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AXISYMMETRIC VIBRATIONS OF PARTIALLY LIQUID-FILLED CYLINDRICAL CONTAINERS

by Rudolf F. Glaser

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DEFINITION OF SYMBOLS

Symbol	Definition
A	excitation amplitude
$\left.\begin{array}{r} a_0, a_1, a_2, \ldots a_n \\ b_1, b_2, \ldots b_n \end{array}\right\}$	unknowns of the linear system of equations (30)
$\left.\begin{array}{r} a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \ldots a_n^{(k)} \\ b_1^{(k)}, b_2^{(k)}, \ldots b_n^{(k)} \end{array}\right\}$	k^{th} eigensolution of the matrix eigenvalue problem (30)
c	wave velocity
c_j	series coefficients defined by equations (71), (72) and (73)
c_j'	series coefficients defined by equations (69)
c_j^*	series coefficients defined by equations (66) and (67)
E	Young's modulus of elasticity
i^{th}	natural frequency
g	acceleration due to gravity
h	shell thickness
H	liquid height
$\mathcal{J}_0(r)$	Bessel function of first kind and order 0
$\mathcal{J}_1(r)$	Bessel function of first kind and order 1
$I_0(r)$	modified Bessel function of first kind and order 0
$I_1(r)$	modified Bessel function of first kind and order 1
i	imaginary unit

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
j, k, l, m, n	indices
$K = \rho c^2$	bulk modulus of elasticity
k_ℓ	constant defined by equation (24)
k_ℓ'	constant defined by equation (38)
k_ℓ^*	constant defined by equation (65)
L	lengths of the container
p_0	external pressure defined by equation (57)
$p(r, z, t)$	liquid pressure
R	container radius
r	cylindrical coordinate
S	domain of the undisturbed liquid surface
$\vec{v}$	velocity vector of the liquid velocity field
u, v, w	shell displacements in direction z, θ, and r
$\overline{w}(z)$	dimensionless shell displacement defined by equation (12)
$\overline{w}_k(z)$	shell displacement component of the k^{th} mode
z	cylindrical coordinate
α_j	j^{th} root of $\mathcal{J}_1(r)$
β	parameter of equation (74)
β_0	root of equation (74)
$i\beta_\ell$	ℓ^{th} imaginary root of equation (74)

DEFINITION OF SYMBOLS (Cont'd)

Symbol	Definition
$\overline{\beta}_0\ \overline{\beta}_\ell$	quantities defined by equations (75)
$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$	Kronecker's symbol
$\gamma,\ \delta$	constants defined by equations (15)
Δ (%)	deviation from test values
ζ	liquid surface elevation
$\overline{\zeta}$	dimensionless surface elevation defined by equations (13)
$\overline{\zeta}_k$	surface elevation component of the k^{th} mode
η	constant defined by equation (63)
θ	cylindrical coordinate
κ_ℓ	constant defined by equation (25)
μ	defined by equation (15)
ρ	liquid density
ρ_s	shell density
$\sigma(r)$	function defined by equations (28)
$\sigma_\ell = \sigma(k_\ell R)$	
$\sigma_\ell' = \sigma(k_\ell' R)$	
$\sigma_\ell^* = \sigma(k_\ell^* R)$	
Σ	domain of the wetted shell wall
$\tau(z)$	function defined by equations (28)

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
$\tau_\ell = \tau(k_\ell z)$	
$\varphi(r, z, t)$	velocity potential of the liquid
$\bar{\varphi}(r, z)$	dimensionless potential defined by equation (11)
$\bar{\varphi}^k(r, z)$	potential of the k^{th} eigensolution
ω	parameter
ω_j	j^{th} (circular) eigenfrequency
Ω	circular excitation frequency

AXISYMMETRIC VIBRATIONS OF PARTIALLY LIQUID-FILLED CYLINDRICAL CONTAINERS

SECTION I. INTRODUCTION

The longitudinal vibration of launch vehicles represents a significant problem. During ignition, lift-off or cutoff, longitudinal vibrations occur as transients and may cause considerable dynamic loads. However, longitudinal vibrations can also be excited during power flight as has been realized in the case of the Titan missile. Coupling of vehicle structure oscillations with pressure oscillations of the propellant system generated thrust vibrations which intensified the original structure vibrations. This created unstable oscillations of the entire system. So, attention must be paid to the dynamic behavior of launch vehicles in the axial direction. In particular the dynamic behavior of the liquid-filled flexible shell container represents a significant problem. The liquid propellant inside the tanks constitutes a high percentage of the vehicle masses. These masses supported by the elastic container walls act like spring mass systems and--coupled--may generate the fundamental frequencies of the entire system. Hence, without a clear understanding of the liquid shell interaction, no exact vibration analysis of the vehicle can be performed. On the other hand, the pressure waves generated by this interaction may be adverse to propellant flow, pump performance and finally, as already mentioned, to the engine performance. Thus, the vibration analysis of partially liquid-filled shell containers is presently considered a problem of particular interest. This interest stimulated considerable scientific work which is cited in reference form [1] and specifically in other publications [2-7]. *

Although the behavior of liquid propellant in flexible containers has been regarded important and considerable theoretical work already has been accomplished, various problems are still left in this area and should be investigated. To discuss, at least, some of these problems, analytical methods have been used in this report based upon a very simple physical model, which is a flexible cylindrical container with a flat rigid bottom. Although this model appears to be rather unrealistic and academic it provides a good understanding of the vibration mechanism and is convenient for computational purposes. The analyses of this paper are restricted to axisymmetric vibrations because of their importance for space vehicle systems. Furthermore, the analyses are based on membrane theory, since--as it is well known--axial bending of the cylinder wall affects only slightly the fundamental frequencies of the system. Comparison of

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* Kana, D. D.; Glaser, R. F.; Eulitz, W. R.; and Abramson, H. N.: Longitudinal Vibration of Spring Supported Cylindrical Membrane Shells Containing Liquid. (in preparation)

experimental and theoretical data as shown in Section II confirms this behavior. See also Kana, Glaser, Eulitz and Abramson (see footnote on page 1) and Kana [5].

Now the special problems to be considered in this report will be discussed. First of all, emphasis is placed on the question of whether or not the effect of the liquid surface oscillations is significant. In other words, is it necessary to consider the combined liquid-shell-surface wave system or only the simpler liquid-shell system assuming zero pressure at the free surface? The modal analysis--applied for both cases (Sections III through VI) --shows that the amount of coupling among the "shell deflection modes" and the "liquid surface modes" is small. This behavior is demonstrated by the numerical example of the Saturn V vehicle (Section II). In Hwang [7] the same behavior is shown for the liquid-filled hemispherical shell. Consequently the analysis based on zero ullage pressure can be considered adequate for predicting natural frequencies and modes. This is also confirmed by the comparison of theoretical and experimental results in Section II. In view of the theory of eigenvalue problems, however, the coupled analysis represents an interesting example of a problem having solutions which consist of pairs of eigenfunctions. Accordingly the orthogonality relations (Section IV) appear somewhat unusual.

Sections VI and VII discuss the influence of liquid compressibility. Although it is recognized that for large flexible containers the compressibility can be neglected it is of some interest to know the conditions under which compressibility becomes effective. This problem is closely related to the theory of Waterhammer. The third table in Section II considers experimental and theoretical values of the first frequency of a liquid-filled flexible steel tube at different filling heights. The theoretical values are calculated for compressible and incompressible liquids. In this way the effect of liquid compressibility is realized. Section VII contains a forced vibration analysis assuming longitudinal excitation of the container. It is shown that the results of this analysis are in agreement with that of Bleich [2].

The above mentioned experiments and many others have been performed by SWRI, San Antonio, Texas. The experiments have been carried out using cylindrical tanks of different materials, unstiffened and stiffened by rings, having flat rigid bottoms and also flexible flat and elliptical bottoms. Emphasis was placed on the axisymmetric vibrations because of their importance to launch vehicle vibrations. The experimental outcomes agree with those of Palmer and Asher [6], i.e., axisymmetric modes are very difficult to determine as they are largely obscured by the strong presence of nonaxisymmetric response.

Furthermore the experimental data show that the nonaxisymmetric response originates from dynamic instability effects of the tank wall. A linear axisymmetric response has been obtained only if considerably stiff containers as tube like shells, brass shells, and ring stiffened shells of other materials have been utilized. In this note, the experimental data for the unstiffened steel and brass shells having flat rigid bottoms are compared with the analytical result using the membrane model discussed above.

SECTION II. NUMERICAL RESULTS, COMPARISON WITH TEST RESULTS

The analytical methods applied in this section are programmed on IBM 709-4. However, some of the numerical results are obtained using approximation formula as shown in Sections V and VI.

Large Aluminum Tank

The analysis applied is that of Section IV based on the assumption of coupled liquid-shell-surface oscillations. It results in two sets of eigensolutions with separated frequency ranges. The k^{th} normal mode consists of a pair of functions: shell deformation $\bar{w}_k(z)$ and liquid surface elevation $\bar{\zeta}_k(r)$. The frequencies of the lower range correspond to modes with predominant surface elevation while the higher frequencies belong to "shell deformation modes." This behavior is demonstrated by Figures 1 through 6 which show the three lowest modes $(\bar{\zeta}_k, \bar{w}_k)$ of each kind for the half-full tank. The order of magnitude of $\bar{\zeta}_k$ and $\bar{w}_k$ proves the weak intercoupling between shell and surface modes.

The coupled frequencies of the full, three-quarter-full, half-full, quarter-full tank up to the seventh are compiled in Table I. To show once more the low coupling effect both kinds of frequencies are also determined without consideration of coupling and presented in Table I. The uncoupled liquid surface frequencies are obtained, assuming the tank is rigid, by the well known formula [8].

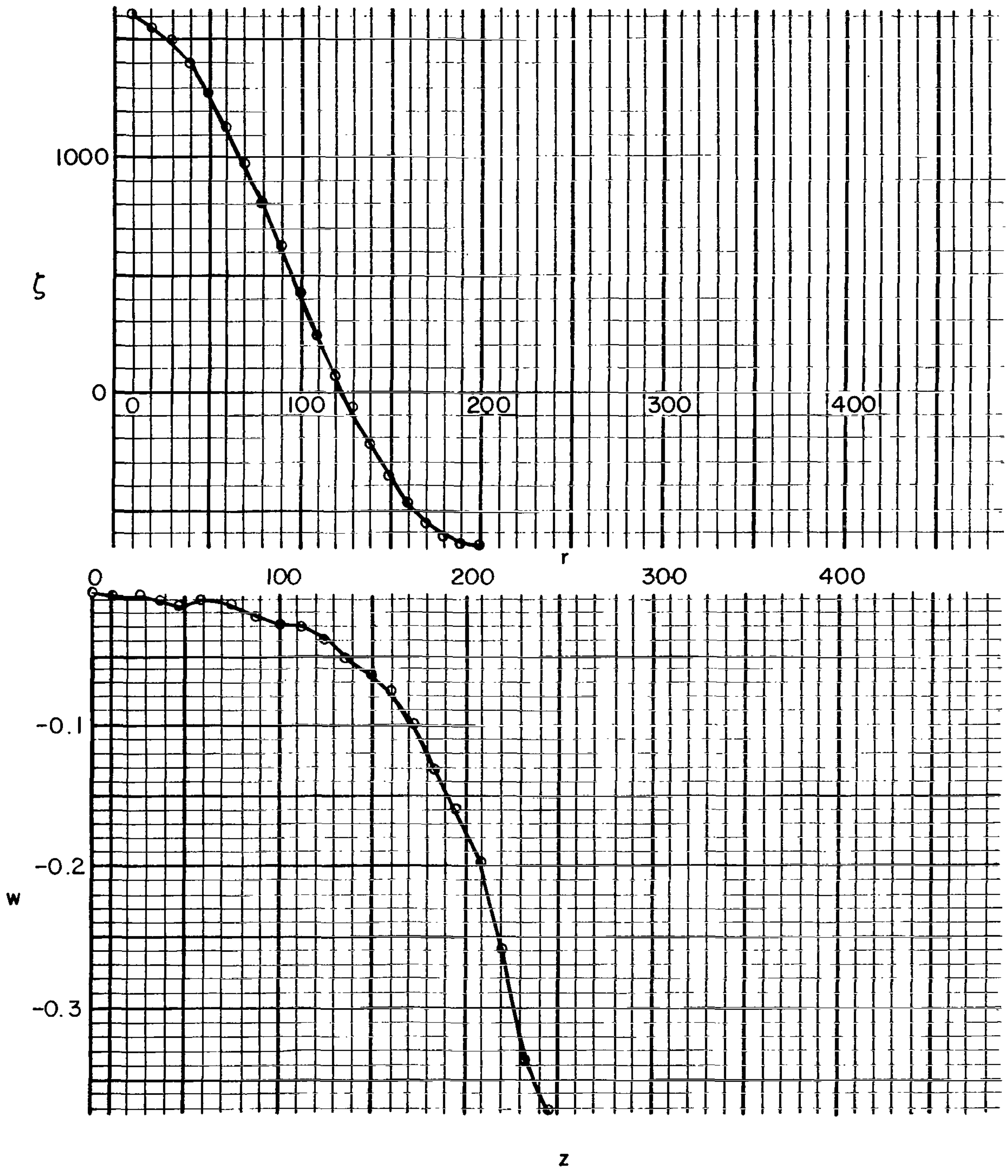

FIGURE 1. LARGE ALUMINUM TANK, H/L = 0.5; FIRST LIQUID SURFACE MODE, $f_1 = 0.44$

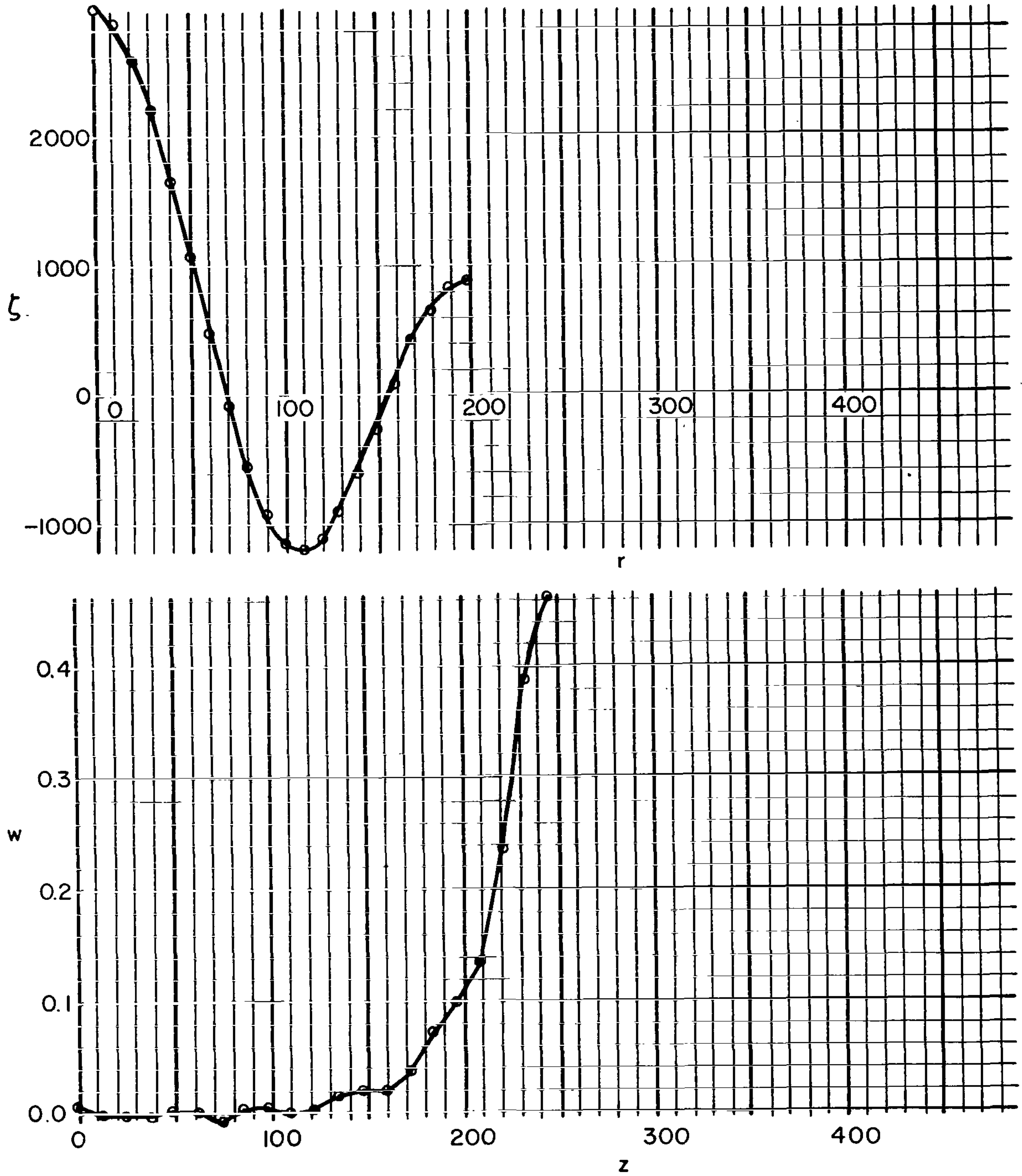

FIGURE 2. LARGE ALUMINUM TANK, H/L = 0.5; SECOND LIQUID SURFACE MODE, $f_2 = 0.59$

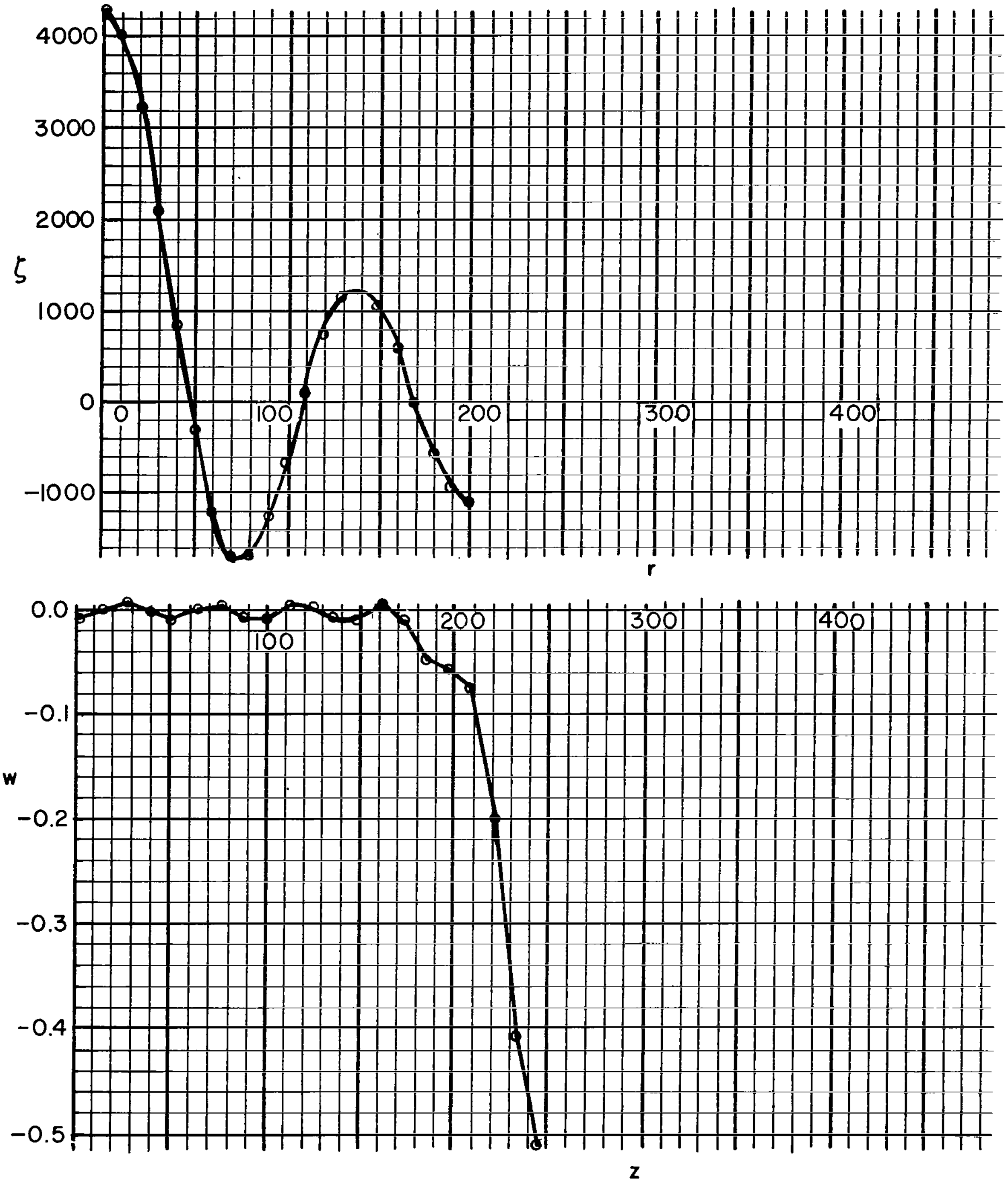

FIGURE 3. LARGE ALUMINUM TANK, H/L = 0.5; THIRD LIQUID SURFACE MODE, $f_3 = 0.70$

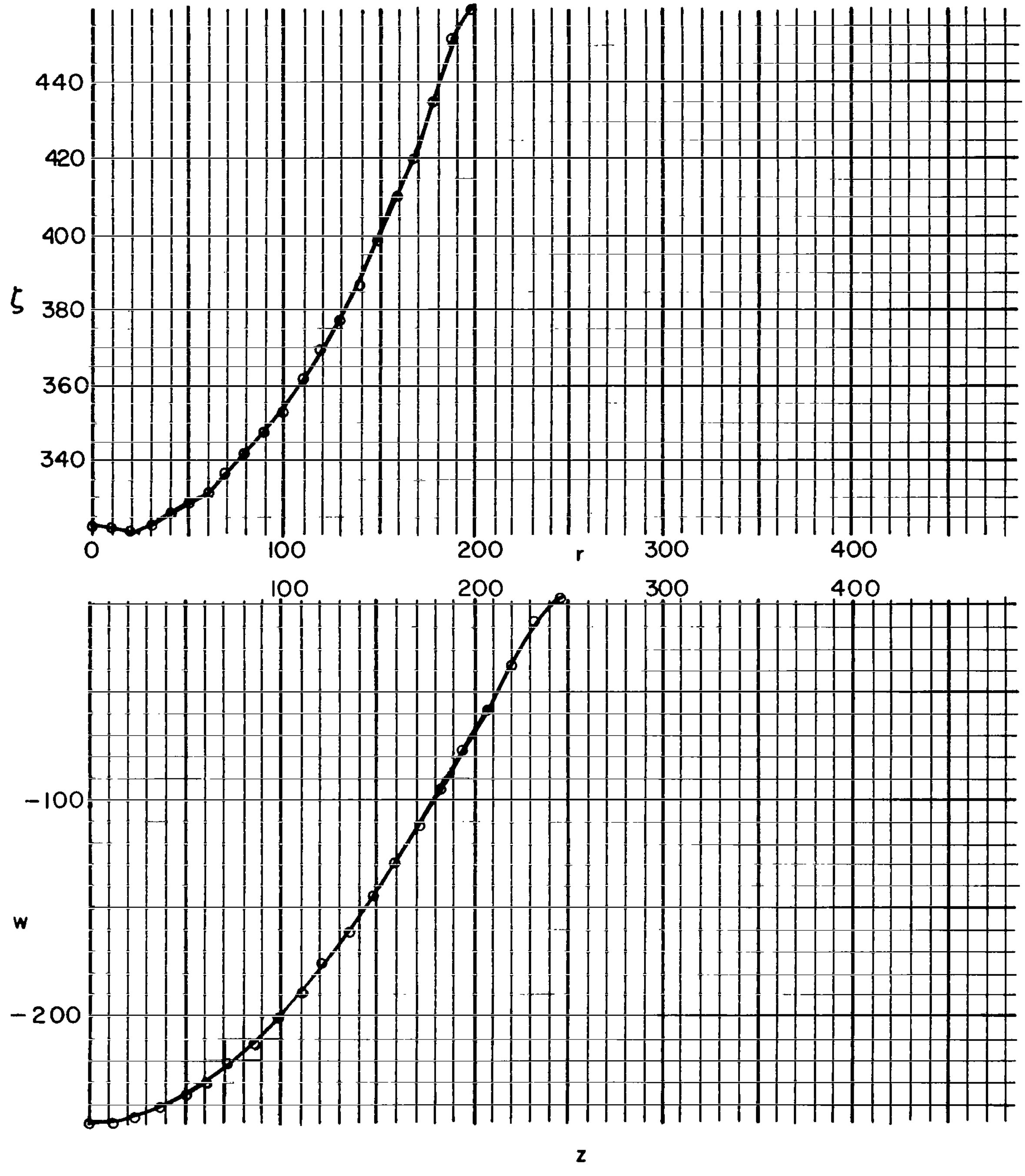

FIGURE 4. LARGE ALUMINUM TANK, H/L = 0.5; FIRST SHELL DEFLECTION MODE, $f_I = 7.3$

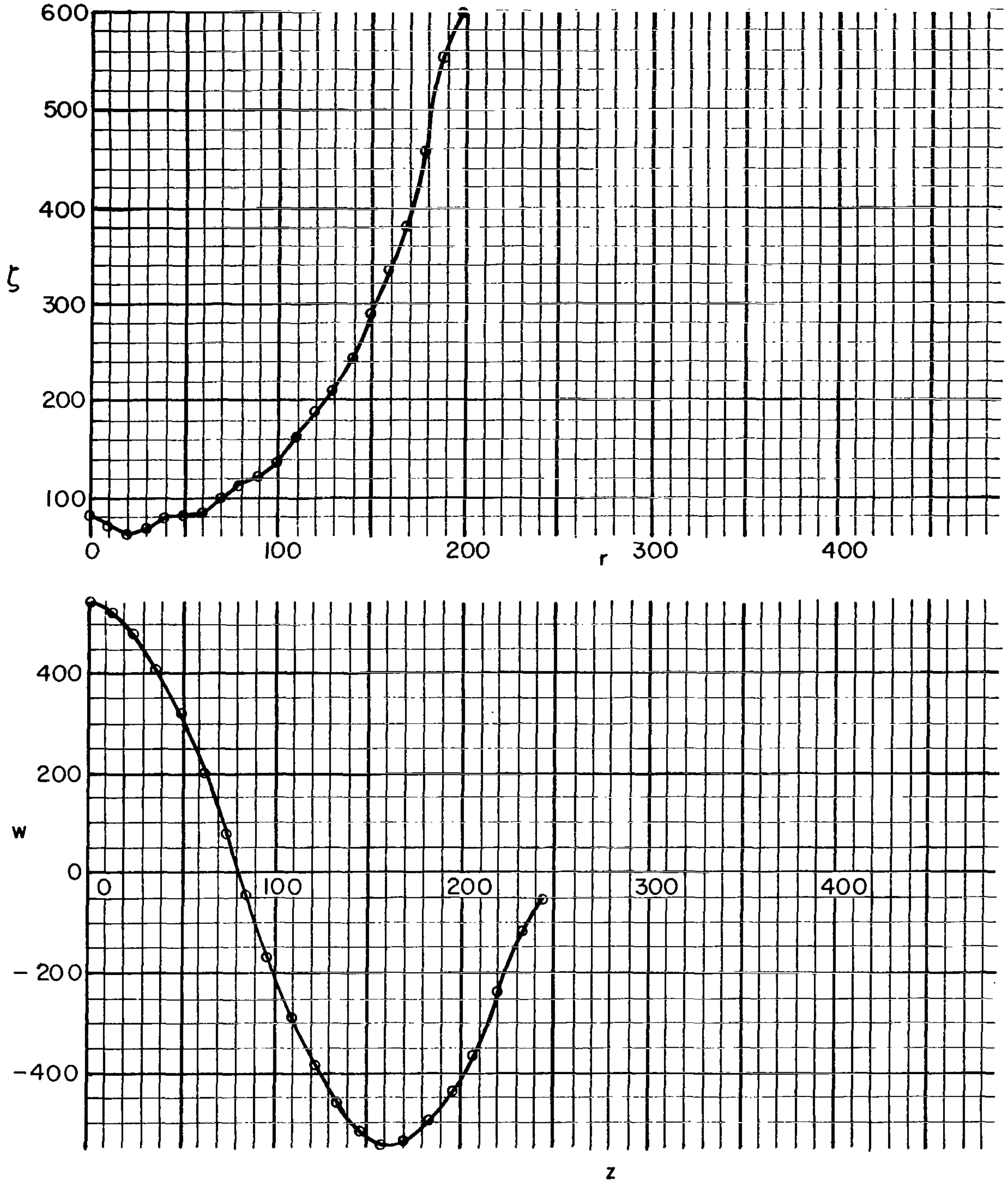

FIGURE 5. LARGE ALUMINUM TANK, H/L = 0.5; SECOND SHELL DEFLECTION MODE, $f_{II} = 16$

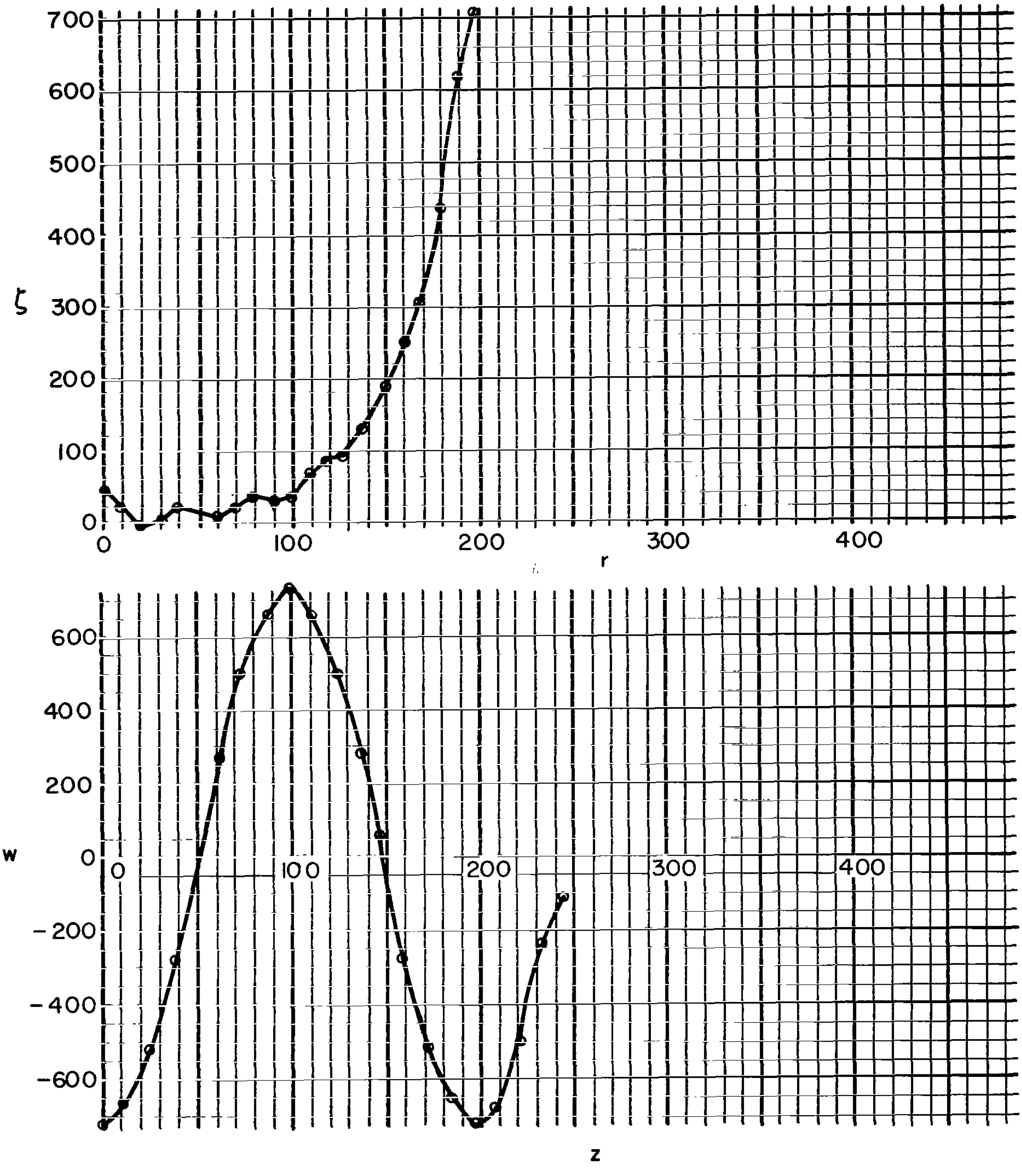

FIGURE 6. LARGE ALUMINUM TANK, H/L = 0.5; THIRD SHELL DEFLECTION MODE, $f_{III} = 21$

TABLE I. LARGE ALUMINUM TANK, COUPLED AND UNCOUPLED FREQUENCIES AT DIFFERENT LIQUID LEVELS

$R = 198.0$ in. (502.92 cm) $h = 0.22$ in. (0.5588 cm)
$L = 488.5$ in. (1240.79 cm) $E = 1.03 \times 10^7$ lb/in.^{2} (7101.85×10^7 N/m^2)
$\rho = 9.359 \times 10^{-5}$ lb sec^2/in.^{4} (0.001×10^4 kg/m^3)
$\rho_s = 2.69 \times 10^{-4}$ lb sec^2/in.^{4} (0.288×10^4 kg/m^3)

	H/L	1.0		0.75		0.50		0.25	
	Number of mode	Coupled	Un-coupled	Coupled	Un-coupled	Coupled	Un-coupled	Coupled	Un-coupled
high frequency range	1	3.90	3.90	5.11	5.11	7.33	7.33	12.4	12.4
	2	10.1	10.1	12.4	12.4	16.0	16.0	23.5	23.5
	3	14.3	14.3	17.0	17.0	21.3	21.3	30.5	30.5
	4	17.5	17.5	20.5	20.5	25.4	25.4	36.0	36.0
	5	20.1	20.1	23.5	23.5	28.9	28.9	40.6	40.7
	6	22.4	22.4	26.1	26.0	32.0	32.0	44.6	44.7
	7	24.5	24.5	28.4	28.4	34.8	34.7	48.2	48.3
low frequency range	1	0.435	0.435	0.435	0.435	0.435	0.435	0.431	0.431
	2	0.589	0.589	0.589	0.589	0.589	0.589	0.589	0.589
	3	0.709	0.709	0.709	0.709	0.709	0.709	0.709	0.709
	4	0.811	0.811	0.811	0.811	0.811	0.811	0.811	0.811
	5	0.902	0.902	0.902	0.902	0.902	0.902	0.902	0.902
	6	0.984	0.984	0.984	0.984	0.984	0.984	0.984	0.984
	7	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06

$$\left.\begin{aligned} \omega_i &= \sqrt{g\,\kappa_i \tanh \kappa_i H} \\ \mathcal{J}_1(\kappa_i R) &= 0 \end{aligned}\right\} \qquad i = 1,\ 2,\ 3,\ \ldots$$

The uncoupled liquid-shell frequencies are determined using the simple analysis of Section V. For the coupling effect to become visible at all, the frequencies of Table I are given by three digitals. However, only in the low frequency range can one observe some slight effects.

Finally, it should be mentioned that application of equation (54) shows that for the case under consideration the influence of the liquid compressibility is negligibly small.

Brass Container

Table II contains data as well as experimental and theoretical results. First and second frequencies at different filling heights are presented. Since application of equation (54) shows that the liquid compressibility can be neglected, the analysis of Section V has been applied. According to the condition (44) the first frequency for the liquid levels H>4 in. has been calculated by first order approximation using the first term of equation (43).

Steel Tube

Data as well as experimental and theoretical results are compiled in Table III. The matter for discussion is the first frequency at different liquid levels. Since condition (44) is satisfied for all levels a first-order approximation for calculation of the frequencies can be used [equation (43)]. On the other hand, application of the condition (55),

$$\frac{h}{2R}\ \frac{E}{K} = 0.32$$

shows that for the case in hand the liquid compressibility cannot be neglected. However, Table III presents both kinds of frequencies, assuming the liquid

TABLE II. BRASS CONTAINER, EXPERIMENTAL AND THEORETICAL FREQUENCIES

$R = 4.5$ in. (11.43 cm) $h = 0.005$ in. (0.0127 cm)
$L = 11.2$ in. (28.448 cm) $E = 15.3 \times 10^6$ lb/in.^{2} ($105\,483.5 \times 10^6$ N/m^2)
$\rho = 9.359 \times 10^{-5}$ lb sec^2/in.^{4} (0.001×10^4 kg/m^3)
$\rho_s = 79.55 \times 10^{-5}$ lb sec^2/in.^{4} (0.851×10^4 kg/m^3)

H	f_1 (Hz)			f_2 (Hz)		
in. (cm)	Test	Analysis	Δ(%)	Test	Analysis	Δ(%)
10.9 (27.686)	217	213.0	1.8	566	548.3	3.1
10.5 (26.670)	226	220.6	2.4	586	563.4	3.9
9.5 (24.13)	250	242.4	3.0	627	604.8	3.5
8.5 (21.59)	276	268.8	2.6	678	652.5	3.8
7.5 (19.05)	309	301.4	2.5	736	707.8	3.8
6.5 (16.51)	352	342.3	2.8	811	773.5	4.5
5.5 (13.97)	404	395.3	2.2			
4.5 (11.43)	478	465.7	2.6			
3.5 (8.89)	576	563.4	2.2			
2.5 (6.35)	739	707.8	4.2			
1.5 (3.81)	996	955.5	4.1			

TABLE III. STEEL TUBE, EXPERIMENTAL AND THEORETICAL FREQUENCIES

$R = 1.5$ in. (3.81 cm)
$L = 16.0$ in. (40.64 cm)
$h = 0.01$ in. (0.0254 cm)
$E = 29 \times 10^6$ lb/in.^{2} ($199\ 955 \times 10^6$ N/m^2)
$\rho = 9.359 \times 10^{-5}$ lb sec^2/in.^{4} (0.100×10^4 kg/m^3)
$c = 0.563 \times 10^5$ in./sec (1.43×10^5 cm/sec)

	Experiment	Analysis			
		Liquid Assumed Compressible		Liquid Assumed Incompressible	
H in. (cm)	f_1 (Hz)	f_1 (Hz)	Deviation from Experiment (%)	f_1 (Hz)	Deviation from Experiment (%)
16 (40.64)	451	436	3.3	502	11.3
15 (38.1)	481	465	3.3	536	11.4
14 (35.56)	518	498	3.9	574	10.8
13 (33.02)	553	537	2.9	618	11.8
12 (30.48)	594	581	2.2	670	12.8
11 (27.94)	652	634	2.8	730	12.0
10 (25.4)	717	698	2.4	803	12.0
9 (22.86)	793	775	2.3	893	12.6
8 (20.32)	894	872	2.5	1004	12.3
7 (17.78)	1003	997	0.6	1148	14.5
6 (15.24)	1176	1163	1.2	1339	13.9
5 (12.7)	1373	1396	1.7	1607	17.0
4 (10.16)	1655	1744	5.4	2009	21.4
3 (7.62)	2133	2326	9.0	2678	25.6

compressible and also incompressible. In this way the significance of the compressibility for the case in question is demonstrated once more.

SECTION III. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The cylindrical shell is referred to a cylindrical system of coordinates r, θ, z (Fig. 7). The shell displacements in directions r, θ, z shall be w, v, u. The surface elevation of the liquid is denoted by ζ. The following simplifying assumptions will be made:

1. The analysis is restricted to the axisymmetric case (v = 0).

2. The components of the shell inertial forces in direction u will be neglected.

3. At the top, the shell is free and hence no axial membrane forces are acting.

4. The liquid is considered inviscous and irrotational.

Under these assumptions the equation governing the motion of the shell wall [9] is given by

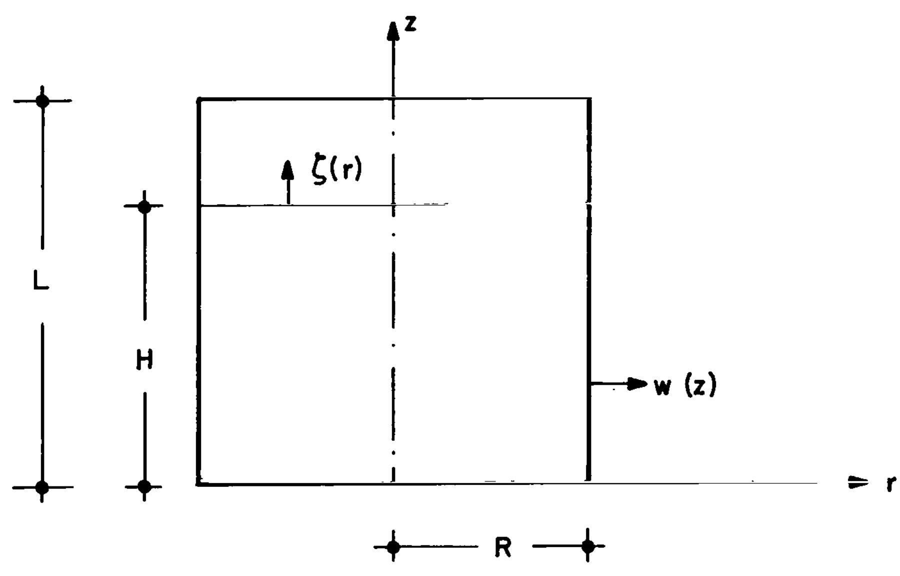

FIGURE 7. PARTIALLY LIQUID-FILLED CYLINDRICAL CONTAINER; COORDINATE SYSTEM; DISPLACEMENTS

$$\frac{Eh}{R^2}\ w(z,t) + \rho_s h \frac{\partial^2 w(z,t)}{\partial t^2} = p(r,z,t) \quad 0 \leq z \leq H \ , \tag{1}$$

while the liquid velocity field can be represented by a gradient field

$$\vec{v} = \nabla \varphi(r,z,t) \quad \begin{matrix} 0 \leq r \leq R \\ 0 \leq z \leq H \end{matrix}$$

where the potential φ is a solution of

$$\nabla^2 \varphi(r,z,t) = 0 \tag{2}$$

or

$$\nabla^2 \varphi(r,z,t) = \frac{1}{c}\ \frac{\partial^2 \varphi}{\partial t^2} \ , \tag{3}$$

depending on whether the liquid is assumed to be incompressible or compressible [2, 3, 10].

Neglecting static and second-order terms the pressure equation is given by

$$p(r,z,t) - p_0 = -\rho \frac{\partial \varphi}{\partial t} \tag{4}$$

where p_0 is an external pressure independent from the potential φ. In the case of free vibrations p_0 can be assumed zero.

The boundary conditions to be satisfied by φ are

$$\frac{\partial \varphi(R,z,t)}{\partial r} = \frac{\partial w}{\partial t} \tag{5}$$

$$\frac{\partial \varphi(r,0,t)}{\partial z} = 0 \quad . \tag{6}$$

The free surface condition depends on whether or not the surface elevation of the liquid is taken into account. In the first case the pressure on the undisturbed liquid surface equals--apart from a constant--the "weight" of the surface wave. In the second case the pressure is constant and can be assumed zero. Hence

$$-\rho \frac{\partial\varphi}{\partial t} = \begin{cases} \rho g \zeta\,(r,t) & (7) \\ 0 \quad \text{alternate condition} \quad . & (8) \end{cases}$$

If the surface elevation ζ is taken into account an additional condition-- equivalent to equation (5) --must be satisfied

$$\frac{\partial\varphi(r,H,t)}{\partial z} = \frac{\partial\zeta}{\partial t} \tag{9}$$

SECTION IV. FREE VIBRATION ANALYSIS OF THE LIQUID-SHELL-FREE SURFACE SYSTEM

The vibrations of this system are governed by equations (1), (2), (4), through (7) and (9). Using matrix notations, these equations can be written as follows:

$$\left.\begin{aligned}
&\begin{bmatrix} \frac{Eh}{R^2} & 0 \\ 0 & \rho g \end{bmatrix} \begin{bmatrix} w(z,t) \\ \zeta(r,t) \end{bmatrix} = - \begin{bmatrix} \rho_s h & 0 \\ 0 & 0 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} w(z,t) \\ \zeta(r,t) \end{bmatrix} - \rho \frac{\partial}{\partial t} \begin{bmatrix} \varphi(R,z,t) \\ \varphi(r,H,t) \end{bmatrix} \\
&\frac{\partial}{\partial t} \begin{bmatrix} w(z,t) \\ \zeta(r,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial\varphi(R,z,t)}{\partial r} \\ \frac{\partial\varphi(r,H,t)}{\partial z} \end{bmatrix} \\
&\nabla^2\varphi(r,z,t) \equiv \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad \begin{array}{l} 0 \le r \le R \\ 0 \le z \le H \end{array} \\
&\frac{\partial\bar{\varphi}(r,0,t)}{\partial z} = 0
\end{aligned}\right\} \tag{10}$$

Assuming

$$\varphi(r,z,t) = H^2\omega\, \overline{\varphi}(r,z) \cos \omega t \tag{11}$$

$$w(z,t) = H\, \overline{w}(z) \sin \omega t \tag{12}$$

$$\zeta(r,t) = H\, \overline{\zeta}(r) \sin \omega t \tag{13}$$

equations (13) reduce to

$$\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \overline{w}(z) \\ \overline{\zeta}(r) \end{bmatrix} = \frac{\omega^2}{\omega_{sh}^2} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{w}(z) \\ \overline{\zeta}(r) \end{bmatrix} + \frac{\mu}{\delta} \begin{bmatrix} \overline{\varphi}(R,z) \\ \overline{\varphi}(r,H) \end{bmatrix} \right\}, \tag{14}$$

where

$$\mu = \frac{H}{R} \; ; \; \gamma = \frac{R^2 \rho g}{Eh} \; ; \; \delta = \frac{\rho_s h}{\rho R} \; ; \; \omega_{sh}^2 = \frac{E}{R^2 \rho_s} \tag{15}$$

$$\begin{bmatrix} \overline{w}(z) \\ \overline{\zeta}(r) \end{bmatrix} = H \begin{bmatrix} \dfrac{\partial \overline{\varphi}(R,z)}{\partial r} \\ \dfrac{\partial \overline{\varphi}(r,H)}{\partial z} \end{bmatrix} \tag{16}$$

$$\nabla^2 \overline{\varphi}(r,z) = \frac{\partial^2 \overline{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{\varphi}}{\partial r} + \frac{\partial^2 \overline{\varphi}}{\partial z^2} = 0 \qquad \begin{matrix} 0 \le r \le R \\ 0 \le z \le H \end{matrix} \tag{17}$$

$$\frac{\partial \overline{\varphi}(r,0)}{\partial z} = 0 \quad . \tag{18}$$

Equations (14) through (18) represent the eigenvalue problem in hand. The eigensolutions consist of the eigenvalues and the corresponding pairs of eigenfunctions:

$$\omega_k^2 \; ; \quad \begin{bmatrix} \overline{w}_k(z) \\ \overline{\zeta}_k(r) \end{bmatrix} \quad ; \quad k = 1,2,3\ldots \tag{19}$$

which can be represented by the normal derivatives of a potential

$$\overline{\varphi}^{(k)}(r,z) \qquad 0 \le r \le R; \qquad 0 \le z \le H \tag{20}$$

in accordance with equations (16) through (18). On the other hand the potential $\bar{\varphi}^{(k)}$ can be derived from the eigensolution (19) in accordance with the second boundary value problem of the potential theory.

In the sequel, the orthogonality of the eigensolutions may be discussed. Let

$$\omega_m^2 ; \quad \begin{bmatrix} \bar{w}_m \\ \bar{\zeta}_m \end{bmatrix} ; \qquad \omega_n^2 ; \quad \begin{bmatrix} \bar{w}_n \\ \bar{\zeta}_n \end{bmatrix}$$

be two different eigensolutions. Then it follows from equation (14) in the usual way

$$\frac{\delta}{\mu}(\omega_m^2 - \omega_n^2) \int_\Sigma \bar{w}_m \bar{w}_n \, ds_1 + \omega_m^2 \left(\int_\Sigma \bar{w}_n \bar{\varphi}^{(m)} \, ds_1 + \int_S \bar{\zeta}_n \bar{\varphi}^{(m)} \, ds_2 \right)$$

$$-\omega_n^2 \left(\int_\Sigma \bar{w}_m \bar{\varphi}^{(n)} \, ds_1 + \int_S \bar{\zeta}_m \bar{\varphi}^{(n)} \, ds_2 \right) = 0$$

where

$$\left.\begin{array}{lllll} \Sigma: & r = R; & 0 \leq \theta \leq 2\pi ; & 0 \leq z \leq H ; & ds_1 = R d\theta \, dz \\ S: & z = H; & 0 \leq \theta \leq 2\pi ; & 0 \leq r \leq R ; & ds_2 = r dr \, d\theta \end{array}\right\} \quad (21)$$

represents wetted shell wall and the free undisturbed liquid surface, respectively. Now, application of the well known Greens theorem shows that the two sums within the brackets are equal, hence the orthogonality relations can be written as:

$$\left.\begin{array}{c} \omega_m \neq \omega_n \\ \int_\Sigma \bar{w}_n \left(\frac{\delta}{\mu} \bar{w}_m + \bar{\varphi}^{(m)} \right) ds_1 + \int_S \bar{\zeta}_n \bar{\varphi}^{(m)} \, ds_2 = 0 \end{array}\right\} . \quad (22)$$

Solving the eigenvalue problem will be performed using Galerkin's method. In doing so the potential $\bar{\varphi}$ will be approximated by

$$\bar{\varphi}(r,z) = \sum_{\ell=1}^{n} a_\ell \frac{I_0(k_\ell r)}{I_0(k_\ell R)} \frac{\cos k_\ell z}{\cos k_\ell H} + \sum_{\ell=1}^{n} b_\ell \frac{\cosh \kappa_\ell z \mathcal{J}_0(\kappa_\ell r)}{\cosh \kappa_\ell H \mathcal{J}_0(\kappa_\ell R)}$$

$$+ a_0 \left(\frac{z^2}{R^2} - \frac{r^2}{2R^2} \right) \tag{23}$$

with

$$k_\ell = \frac{\ell \pi}{H} \tag{24}$$

$$\ell = 1, 2, \ldots .$$

$$\mathcal{J}_1(\kappa_\ell R) = 0 \tag{25}$$

As it can be seen by substitution the series (23) satisfies Laplace's equation (17) and the boundary condition (18). The three terms of the right side of equation (23) in succession represent the velocity potentials of:

1. The liquid-filled container with flexible wall, rigid flat bottom, and rigid flat top on the undisturbed surface.

2. The liquid-filled container with rigid wall and bottom and free liquid surface.

3. The liquid-filled container with rigid bottom, free surface, and a ring-like flexibility of the wall. This potential governs the mean flow through the container wall and the undisturbed liquid surface.

From equations (23) and (16) it follows

$$\begin{bmatrix} \bar{\varphi}(R,z) \\ \bar{\varphi}(r,H) \end{bmatrix} = \sum_{\ell=1}^{n} a_\ell \begin{bmatrix} \cos k_\ell z / \cos k_\ell H \\ I_0(k_\ell r)/I_0(k_\ell R) \end{bmatrix} + \sum_{\ell=1}^{n} b_\ell \begin{bmatrix} \cosh \kappa_\ell z / \cosh \kappa_\ell H \\ \mathcal{J}_0(\kappa_\ell r)/\mathcal{J}_0(\kappa_\ell R) \end{bmatrix}$$

$$+ a_0 \begin{bmatrix} \dfrac{z^2}{R^2} - \dfrac{1}{2} \\ \mu^2 - \dfrac{r^2}{2R^2} \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} \overline{w}(z) \\ \overline{\zeta}(r) \end{bmatrix} = \mu \begin{bmatrix} R\,\overline{\varphi}_r(R,z) \\ R\,\overline{\varphi}_z(r,H) \end{bmatrix} = \mu \sum_{\ell=1}^{n} a_\ell \begin{bmatrix} \sigma_\ell \cos k_\ell z / \cos k_\ell H \\ 0 \end{bmatrix}$$

$$+ \sum_{\ell=1}^{n} b_\ell \begin{bmatrix} 0 \\ \tau_\ell\, \mathcal{J}_0(\kappa_\ell r)/\mathcal{J}_0(\kappa_\ell R) \end{bmatrix} + a_0\mu \begin{bmatrix} -1 \\ 2\mu \end{bmatrix} , \qquad (27)$$

where

$$\left.\begin{aligned} \sigma(r) &= \frac{r\, I_1(r)}{I_0(r)} \quad ; \quad \tau(z) = z \tanh z \\ \sigma_\ell &= \sigma(k_\ell R) \quad ; \quad \tau_\ell = \tau(\kappa_\ell H) \quad ; \quad \ell = 1,\ 2,\ 3, \ldots \end{aligned}\right\} . \qquad (28)$$

From equation (27) the pairs of coordinate functions of the Galerkin approach may be recognized as

$$\begin{bmatrix} \sigma_j \cos k_j z / \cos k_j H \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ \frac{\tau_j}{\mu} \mathcal{J}_0(\kappa_j r)/\mathcal{J}_0(\kappa_j R) \end{bmatrix} ; \begin{bmatrix} -1 \\ 2\mu \end{bmatrix}$$

$$j = 1, 2, 3, \ldots n .$$

To apply Galerkin's method equations (26) and (27) must be substituted into equation (14). Now successive scalar (left) multiplication of equation (14) with the above pairs of functions and integration over wetted shell wall and liquid surface as defined by the domains (21) yields a linear homogeneous system of equations in the unknowns:

$$a_1;\ a_2;\ a_3; \ldots a_n;\ b_1;\ b_2;\ b_3; \ldots b_n;\ a_0 . \qquad (29)$$

In doing so, some of the integrations to be performed are the following:

$$\int_\Sigma \cos k_i z \cos k_j z\, ds_1 = R\pi \int_0^H \cos k_i z \cos k_j z\, dz = R\pi H\, \delta_{ij}$$

$$\int_S \mathcal{J}_0(\kappa_i r)\,\mathcal{J}_0(\kappa_j r)\,ds_2 = 2\pi \int_0^R \mathcal{J}_0(\kappa_i r)\,\mathcal{J}_0(\kappa_j r)\,r\,dr = R^2\pi\left[\mathcal{J}_0(\kappa_j R)\right]^2 \delta_{ij} \quad [11]$$

(Orthogonality conditions)

$$\int_\Sigma \sigma_j \frac{\cos k_j z}{\cos k_j H}\,\frac{\cosh \kappa_\ell z}{\cosh \kappa_\ell H}\,ds_1 = \frac{1}{H^2}\,\frac{2\sigma_j\,\tau_\ell}{k_j^2 + \kappa_\ell^2}\,R\pi H$$

$$\int_\Sigma \sigma_j \frac{z^2}{R^2}\,\frac{\cos k_j z}{\cos k_j H}\,ds_1 = 4\mu^2\,\frac{\sigma_j}{(k_j H)^2}\,R\pi H$$

$$\int_S \frac{\tau_j}{\mu}\,\frac{\mathcal{J}_0(\kappa_j r)}{\mathcal{J}_0(\kappa_j R)}\,\frac{I_0(k_\ell r)}{I_0(k_\ell R)}\,ds_2 = \frac{1}{H^2}\,\frac{2\sigma_\ell\,\tau_j}{k_\ell^2 + \kappa_j^2}\,R\pi H \quad [11]$$

$$\int_S 2\mu\,\frac{I_0(k_\ell r)}{I_0(k_\ell R)}\,ds_2 = 4\mu^2\,\frac{\sigma_\ell}{(k_\ell H)^2}\,R\pi H \quad [11]$$

$$\int_S \frac{\tau_j}{\mu}\left(\mu^2 - \frac{1}{2}\,\frac{r^2}{R^2}\right)\frac{\mathcal{J}_0(\kappa_j r)}{\mathcal{J}_0(\kappa_j R)}\,ds_2 = -\frac{2}{\mu^2}\,\frac{\tau_j}{(\kappa_j R)^2}\,R\pi H \quad . [11]$$

In this way the problem will be reduced to the following symmetrical matrix eigenvalue problem

$$
\left\{
\left[
\begin{array}{ccc|ccc|c}
\sigma_1^2 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 & \sigma_2^2 & 0 & 0 & \cdots & 0 & \vdots \\
 & \ddots & & & & & \vdots \\
0 & \cdots & \sigma_n^2 & 0 & & 0 & 0 \\
\hline
0 & \cdots & 0 & \frac{\tau_1^2}{\mu^3}\gamma & & 0 & 0 \\
0 & \cdots & 0 & 0 & \frac{\tau_2^2}{\mu^3}\gamma & 0 & \vdots \\
 & & & & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \frac{\tau_n^2}{\mu^3}\gamma & 0 \\
\hline
0 & \cdots & 0 & 0 & \cdots & 0 & 2 + 4\mu\gamma
\end{array}
\right]
\right.
$$

(30)

$$
\left.
-\frac{\omega^2}{\omega_{sh}^2}\frac{1}{\delta}
\left[
\begin{array}{ccc|ccc|c}
\delta\sigma_1^2 + \sigma_1 & \cdots & 0 & & & & 4\mu^2\frac{\sigma_1}{\pi^2} \\
0 & \delta\sigma_2^2 + \sigma_2 & 0 & & \frac{2\sigma_j\tau_\ell}{(j\pi)^2 + (\mu\alpha_\ell)^2} & & 4\mu^2\frac{\sigma_2}{(2\pi)^2} \\
 & \ddots & & & & & \vdots \\
0 & \cdots & \delta\sigma_n^2 + \sigma_n & & & & 4\mu^2\frac{\sigma_n}{(n\pi)^2} \\
\hline
 & & & \frac{\tau_1}{\mu^2} & 0 \cdots & 0 & -\frac{2}{\mu^2}\frac{\tau_1}{\alpha_1^2} \\
 & \frac{2\tau_j\sigma_\ell}{(\ell\pi)^2 + (\mu\alpha_j)^2} & & 0 & \frac{\tau_2}{\mu^2} & 0 & -\frac{2}{\mu^2}\frac{\tau_2}{\alpha_2^2} \\
 & & & & \ddots & & \vdots \\
 & & & 0 & \cdots & \frac{\tau_n}{\mu^2} & -\frac{2}{\mu^2}\frac{\tau_n}{\alpha_n^2} \\
\hline
4\mu^2\frac{\sigma_1}{\pi^2} & \cdots & 4\mu^2\frac{\sigma_n}{(n\pi)^2} & -\frac{2}{\mu^2}\frac{\tau_1^2}{\alpha_1^2} & \cdots & -\frac{2}{\mu^2}\frac{\tau_n}{\alpha_n^2} & 2\delta + \frac{4}{3}\mu^2 + \frac{1}{2}
\end{array}
\right]
\right\}
\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ b_1 \\ b_2 \\ \vdots \\ b_n \\ a_0 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}
$$

where

$$\alpha_j = \kappa_j R \qquad j = 1, 2, \ldots \quad .$$

Let

$$\omega_k^2 \; ; \qquad \begin{bmatrix} a_1^{(k)} \\ a_2^{(k)} \\ \\ a_n^{(k)} \\ b_1^{(k)} \\ b_2^{(k)} \\ \\ b_n^{(k)} \\ a_0^{(k)} \end{bmatrix} \tag{31}$$

be the eigensolution of the matrix eigenvalue problem (30). Then from equations (23) and (27) the eigensolution (19) (20) follows as

$$\omega_k^2; \begin{cases} \overline{w}_k = \mu \left(\sum_{\ell=1}^{n} a_\ell^{(k)} \sigma_\ell \dfrac{\cos k_\ell z}{\cos k_\ell H} - a_0^{(k)} \right) \\ \overline{\zeta}_k = \sum_{\ell=1}^{n} b_\ell^{(k)} \tau_\ell \dfrac{\mathcal{J}_0(\kappa_\ell r)}{\mathcal{J}_0(\kappa_\ell R)} + 2\mu^2 a_0^{(k)} \end{cases}$$

$$\overline{\varphi}_k(r,z) = \sum_{\ell=1}^{n} a_\ell^{(k)} \frac{I_0(k_\ell r)}{I_0(k_\ell R)} \; \frac{\cos k_\ell z}{\cos k_\ell H} + \sum_{\ell=1}^{n} b_\ell^{(k)} \frac{\cosh \kappa_\ell z \, \mathcal{J}_0(\kappa_\ell r)}{\cosh \kappa_\ell H \, \mathcal{J}_0(\kappa_\ell R)}$$

$$+ a_0^{(k)} \left(\frac{z^2}{R^2} - \frac{r^2}{2R^2} \right)$$

$$0 \le r \le R \; ; \qquad 0 \le z \le H \, .$$

The k^{th} natural frequency may be obtained from the eigenvalue (31) as

$$f_k = \frac{\omega_k}{2\pi} \quad .$$

From equation (30) the orthogonality conditions of the eigenvectors (29) may be obtained easily.

SECTION V. FREE VIBRATION ANALYSIS BASED ON THE ASSUMPTION OF ZERO ULLAGE PRESSURE

The vibration is determined by equations (1), (2), (4) through (6) and (8). Using the notation (11), (12) and (15) these equations can be written as:

$$\overline{w}(z) = \frac{\omega^2}{\omega_{sh}^2}\left[\overline{w}(z) + \frac{\mu}{\delta}\,\overline{\varphi}\right] \tag{32}$$

$$\nabla^2\overline{\varphi}(r,z) \equiv \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\overline{\varphi}}{\partial r} + \frac{\partial^2\overline{\varphi}}{\partial z^2} = 0 \tag{33}$$

$$\overline{w}(z) = H\,\frac{\partial\overline{\varphi}(R,z)}{\partial r} \tag{34}$$

$$\overline{\varphi}(r,H) = 0 \tag{35}$$

$$\frac{\partial\overline{\varphi}(r,0)}{\partial z} = 0 \quad . \tag{36}$$

Equations (32) through (36) represent the eigenvalue problem in hand. Solutions of equation (33) satisfying the boundary conditions (35) and (36) are given by

$$\overline{\varphi}(r,z) = I_0(k_\ell' r)\,\cos k_\ell' z \tag{37}$$

$$\ell = 1, 2, \ldots \quad .$$

$$k_\ell' = \frac{(2\ell - 1)\pi}{2H} \tag{38}$$

From equations (32) (34) (37) the eigenvalues may be concluded as

$$\omega_\ell^2 = \omega_{sh}^2 \frac{\delta\sigma_\ell'}{1 + \delta\sigma_\ell'} \tag{39}$$

$$\sigma_\ell' = \sigma(k_\ell' R) \tag{40}$$

where σ is defined by equations (28).

From equations (34) and (37) the correspondent mode follows as

$$\overline{w}_j(z) = k_j' H \, I_1(k_j' R) \cos k_j' z \qquad j = 1, 2, \ldots \quad .$$

From equations (28) and (40) one recognizes

$$\lim_{\ell \to \infty} \sigma_\ell' = \infty \; .$$

Hence equation (39) shows that the empty shell frequency ω_{sh} represents the upper limit of the frequencies ω_ℓ.

In general the shell wall mass (as represented by ρ_{sh}) is small. Hence, for the lowest frequencies,

$$\delta\sigma_\ell' \ll 1$$

and can therefore be neglected against 1. In that case equation (39) reduces to

$$\omega_\ell^2 = \omega_{sh}^2 \, \delta \, \sigma_\ell' \quad . \tag{41}$$

Using the power series expansions of $I_0(r)$ and $I_1(r)$ the first terms of the series expansion of $\sigma(r)$ can easily be found. One obtains

$$\sigma(r) = \frac{r^2}{2}\left[1 - \frac{1}{2}\left(\frac{r}{2}\right)^2 + \frac{1}{3}\left(\frac{r}{2}\right)^4 - \frac{11}{48}\left(\frac{r}{2}\right)^6 + \ldots\right]. \tag{42}$$

Although $\sigma(r)$ is defined for all values of r the convergence of the series (42) is sure if r is at least smaller than or equal to 2.

From equations (41) and (42) it follows:

$$\omega_\ell' \cong \omega_{sh}^2 \delta \frac{k_\ell'^2 R^2}{2} \left[1 - \frac{1}{2}\left(\frac{k_\ell' R}{2}\right)^2 + \frac{1}{3}\left(\frac{k_\ell' R}{2}\right)^4 - \frac{11}{48}\left(\frac{k_\ell' R}{2}\right)^6\right]. \quad (43)$$

Formula (43) can be used for calculation of ω_ℓ if the shell wall inertia is negligible (low frequency range) and if $k_\ell' R$ is sufficiently smaller as two. If, in addition

$$\frac{1}{4}\left(\frac{k_\ell' R}{2}\right)^2 = \frac{1}{16}\left[\frac{(2\ell - 1)\pi}{2} \frac{R}{H}\right]^2 \ll 1, \quad (44)$$

a first-order approximation can be used.

SECTION VI. FREE VIBRATION ANALYSIS BASED ON THE ASSUMPTIONS OF ZERO ULLAGE PRESSURE AND LIQUID COMPRESSIBILITY

Equations of motion and boundary conditions are given by equations (1), (3), (4) through (6) and (8). Using the notations (11), (12) and (15) these equations can be represented as

$$\overline{w}(z) = \frac{\omega^2}{\omega_{sh}^2}\left[\overline{w}(z) + \frac{\mu}{\delta}\overline{\varphi}(R,z)\right] \quad (45)$$

$$\nabla^2\overline{\varphi}(r,z) \equiv \frac{\partial^2\overline{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\varphi}}{\partial r} + \frac{\partial^2\overline{\varphi}}{\partial z^2} = -\frac{\omega^2}{c^2}\overline{\varphi} \quad (46)$$

$$\overline{w}(z) = H\frac{\partial\overline{\varphi}(R,z)}{\partial r} \quad (47)$$

$$\overline{\varphi}(r,H) = 0 \quad (48)$$

$$\frac{\partial\overline{\varphi}(r,0)}{\partial z} = 0 \quad . \quad (49)$$

Solutions of equations (46) which satisfy the boundary conditions (48) and (49) are given by

$$I_0\left(r\sqrt{k_\ell'^2 - \omega_\ell^2/c^2}\right) \cos k_\ell' z \qquad \ell = 1,2,\ldots, \tag{50}$$

where k_ℓ' is given by equation (38). This can be proved by substitution.

The additional equations (45) and (47) can be satisfied by selecting

$$\frac{\omega_\ell^2}{\omega_{sh}^2} = \frac{\delta\ \sigma\left(R\sqrt{k_\ell'^2 - \omega_\ell^2/c^2}\right)}{1 + \delta\ \sigma\left(R\sqrt{k_\ell'^2 - \omega_\ell^2/c^2}\right)} , \tag{51}$$

where σ is defined by equation (28).

As in Section V it can be concluded that for the lowest frequencies, ω_ℓ, it may be

$$\delta\ \sigma\left(R\sqrt{k_\ell'^2 - \omega_\ell^2/c^2}\right) \ll 1 .$$

Hence, in that case equation (51) can be simplified for

$$\omega_\ell^2 \cong \omega_{sh}^2\, \delta\ \sigma\left(R\sqrt{k_\ell'^2 - \omega_\ell^2/c^2}\right) . \tag{52}$$

Now, if equation (44) is valid, that means low frequency range of cases with large H (for instance, liquid-filled pipes) then, in accordance with the series expansion (42), one may replace equation (51) by the first-order approximation

$$\omega_\ell^2 \cong \frac{\omega_{sh}^2\,\delta}{2}\ R^2\left(k_\ell'^2 - \omega_\ell^2/c^2\right)$$

Elimination of ω_ℓ leads to

$$\omega_\ell^2 = k_\ell'^2 \frac{1}{\rho\left(\frac{1}{K} + \frac{2R}{Eh}\right)} , \tag{53}$$

where $K = \rho c^2$ represents the bulk modulus of elasticity of the liquid and

$$\sqrt{\frac{1}{\rho\left(\frac{1}{K} + \frac{2R}{Eh}\right)}}$$

the pressure wave velocity of the liquid within the elastic container [12].

Comparison of equations (39) and (57) shows that the compressibility of the liquid can be neglected if

$$\frac{\omega_\ell^2}{c^2 k_\ell'^2} \ll 1 . \tag{54}$$

Application of this inequality to the cases covered by condition (44) means, in accordance with equation (52),

$$\frac{h}{2R} \frac{E}{K} \ll 1 \tag{55}$$

SECTION VII. FORCED VIBRATIONS, COMPARISON WITH BLEICH'S RESULT

In the following a forced vibration analysis of partially liquid-filled cylindrical shell containers with flat rigid bottoms will be presented. The liquid is assumed to be compressible, the ullage pressure shall be zero.

Let

$$x = A \sin \Omega t \tag{56}$$

be the excitation of the container. If the body-fixed coordinate system of Figure 7 is used, the external pressure field created by the excitation (56) is given by

$$p_0 = \rho \, (H - z) \, \ddot{x} \quad . \tag{57}$$

The equations of motion and boundary conditions are given by equations (1), (3) through (6) and (8) with p_0 given by equation (57). Using the notations (11) and (12) these equations can be written as:

$$\overline{w}(z) - \frac{\Omega^2}{\omega_{sh}^2} \overline{w}(z) = \frac{\Omega^2}{\omega_{sh}^2} \frac{\mu}{\delta} \left[\overline{\varphi} - \eta \left(1 - \frac{z}{H} \right) \right] \tag{58}$$

$$\nabla^2 \overline{\varphi}(r,z) \equiv \frac{\partial^2 \overline{\varphi}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{\varphi}}{\partial r} + \frac{\partial^2 \overline{\varphi}}{\partial z^2} = - \frac{\Omega^2}{c^2} \overline{\varphi} \tag{59}$$

$$\overline{w}(z) = H \frac{\partial \overline{\varphi}(R,z)}{\partial r} \tag{60}$$

$$\overline{\varphi}(r,H) = 0 \tag{61}$$

$$\frac{\partial \overline{\varphi}(r,0)}{\partial z} = 0 \tag{62}$$

$$0 \leq r \leq R, \qquad 0 \leq z \leq H \quad ,$$

where

$$\eta = \frac{A}{H} \tag{63}$$

and δ, μ, ω_{sh} are defined by equation (15).

Solutions of equation (59) which satisfy the conditions (61) and (62) can be represented by

$$I_0(k_j^* r) \cos k_j' z \qquad j = 1, 2, \ldots \quad , \tag{64}$$

where

$$k_j^{*2} = k_j'^2 - \frac{\Omega^2}{c^2} \tag{65}$$

and k_j' is given by equation (38). Proof can be made by substitution.

To satisfy the remaining equations (58) and (60) the following series expansion in terms of the function set (64)

$$\bar{\varphi}^*(r,z) = \eta \sum_{j=1}^{\infty} c_j^* \, I_0(k_j^* r) \cos k_j' z \tag{66}$$

will be used. Substitution of this expression into equation (58) and subsequent integration over z from zero to H, taking into account the orthogonality conditions

$$\int_0^H \cos k_i z \cos k_j z \, dz = \frac{H}{2} \delta_{ij} \qquad i,j = 1,2,\dots,$$

yields

$$c_\ell^* = \frac{2}{\delta} \frac{\Omega^2}{\omega_{sh}^2} \frac{1}{(k_\ell' H)^2 k_\ell^* R \, I_1(k_\ell^* R)} \frac{1}{\dfrac{\Omega^2}{\omega_{sh}^2} \dfrac{1 + \delta\sigma_\ell^*}{\delta\sigma_\ell^*} - 1} \, , \tag{67}$$

where:

$$\sigma_\ell^* = \sigma(k_\ell^* R) \qquad \ell = 1,2,\dots \tag{68}$$

and σ is defined by equations (28).

Equations (65), (67) and (68) show that resonance will occur if

$$\Omega = \omega_\ell \qquad \ell = 1,2,3,\dots \, ,$$

where ω_ℓ is defined by equation (51).

From equation (37), (64) and (65) one concludes that the liquid compressibility can be neglected if

$$\left(\frac{\Omega^2}{c\,k_\ell'}\right)^2 \ll 1\,.$$

In that case equations (66) and (67) reduce to:

$$\left.\begin{aligned}\bar{\varphi}'(r,z) &= \eta \sum_{\ell=1}^{\infty} c_\ell' \, I_0(k_\ell' r)\cos k_\ell' z \\ c_\ell' &= \frac{2}{\delta}\,\frac{\Omega^2}{\omega_{sh}^2}\,\frac{1}{(k_\ell' H)^2\, k_\ell' R\, I_1(k_\ell' R)}\,\frac{1}{\dfrac{\Omega^2}{\omega_\ell^2} - 1}\end{aligned}\right\} \tag{69}$$

where ω_ℓ is given by equation (39).

The total pressure as indicated between the brackets of the right side of equation (58) follows from equation (66) as

$$\begin{aligned}&\bar{\varphi} - \eta\left(1 - \frac{z}{H}\right) \\ &= \eta\left[-\left(1 - \frac{z}{H}\right) + \sum_{1}^{\infty} c_\ell^* \, I_0(k_\ell^* r)\cos k_\ell' z\right].\end{aligned} \tag{70}$$

H. H. Bleich [2] derives the total pressure applying another method of solution. He uses a space-fixed system of coordinates with the origin at the liquid surface at rest. In the following, Bleich's solution will be compared with that given above. To facilitate the comparison, Bleich's solution will be referred to a coordinate system having its origin at the center of the container bottom at rest. Also sign rules and some notations of the paper in hand will be utilized. Bleich's potential is then given by

$$\eta\left\{c_0 \sin\bar{\beta}_0 (H - z)\, I_0(\beta_0 r) + \sum_{1}^{\infty} c_j \sinh\left[\bar{\beta}_j (H - z)\right] \mathcal{J}_0(\beta_j r)\right\} \tag{71}$$

$$c_0 = -\frac{2 I_1(\beta_0 R)}{\mu \left[I_0^2(\beta_0 R) - I_1^2(\beta_0 R) \right] \beta_0 \overline{\beta}_0 R^2 \cos \overline{\beta}_0 H} \tag{72}$$

and

$$c_\ell = -\frac{2 \mathcal{J}_1(\beta_\ell R)}{\mu \left[\mathcal{J}_0^2(\beta_\ell R) + \mathcal{J}_1^2(\beta_\ell R) \right] \beta_\ell \overline{\beta}_\ell R^2 \cosh \overline{\beta}_\ell H} \, . \tag{73}$$

β_0 is the only real root of

$$\Omega^2 = \omega_{sh}^2 \frac{\delta \sigma(\beta R)}{1 + \delta \sigma(\beta R)} \tag{74}$$

where μ, δ, ω_{sh} and σ are defined by equations (15) and (28).

$$i\beta_\ell \qquad (\beta_\ell > 0) \qquad \ell = 1, 2, 3, \ldots$$

represent the imaginary roots of equation (74). Between the $\overline{\beta}$ and the β the following relations exist:

$$\overline{\beta}_0^2 = \beta_0^2 + \frac{\Omega^2}{c^2} \qquad \overline{\beta}_\ell^2 = \beta_\ell^2 - \frac{\Omega^2}{c^2} \qquad \ell = 1, 2, \ldots \quad . \tag{75}$$

Equation (72) shows that the denominator of the right side vanishes if

$$\overline{\beta}_0 = \frac{(2j - 1)\pi}{2H} = k_j'$$

or

$$\beta_0 = \sqrt{k_j'^2 - \frac{\Omega^2}{c^2}} \, .$$

Substitution of this expression into equation (74) and comparison with equation (51) shows that in this case

$$\Omega = \omega_j$$

and this represents resonance.

To prove that the expressions (70) and (71) represent one and the same quantity one first assumes the equality of both expressions then multiplies both series by

$$r\, I_0\, (\beta_0 r) \cos k_j' z \qquad j = 1, 2, 3, \ldots$$

or

$$r\, \mathcal{J}_0 (\beta_\ell r) \cos k_j' z \qquad \ell = 1, 2, 3, \ldots$$

respectively, and integration over the domains

$$0 \le z \le H; \quad 0 \le r \le R,$$

taking into account the orthogonality conditions of the trigonometric and Bessel functions, results in equations (72) and (73).

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National Aeronautics and Space Administration
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