m_o y_{cg}\,(P^2 + R^2) - m_o z_{cg}\,(\dot{P} - QR) \right] \bar{j} \\
&+ \left[ m_o\,(\dot{W} + PV - QU) - m_o x_{cg}\,(\dot{Q} - PR) + m_o y_{cg}\,(\dot{P} + QR) - m_o z_{cg}\,(P^2 + Q^2) \right] \bar{k}
\end{aligned} \tag{11}$$

We need an expression for the rate of change of angular momentum about a moving point. We derive this as follows. Let a particle of mass "$m_i$" have a radius vector "$\bar{\rho}_i$" with respect to a moving point S', and let the radius vector from S' to a fixed point O be denoted by $\bar{\lambda}$. Then the angular momentum about the fixed point O is

$$\begin{aligned}
\bar{H}_o &= \sum_i \left( \bar{\lambda} + \bar{\rho}_i \right) \times \left( \dot{\bar{\lambda}} + \dot{\bar{\rho}}_i \right) m_i \\
&= \sum_i \left[ \bar{\lambda} \times \dot{\bar{\lambda}} + \bar{\lambda} \times \dot{\bar{\rho}}_i + \bar{\rho}_i \times \dot{\bar{\lambda}} + \bar{\rho}_i \times \dot{\bar{\rho}}_i \right] m_i
\end{aligned} \tag{12}$$

Putting

$$\bar{H}_{S'} = \sum_i \left( \bar{\rho}_i \times \dot{\bar{\rho}}_i \right) m_i \tag{13}$$

we have

$$\bar{H}_o = \sum_i \left[ \bar{\lambda} \times \dot{\bar{\lambda}} + \bar{\lambda} \times \dot{\bar{\rho}}_i + \bar{\rho}_i \times \dot{\bar{\lambda}} \right] m_i + \bar{H}_{S'} \tag{14}$$

Now

$$\frac{d\bar{H}_o}{dt} = \sum_i \left[ \dot{\bar{\lambda}} \times \dot{\bar{\lambda}} + \bar{\lambda} \times \ddot{\bar{\lambda}} + \dot{\bar{\lambda}} \times \dot{\bar{\rho}}_i + \bar{\lambda} \times \ddot{\bar{\rho}}_i + \dot{\bar{\rho}}_i \times \dot{\bar{\lambda}} + \bar{\rho}_i \times \ddot{\bar{\lambda}} \right] m_i + \frac{d\bar{H}_{S'}}{dt}$$

Noting that

$$\dot{\bar{\lambda}} \times \dot{\bar{\lambda}} = 0$$

$$\dot{\bar{\lambda}} \times \dot{\bar{\rho}} + \dot{\bar{\rho}}_i \times \dot{\bar{\lambda}} = 0$$

$$\sum_i m_i \bar{\rho}_i = m_o \bar{\rho}_c$$

we obtain

$$\frac{d\bar{H}_o}{dt} = \sum_i \bar{\lambda} \times (\ddot{\bar{\lambda}} + \ddot{\bar{\rho}}_i) m_i + m_o \bar{\rho}_c \times \ddot{\bar{\lambda}} + \frac{d\bar{H}_{S'}}{dt} . \tag{15}$$

Denoting by $\bar{G}_o$ the moment of the external forces about O, we have

$$\bar{G}_o = \sum_i (\bar{\rho}_i + \bar{\lambda}) \times \bar{F}_i = \sum_i \bar{\rho}_i \times \bar{F}_i + \bar{\lambda} \times \sum \bar{F}_i \tag{16}$$

where $\bar{F}_i$ is the force acting on the particle of mass $m_i$.

Writing

$$\bar{G}_{S'} = \sum_i \bar{\rho}_i \times \bar{F}_i \tag{17}$$

Eq. (16) becomes

$$\bar{G}_o = \bar{G}_{S'} + \bar{\lambda} \times \sum_i \bar{F}_i \tag{18}$$

But

$$\bar{G}_o = \frac{d\bar{H}_o}{dt} \tag{19}$$

Hence, combining Eqs. (15) and (18) yields

$$\bar{G}_{S'} + \bar{\lambda} \times \sum_i \bar{F}_i = \bar{\lambda} \times \sum_i m_i (\ddot{\bar{\lambda}} + \ddot{\bar{\rho}}_i) + m_o \bar{\rho}_c \times \ddot{\bar{\lambda}} + \frac{d\bar{H}_{S'}}{dt} \tag{20}$$

However

$$\sum_i \bar{F}_i = \sum_i m_i (\ddot{\bar{\lambda}} + \ddot{\bar{\rho}}_i) \tag{21}$$

so that

$$\bar{G}_{S'} = \frac{d\bar{H}_{S'}}{dt} + m_o \bar{\rho}_c \times \ddot{\bar{\lambda}} \tag{22}$$

This is the relation sought.

Eq. (22) may be expressed in the form

$$\overline{G}_{S'} = \frac{d}{dt}\int \bar{\rho} \times \dot{\bar{\rho}}\, dm + m_o \bar{\rho}_c \times \ddot{\bar{\lambda}} \tag{23}$$

where, in the problem under consideration

$$\dot{\bar{\rho}} = \bar{\omega} \times \bar{\rho}$$

$$\dot{\bar{\lambda}} = \bar{\mu}$$

$$\ddot{\bar{\lambda}} = \frac{d\bar{\mu}}{dt}$$

We may write therefore

$$\overline{G}_{S'} = \frac{d}{dt}\int [\bar{\rho} \times (\bar{\omega} \times \bar{\rho})]\, dm + m_o \bar{\rho}_c \times \frac{d\bar{\mu}}{dt} \tag{24}$$

The expression for the angular momentum about S' is

$$\overline{H}_{S'} = \int [\bar{\rho} \times (\bar{\omega} \times \bar{\rho})]\, dm$$

$$= \int [\bar{\omega}\rho^2 - \bar{\rho}(\bar{\omega} \cdot \bar{\rho})]\, dm \tag{25}$$

In components,

$$\begin{cases} H_x = I_{xx} P - I_{xy} Q - I_{xz} R \\ H_y = -I_{xy} P + I_{yy} Q - I_{yz} R \\ H_z = -I_{xz} P - I_{yz} Q + I_{zz} R \end{cases}$$

where

$$I_{xx} = \int (y^2 + z^2)\, dm \qquad I_{xy} = \int xydm$$

$$I_{yy} = \int (x^2 + z^2)\, dm \qquad I_{yz} = \int yzdm$$

$$I_{zz} = \int (x^2 + y^2)\, dm \qquad I_{xz} = \int xzdm$$

Noting that

$$\overline{H}_{S'} = H_x \overline{i} + H_y \overline{j} + H_z \overline{k}$$

we have on differentiation

$$\frac{d\overline{H}_{S'}}{dt} = \frac{dH_x}{dt}\overline{i} + \frac{dH_y}{dt}\overline{j} + \frac{dH_z}{dt}\overline{k} + H_x \frac{d\overline{i}}{dt} + H_y \frac{d\overline{j}}{dt} + H_z \frac{d\overline{k}}{dt}$$

$$= (\dot{H}_x \overline{i} + \dot{H}_y \overline{j} + \dot{H}_z \overline{k}) + \overline{\omega} \times \overline{H}_{S'} \tag{26}$$

We obtain finally

$$\begin{aligned}
\overline{G}_{S'} = &\Big[ I_{xx}\dot{P} - I_{xy}(\dot{Q} - PR) - I_{xz}(\dot{R} + PQ) + I_{yz}(R^2 - Q^2) + (I_{zz} - I_{yy})\,QR \\
&+ m_o y_{cg}(\dot{W} + PV - QU) - m_o z_{cg}(\dot{V} + RU - PW)\Big]\overline{i} \\
&+ \Big[ -I_{xy}(\dot{P} + QR) + I_{yy}\dot{Q} - I_{yz}(\dot{R} - PQ) + I_{xz}(P^2 - R^2) + (I_{xx} - I_{zz})\,PR \\
&- m_o x_{cg}(\dot{W} + PV - QU) + m_o z_{cg}(\dot{U} + QW - RV)\Big]\overline{j} \\
&+ \Big[ -I_{xz}(\dot{P} - QR) - I_{yz}(\dot{Q} + PR) + I_{zz}\dot{R} + I_{xy}(Q^2 - P^2) + (I_{yy} - I_{xx})\,PQ \\
&+ m_o x_{cg}(\dot{V} + RU - PW) - m_o y_{cg}(\dot{U} + QW - RV)\Big]\overline{k}
\end{aligned} \tag{27}$$

Eqs. (11) and (27) completely describe the motion.

In order to study the stability of the short period motion, assume that the components of linear and angular velocity may be expressed as the sum of a steady-state value and disturbance component as follows:

$$\begin{aligned}
U &= U_o + u & \qquad P &= P_o + p \\
V &= V_o + v & \qquad Q &= Q_o + q \\
W &= W_o + w & \qquad R &= R_o + r
\end{aligned} \tag{28}$$

The components of the airspeed resolved along body axes are

$$U - U_w$$

$$V - V_w$$

$$W - W_w$$

We now assume that the quantities V, W, u, v, w, $U_w$, $V_w$, and $W_w$ are small compared to $U_o$. Defining

$$\alpha = \frac{w}{U_o} + \alpha_w \tag{29}$$

$$\beta = \frac{v}{U_o} + \beta_w \tag{30}$$

where

$$\alpha_w = -\frac{W_w}{U_o} \tag{31}$$

$$\beta_w = -\frac{V_w}{U_o} \tag{32}$$

and substituting Eqs. (29) and (30) into (11) and (27), we find, after eliminating steady-state and higher order terms,

$$\begin{aligned} \Sigma F_x = {} & m_o [\dot{u} - V_o r + W_o q + U_o (Q_o \alpha - R_o \beta)] \\ & - 2 m_o x_{cg} (Q_o q + R_o r) - m_o y_{cg} (\dot{r} - P_o q - Q_o p) \\ & + m_o z_{cg} (\dot{q} + P_o r + R_o p) \end{aligned} \tag{33}$$

$$\begin{aligned} \Sigma F_y = {} & m_o [\dot{v} + R_o u + U_o r - W_o p - P_o U_o \alpha] \\ & + m_o x_{cg} (\dot{r} + P_o q + Q_o p) - 2 m_o y_{cg} (P_o p + R_o r) \\ & - m_o z_{cg} (\dot{p} - Q_o r - R_o q) \end{aligned} \tag{34}$$

$$\Sigma F_z = m_o\,[\dot{w} + V_o p - Q_o u - U_o q + P_o U_o \beta]$$

$$- m_o x_{cg}\,(\dot{q} - P_o r - R_o p) + m_o y_{cg}\,(\dot{p} + Q_o r + R_o q)$$

$$- 2\, m_o z_{cg}\,(P_o p + Q_o q) \tag{35}$$

$$\Sigma M_x = I_{xx}\dot{p} + (I_{zz} - I_{yy})\,(Q_o r + R_o q) - I_{xy}\,(\dot{q} - P_o r - R_o p)$$

$$- I_{xz}\,(\dot{r} + P_o q + Q_o p) - 2\, I_{yz}\,(R_o r - Q_o q)$$

$$+ m_o y_{cg}\,(\dot{w} + V_o p - Q_o u - U_o q + P_o U_o \beta)$$

$$- m_o z_{cg}\,(\dot{v} + R_o u + U_o r - W_o p - P_o U_o \alpha) \tag{36}$$

$$\Sigma M_y = (I_{xx} - I_{zz})\,(P_o r + R_o p) + I_{yy}\,\dot{q} - I_{xy}\,(\dot{p} + Q_o r + R_o q)$$

$$- I_{yz}\,(\dot{r} - P_o q - Q_o p) + 2\, I_{xz}\,(P_o p - R_o r)$$

$$- m_o x_{cg}\,(\dot{w} + V_o p - Q_o u - U_o q + P_o U_o \beta)$$

$$+ m_o z_{cg}\,[\dot{u} - V_o r + W_o q + U_o\,(Q_o \alpha - R_o \beta)] \tag{37}$$

$$\Sigma M_z = (I_{yy} - I_{xx})\,(P_o q + Q_o p) + I_{zz}\,\dot{r} - I_{xz}\,(\dot{p} - Q_o r - R_o q)$$

$$- I_{yz}\,(\dot{q} + P_o r + R_o p) + 2\, I_{xy}\,(Q_o q - P_o p)$$

$$+ m_o x_{cg}\,(\dot{v} + R_o u + U_o r - W_o p - P_o U_o \alpha)$$

$$- m_o y_{cg}\,[\dot{u} - V_o r + W_o q + U_o\,(Q_o \alpha - R_o \beta)] \tag{38}$$

The external forces and moments in the above equations are due to gravity, thrust, aerodynamics, propellant sloshing, and engine inertia. We write these as follows (resolved along body axes).

$$\Sigma F_x = F_{xg} + F_{xT} + F_{xa} + F_{xs} + F_{xE} \tag{39}$$

$$\Sigma F_y = F_{yg} + F_{yT} + F_{ya} + F_{ys} + F_{yE} \tag{40}$$

$$\Sigma F_z = F_{zg} + F_{zT} + F_{za} + F_{zs} + F_{zE} \tag{41}$$

$$\Sigma M_x = M_{xg} + M_{xT} + M_{xa} + M_{xs} + M_{xE} \tag{42}$$

$$\Sigma M_y = M_{yg} + M_{yT} + M_{ya} + M_{ys} + M_{yE} \tag{43}$$

$$\Sigma M_z = M_{zg} + M_{zT} + M_{za} + M_{zs} + M_{zE} \tag{44}$$

Since $V_O$, $W_O$, $P_O$, $Q_O$, and $R_O$ are small quantities, of the same order of magnitude as the perturbation variables, Eqs. (33) through (38) reduce to

$$\Sigma F_x = m_o (\dot{u} - y_{cg} \dot{r} + z_{cg} \dot{q}) \tag{45}$$

$$\Sigma F_y = m_o (\dot{v} + U_o r + x_{cg} \dot{r} - z_{cg} \dot{p}) \tag{46}$$

$$\Sigma F_z = m_o (\dot{w} - U_o q - x_{cg} \dot{q} + y_{cg} \dot{p}) \tag{47}$$

$$\Sigma M_x = I_{xx} \dot{p} - I_{xy} \dot{q} - I_{xz} \dot{r} + m_o y_{cg} (\dot{w} - U_o q) - m_o z_{cg} (\dot{v} + U_o r) \tag{48}$$

$$\Sigma M_y = I_{yy} \dot{q} - I_{xy} \dot{p} - I_{yz} \dot{r} - m_o x_{cg} (\dot{w} - U_o q) + m_o z_{cg} \dot{u} \tag{49}$$

$$\Sigma M_z = I_{zz} \dot{r} - I_{xz} \dot{p} - I_{yz} \dot{q} + m_o x_{cg} (\dot{v} + U_o r) - m_o y_{cg} \dot{u} \tag{50}$$

### 3.1.1 <u>Euler Angles</u>

Let $S_O'$ denote the vehicle body axes in the steady-state condition. The disturbed orientation, $S'$, is then related to $S_O'$ by three Euler angles -- $\psi$, $\theta$, $\varphi$ -- defined as follows.

a. Rotate $S_O'$ about the $Z'$ axis by an angle $\psi$ in the positive direction*

b. Then rotate about the $Y'$ axis by an angle $\theta$ in the positive direction

c. Finally, rotate about the $X'$ axis by an angle $\varphi$ in the positive direction.

This brings $S_O'$ into $S'$. We then have**

$$S' = A\, S_0' \tag{51}$$

---

*Positive direction is determined by the usual right-hand rule.

**We will let $S'$ denote either the body axis frame itself or some vector in $S'$ (similarly for $S_O'$). This should cause no confusion since the meaning will be clear from the context.

where A is the transformation matrix given by*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & s\varphi \\ 0 & -s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$A = \left[\begin{array}{c|c|c} c\theta\ c\psi & c\theta\ s\psi & -s\theta \\ s\varphi\ s\theta\ c\psi - c\varphi\ s\psi & s\varphi\ s\theta\ s\psi + c\varphi\ c\psi & s\varphi\ c\theta \\ c\varphi\ s\theta\ c\psi + s\varphi\ s\psi & c\varphi\ s\theta\ s\psi - s\varphi\ c\psi & c\varphi\ c\theta \end{array}\right] \tag{52}$$

Note that $A^T = A^{-1}$. By direct resolution of vectors, we find that the components of angular velocity in the S' frame are given by

$$\omega_x = \dot{\varphi} - \dot{\psi} \sin\theta \tag{53}$$

$$\omega_y = \dot{\theta} \cos\varphi + \dot{\psi} \cos\theta \sin\varphi \tag{54}$$

$$\omega_z = \dot{\psi} \cos\theta \cos\varphi - \dot{\theta} \sin\varphi \tag{55}$$

We now assume that the quantities $\psi$, $\theta$, $\varphi$, and $\dot{\psi}$, $\dot{\theta}$, $\dot{\varphi}$ are small, so that the above equations reduce to

$$A = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \varphi \\ \theta & -\varphi & 1 \end{bmatrix} \tag{56}$$

$$\omega_x = \dot{\varphi} = p$$

$$\omega_y = \dot{\theta} = q$$

$$\omega_z = \dot{\psi} = r \tag{57}$$

---

*Occasionally, we will write $s\theta$ for $\sin\theta$, $c\theta$ for $\cos\theta$, etc., for brevity.

## 3.2 EQUATIONS OF ELASTIC VIBRATIONS

The equations of motion for the forced vibrations of a free-free elastic vehicle are developed in the monograph entitled, "Elastic Body Equations," which is part 2 of Vol. I in this series. Only the main results are summarized here for a complete description of the short period dynamics of the vehicle. A schematic of the deflected shape of the vehicle in the pitch plane is shown in Fig. 2. The elastic deflection at any point along the vehicle is given by

$$\xi_p(\ell, t) = \sum_{i=1}^{\infty} q_p^{(i)}(t)\, \varphi_p^{(i)}(\ell) \tag{58}$$

Here $\varphi_p^{(i)}(\ell)$ denotes the normalized mode shape of the $i^{th}$ mode in the pitch plane and is a function only of the beam stiffness and mass distribution. $q_p^{(i)}(t)$ is the generalized coordinate due to elasticity, for the $i^{th}$ mode in the pitch plane. It satisfies the equation

$$\ddot{q}_p^{(i)} + 2\,\zeta_p^{(i)}\,\omega_p^{(i)}\,\dot{q}_p^{(i)} + \left[\omega_p^{(i)}\right]^2 q_p^{(i)} = \frac{Q_p^{(i)}}{M_p^{(i)}} \tag{59}$$

where $Q_p^{(i)}$ and $M_p^{(i)}$ are the generalized force and mass, respectively, and are given by*

$$Q_p^{(i)} = \int_0^L f_p(\ell, t)\, \varphi_p^{(i)}(\ell)\, d\ell \tag{60}$$

$$M_p^{(i)} = \int_0^L m(\ell) \left[\varphi_p^{(i)}(\ell)\right]^2 d\ell \tag{61}$$

and $\omega_p^{(i)}$ represents the natural frequency of the $i^{th}$ mode.

The forced vibration equations for the yaw plane are completely analogous in form to those of the pitch plane; viz. (see Fig. 3)

$$\xi_y(\ell, t) = \sum_{i=1}^{\infty} q_y^{(i)}(t)\, \varphi_y^{(i)}(\ell) \tag{62}$$

$$\ddot{q}_y^{(i)} + 2\,\zeta_y^{(i)}\,\omega_y^{(i)}\,\dot{q}_y^{(i)} + \left[\omega_y^{(i)}\right]^2 q_y^{(i)} = \frac{Q_y^{(i)}}{M_y^{(i)}} \tag{63}$$

---

* A thorough discussion of the quantities contained in these equations may be found in the monograph on Elastic Body Equations referred to above.

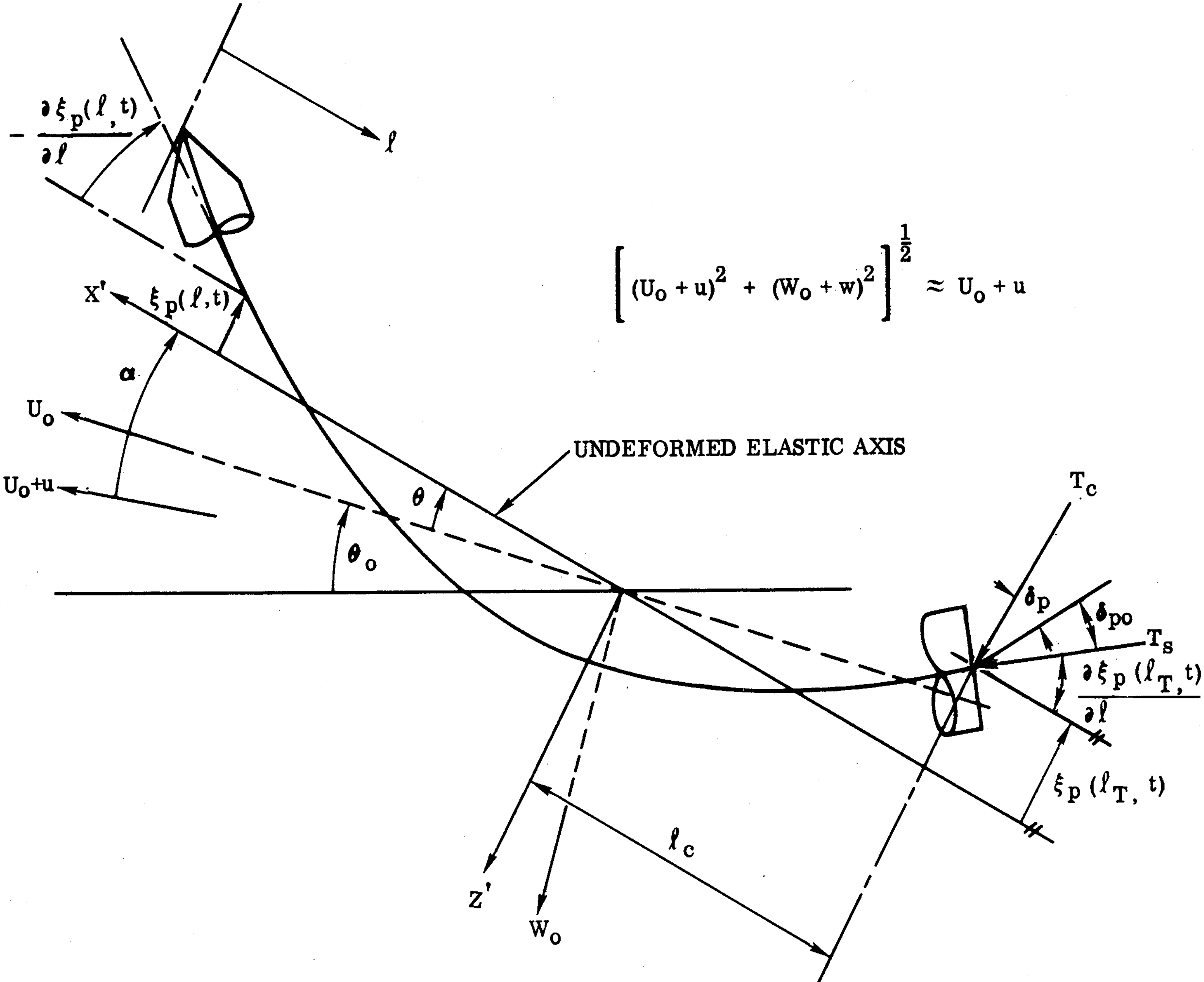


Figure 2. Elastic Vehicle in the Pitch Plane

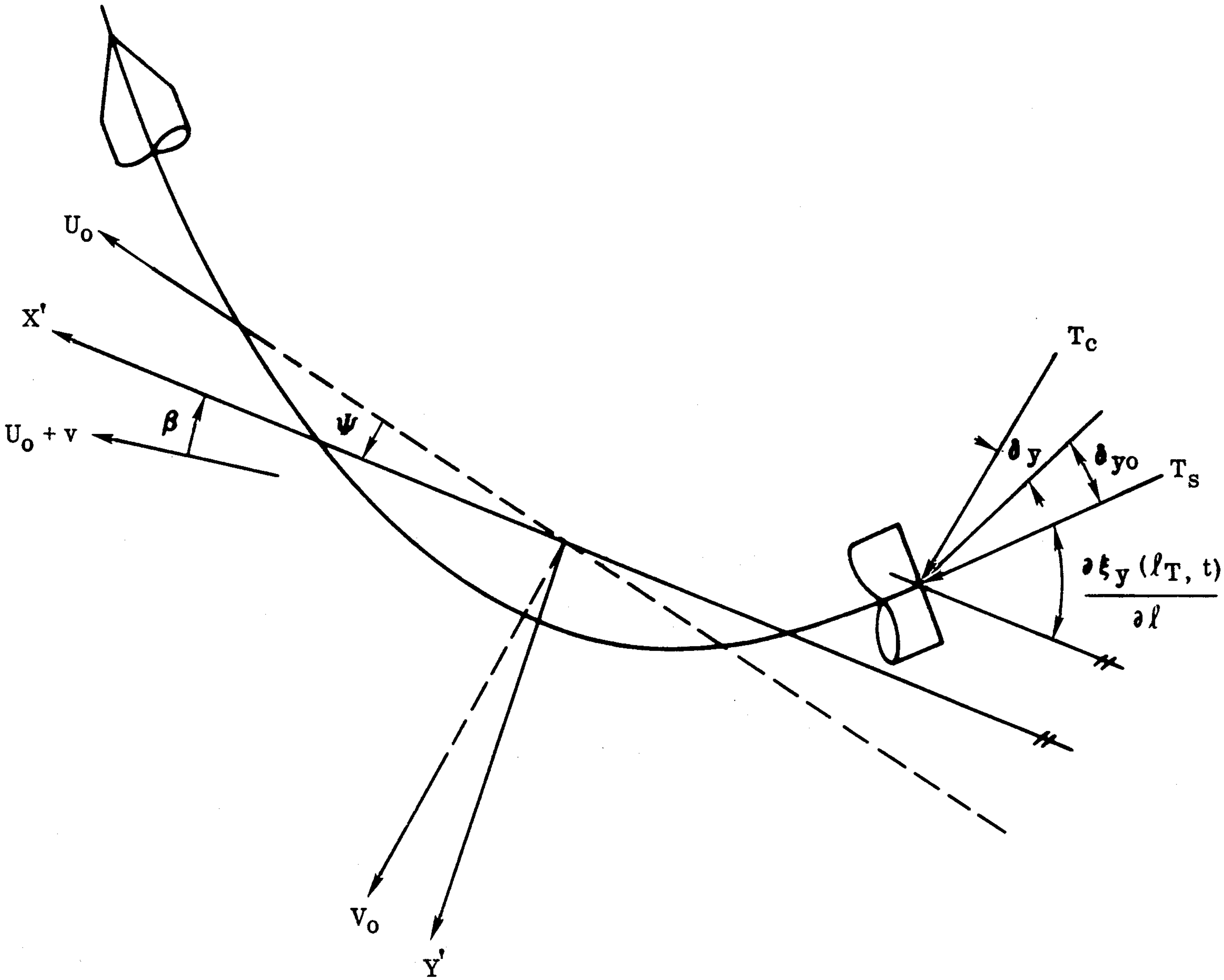


Figure 3. Elastic Vehicle in Yaw Plane

$$Q_y^{(i)} = \int_0^L f_y(\ell, t)\, \varphi_y^{(i)}(\ell)\, d\ell \tag{64}$$

$$M_y^{(i)} = \int_0^L m(\ell) \left[\varphi_y^{(i)}(\ell)\right]^2 d\ell \tag{65}$$

The torsional vibration modes about the longitudinal axis may also be written in a form analogous to the above.

$$\xi_r(\ell, t) = \sum_{i=1}^{\infty} q_r^{(i)}(t)\, \varphi_r^{(i)}(\ell) \tag{66}$$

$$\ddot{q}_r^{(i)} + 2\, \zeta_r^{(i)}\, \omega_r^{(i)}\, \dot{q}_r^{(i)} + \left[\omega_r^{(i)}\right]^2 q_r^{(i)} = \frac{Q_r^{(i)}}{M_r^{(i)}} \tag{67}$$

$$Q_r^{(i)} = \int_0^L f_r(\ell, t)\, \varphi_r^{(i)}(\ell)\, d\ell \tag{68}$$

$$M_r^{(i)} = \int_0^L I_r(\ell) \left[\varphi_r^{(i)}\right]^2 d\ell \tag{69}$$

The modal slopes are defined in the following manner

$$\frac{\partial \xi_p(\ell, t)}{\partial \ell} = \sum_i q_p^{(i)}(t)\, \frac{\partial \varphi_p^{(i)}(\ell)}{\partial \ell}$$

$$= - \sum_i q_p^{(i)}(t)\, \sigma_p^{(i)}(\ell) \tag{70}$$

with similar expressions for yaw and roll.

As has been pointed out earlier, the equations of motion for the complete system are derived on the basis of artificially uncoupling the modes. This means that the coupling between these orthogonal modes arises only through a dependency of the external forces upon the motions themselves. Furthermore, in writing the equations of motion for each of these modes (sloshing, elasticity, engine inertia, etc.), the forcing functions must include inertia forces (in the sense of D'Alembert) in some appropriate manner.

We turn now to a description of these forces.

## 3.3 FORCES AND MOMENTS

The forces and moments to be derived in this section are essentially as listed in Eqs. (39) through (44) and denote perturbations from the steady-state condition. These quantities when combined with the equations of motion and the autopilot feedback loops will yield a complete description of the short period dynamics of the vehicle.

### 3.3.1 Gravity

We assume that in the steady state, the vehicle configuration is as shown in Fig. 4 where $\psi_o = \varphi_o = 0$. In this case, the force of gravity resolved along the body axes is

$$F_{xg}^{(o)} = -M_t g \cos \theta_o$$

$$F_{yg}^{(o)} = 0$$

$$F_{zg}^{(o)} = -M_t g \sin \theta_o$$

We may use Eq. (51) to obtain the components of the gravity force vector along the body axes in the disturbed condition. However we write

$$\begin{bmatrix} F_{xg} \\ F_{yg} \\ F_{zg} \end{bmatrix} = (A-I) \begin{bmatrix} -\cos \theta_o \\ 0 \\ -\sin \theta_o \end{bmatrix} M_t g$$

to ensure the removal of the steady-state components. We use the transformation matrix given by Eq. (56) and let I denote the unit matrix. Expanding

$$F_{xg} = M_t g \theta \sin \theta_o \tag{71}$$

$$F_{yg} = M_t g (\psi \cos \theta_o - \varphi \sin \theta_o) \tag{72}$$

$$F_{zg} = -M_t g \theta \cos \theta_o \tag{73}$$

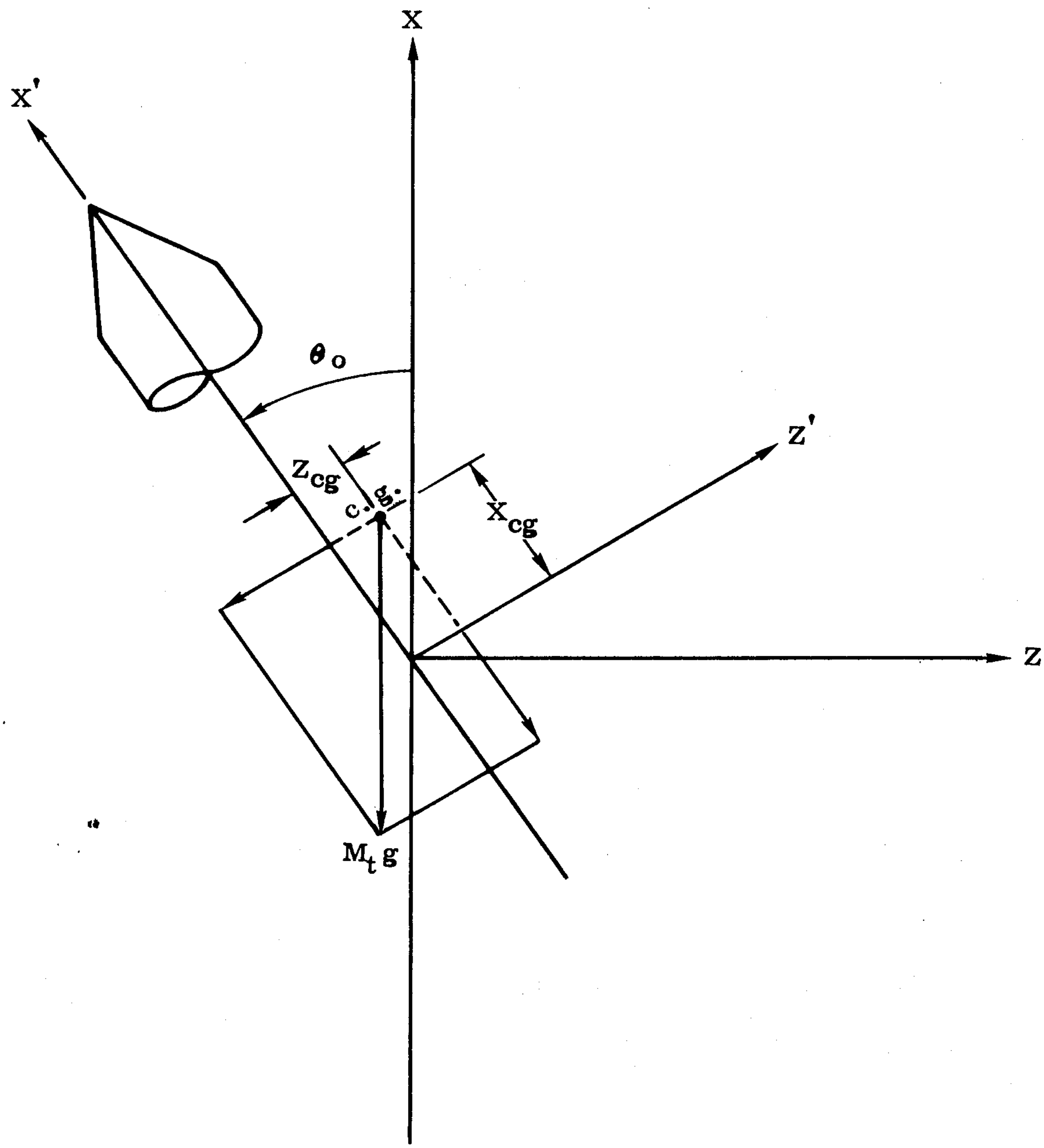


Figure 4. Gravity Force in the Steady-State Condition, Pitch Plane

Because the vehicle c.g. is displaced relative to the geometric center, there will be moments due to gravity about the body axes. In the steady-state position, these moments are

$$M_{xg}^{(o)} = -M_t g\, y_{cg} \sin\theta_o$$

$$M_{yg}^{(o)} = M_t g\,(x_{cg} \sin\theta_o - z_{cg} \cos\theta_o)$$

$$M_{zg}^{(o)} = M_t g\, y_{cg} \cos\theta_o$$

The components of the perturbation moments resolved along body axes in the disturbed condition may be computed from

$$\begin{bmatrix} M_{xg} \\ M_{yg} \\ M_{zg} \end{bmatrix} = (A-I)\begin{bmatrix} -y_{cg} \sin\theta_o \\ x_{cg} \sin\theta_o - z_{cg} \cos\theta_o \\ y_{cg} \cos\theta_o \end{bmatrix} M_t g$$

or

$$M_{xg} = M_t g\,(x_{cg}\,\psi \sin\theta_o - y_{cg}\,\theta \cos\theta_o - z_{cg}\,\psi \cos\theta_o) \tag{74}$$

$$M_{yg} = M_t g\; y_{cg}\,(\varphi \cos\theta_o + \psi \sin\theta_o) \tag{75}$$

$$M_{zg} = M_t g\,(-y_{cg}\,\theta \sin\theta_o - x_{cg}\,\varphi \sin\theta_o + z_{cg}\,\varphi \cos\theta_o) \tag{76}$$

### 3.3.2 Thrust

The perturbation thrust forces and moments listed in Eqs. (39) through (44) may be obtained from Figs. 2 and 3 by direct resolution of vectors and the elimination of steady-state components. The final result is

$$F_{xT} \approx 0 \tag{77}$$

since there is negligible perturbation of motion in the longitudinal direction.*

$$F_{yT} = T_c\,\delta_y - (T_c + T_s) \sum_i q_y^{(i)}(t)\; \sigma_y^{(i)}(\ell_T) \tag{78}$$

---

*By definition, $\sigma^{(i)}(\ell) = -\dfrac{\partial \varphi^{(i)}}{\partial \ell}$. (See list of symbols.)

$$F_{zT} = T_c \delta_p - (T_c + T_s) \sum_i q_p^{(i)}(t) \; \sigma_p^{(i)}(\ell_T) \tag{79}$$

$$M_{yT} = \ell_c \left[ T_c \delta_p - (T_c + T_s) \sum_i q_p^{(i)}(t) \; \sigma_p^{(i)}(\ell_T) \right]$$

$$- (T_c + T_s) \sum_i q_p^{(i)}(t) \; \varphi_p^{(i)}(\ell_T) \tag{80}$$

$$M_{zT} = -\ell_c \left[ T_c \delta_y - (T_c + T_s) \sum_i q_y^{(i)}(t) \; \sigma_y^{(i)}(\ell_T) \right]$$

$$+ (T_c + T_s) \sum_i q_y^{(i)}(t) \; \varphi_y^{(i)}(\ell_T) \tag{81}$$

For roll control, we assume that separate roll control engines are available which produce a torque proportional to some signal $\delta_r$, which in turn is a function of the perturbation roll angle, $\varphi$. We have therefore

$$M_{xT} = T_r \delta_r \tag{82}$$

### 3.3.3 Sloshing

It is well known that the dynamic effects of a sloshing liquid can be closely approximated by replacing the liquid mass with a rigid mass plus a harmonic oscillator (spring mass or pendulum). The pendulum parameters are a function of tank shape, liquid level, etc. Ref. 2 contains a detailed discussion of this so-called, "hydrodynamic analogy" together with an extensive bibliography. For present purposes, assume that these pendulum parameters (mass, length, hinge point location) are given, and we now seek to derive the equation of motion of the $i^{th}$ pendulum. Following this, we will obtain the forces and moments, due to sloshing, which act on the vehicle. We consider the schematic of the $i^{th}$ pendulum, shown in Fig. 5. The velocity of this pendulum relative to inertial space is given by

$$\bar{\mu}_{pi} = \bar{\mu} + \frac{d\bar{\rho}_{pi}}{dt}$$

$$= \bar{\mu} + \frac{\delta \bar{\rho}_{pi}}{\delta t} + \bar{\omega} \times \bar{\rho}_{pi}$$

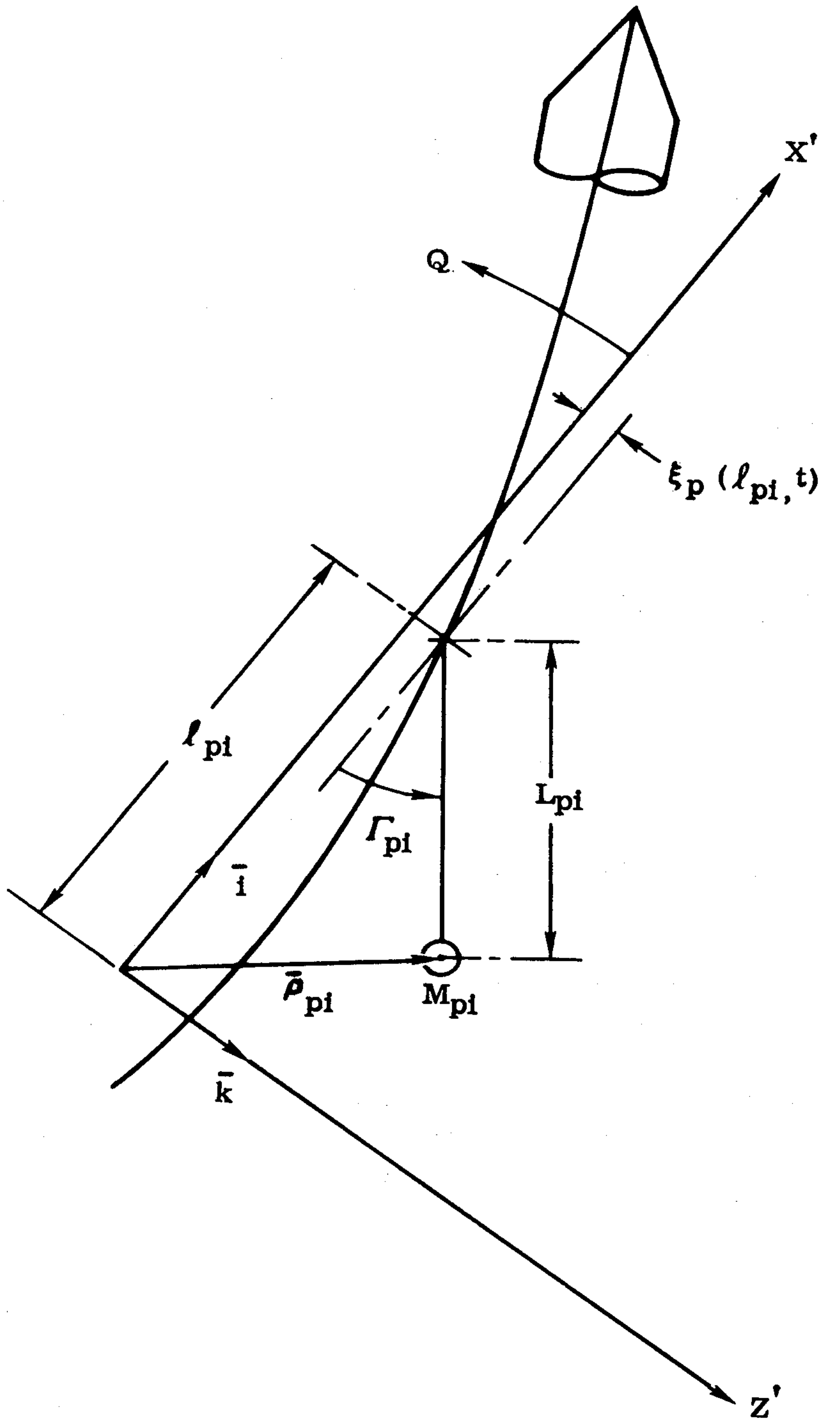


**Figure 5. Schematic of Sloshing Pendulum, Pitch Plane**

Here

$$\omega = Q\bar{j}$$

$$\bar{\mu} = U\bar{i} + W\bar{k}$$

and

$$\bar{\rho}_{pi} = (\ell_{pi} - L_{pi} \cos \Gamma_{pi}) \bar{i} + (L_{pi} \sin \Gamma_{pi} + \xi_{pi}) \bar{k}$$

where $\xi_{pi}$ stands for $\xi_p (\ell_{pi}, t)$

Carrying out the indicated operations, we find

$$\bar{\mu}_{pi} = [ U + L_{pi} \dot{\Gamma}_{pi} \sin \Gamma_{pi} + Q (L_{pi} \sin \Gamma_{pi} + \xi_{pi}) ] \bar{i}$$

$$+ [ W + (L_{pi} \dot{\Gamma}_{pi} \cos \Gamma_{pi} + \dot{\xi}_{pi}) - Q (\ell_{pi} - L_{pi} \cos \Gamma_{pi}) ] \bar{k}$$

The kinetic energy is

$$T' = \frac{1}{2} M_{pi} \bar{\mu}_{pi} \cdot \bar{\mu}_{pi}$$

There is no potential energy since the system is in free fall. Denoting the kinetic potential by L' (= T'), the equation of motion expressed in Lagrangian form is

$$\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{\Gamma}_{pi}} \right) - \frac{\partial L'}{\partial \Gamma_{pi}} = 0$$

After performing the indicated operations and assuming that $\Gamma_{pi}$ and $\dot{\Gamma}_{pi}$ are small quantities, we find

$$\ddot{\Gamma}_{pi} + \frac{\dot{U}}{L_{pi}} \Gamma_{pi} = - \frac{1}{L_{pi}} [ \dot{W} - UQ - \dot{Q} (\ell_{pi} - L_{pi}) + \ddot{\xi}_{pi} ]$$

Now put

$$\omega_{pi}^2 = \frac{\dot{U}}{L_{pi}}$$

$$W = W_o + w$$

$$Q = Q_o + q$$

$$U = U_o + u$$

and noting that $\dot{W}_o = \dot{Q}_o = 0$, we obtain after subtracting out the steady-state components

$$\ddot{\Gamma}_{pi} + \omega_{pi}^2 \Gamma_{pi} = \frac{1}{L_{pi}} \left[ U_o \dot{\theta} - \dot{w} + \ddot{\theta} (\ell_{pi} - L_{pi}) - \sum_j \ddot{q}_p^{(j)} \varphi_p^{(j)} (\ell_{pi}) \right] \tag{83}$$

The equation of motion of the $i^{th}$ pendulum in the yaw plane is obtained in analogous fashion using the schematic of Fig. 6. The final result is

$$\ddot{\Gamma}_{yi} + \omega_{yi}^2 \Gamma_{yi} = \frac{1}{L_{pi}} \left[ -U_o \dot{\psi} - \dot{v} - \ddot{\psi} (\ell_{pi} - L_{pi}) - \sum_j \ddot{q}_y^{(j)} \varphi_y^{(j)} (\ell_{pi}) \right] \tag{84}$$

The sloshing forces and moments now appear as

$$F_{xs} \cong 0 \tag{85}$$

$$F_{ys} = \sum_i M_{pi} \dot{U}_o \Gamma_{yi} \tag{86}$$

$$F_{zs} = \sum_i M_{pi} \dot{U}_o \Gamma_{pi} \tag{87}$$

$$M_{xs} = 0 \tag{88}$$

$$M_{ys} = - \sum_i M_{pi} \ell_{pi} \dot{U}_o \Gamma_{pi} \tag{89}$$

$$M_{zs} = \sum_i M_{pi} \ell_{pi} \dot{U}_o \Gamma_{yi} \tag{90}$$

It has been assumed that there are no sloshing effects about the roll axis.

### 3.3.4 Engine Inertia

The equation of motion of the rocket engine is best obtained using the Lagrangian formulation. Referring to Fig. 7, we find that the location of the engine c.g. relative to inertial space is given by

$$X_e = X_G - \ell_R \cos \left[ \delta_{po} + \delta_p - \theta_o - \theta + \frac{\partial \xi_p (\ell_T, t)}{\partial \ell} \right] \tag{91}$$

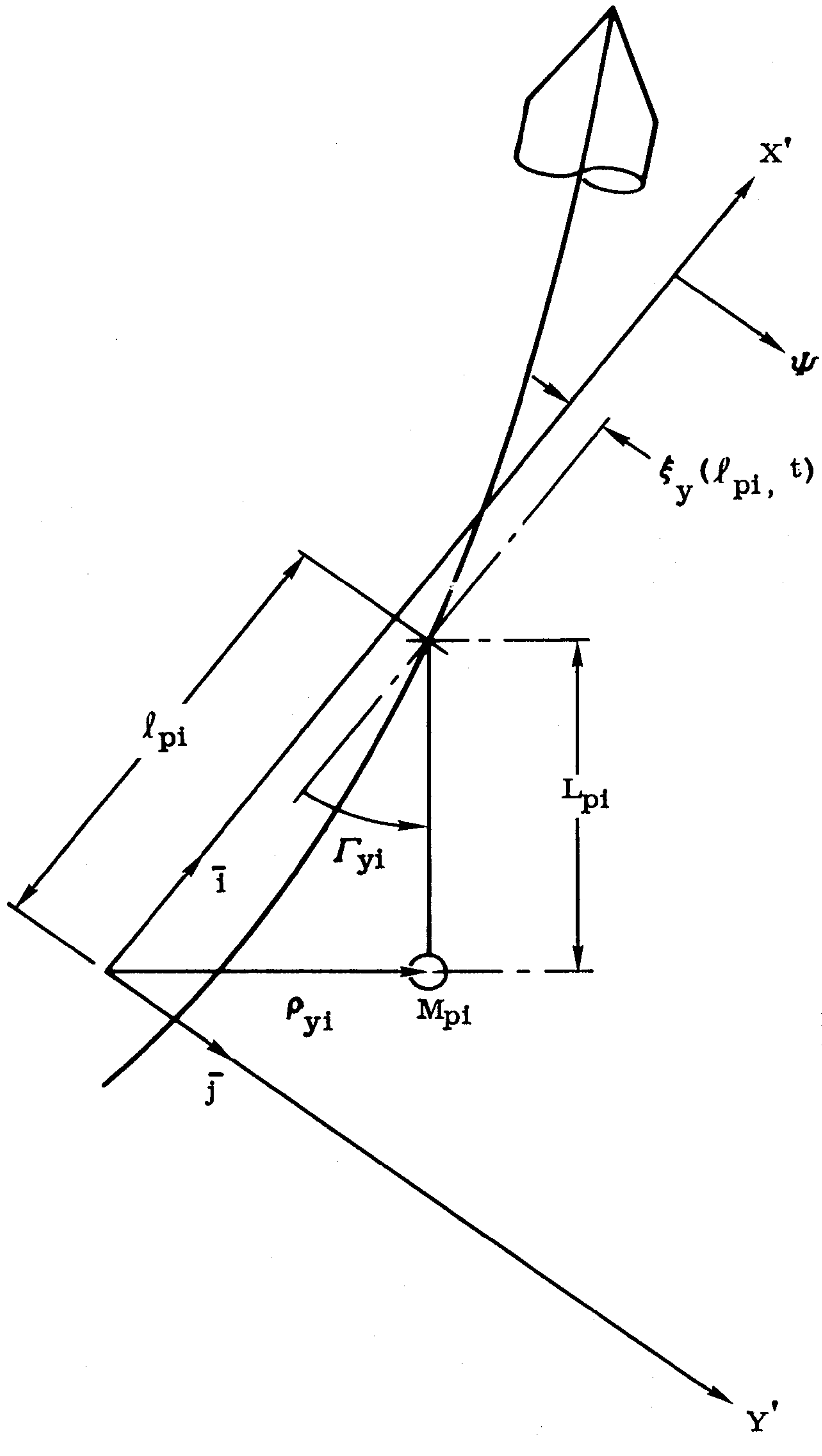


Figure 6. Schematic of Sloshing Pendulum, Yaw Plane

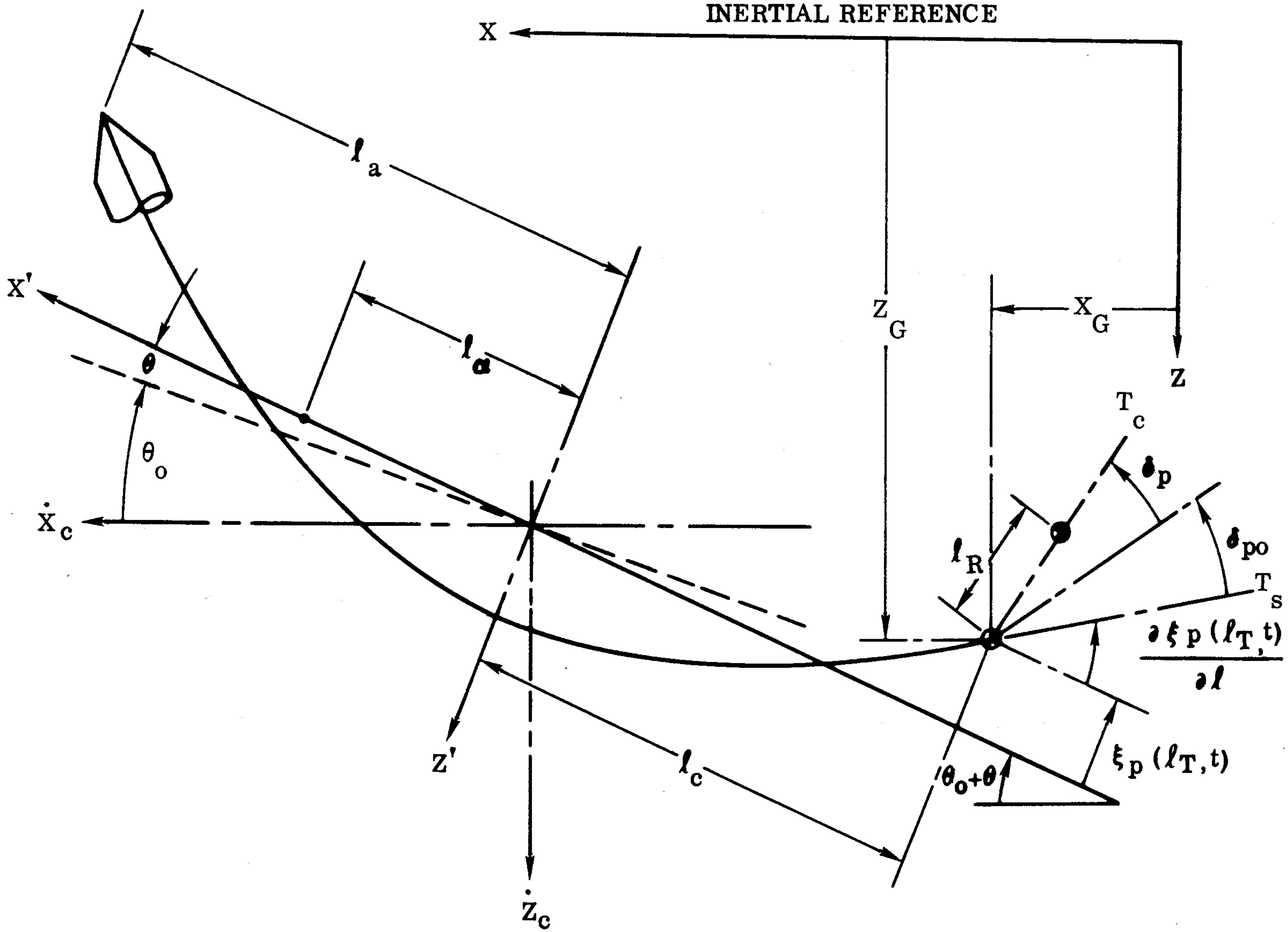


Figure 7. Schematic of Elastic Vehicle Relative to Inertial Space, Pitch Plane

$$Z_e = Z_G - \ell_R \sin\left[\delta_{po} + \delta_p - \theta_o - \theta + \frac{\partial \xi_p(\ell_T, t)}{\partial \ell}\right] \tag{92}$$

The location of the origin of the vehicle body axes relative to inertial space may be obtained with reference to Fig. 7; viz.

$$X_c = X_G + \ell_c \cos(\theta_o + \theta) + \xi_{pT} \sin(\theta_o + \theta) \tag{93}$$

$$Z_c = Z_G + \ell_c \sin(\theta_o + \theta) - \xi_{pT} \cos(\theta_o + \theta) \tag{94}$$

Here $\xi_{pT}$ stands for $\xi_p(\ell_T, t)$.

Substituting Eqs. (93) and (94) into (91) and (92), we find

$$X_e = X_c - \ell_c \cos(\theta_o + \theta) - \xi_{pT} \sin(\theta_o + \theta) - \ell_R \cos\left(\delta_{po} + \delta_p - \theta_o - \theta + \frac{\partial \xi_{pT}}{\partial \ell}\right)$$

$$Z_e = Z_c - \ell_c \sin(\theta_o + \theta) + \xi_{pT} \cos(\theta_o + \theta) - \ell_R \sin\left(\delta_{po} + \delta_p - \theta_o - \theta + \frac{\partial \xi_{pT}}{\partial \ell}\right)$$

Differentiating with respect to time yields

$$\dot{X}_e = \dot{X}_c + \ell_c \dot{\theta} \sin(\theta_o + \theta) - \xi_{pT} \dot{\theta} \cos(\theta_o + \theta) - \dot{\xi}_{pT} \sin(\theta_o + \theta)$$
$$+ \ell_R \left(\dot{\delta}_p - \dot{\theta} + \frac{\partial^2 \xi_{pT}}{\partial \ell \partial t}\right) \sin\left(\delta_{po} + \delta_p - \theta_o - \theta + \frac{\partial \xi_{pT}}{\partial \ell}\right) \tag{95}$$

$$\dot{Z}_e = \dot{Z}_c - \ell_c \dot{\theta} \cos(\theta_o + \theta) - \xi_{pT} \dot{\theta} \sin(\theta_o + \theta) + \dot{\xi}_{pT} \cos(\theta_o + \theta)$$
$$- \ell_R \left(\dot{\delta}_p - \dot{\theta} + \frac{\partial^2 \xi_{pT}}{\partial \ell \partial t}\right) \cos\left(\delta_{po} + \delta - \theta_o - \theta + \frac{\partial \xi_{pT}}{\partial \ell}\right) \tag{96}$$

The kinetic energy is

$$T'' = \frac{1}{2} M_R \left[\dot{X}_e^2 + \dot{Z}_e^2\right] + \frac{1}{2} I_o \left(\dot{\delta}_p - \dot{\theta} + \frac{\partial^2 \xi_{pT}}{\partial \ell \partial t}\right)^2 \tag{97}$$

There is no potential energy. The equations of motion then take the form

$$\frac{d}{dt}\left(\frac{\partial T''}{\partial \dot{X}_c}\right) - \frac{\partial T''}{\partial X_c} = F_{xE}^{(o)} \tag{98}$$

$$\frac{d}{dt}\left(\frac{\partial T''}{\partial \dot{Z}_c}\right) - \frac{\partial T''}{\partial Z_c} = F_{zE}^{(o)} \tag{99}$$

The generalized forces in the above equations are exerted on the engine by the vehicle and are in the direction of positive $X_c$ and $Y_c$ respectively.

To keep the engine equations reasonably tractable, it will be assumed that the steady-state pitch angle, $\theta_o$, is also a small quantity. This means that the inertial axis, X, is vertical, and that during the range of validity of the perturbation equations here developed, the vehicle is rising vertically. (See Fig. 4.)

Using Eq. (97) and performing the operations indicated in (98) and (99), we find after eliminating steady-state and higher order terms

$$F_{xE}^{(1)} = M_R \dot{u}$$

$$F_{zE}^{(1)} = M_R \left[\dot{w} - \dot{U}_o \theta - \ell_c \ddot{\theta} + \ddot{\xi}_{pT} - \ell_R \left(\ddot{\delta}_p - \ddot{\theta} + \frac{\partial^3 \xi_{pT}}{\partial \ell \, \partial t^2}\right)\right]$$

Here we have made use of the relations

$$\dot{U} = \ddot{X}_c \cos(\theta_o + \theta) - \ddot{Z}_c \sin(\theta_o + \theta)$$

$$\dot{W} = \ddot{X}_c \sin(\theta_o + \theta) + \ddot{Z}_c \cos(\theta_o + \theta)$$

which for small $\theta_o$ and $\theta$, reduce to

$$\dot{U} = \ddot{X}_c - \ddot{Z}_c(\theta_o + \theta) \tag{100}$$

$$\dot{W} = \ddot{X}_c(\theta_o + \theta) + \ddot{Z}_c \tag{101}$$

The forces exerted by the engine on the vehicle now become

$$F_{xE} = -\left[F_{xE}^{(1)} \cos(\theta_o + \theta) - F_{zE}^{(1)} \sin(\theta_o + \theta)\right]$$

$$F_{zE} = -\left[F_{xE}^{(1)} \sin(\theta_o + \theta) + F_{zE}^{(1)} \cos(\theta_o + \theta)\right]$$

or, since $\theta_o$ and $\theta$ are small

$$F_{xE} = - M_R \dot{u} \tag{102}$$

$$\begin{aligned} F_{zE} = M_R \Bigg\{ & \ell_R \ddot{\delta}_p + (\ell_c - \ell_R) \ddot{\theta} - \dot{w} + \dot{U}_o \theta \\ & - \sum_i \left[ \varphi_p^{(i)} (\ell_T) + \ell_R \, \sigma_p^{(i)} (\ell_T) \right] \ddot{q}_p^{(i)} \Bigg\} \end{aligned} \tag{103}$$

We have also

$$\frac{d}{dt} \left( \frac{\partial T''}{\partial \dot{\delta}_p} \right) - \frac{\partial T''}{\partial \delta_p} = M_{yE}^{(o)} \tag{104}$$

Carrying out the indicated operations, while making use of Eqs. (97), (100), and (101), results in

$$\begin{aligned} M_{yE}^{(1)} = {} & M_R \ell_R \left[ \dot{U}_o \left( \delta_p + \frac{\partial \xi_{pT}}{\partial \ell} \right) - \dot{w} + \ell_c \ddot{\theta} - \ddot{\xi}_{pT} \right] \\ & + I_R \left( \ddot{\delta}_p - \ddot{\theta} \quad \frac{\partial^3 \xi_{pT}}{\partial \ell \, \partial t^2} \right) \end{aligned} \tag{105}$$

after dropping steady-state and higher order terms.

The quantity $M_{yE}^{(1)}$ represents an external torque applied to the engine. This torque is generally supplied by a hydraulic servo actuator. The result is that a total torque of magnitude $(M_{yE}^{(1)} + F_{zE} \, \ell_c)$ is applied to the vehicle in the positive $\theta$ direction due to engine inertia forces; viz.

$$\begin{aligned} M_{yE} = {} & (I_R + M_R \ell_R \ell_c) \ddot{\delta}_p + M_R \ell_R \dot{U}_o \delta_p - (I_R - M_R \ell_c^2) \ddot{\theta} \\ & - M_R (\ell_R + \ell_c) \dot{w} + M_R \ell_c \dot{U}_o \theta - M_R \ell_R \dot{U}_o \sum_i \sigma_p^{(i)} (\ell_T) q_p^{(i)} \\ & - \sum_i \left[ M_R (\ell_R + \ell_c) \varphi_p^{(i)} (\ell_T) + (I_R + M_R \ell_R \ell_c) \sigma_p^{(i)} (\ell_T) \right] \ddot{q}_p^{(i)} \end{aligned} \tag{106}$$

Proceeding in completely analogous fashion, and using Fig. 8, we find that in the yaw plane

$$F_{yE} = M_R \left\{ \ell_R \ddot{\delta}_y + (\ell_c + \ell_R) \ddot{\psi} - \dot{v} - \dot{U}_o \psi \right.$$
$$\left. + \sum_i \left[ \varphi_y^{(i)} (\ell_T) + \ell_R \sigma_y^{(i)} (\ell_T) \right] \ddot{q}_y^{(i)} \right\} \tag{107}$$

The engine inertia torque is

$$M_{zE}^{(1)} = M_R \ell_R \left[ \dot{U}_o \left( \delta_y + \frac{\partial \xi_{yT}}{\partial \ell} \right) - \dot{v} + \ell_c \ddot{\psi} + \ddot{\xi}_{yT} \right]$$
$$+ I_R \left( \ddot{\delta}_y + \ddot{\psi} + \frac{\partial^3 \xi_{yT}}{\partial \ell \, \partial t^2} \right) \tag{108}$$

The total torque applied to the vehicle in a positive $\psi$ direction due to the engine inertia forces is $-(M_{zE}^{(1)} + F_{zE} \ell_c)$; or in expanded form

$$M_{zE} = -(I_R + M_R \ell_R \ell_c) \ddot{\delta}_y - M_R \ell_R \dot{U}_o \delta_y - \left[ I_o + M_R (\ell_c + \ell_R)^2 \right] \ddot{\psi}$$
$$+ M_R (\ell_R + \ell_c) \dot{v} + M_R \ell_c \dot{U}_o \psi + M_R \ell_R \dot{U}_o \sum_i \sigma_y^{(i)} (\ell_T) q_y^{(i)}$$
$$- \sum_i \left[ M_R (\ell_R + \ell_c) \varphi_y^{(i)} (\ell_T) - (I_R + M_R \ell_R \ell_c) \sigma_y^{(i)} (\ell_T) \right] \ddot{q}_y^{(i)} \tag{109}$$

### 3.3.5 Aerodynamics

In developing the forces and moments due to aerodynamic loads, the results of quasi-steady-state aerodynamic theory will be employed. This approach is valid for low frequencies of oscillation, which is the case for large booster vehicles.

The forces and moments are expressed in the usual manner as follows.

Axial Force: $$F_{XA} = \frac{1}{2} \rho U^2 A_1 C_A \tag{110}$$

Side Force: $$F_{YA} = \frac{1}{2} \rho U^2 A_2 C_Y \tag{111}$$

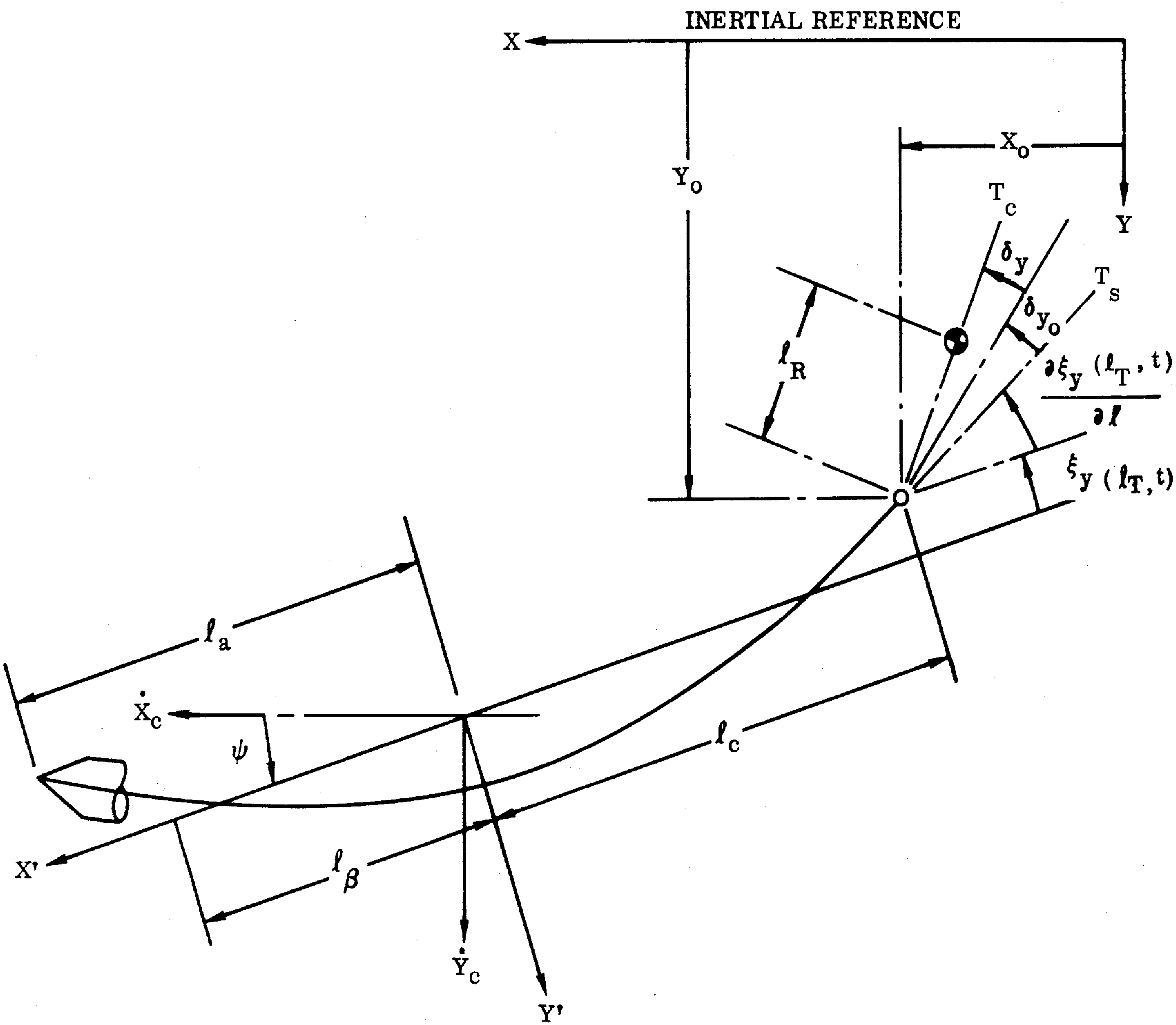


Figure 8. Schematic of Elastic Vehicle Relative to Inertial Space, Yaw Plane

Normal Force: $$F_{ZA} = \frac{1}{2} \rho U^2 A_3 C_N \tag{112}$$

Rolling Moment: $$L_A = \frac{1}{2} \rho U^2 A_4 \ell_1 C_1 \tag{113}$$

Pitching Moment: $$M_A = \frac{1}{2} \rho U^2 A_5 \ell_2 C_m \tag{114}$$

Yawing Moment: $$N_A = \frac{1}{2} \rho U^2 A_6 \ell_3 C_n \tag{115}$$

Here the $A_i$ and $\ell_i$ represent reference areas and lengths respectively. It has been assumed that V and W are small compared to U such that

$$(U^2 + V^2 + W^2)^{1/2} \approx U$$

It is often convenient to express the pitching and yawing moments in terms of normal and side forces as follows.

$$M_A = \frac{1}{2} \rho U^2 A_3 \ell_\alpha C_N \tag{116}$$

$$N_A = \frac{1}{2} \rho U^2 A_2 \ell_\beta C_Y \tag{117}$$

We now assume that each of the C coefficients in Eqs. (110) through (117) is a function of the variables, $\alpha$, $\beta$, $\dot{\alpha}$, $\dot{\beta}$, p, q, and r, and that this C function may be expanded in a Taylor series about some steady-state condition. The general expression for any $C_\lambda$, where $\lambda$ stands for A, Y, N, 1, m, or n, is

$$\begin{aligned} C_\lambda = {} & C_{\lambda o} + C_{\lambda\alpha}\, \alpha + C_{\lambda\beta}\, \beta + \frac{\ell_\nu}{2U} C_{\lambda\dot{\alpha}}\, \dot{\alpha} + \frac{\ell_\nu}{2U} C_{\lambda\dot{\beta}}\, \dot{\beta} \\ & + \frac{\ell_\nu}{2U} C_{\lambda p}\, p + \frac{\ell_\nu}{2U} C_{\lambda q}\, q + \frac{\ell_\nu}{2U} C_{\lambda r}\, r \end{aligned} \tag{118}$$

$C_{\lambda o}$ represents the value of $C_\lambda$ in the steady state while

$$C_{\lambda\alpha} \equiv \frac{\partial C_\lambda}{\partial \alpha} \text{ etc.}$$

The subscript $\nu$ on $\ell_\nu$ equals 1, 2, or 3 corresponding to $\lambda$ equals 1, m, or n.

It is further assumed that the coefficients $C_{\lambda\alpha}$, $C_{\lambda\beta}$, etc. (which correspond to the usual stability derivatives) are independent of the variables, $\alpha$, $\beta$, $\dot{\alpha}$, $\dot{\beta}$, p, q, and r. The truncation of the Taylor expansion (118) implies that these variables are small quantities. There may indeed be other terms of the form

$$C_{\lambda\delta_f}\ \delta_f\,,\quad \frac{\ell_\nu}{2U}\ C_{\lambda\dot{\delta}_f}\ \dot{\delta}_f\,,\qquad \text{etc.}$$

in the expansion (118) if aerodynamic control surfaces are present; $\delta_f$ denotes the deflection of this control surface.

Unlike the subsonic case, the above stability derivatives are a function of Mach number for transonic and supersonic speeds.

Of the 48 stability derivatives appearing in the general expression (118) for $C_\lambda$, many vanish due to symmetry of the vehicle. Furthermore, for large booster vehicles having little or no lifting surfaces, the damping derivatives are negligible, and the only stability derivatives of significance are $C_{N\alpha}$ and $C_{Y\beta}$. In the case of long slender configurations, these are generally a function of position along the vehicle and we therefore write $C_{N\alpha}(\ell)$ and $C_{Y\beta}\ (\ell)$ to emphasize this fact.

Consider now the perturbation component of the normal force. We have

$$\Delta F_{ZA} = \frac{\partial F_{ZA}}{\partial \alpha} \cdot \alpha = \frac{1}{2}\rho\, U^2 A_3 \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha}\ \alpha'\, d\ell \tag{119}$$

where*

$$\alpha' = \alpha - \frac{(\ell_a - \ell)}{U}\dot{\theta} - \frac{\partial \xi_p(\ell, t)}{\partial \ell} - \frac{\dot{\xi}_p(\ell, t)}{U} + \alpha_w \tag{120}$$

In this formulation we take account of rigid body rotation and aerodynamic damping resulting from angle of attack caused by local vehicle velocities normal to the aerodynamic velocity vector; the influence of vehicle elasticity results in similar effects yielding the last two terms of Eq. (120).

In accordance with common practice, we assume that $F_{ZA}$ acts in the negative $Z'$ direction, and that $F_{XA}$ acts in the negative $X'$ direction. The latter is simply the aerodynamic drag which is essentially independent of perturbations in pitch and yaw. Substituting Eq. (120) into (119) and making use of Eqs. (58) and (70), we find

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* $\alpha_w$ is defined by Eq. (31).

$$F_{za} = -\frac{1}{2}\rho U^2 A_3 \left[ \alpha \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} d\ell - \frac{\dot{\theta}}{U} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) d\ell \right.$$

$$\left. + \sum_i q_p^{(i)}(t) \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} \sigma_p^{(i)}(\ell) d\ell - \sum_i \frac{\dot{q}_p^{(i)}(t)}{U} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} \varphi_p^{(i)}(\ell) d\ell \right] \tag{121}$$

Similarly

$$\frac{\partial F_{YA}}{\partial \boldsymbol{\beta}} \cdot \beta = \frac{1}{2}\rho U^2 A_2 \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} \beta' d\ell \tag{122}$$

where*

$$\beta' = \beta - \frac{(\ell_a - \ell)}{U} \dot{\psi} - \frac{\partial \xi_y(\ell, t)}{\partial \ell} - \frac{\dot{\xi}_y(\ell, t)}{U} + \boldsymbol{\beta}_w \tag{123}$$

So that

$$F_{ya} = \frac{1}{2}\rho U^2 A_2 \left[ \beta \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} d\ell - \frac{\dot{\psi}}{U} \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} (\ell_a - \ell) d\ell \right.$$

$$\left. + \sum_i q_y^{(i)}(t) \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} \sigma_y^{(i)}(\ell) d\ell - \sum_i \frac{\dot{q}_y^{(i)}(t)}{U} \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} \varphi_y^{(i)}(\ell) d\ell \right] \tag{124}$$

The perturbation moments in pitch and yaw may be expressed as

$$M_A = \frac{1}{2}\rho U^2 A_3 \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) \alpha' d\ell \tag{125}$$

$$N_A = \frac{1}{2}\rho U^2 A_2 \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} (\ell_a - \ell) \beta' d\ell \tag{126}$$

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* $\boldsymbol{\beta}_w$ is defined by Eq. (32).

Using Eqs. (120) and (123), we obtain for the perturbation moments acting on the vehicle

$$M_{ya} = \frac{1}{2}\rho U^2 A_3 \left[ \alpha \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)\, d\ell - \frac{\dot{\theta}}{U} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)^2\, d\ell \right.$$

$$\left. + \sum_i q_p^{(i)}(t) \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)\, \sigma_p^{(i)}(\ell)\, d\ell - \sum_i \frac{\dot{q}_p^{(i)}(t)}{U} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)\, \varphi_p^{(i)}(\ell)\, d\ell \right] \tag{127}$$

$$M_{za} = \frac{1}{2}\rho U^2 A_2 \left[ \beta \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} (\ell_a - \ell)\, d\ell - \frac{\dot{\psi}}{U} \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} (\ell_a - \ell)^2\, d\ell \right.$$

$$\left. + \sum_i q_y^{(i)}(t) \int_0^L \frac{\partial C_Y(\ell)}{\partial \beta} (\ell_a - \ell)\, \sigma_y^{(i)}(\ell)\, d\ell - \sum_i \frac{\dot{q}_y^{(i)}(t)}{U} \int_0^L \frac{C_Y(\ell)}{\partial \beta} (\ell_a - \ell)\, \varphi_y^{(i)}(\ell)\, d\ell \right] \tag{128}$$

For the case of a long slender booster vehicle, the aerodynamic perturbation torque in roll is of negligible magnitude. We have therefore

$$M_{xa} \approx 0 \tag{129}$$

and also

$$F_{xa} \approx 0 \tag{130}$$

## 3.4 COMPLETE EQUATIONS OF PERTURBED MOTION

In collecting and summarizing the results obtained thus far, it becomes evident, because of the assumptions made, that the forward motion of the vehicle remains essentially unaffected by small perturbations. Accordingly, we may write for the forward acceleration of the origin of the vehicle body axes

$$\alpha_T = \frac{T_c + T_s - D}{m_o + \sum_i M_{pi}} \approx \dot{U}_o \tag{131}$$

When the vehicle center of mass coincides with the origin of these body axes, then Eq. (131) in fact represents the acceleration of the mass center. Otherwise, in the steady-state condition, the thrust vector passes through the mass center, which means that $\dot{V}_o$ and $\dot{W}_o$ have nonzero values.

With these observations we drop the equations of motion in the X' direction from further consideration.

The complete equations of perturbed motion may now be summarized as follows.

From Eqs. (46) through (50), making use of Eqs. (37), (38), and (57),

Rigid Body

$$\dot{\beta} = \frac{1}{U_o}\left(\frac{\Sigma F_y}{m_o} - U_o\dot{\psi} - x_{cg}\ddot{\psi} + z_{cg}\ddot{\varphi}\right) \tag{132}$$

$$\dot{\alpha} = \frac{1}{U_o}\left(\frac{\Sigma F_z}{m_o} + U_o\dot{\theta} + x_{cg}\ddot{\theta} - y_{cg}\ddot{\varphi}\right) \tag{133}$$

$$\begin{aligned} I_{xx}\ddot{\varphi} &= \Sigma M_x + I_{xy}\ddot{\theta} + I_{xz}\ddot{\psi} - m_o y_{cg} U_o(\dot{\alpha} - \dot{\theta}) \\ &\quad + m_o z_{cg} U_o(\dot{\beta} + \dot{\psi}) \end{aligned} \tag{134}$$

$$I_{yy}\ddot{\theta} = \Sigma M_y + I_{xy}\ddot{\varphi} + I_{yz}\ddot{\psi} + m_o x_{cg} U_o(\dot{\alpha} - \dot{\theta}) \tag{135}$$

$$I_{zz}\ddot{\psi} = \Sigma M_z + I_{xz}\ddot{\varphi} + I_{yz}\ddot{\theta} - m_o x_{cg} U_o(\dot{\beta} + \dot{\psi}) \tag{136}$$

In the above equations, the quantities $\Sigma F$ and $\Sigma M$ are obtained from Eqs. (40) through (44) where the terms on the right-hand side of the latter are computed in Sec. 2.3.

The elastic body degrees of freedom in the pitch, yaw, and roll planes are obtained directly from Eqs. (59), (63), and (67); viz.

Elastic Body

$$\ddot{q}_p^{(i)} + 2\xi_p^{(i)}\omega_p^{(i)}\dot{q}_p^{(i)} + \left[\omega_p^{(i)}\right]^2 q_p^{(i)} = \frac{Q_p^{(i)}}{M_p^{(i)}} \tag{137}$$

$$\ddot{q}_y^{(i)} + 2\,\zeta_y^{(i)}\,\omega_y^{(i)}\,\dot{q}_y^{(i)} + \left[\omega_y^{(i)}\right]^2 q_y^{(i)} = \frac{Q_y^{(i)}}{M_y^{(i)}} \tag{138}$$

$$\ddot{q}_r^{(i)} + 2\,\zeta_r^{(i)}\,\omega_r^{(i)}\,\dot{q}_r^{(i)} + \left[\omega_r^{(i)}\right]^2 q_r^{(i)} = \frac{Q_r^{(i)}}{M_r^{(i)}} \tag{139}$$

Eqs. (83) and (84) give the sloshing modes in pitch and yaw as follows.

Sloshing

$$\ddot{\Gamma}_{pi} + \omega_{pi}^2\,\Gamma_{pi} = \frac{1}{L_{pi}}\left[U_o\dot{\theta} - \dot{w} + \ddot{\theta}\,(\ell_{pi} - L_{pi}) - \sum_j \ddot{q}_p^{(j)}\,\varphi_p^{(j)}(\ell_{pi})\right] \tag{140}$$

$$\ddot{\Gamma}_{yi} + \omega_{yi}^2\,\Gamma_{yi} = \frac{-1}{L_{pi}}\left[U_o\dot{\psi} + \dot{v} + \ddot{\psi}\,(\ell_{pi} - L_{pi}) + \sum_j \ddot{q}_y^{(j)}\,\varphi_y^{(j)}(\ell_{pi})\right] \tag{141}$$

The torque which must be supplied by the engine servo system (in the pitch plane) is

$$T_{Ep} = M_{yE}^{(1)} + T_{fp}$$

where $T_{fp}$ represents the friction torque at the gimbal point. This is generally a combination of viscous and coulomb friction as follows

$$T_{fp} = C_{vp}\,\dot{\delta}_p + C_{Bp}\,\mathrm{sgn}\,\dot{\delta}_p$$

Using Eq. (105) in combination with the above, we find

$$T_{Ep} = (I_R + M_R\ell_R\ell_c)\,\ddot{\delta}_p + C_{vp}\,\dot{\delta}_p + C_{Bp}\,\mathrm{sgn}\,\dot{\delta}_p + M_R\ell_R\dot{U}_o\,\delta_p + T_{Lp}$$

where $T_{Lp}$ is the load torque which contains all the inertial load torques due to acceleration of the various body modes; viz.

$$\begin{aligned} T_{Lp} = &-(I_R - M_R\ell_c^2)\,\ddot{\theta} - M_R(\ell_R + \ell_c)\,\dot{w} + M_R\ell_c\dot{U}_o\theta \\ &- \sum_i \left[M_R(\ell_R + \ell_c)\,\varphi_p^{(i)}(\ell_T) + (I_R + M_R\ell_R\ell_c)\,\sigma_p^{(i)}(\ell_T)\right]\ddot{q}_p^{(i)} \\ &- M_R\ell_R\dot{U}_o\sum_i \sigma_p^{(i)}(\ell_T)\,q_p^{(i)} \end{aligned}$$

The engine servo torque, $T_{Ep}$, is a complicated function of the input signal, $\delta_{cp}$, and involves the nonlinear electro-hydraulic dynamic effects of the rocket engine servo system. A linearization procedure may be employed, as described in part 11, Vol. III of this series of monographs. The end result is (in Laplace transform notation)

Engine Servo

$$(s^3 + 2\,\zeta_{ep}\,\omega_{ep}\,s^2 + \omega_{ep}^2\,s + K_{cp}\,\omega_{ep}^2)\,\delta_p\,(s)$$

$$= K_{cp}\,\omega_{ep}^2\,\delta_{pc}\,(s) - \frac{(s + K_{bp})}{I_R}\,T_{Lp}\,(s) \tag{142}$$

A similar equation holds for the yaw plane; viz.

$$(s^3 + 2\,\zeta_{ey}\,\omega_{ey}\,s^2 + \omega_{ey}^2\,s + K_{cy}\,\omega_{ey}^2)\,\delta_y\,(s)$$

$$= K_{cy}\,\omega_{ey}^2\,\delta_{yc}\,(s) - \frac{(s + K_{by})}{I_R}\,T_{Ly}\,(s) \tag{143}$$

The various constants in the above two equations are defined in the aforementioned reference in terms of the engine servo system parameters.

To complete the description of the short period dynamics it is necessary to incorporate the equations describing the gyro feedback loops. These generate a feedback signal of the following form.

Feedback Signal For Attitude Control

$$\theta_F = \theta_{RG} + \theta_{PG} + \theta_\alpha + \theta_a \tag{144}$$

$$\psi_F = \psi_{RG} + \psi_{PG} + \psi_\beta + \psi_a \tag{145}$$

$$\varphi_F = \varphi_{RG} + \varphi_{PG} \tag{146}$$

where (in Laplace transform notation)

Rate Gyro Transfer Function

$$(s^2 + 2\,\zeta_{Rp}\,\omega_{Rp}\,s + \omega_{Rp}^2)\,\theta_{RG}$$

$$= \omega_{Rp}^2\,K_{Rp}\,s\left[\theta + \sum_i \sigma_p^{(i)}(\ell_G)\,q_p^{(i)}\right] \tag{147}$$

$$(s^2 + 2\,\zeta_{Ry}\,\omega_{Ry}\,s + \omega_{Ry}^2)\,\psi_{RG}$$

$$= \omega_{Ry}^2\,K_{Ry}\,s\left[\psi + \sum_i \sigma_y^{(i)}(\ell_G)\,q_y^{(i)}\right] \tag{148}$$

$$(s^2 + 2\,\zeta_{Rr}\,\omega_{Rr}\,s + \omega_{Rr}^2)\,\varphi_{RG}$$

$$= \omega_{Rr}^2\,K_{Rr}\,s\left[\varphi + \sum_i \sigma_r^{(i)}(\ell_G)\,q_r^{(i)}\right] \tag{149}$$

Position Gyro Transfer Function

$$(\tau_p\,s + 1)\,\theta_{PG} = \theta + \sum_i \sigma_p^{(i)}(\ell_G)\,q_p^{(i)} \tag{150}$$

$$(\tau_y\,s + 1)\,\psi_{PG} = \psi + \sum_i \sigma_y^{(i)}(\ell_G)\,q_y^{(i)} \tag{151}$$

$$(\tau_r\,s + 1)\,\varphi_{RG} = \varphi + \sum_i \sigma_r^{(i)}(\ell_G)\,q_r^{(i)} \tag{152}$$

Accelerometer Transfer Function

$$(s^2 + 2\,\zeta_a\,\omega_a\,s + \omega_a^2)\,\theta_a = \omega_a^2\,K_a^{(p)}\left\{\frac{\Sigma F_z}{M_t} - \ell_A\,\ddot{\theta} + \alpha_T\,\theta - \sum_i\left[\varphi_p^{(i)}(\ell_A)\,\ddot{q}_p^{(i)} + \alpha_T\,\sigma_p^{(i)}(\ell_A)\,q_p^{(i)}\right]\right\} \tag{153}$$

$$(s^2 + 2\zeta_a\,\omega_a\,s + \omega_a^2)\,\psi_a = \omega_a^2\,K_a^{(y)}\left\{\frac{\Sigma F_y}{M_t} + \ell_A\,\ddot{\psi} - \alpha_T\,\psi - \sum_i\left[\varphi_y^{(i)}(\ell_A)\,\ddot{q}_y^{(i)} - \alpha_T\,\sigma_y^{(i)}(\ell_A)\,q_y^{(i)}\right]\right\} \tag{154}$$

Angle of Attack Meter (Vane Sensor) Transfer Function

$$(s^2 + 2\zeta_\alpha \omega_\alpha s + \omega_\alpha^2)\,\theta_\alpha = \omega_\alpha^2 K_\alpha \left[\alpha - \sum_i \sigma_p^{(i)} (\ell_m)\, q_p^{(i)}\right] \tag{155}$$

$$(s^2 + 2\zeta_\alpha \omega_\alpha s + \omega_\alpha^2)\,\psi_\beta = \omega_\alpha^2 K_\beta \left[\beta - \sum_i \sigma_y^{(i)} (\ell_m)\, q_y^{(i)}\right] \tag{156}$$

Engine Command Signal

$$\delta_{pc} = K_{Ap}\left(1 + \frac{K_{Ip}}{s}\right) G_p(s)\,(\theta_c - \theta_F) \tag{157}$$

$$\delta_{yc} = K_{Ay}\left(1 + \frac{K_{Iy}}{s}\right) G_y(s)\,(\psi_c - \psi_F) \tag{158}$$

$$\delta_{rc} = K_{Ar}\left(1 + \frac{K_{Ir}}{s}\right) G_r(s)\,(\varphi_c - \varphi_F) \tag{159}$$

The G(s) terms denote the transfer functions of some appropriate filter.

The set of equations (132) through (159) represents the complete description of the short period dynamics of the vehicle.

## 3.5 SIMPLIFIED TRANSFER FUNCTIONS

The conventional type of launch vehicle has a high degree of symmetry about the longitudinal axis, which means that the inertial and aerodynamic cross coupling terms between pitch, roll, and yaw are negligible. This affords a crucial simplification in that pitch, roll, and yaw control systems may be analyzed separately. Beyond this, very little can be said, in general, regarding further simplifications. A detailed review of this subject is contained in Sec. 4 of Ref. 2. The validity of various approximations is generally a function of a particular vehicle, and in this sense various simplifying assumptions can be made, which must however be interpreted with due regard for inherent limitations. Since this topic is a rather extensive one, a detailed discussion is relegated to part 1 of Vol. III in this series, which treats the general problem of attitude control of launch vehicles in a more comprehensive manner.

The main objective in this monograph has been to derive a fairly complete representation of the short period dynamics of a launch vehicle, starting from first principles. In order to analyze the stability properties of the complete system, the use of a computer is mandatory.

However, for purposes of investigating the general features of a highly simplified version of the control system, the set of equations (132) through (159) may be reduced to any desired degree of simplicity. To illustrate this reduction, let us consider the rigid body dynamics of a vehicle whose mass center coincides with the geometric center. For motion in the pitch plane, we have from Eqs. (133) and (135)

$$m_o U_o \dot{\gamma} = \Sigma F_z \tag{160}$$

$$I_{yy} \ddot{\theta} = \Sigma M_y \tag{161}$$

where $\gamma$ is the perturbation flight path angle defined by

$$\gamma = \alpha - \theta \tag{162}$$

The quantities $\Sigma F_z$ and $\Sigma M_y$ are defined by Eqs. (41) and (43) respectively. If it is assumed that the engine and slosh pendulum masses are negligible compared to the vehicle mass, then

$$F_{zs} = F_{zE} = M_y = M_{yE} \cong 0$$

We then have*

$$F_{zg} = -M_t g \theta$$

$$M_{yg} = 0$$

$$F_{zT} = T_c \delta_p$$

$$M_{yT} = T_c \ell_c \delta_p$$

$$F_{za} = -\frac{1}{2}\rho U^2 A_3 \left[ \alpha \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} d\ell - \frac{\dot{\theta}}{U} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) d\ell \right]$$

$$M_{ya} = \frac{1}{2}\rho U^2 A_3 \left[ \alpha \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) d\ell - \frac{\dot{\theta}}{U} \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell)^2 d\ell \right]$$

from Eqs. (73), (75), (79), (80), (121), and (127).

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*Assuming that the steady-state attitude angle, $\theta_o$, is small.

The term containing $\frac{\dot{\theta}}{U}$ in the last two equations is generally negligible. By writing

$$L_{\alpha} = \frac{1}{2} \rho U^2 A_3 \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} d\ell \tag{163}$$

and denoting by $\ell_{\alpha}$ the distance from the geometric center to the center of pressure, we may write also

$$\ell_{\alpha} L_{\alpha} = \frac{1}{2} \rho U^2 A_3 \int_0^L \frac{\partial C_N(\ell)}{\partial \alpha} (\ell_a - \ell) d\ell \tag{164}$$

Therefore

$$F_{za} = - L_{\alpha} \alpha$$

and

$$M_{ya} = L_{\alpha} \ell_{\alpha} \alpha$$

Consequently, Eqs. (160) and (161) may be expressed as*

$$M_t U_o (\dot{\alpha} - \dot{\theta}) = T_c \delta_p - L_{\alpha} \alpha - M_t g \theta$$

$$I_{yy} \ddot{\theta} = T_c \ell_c \delta_p + L_{\alpha} \ell_{\alpha} \alpha$$

The transfer function relating attitude angle to engine deflection angle is now readily obtained as follows

$$\frac{\theta(s)}{\delta_p(s)} = \frac{\mu_c \left( s + \frac{L_{\alpha}}{M_t U_o} + \frac{\mu_{\alpha} T_c}{\mu_c M_t U_o} \right)}{\left( s^3 + \frac{L_{\alpha}}{M_t U_o} s^2 - \mu_{\alpha} s + \frac{\mu_{\alpha} g}{U_o} \right)}$$

where

$$\mu_c = \frac{T_c \ell_c}{I_{yy}}$$

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* When the sloshing masses are negligible, $M_t \approx m_o$.

$$\mu_\alpha = \frac{L_\alpha \ell_\alpha}{I_{yy}}$$

Using parameters for large booster vehicles, the following approximations are valid in the ranges noted.

Near maximum dynamic pressure:

$$\frac{\theta(s)}{\delta_p(s)} \approx \frac{\mu_c}{s^2 - \mu_\alpha}$$

At low dynamic pressure:

$$\frac{\theta(s)}{\delta_p(s)} \approx \frac{\mu_c}{s^2}$$

Progressively more complex transfer functions may be developed by including more effects; i.e., sloshing, engine inertia, bending, etc. As noted earlier, the development and interpretation of such transfer functions and how they influence the selection of compensation filters, sensor location, etc. is reserved for a later monograph in this series.*

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* See part 1, Vol. III, "Attitude Control During Launch."

## 4. REFERENCES

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