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ON THE TRANSFORMATION TO PHASE-VARIABLE CANONICAL FORM

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INTRODUCTION

Consider the control system defined by

$$\dot{x} = Ax + u(t)f \qquad (\dot{} = \frac{d}{dt}), \tag{1}$$

where $x = (x_1, \ldots, x_n)$ denotes an n-dimensional state vector, A is an nxn constant matrix, $f = (f_1, \ldots, f_n)$ a constant vector, and $u(t)$ a scalar control function. If it is assumed that (A, f) is controllable, then there is known to exist [1] a non-singular linear transformation

$$x = Ky$$

which reduces equation (1) to the canonical (phase-variable) form

$$\dot{y} = A_0 y + u(t) f_0 , \tag{2}$$

where

$$A_0 = \begin{vmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & . & . & . & -a_n \end{vmatrix} \tag{3}$$

is the companion matrix of the characteristic polynomial of A, and

$$f_0 = \begin{vmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{vmatrix} . \tag{4}$$

In Johnson and Wonham [2] an explicit expression for K was obtained in terms of the Vandermonde matrix and a modal matrix of A. To obtain this result, however, the authors found it necessary to require that the eigenvalues of A be distinct. This report provides an equally convenient expression for K with no restrictions placed upon A, other than the controllability of (A, f), and which includes the result of Johnson and Wonham [2] as a special case.

The essential difference between the result of this report and that given in Mufti [3] is a computational one. However, in both cases the final result depends crucially on the existence of the inverse of certain transformation matrices. No proof was given in Mufti [3] that these inverse matrices do exist. However, the present result shows that their existence is, in fact, an implication of the controllability assumption.

THE JORDAN FORM OF A

Let

$$J_k = \left.\begin{vmatrix} \lambda_k & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_k & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \lambda_k \end{vmatrix}\right\} p_k \tag{5}$$

be the Jordan block corresponding to an elementary divisor $(\lambda - \lambda_k)^{p_k}$ of matrix A. In all that follows, it will be assumed that (A, f) is controllable. Consequently, the minimal and characteristic polynomials of A coincide, and it follows that A must have pairwise co-prime elementary divisors

$$(\lambda - \lambda_1)^{p_1},\ (\lambda - \lambda_2)^{p_2},\ \ldots,\ (\lambda - \lambda_s)^{p_s} \qquad \sum_{i=1}^{s} p_i = n,\ \lambda_i \neq \lambda_j.$$

In fact, a necessary and sufficient condition that (A, f) be controllable [1] is that equation (1) be representable in the form

$$\dot{z} = Jz + u(t)g, \tag{6}$$

where no two diagonal blocks J_k of the Jordan form

$$J = \operatorname{diag}\{J_1, \ldots, J_s\} \tag{7}$$

correspond to the same eigenvalue λ_i and where the p_k th element of the vector g does not vanish for any $k = 1, 2, \ldots, s$.

THE TRANSFORMING MATRICES

Lemma I

$$\underline{\text{Let}}\ T_{II} = (T_1, T_2, \ldots, T_s) \tag{8}$$

where

$$T_j = \| t_{ik}^{(j)} \| = \| t_{i-1,k}^{(j)} \lambda_j + t_{i-1,k+1}^{(j)} \| \quad \begin{array}{l} i = 2, \ldots, n \\ k = 1, \ldots, p_j \end{array} \tag{9}$$

with

$$t_{1k}^{(j)} = \left. \begin{array}{l} 1 \\ 0 \end{array} \right\} \quad \begin{array}{l} k = p_1, \ldots, p_s \\ k \neq p_1, \ldots, p_s \end{array} \tag{10}$$

then

$$A_0 = (T_{II}F)\, J\, (T_{II}F)^{-1} \tag{11}$$

where A_0 is the matrix (3) and F is a quasi-diagonal matrix

$$F = \operatorname{diag}\ \{F_1, F_2, \ldots, F_s\}$$

with

$$
F_j = \begin{array}{c} \overbrace{\qquad\qquad}^{p_j} \\ \left|\begin{matrix} 0 & 0 & . & . & . & 0 & 1 \\ 0 & 0 & . & . & . & 1 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & 0 & 0 \\ 1 & 0 & . & . & . & 0 & 0 \end{matrix}\right| \end{array} \left.\vphantom{\begin{matrix}0\\0\\0\\0\\0\end{matrix}}\right\} p_j . \qquad (12)
$$

The proof of the lemma follows from choosing the first row vector, given by equation (10), as the generating vector in Krylov's method of determining the transforming matrix ([4] pp. 202-214). In this way, it is found that

$$
A_0^T = (T_{II}^T)^{-1} J \, T_{II}^T \qquad (13)
$$

providing T_{II}^{-1} exists. To show that this inverse does exist, observe that T_{II}^T may be written as

$$
T_{II}^T = (e, \ Je, \ J^2e, \ \ldots, \ J^{n-1}e), \qquad (14)
$$

where J is given by equation (7) and where

$$
e_i = \begin{cases} 1 & i = \sum\limits_{k=1}^{q} p_k \\ 0 & i \neq \sum\limits_{k=1}^{q} p_k \end{cases} . \ (q = 1, \ 2, \ \ldots, \ s) .
$$

The necessary and sufficient condition for controllability referred to above shows (J, e) controllable; this implies $\det(T_{II}) \neq 0$. Consequently, $(T_{II})^{-1}$ exists. Equation (11) is then obtained by taking the transpose of equation (13) to obtain

$$
A_0 = T_{II} J^T T_{II}^{-1} \qquad (15)
$$

and using the transformation

$$J^T = FJF. \tag{16}$$

In the case that the eigenvalues of A are distinct $(s = n)$, the matrix T_{II} becomes the Vandermonde matrix of A. However, even in the cases that $s \neq n$, the structure of T_{II} remains simple and, in concrete problems, can be obtained by the recursion formula of the lemma, by matrix multiplication indicated by equation (14), or by recognizing that the elements of each (nxp_j) submatrix T_j are readily available through use of Pascal's triangle.

In the next lemma, the matrix T_I is the transforming matrix which takes A into the Jordan form, J, of equation (7); i.e.,

$$J = T_I^{-1} A T_I . \tag{17}$$

Since the elementary divisors of A are co-prime, the transforming matrix T_I may be determined with little difficulty (e.g., the construction procedure described on pages 166-167 of Gantmacher [4]). The vector f_0 of the lemma is given by equation (4), and matrices T_{II} and f are as defined by the preceding lemma.

<u>Lemma II</u>

<u>There exists a non-singular upper triangular, quasi-diagonal, matrix</u>

$$B = (B_1, \ldots, B_s)$$

<u>where</u>

$$B_j = \begin{vmatrix} b_{1j} & b_{2j} & b_{3j} & \cdot & \cdot & b_{p_j j} \\ 0 & b_{1j} & b_{2j} & \cdot & \cdot & b_{p_j-1 j} \\ 0 & 0 & b_{1j} & \cdot & \cdot & b_{p_j-2 j} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & b_{2j} \\ 0 & 0 & 0 & \cdot & \cdot & b_{1j} \end{vmatrix} \tag{18}$$

such that

$$(T_{II} F B\, T_I^{-1})\, f = f_0 \tag{19}$$

and

$$J = BJB^{-1}. \tag{20}$$

To prove the lemma, first let us notice that equation (20) will be satisfied by any matrix B which commutes with J and whose inverse exists. Taking into account that the elementary divisors of J are co-prime, it follows from basic theorems on commuting matrices ([4], pp. 220-225) that any matrix of the form given by equation (18) will commute with J. Consequently, it must be shown that the n elements b_{ik} of B can be chosen such that equation (19) is satisfied and $|B| \neq 0$. By expanding equation (19), it follows that the b_{ik} can be so chosen if the minor of each element t_{nk} $(k = 1, p_1 + 1, \ldots, n - p_s + 1)$ of matrix T_{II} is non-zero. Let

$$J'_k = \left| \begin{array}{ccccccc} \lambda_k & \overbrace{1 \quad 0 \quad \cdot \quad \cdot}^{p_k - 1} & & & & \cdot & 0 \\ 0 & \lambda_k & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_k & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \vdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_k \end{array} \right| \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} p_k - 1 \text{ rows} \tag{21}$$

so that J'_k is a Jordan block corresponding to an elementary divisor $(\lambda - \lambda_k)^{p_k - 1}$.

Replacing the kth diagonal Jordan block in equation (7) by J'_k of equation (21) results in an $(n - 1)x(n - 1)$ Jordan matrix J' corresponding to elementary divisors

$$(\lambda - \lambda_1)^{p_1},\ (\lambda - \lambda_2)^{p_2},\ \ldots,\ (\lambda - \lambda_k)^{p_k - 1}\ ,\ \ldots,\ (\lambda - \lambda_s)^{p_s}. \tag{22}$$

Further, let e' be a column vector of n - 1 elements obtained from vector e of equation (14) by deleting the element e_j, where $j = \sum_{i=0}^{k-1} (1 + p_i), (p_0 = 0)$. Then (J', e') satisfies the conditions for controllability, and the (n - 1)x(n - 1) matrix

$$(e', J'e', (J')^2 e', \ldots, (J')^{n-1} e') \tag{23}$$

has a non-zero determinant. But it can be easily verified that the matrix of equation (23) is exactly the minor of element t_{nk} $(k = 1, p_i + 1, \ldots, n - p + 1)$ of T_{II}.

We are now able to provide an explicit expression for the transforming matrix K.

Theorem

Let (A, f) of equation (1) be controllable and let matrices T_I, T_{II}, f, and B be defined as in lemmas I and II. Then, the transformation

$$x = Ky$$

$$K = T_I B^{-1} F T_{II}^{-1}$$

reduces (1) to the canonical (phase-variable) form of equation (2).

The proof follows immediately upon carrying out the indicated change of variables and reduction according to lemmas I and II.

EXAMPLE

Let

$$A = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix}, \quad f = \begin{vmatrix} 0 \\ 1 \\ 2 \\ 1 \end{vmatrix}.$$

Since A is already in Jordan form $T_I = E$, where E is the unit matrix, and, by inspection,

$$\lambda_1 = 1,\ \lambda_2 = 2,\ p_1 = 2,\ p_2 = 2.$$

By lemma I,

$$T_{II} = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & \lambda_1 & 1 & \lambda_2 \\ 2\lambda_1 & \lambda_1^2 & 2\lambda_2 & \lambda_2^2 \\ 3\lambda_1^2 & \lambda_1^3 & 3\lambda_2^2 & \lambda_2^3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 4 \\ 3 & 1 & 12 & 8 \end{vmatrix}$$

$$T_{II}F = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ 1 & 3 & 8 & 12 \end{vmatrix}, \quad T_{II}^{-1} = \begin{vmatrix} -4 & 8 & -5 & 1 \\ -4 & 12 & -9 & 2 \\ -2 & 5 & -4 & 1 \\ 5 & -12 & 9 & -2 \end{vmatrix}$$

By lemma II,

$$B = \begin{vmatrix} b_{11} & b_{21} & 0 & 0 \\ 0 & b_{11} & 0 & 0 \\ 0 & 0 & b_{12} & b_{22} \\ 0 & 0 & 0 & b_{12} \end{vmatrix}.$$

Solving equation (19),

$$BT_I^{-1}f = \begin{vmatrix} b_{21} \\ b_{11} \\ 2b_{12}+b_{22} \\ b_{12} \end{vmatrix}, \quad FT_{II}^{-1}f_0 = \begin{vmatrix} 2 \\ 1 \\ -2 \\ 1 \end{vmatrix}$$

so that

$$B = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad B^{-1} = \begin{vmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

From the theorem

$$K = T_I B^{-1} F T_{II}^{-1} = \begin{vmatrix} 4 & -4 & 1 & 0 \\ -4 & 8 & -5 & 1 \\ -3 & 8 & -7 & 2 \\ -2 & 5 & -4 & 1 \end{vmatrix} .$$

Checking,

$$K^{-1} = T_{II} F B \cdot T_I^{-1} = \begin{vmatrix} 1 & 2 & 1 & -4 \\ 1 & 3 & 2 & -7 \\ 1 & 4 & 4 & -12 \\ 1 & 5 & 8 & -20 \end{vmatrix}$$

$$K^{-1} A K = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{vmatrix} = A_0, \qquad K^{-1} f = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} = f_0 .$$

CONCLUSIONS

The example illustrates both the advantages and disadvantages of the transformation obtained through use of lemmas I and II. On the one hand, the forms of the matrices are quite convenient and computation of the transforming matrices themselves is correspondingly simple. On the other hand, it is necessary to invert the three matrices T_I, T_{II}, and B. As shown in the report, these inversions are, however, only a computational inconvenience, since the existence of the inverses follows from the controllability assumption.

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