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ON MIXED REFLECTION OF SUNLIGHT

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ABSTRACT

Light falling on an engineering surface will be reflected partly in a preferred direction (specularly), partly in a diffuse manner. The scattering of the light energy is assumed to follow Lambert's cosine rule. The force differential acting on an elemental surface is set up for such cases of mixed reflection. A smoothness coefficient (≤ 1) accounts for that portion of the reflected energy that is radiated specularly. A loss coefficient, also ≤ 1, gives the fraction of the incident energy that enters into the reflected light.

Applications are made to the sphere and its motion under the impact of light, and to the circular cone with fully illuminated curved surface, some results being added when a shadow region exists on it.

NASA - GEORGE C. MARSHALL SPACE FLIGHT CENTER

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TECHNICAL AND SCIENTIFIC STAFF
AERO-ASTRODYNAMICS LABORATORY
RESEARCH AND DEVELOPMENT OPERATIONS

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ON MIXED REFLECTION OF SUNLIGHT

SUMMARY

If an elemental surface is exposed to sunlight the total light force on it is the compound effect of three intensities of which that of incident light can be taken as the basic measure. Only a portion of it will be reflected (described by a loss coefficient, κ). This portion in turn will usually be radiated back in part specularly, in part diffusely. A smoothness coefficient, q, identifies the fraction of it that follows the familiar mirror reflection law. Lambert's cosine rule is assumed for the energy flux distribution in diffusion. On this basis an expression for the differential of the light force is derived which must be integrated over the illuminated part of the surface. This is more or less easily done if the surface equation is known in analytic terms and if the loss and smoothness coefficients can be taken as constants.

An application to the sphere shows that, unless diffuse reflection is present, the force on it is that of incidence alone no matter what portion of the energy is reradiated specularly. A non-homogeneous sphere will, aside from following a straight translation course, in general carry out a swinging motion about an axis through the mass center, except in singular circumstances when the initial torque is zero so that a rotatory motion cannot develop.

The force on a fully illuminated circular cone can be larger or smaller than the incident force alone, depending mainly on whether the opening angle of the cone is large or small. The force moment will tend to align a homogeneous cone's axis with the light direction, provided that the product $q\kappa$ is smaller than 0.6. If it is larger, sufficiently slender cones are unstable. (The Apollo capsule is likely to be stable if the conical part is fully illuminated.)

A numerical computation was carried out for loss-free reflection ($\kappa = 1$), full illumination, and the opening angle of the Apollo capsule. If the axis inclination towards light direction is increased, the force's inclination increases also, and does so at a faster pace with smoother surfaces. With these, it also becomes larger in magnitude. At a fixed attitude, the force depends on the surface quality, growing stronger and less inclined as diffusion grows more preponderant.

In a concluding section approaches other than Lambert's rule to account for diffuse reflection are briefly discussed.

I. INTRODUCTION

The energy of reflected light is smaller as a rule than that of the incoming light. Part of the latter may go into different forms (heat; electrical energy), another part may be transmitted through a transparent surface. All losses of radiant energy will be consolidated into one fractional loss coefficient, κ, so that

$$I_r = \kappa I_i \tag{1}$$

is the fraction of the incident intensity that is preserved in the reflected light ($0 \leqq \kappa \leqq 1$).

Except in ideal circumstances the surface is rough relative to photon size. Experience shows that a portion only of the reflected energy resides in light reemitted specularly, that is, in such a manner that the surface normal bisects the angle between the directions of incoming and reflected rays. A smoothness coefficient $q (0 \leqq q \leqq 1)$ will be used to describe the intensity, I_s, of light reflected specularly:

$$I_s = qI_r. \tag{2}$$

With $q = 1$, the reflection is ideal; with $q = 0$, it is diffuse in its entirety. Neither of these limiting values will presumably be found with engineering surfaces.

Diffuse reflection is usually taken as obeying Lambert's rule (which, by the way, stems from 1760). The total solid angle ($\Omega = 2\pi$) above an elemental surface is divided into elements $d\Omega$ surrounding the surface normal. If one of these elemental solid angles makes the angle ϑ with the surface normal, the flux of the light energy in it is assumed proportional to $\cos \vartheta$. It may be written as

$$d^2\phi = (\lambda \cos \vartheta \, d\Omega) \, dS$$

where dS is the area of the surface element and λ is a factor of proportionality. For the integration one best uses the system of spherical coordinates φ, ϑ, r as depicted in figure 1. In it

$$d\Omega = \sin \vartheta \, d\vartheta \, d\varphi,$$

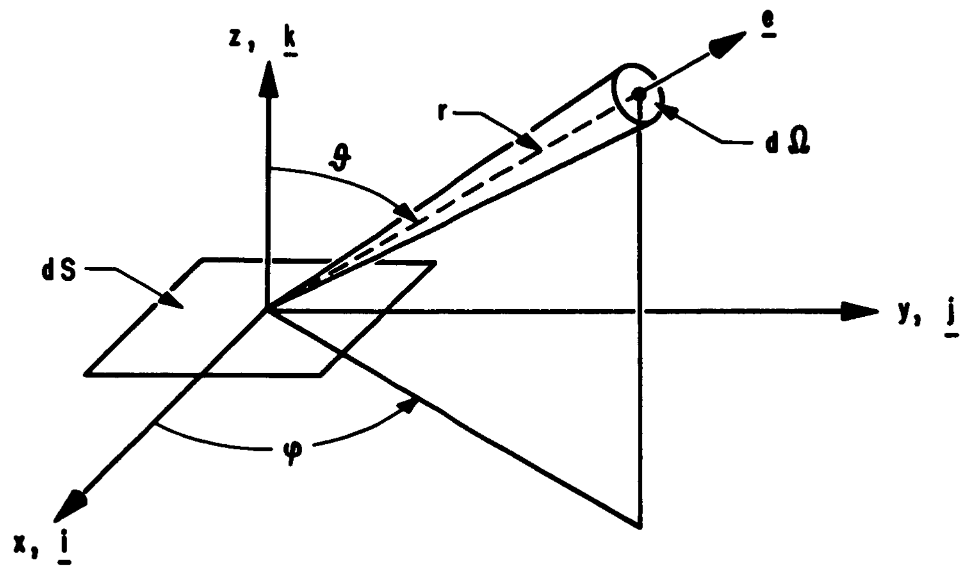

Figure 1. Spherical Coordinates

so that the total flux given out by the element dS becomes

$$\phi_t = \lambda dS \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi/2} \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi = \pi\lambda \, dS. \tag{3}$$

Note that it is proportional to dS.

In being reemitted light moves in the general direction (figure 1)

$$\underline{e} = \underline{i} \sin\vartheta \cos\varphi + \underline{j} \sin\vartheta \sin\varphi + \underline{k} \cos\vartheta;^{*}$$

*Vectors are denoted by underlined symbols. For example, $\underline{i}$, $\underline{j}$, $\underline{k}$ are the unit vectors in the x,y,z-directions, respectively.

it exerts a force whose direction is opposite to $\underline{e}$ and whose magnitude is given by

$$\frac{1}{c} d^2\phi,$$

where c is the speed of light. The total force acting through diffuse reemission on the element dS becomes

$$\underline{f}_d = -\frac{\lambda}{c} dS \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi/2} \sin\vartheta \cos\vartheta \, d\vartheta \, d\varphi \, (\underline{i} \sin\vartheta \cos\varphi + \underline{j} \sin\vartheta \sin\varphi$$

$$+ \underline{k} \cos\vartheta).$$

On integrating over φ the $\underline{i}$- and $\underline{j}$-components cancel. Thus,

$$\underline{f}_d = -\frac{2\pi\lambda}{c} \underline{k} \, dS \int_{\vartheta=0}^{\pi/2} \sin\vartheta \cos^2\vartheta \, d\vartheta = -\frac{2\pi\lambda}{3c} \underline{k} \, dS.$$

The "diffusion" force is in the direction, $-\underline{k}$, of the interior normal of the surface element. Its absolute value,

$$\frac{2\pi\lambda}{3} dS,$$

can be connected with an energy flux normal to dS of magnitude

$$\phi_d = \frac{2\pi\lambda}{3} dS = \frac{2}{3} \phi_t,$$

according to expression (3). On dividing through by dS one obtains an intensity related to $\underline{f}_d$:

$$I_d = \frac{2}{3} I_t = \frac{2}{3} (1 - q) I_r. \tag{4}$$

The unknown factor λ is eliminated in this expression. Note that while in fact the entire fraction $I_t = (1 - q) I_r$ of the reflected intensity is emitted diffusely, only part of it contributes to the accelerating force on dS, since the lateral force components cancel (and thus may be said to merely produce stresses in the surface element). This is consequent on the assumption made in the derivation that Lambert's intensities give rise to actual forces attacking the element in the directions $-\underline{e}$.

II. THE LIGHT FORCE ACTING ON DIFFERENTIAL AND INTEGRAL SURFACES

The intensity of parallel (sunlight) radiation is numerically identical with the pressure of light on a surface placed normal to ray direction. For such surfaces, expressions (1), (2), and (4) may also be written as

$$\left.\begin{aligned} p_r &= \kappa p_o \\ p_s &= \kappa q\ p_o \\ p_d &= \frac{2}{3}\kappa(1 - q)\ p_o. \end{aligned}\right\} \tag{5}$$

The pressure, p_o, of the incident light must be taken from observation. In nearby space its value is very small:

$$p_o = 0.4583 \times 10^{-4}\ \frac{\text{dyne}}{\text{cm}^2}\ . \tag{6}$$

This figure is an average, for in fact, it varies slightly with the earth's distance from the sun.

An element, dS, sitting on an extended surface will not in general be perpendicular to ray direction. If its normal makes the local angle, α, of incidence with that direction, the magnitude of the force on it is calculated as the product of the pressure with the projected area, dS cos α. With incident parallel light the direction of the force is that of the rays which we may designate by the unit vector $\underline{s}$. Thus, the force element engendered by the light incoming at dS is given by

$$d^2\underline{F}_i = p_o\ dS\ \cos\ \alpha\ \underline{s}; \tag{7}$$

that is, the force is largest at normal incidence ($\alpha = 0$). It is zero with grazing incidence ($\alpha = 90°$). This is not strictly correct with rough surfaces where a few peaks will still be illuminated. However, their projected area is small at least of second order. The effect can be disregarded (here as well as with $\alpha \neq 90°$)

With specular reflection, the force is in a direction, $\underline{s}'$, opposite to that of the outgoing light and again makes the angle α with the interior normal unit vector $\underline{n}'$. Since $\underline{s}$ and $\underline{s}'$ are also unit vectors, the diagonal's length in figure 2 is 2 cos α. It follows that

$$\underline{s}' = 2\underline{n} \cos \alpha - \underline{s}.$$

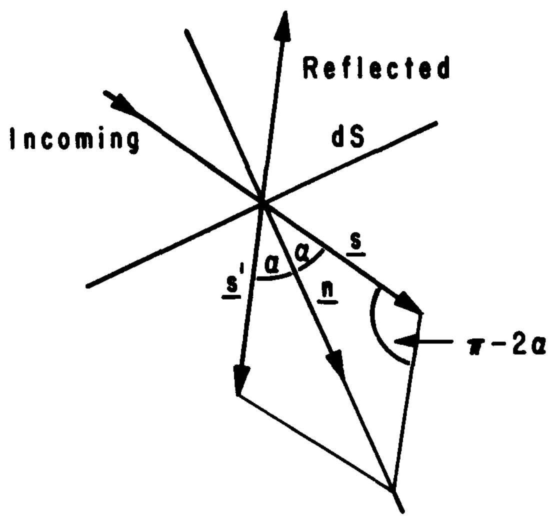

Figure 2. The Direction $\underline{s}'$

Finally, the pressure in the outgoing light being p_s, the elemental force produced by it becomes

$$d^2\underline{F}_s = q\kappa\, p_o \cos \alpha \, dS \, (2\underline{n} \cos \alpha - \underline{s}). \tag{8}$$

Earlier it was found that, with diffuse reflection, the force points into the direction of the interior normal ($\alpha = 0°$). The third relation (5) then yields

$$d^2\underline{F}_d = \frac{2}{3}\, \kappa(1 - q)\, p_o \, dS \, \underline{n} \,. \tag{9}$$

Expressions (7), (8), (9) add up to give the total force acting on the surface element:

$$d^2\underline{F} = p_o dS \left[\underline{s}(1 - q\kappa) \cos \alpha + 2\underline{n} \, \kappa \left(\frac{1 - q}{3} + q \cos^2\alpha\right)\right] . \tag{10}$$

This formula simplifies considerably in the limiting cases of no reflection at all ($\kappa = 0$) and of ideal specular reflection ($\kappa = q = 1$), less so in the case of purely diffuse reflection ($q = 0$). It will be noticed that even with grazing incidence ($\cos \alpha = 0$), there is still a non-vanishing force. It is entirely produced by Lambert reflection, diminishes with it and disappears in the ideal case $q = 1$. One may argue that with strongly oblique incidence diffuse reflection is not likely to follow the rotational symmetry of the cosine law. This possible defect, however, would seem to carry weight with plane surfaces only where such incidence will extend over finite areas.

For application of the differential (10) the surface in question is best given a pointwise representation:

$$x = x(\sigma,\tau), \quad y = y(\sigma,\tau), \quad z = z(\sigma,\tau). \tag{11}$$

The intervals in which the parameters σ and τ move are determined by the geometrical shape of the surface. On it, the curves σ = const. τ = const. are coordinate lines (not necessarily orthogonal to each other). If the surface is composite, several of such sets (11) will have to be used.

Once expressions (11) are known the components in the (x,y,z)-system of the surface normal at any point (σ,τ) are determinable:

$$n_1 = \pm \frac{1}{N} \begin{vmatrix} y_\sigma & z_\sigma \\ y_\tau & z_\tau \end{vmatrix}, \quad n_2 = \pm \frac{1}{N} \begin{vmatrix} z_\sigma & x_\sigma \\ z_\tau & x_\tau \end{vmatrix}, \quad n_3 = \pm \frac{1}{N} \begin{vmatrix} x_\sigma & y_\sigma \\ x_\tau & z_\tau \end{vmatrix} \tag{12}$$

with

$$N = \sqrt{\begin{vmatrix} y_\sigma & z_\sigma \\ y_\tau & z_\tau \end{vmatrix}^2 + \begin{vmatrix} z_\sigma & x_\sigma \\ z_\tau & x_\tau \end{vmatrix}^2 + \begin{vmatrix} x_\sigma & y_\sigma \\ x_\tau & z_\tau \end{vmatrix}^2} . \tag{13}$$

The indices here signify partial differentiation ($y_\sigma = \partial y/\partial\sigma$, etc.). From geometrical inspection, the signs in formulas (12) can be chosen such that the vector $\underline{n} = n_1\underline{i} + n_2\underline{j} + n_3\underline{k}$ points toward the interior, that is, toward the region shielded from light rays by the elemental surface itself.

The surface differential dS is given by

$$dS = N\,d\sigma\,d\tau\ . \tag{14}$$

In integrating, $d\sigma$ and $d\tau$ must be defined as positive increments, because N and dS are positive.

All geometric quantities appearing in the force differential (10) are known from expressions (12), (13), (14), including the cosine of the local angle of incidence, as it is the scalar product

$$\cos\alpha = \underline{n}\cdot\underline{s}\ . \tag{15}$$

The light direction $\underline{s}$ is of course a given unit vector.

Shadow boundaries will often exist on the surface; the integration then may use only those parts of the σ- and τ-intervals that describe the illuminated regions. More detail on this can be found in Aero-Astrodynamics Research Review No. 4.

Calculation of torques requires the determination of the center of pressure for which a general method is set forth in the same paper.*

By way of examples, the differential (10) will be integrated for the sphere and the circular cone. The surfaces are considered non-transparent to avoid the complications arising through backside reflection. Furthermore, it will be assumed that the loss and smoothness coefficients are constant over the entire surface; this means especially that they are independent of the angle of incidence. Otherwise, numerical integration is called for.

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*W. Heybey, "On Radiation Pressure and Its Effects on Satellite Motion," NASA TM X-53462, April 1966.

III. THE SPHERE

If the sphere's center is at the origin, the parameters σ and τ can be chosen as the angles φ and ϑ in the system of spherical coordinates (figure 1). The representation (11) assumes the form

$$x = R \sin \vartheta \cos \varphi$$

$$y = R \sin \vartheta \sin \varphi$$

$$z = R \cos \vartheta$$

where R is the radius of the sphere. All points of the sphere are accounted for if the parameters φ and ϑ move in the intervals

$$0 \leqq \varphi \leqq 2\pi$$

$$0 \leqq \vartheta \leqq \pi.$$

The coordinate lines, φ = const. and ϑ = const, are the circles of longitude and latitude, respectively.

Application of expressions (12) and (13) yields

$$n_1 = -\sin \vartheta \cos \varphi, \quad n_2 = -\sin \vartheta \sin \varphi, \quad n_3 = -\cos \vartheta$$

$$N = R^2 \sin \vartheta.$$

Let the parallel light arrive directly from above:

$$\underline{s} = -\underline{k}.$$

Then

$$\cos \alpha = \underline{n} \cdot \underline{s} = \cos \vartheta.$$

The differential (10) becomes

$$d^2\underline{F} = p_o R^2 \sin \vartheta \, d\vartheta \, d\varphi \left[- \underline{k} (1 - q\kappa) \cos \vartheta - 2\kappa \left(\frac{1 - q}{3} + q \cos^2\vartheta \right) \right.$$

$$\left. \cdot \left(\underline{i} \sin \vartheta \cos \varphi + \underline{j} \sin \vartheta \sin \varphi + \underline{k} \cos \vartheta \right) \right] . \qquad (16)$$

The upper half of the sphere only is illuminated. While one has to integrate over the full natural interval $0 \leqq \varphi \leqq 2\pi$, that of ϑ must be curtailed into $0 \leqq \vartheta \leqq \pi/2$, because a shadow boundary exists at $\vartheta = \pi/2$.

On integrating over φ, the $\underline{i}$- and $\underline{j}$- constants of the force vanish, so that

$$d\underline{F} = -2\pi\, p_o\, R^2 \sin\vartheta \left[(1 - q\kappa)\cos\vartheta + 2\kappa\left(\frac{1-q}{3} + q\cos^2\vartheta\right)\cos\vartheta\right]\underline{k}\, d\vartheta.$$

Integration with respect to ϑ yields

$$\underline{F} = -\pi\, p_o\, R^2\left(1 + 2\kappa\,\frac{1-q}{3}\right)\underline{k} \equiv -F\,\underline{k}\ . \tag{17}$$

This force has the magnitude F and is in the direction $-\underline{k}$ of the incoming light. If m is the mass of the sphere, the center of gravity moves with the speed $v = (F/m)t$ in the direction $\underline{s}$ ($v = 0$ for $t = 0$).

For symmetry reasons the line of attack coincides with the zenith-nadir-line. Any sphere whose mass center is on that line experiences no torque; a homogeneous sphere therefore cannot be set into rotatory motion at all through the force exerted by the light, no matter from which direction it arrives.

Suppose, however, that the sphere is divided by a plane, $y = y^*$, parallel to the (z,x)-plane such that left of it the mass density is smaller than right of it (figure 3). The mass center, G, will then be at a distance $y_g > 0$ on the y-axis. At $t = 0$, the sphere is subject to a moment

$$\underline{M} = (-\underline{j}y_g) \times (-F\underline{k}) = \underline{i}y_g F,$$

which seeks to rotate the body about an $\underline{i}$-axis parallel through the mass center and thus moves the geometric center, C, of the sphere through, say, an angle $\Delta\psi$ from its original seat at $\underline{O}$. The force (17) retains magnitude and direction; however, its arm GC is now

$$-\underline{j}y_g \cos\Delta\psi - \underline{k}y_g \sin\Delta\psi,$$

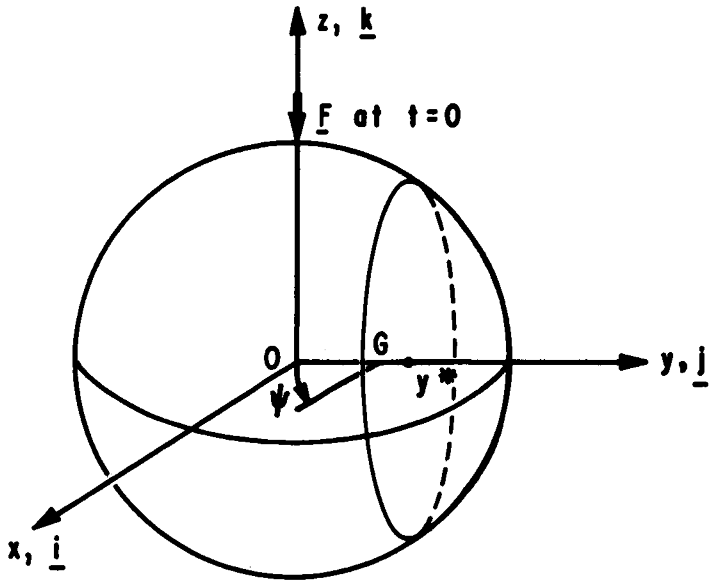

Figure 3. Inhomogeneous Sphere

so that the moment becomes

$$\underline{M} = \underline{i} y_g \; F \cos \Delta\psi \; .$$

It is still in the $\underline{i}$-direction, which is also the direction of a principal axis of inertia through the mass center G. The motion will therefore be a mere rotation about this axis (no tumbling occurring). Let T denote the moment of inertia relative to the axis of rotation.

With a finite angle ψ, that is, at a given time t during the motion, the absolute value of the moment, $M = y_g \; F \cos \psi$, appears in the equation of motion

$$\frac{d^2\psi}{dt^2} = A \cos \psi$$

where $A = y_g F/T$ is a positive quantity. Putting $d\psi/dt = \omega$, this equation may be written as

$$\frac{d}{dt}\,\omega^2 = 2A\,\frac{d\psi}{dt}\,\cos\,\psi$$

so that a first integral, with $\psi = 0$ and $\omega = 0$ for $t = 0$, becomes

$$\omega = \pm\sqrt{2A\,\sin\,\psi} = \frac{d\psi}{dt}\ .$$

Further integration shows t to be an elliptic integral similar to the integral that occurs with the motion of a pendulum. In writing the solution down let it be recalled that the Jacobian elliptic function sinus amplitudinis (sn z) has the real period 4K and takes on the values 0,1,0,-1 for z = 0,K,2K,3K, respectively. With its aid the dependence of the angle ψ on time may be given as

$$\sin\,\psi = \frac{\mathrm{sn}^2\,(\sqrt{A}\,t)}{2 - \mathrm{sn}^2\,(\sqrt{A}\,t)}\ .$$

The angular velocity then becomes

$$\dot{\psi} = \omega = \sqrt{\frac{2A}{2 - \mathrm{sn}^2\,(\sqrt{A}\,t)}}\ \mathrm{sn}\,(\sqrt{A}\,t).$$

The modulus characteristic for the function here is $k = 1/\sqrt{2}$, so that

$$K = 1.8541,$$

as can be found from a table of the complete elliptic integrals.

Now, at $t = 0$, $\psi = 0$ and $\omega = 0$. As time proceeds, the angular velocity attains positive values up to a maximum $\sqrt{2A}$ at $\psi = 90°$ ($\sqrt{A}\,t = K$), when the sphere's original right-hand extremity is now on top and the torque $\underline{M}$ has become zero (the mass center being on the line of attack; $\cos\,\psi = 0$). Inertia carries the motion on; a torque then appears in the negative $\underline{i}$-direction and causes deceleration. The angular velocity decreases to the value $\omega = 0$ at $\psi = 180°$ ($\sqrt{A}\,t = 2K$), where the "negative" torque has its greatest absolute value and succeeds in reverting the sign of the angular velocity. A return half-cycle follows which takes the velocity through negative values, until at $\sqrt{A}\ t = 4K$ the initial position and velocity are attained a second time.

Since the quantity

$$A = \frac{y_g}{T} \cdot \pi p_o R^2 \left(1 + 2\kappa \frac{1 - q}{3}\right)$$

can be expected to be very small, especially with appreciable sphere mass, this swinging motion will often be slow. If, for example, $\sqrt{A}$ is of the order 10^{-5} 1/sec, the time needed to accomplish a full cycle would be $4K \times 10^5$ sec or somewhat more than 8 days.

In addition, other small torques may be operative (through micrometeoric impact, the gravity gradient, aerodynamic forces) which may lead to a slow gyroscopic (tumbling) motion after all.

If the force has the general direction $\underline{s}$ (instead of $-\underline{k}$), we may introduce the $\underline{k}$-direction in such a way that $\underline{s}$ is in the (y,z)-plane. It is then not hard to show that qualitatively the motion is the same as before (assuming again that $\psi = \omega = 0$ for $t = 0$). In the exceptional cases where $\underline{s} = \pm \underline{j}$, no rotatory motion at all is initiated for lack of a moment at $t = 0$.

Expression (17) shows that the force takes on a particularly simple form for a totally absorbing surface ($\kappa = 0$). This is to be expected, since the sphere then moves under the sole impact of incident light. More remarkable is the fact that the same result obtains if the surface is a perfect mirror ($q = 1$, κ arbitrary). Quite regardless of what part of the energy goes into the reflected light, the force is always due to incidence alone. One can indeed see from expression (16) that the vertical ($\underline{k}$-) components of the forces $d^2\underline{F}_r$ cancel at surface points that lie 45° apart on a meridian. The lateral components cancel at points 180° apart on a circle of latitude. The net force exerted through specular reflection is therefore zero.

IV. THE CIRCULAR CONE

In figure 4 it is assumed that the light, arriving from the left and back, determines the (x,y)-plane with the cone (y-) axis. Its

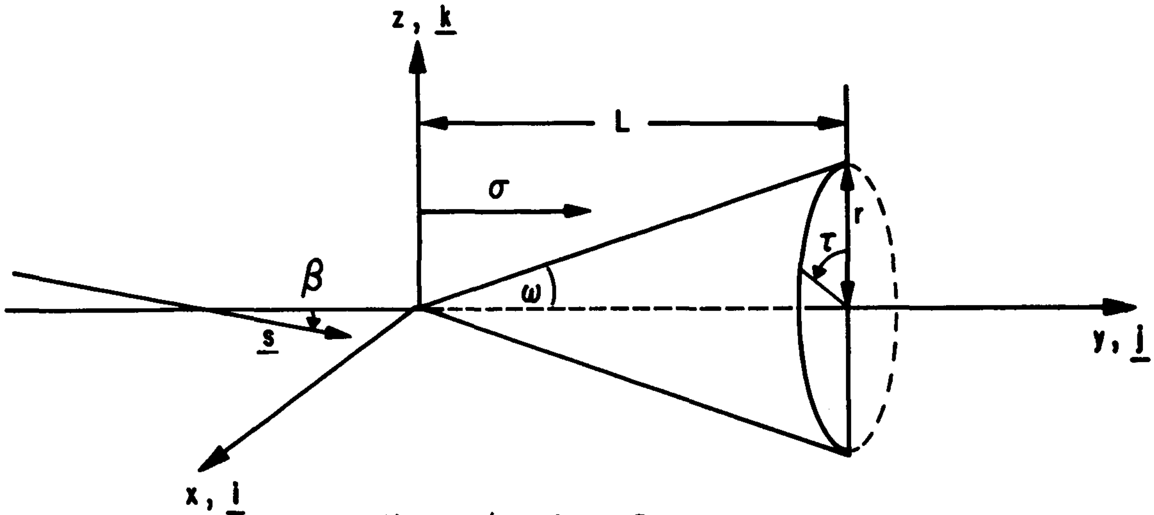

Figure 4. Cone Geometry

direction is then given by the unit vector

$$\underline{s} = \underline{i} \sin \beta + \underline{j} \cos \beta$$

where β is counted a positive angle (negative values would assign a backward component to the light direction with no essential consequences for the description). Let us consider the case of full illumination:

$$\beta \leqq \omega.$$

The parametric representation of the curved surface may be written as

$$x = R\sigma \sin \tau$$

$$y = L\sigma$$

$$z = R\sigma \cos \tau$$

where

$$0 \leqq \sigma \leqq 1, \qquad 0 \leqq \tau \leqq 2\pi$$

are the natural intervals needed for full coverage of the surface. The coordinate lines σ = const., τ = const. are circles normal to the cone axis and the generating lines, respectively.

Since

$$\begin{vmatrix} y_\sigma & z_\sigma \\ y_\tau & z_\tau \end{vmatrix} = -LR\sigma \sin \tau, \qquad \begin{vmatrix} z_\sigma & x_\sigma \\ z_\tau & x_\tau \end{vmatrix} = R^2\sigma, \qquad \begin{vmatrix} x_\sigma & y_\sigma \\ x_\tau & y_\tau \end{vmatrix} = -LR\sigma \cos \tau,$$

it follows from the general expressions (13), (12) and (15) that

$$N = R\sigma \sqrt{L^2 + R^2} = \frac{RL\sigma}{\cos \omega}$$

$$n_1 = -\sin \tau \cos \omega, \qquad n_2 = \sin \omega, \qquad n_3 = -\cos \tau \cos \omega$$

$$\cos \alpha = -\sin \tau \cos \omega \sin \beta + \sin \omega \cos \beta.$$

With these results and the area differential (14) the force element (10) becomes

$$d^2\underline{F} = \frac{p_o \, RL\sigma}{\cos \omega} \, d\sigma \, d\tau \Big\{ (1 - qk)(\underline{i} \sin \beta + \underline{j} \cos \beta)(-\sin \tau \cos \omega \sin \beta$$

$$+ \sin \omega \cos \beta) + 2\kappa(-\underline{i} \sin \tau \cos \omega + \underline{j} \sin \omega - \underline{k} \cos \tau \cos \omega)$$

$$\cdot \left[\frac{1 - q}{3} + q(-\sin \tau \cos \omega \sin \beta + \sin \omega \cos \beta)^2 \right] \Big\}.$$

In integrating over the natural σ-interval the factor 1/2 appears. Since the parameter τ also moves in its full interval $0 \leqq \tau \leqq 2\pi$ (no shadow boundary being admitted), all τ-quadratures leading to trigonometric functions alone result in zero. The final outcome is less involved than might have been expected:

$$\underline{F} = \pi p_o R^2 \left\{ \underline{i} \sin \beta \cos \beta \, (1 + qk \cos 2\omega) \right.$$

$$\left. + \underline{j} \left[\cos^2\beta + \frac{2 + q}{3} \kappa - \kappa q \, (\sin^2\beta \sin^2\omega + 2 \cos^2\beta \cos^2\omega) \right] \right\} . \qquad (18)$$

The cone length L appearing in the force differential has been replaced here by R cotg ω.

It is seen that with $q = 1$, expression (18) still contains terms involving the loss coefficient κ and that therefore, other than with the sphere, the force continues to depend on the quality of the reflection. It is not equal to the force through incidence alone:

$$\underline{F}_i = \pi p_o R^2 \cos \beta \, (\underline{i} \sin \beta + \underline{j} \cos \beta) \qquad (19)$$

except in the trivial case where there is no reflection ($\kappa = 0$). Its absolute value is often smaller than that of $\underline{F}_i$, especially with slender cones, with which the force through specular reflection has a strong component opposite to the axial component of the incident force. Consider zero incidence where

$$|\underline{F}_i| = \pi p_o R^2$$

while, by expression (18)

$$|\underline{F}| = \pi p_o R^2 \left[1 + \kappa \left(\frac{2 + q}{3} - 2q \cos^2\omega \right) \right] .$$

When

$$\cos^2\omega > \frac{2 + q}{6q},$$

$|\underline{F}| < |\underline{F}_i|$. In the ideal case $q = 1$ the condition becomes $\omega < 45°$.

Clearly, reflection not always strengthens the force as one might be led to believe by incorrectly generalizing flat plate results.

Fairly extended calculation is necessary to locate the center of pressure. It is found at

$$x^* = z^* = 0, \qquad y^* = L \frac{1 + \frac{q\kappa}{3}}{1 + q\kappa \cos 2\omega} .$$

Its position does not depend on the angle β, but it is variable with the cone angle. When the cone length is fixed, it moves away from the tip with increasing base radius; if ω reaches a value $\approx 35°15'$ ($\cos 2\omega = 1/3$), it resides at base center.

For a homogeneous cone the mass center is at

$$x_g = z_g = 0, \qquad y_g = \frac{3}{4} L.$$

The stability condition, $y^* > y_g$, tells that most cones are stable except the slender ones for which

$$\frac{1 + \frac{q\kappa}{3}}{1 + 2\kappa \cos 2\omega} < \frac{3}{4} . \tag{20}$$

As long as $q\kappa < 0.6$, the cone motion is stable, in particular with a non-reflecting surface ($\kappa = 0$) and with completely diffuse reflection ($q = 0$). Ideal specular reflection ($\kappa = q = 1$) renders a cone unstable provided its tip angle is sufficiently small ($\sin \omega < 1/3$, $\omega < 19.5°$).

The torque about the center of pressure is found from

$$\underline{M} = -j(y_g - y^*) \times \underline{F}$$

which expression gives

$$\underline{M} = \pi p_o R^2 L \sin \beta \cos \beta \left[\frac{3}{4} (1 + q\kappa \cos 2\omega) - \left(1 + \frac{q\kappa}{3}\right) \right] . \tag{21}$$

This formula, as the force expression (18), is valid only with a fully illuminated cone, i.e., for $\beta \leqq \omega$. Once the angle β is inside this region, it will remain there provided the motion is stable. Otherwise, β will eventually exceed ω. The much more complicated expressions which arise with the presence of a shadow boundary on the cone simplify for a non-reflecting surface; one finds that, with $\beta \geqq \omega$, the motion is stable here as long as $\tan \omega \operatorname{cotg} \beta > 0.17$.

In the case of the Apollo capsule where $\omega \approx 33°$, this complication is not likely to occur*, although it is not a homogeneous body. Once the tip points in the general direction of the sun so that the conical part of the surface is entirely exposed to sunlight, the torque provided by the latter should tend to align the cone axis with light direction. It would be expected that the same tendency prevails if a narrow triangular shadow zone exists on the surface. However, with a wider such zone, a detailed mathematical investigation is called for, which also would have to allow for the forces that attack the illuminated parts of the base cap.

In order to study the behavior of the light force acting on a fully illuminated cone, a computer program** was run which assumed lossless reflection ($\kappa = 1$). The angle ω was taken as 33° (Apollo capsule). The remaining parameters then are the smoothness coefficient q and the angle β ($\leqq \omega$) of overall incidence. The results can be summarized as follows:

(a) β = const. The force increases with the smoothness coefficient decreasing from 1 to zero (in the approximate ratios 1:3, 3:5 at $\beta = 0°$, $\beta = 33°$). Its inclination towards the cone axis decreases (except at $\beta = 0$ where it is always zero). The maximum decrease occurs at $\beta = 33°$ (from 45.75° to 18.44°). In short, increasing diffusivity renders the force stronger and less inclined towards the cone axis.

(b) q = const. The inclination of the force increases with β (from 0 to 18.44° at $q = 0$, from 0 to 45.75° at $q = 1$). Its magnitude decreases if $q < 0.5$ (in the ratio 7:6 for $q = 0$); it increases if $q > 0.5$ (in the ratio 2:3 for $q = 1$). With $q = 0.5$ there is an insignificant force maximum at $\beta \approx 25°$, the whole variation amounting to less than 1 percent in the full range $0 \leqq \beta \leqq \omega$.

* There is no value of $q\kappa$ ($\leqq 1$) that would satisfy condition (20), as $\cos 2\omega \approx 0.4$.

** Set up by James E. Mabry.

With the opposite extreme of κ ($\kappa = 0$), the behavior of the force does not depend on ω and is directly seen from expression (19). Its angle with the cone axis is always β, and its absolute value decreases as β moves from 0° to ω.

If β exceeds ω, expression (19) for $\omega = 0$ must be replaced by

$$\underline{F} \equiv \underline{F}_i = p_o R^2(\underline{i} \sin \beta + \underline{j} \cos \beta) \left[\frac{\pi + 2\tau_s}{2} \cos \beta + \sin \beta \operatorname{cotg} \omega \cos \tau_s\right]$$

where the relation

$$\sin \tau_s = \operatorname{tg} \omega \operatorname{cotg} \beta$$

defines the two straight shadow boundaries $\tau = \tau_s$ and $\tau = \pi - 2\tau_s$.

In the limiting cases of the half and fully illuminated cone ($\beta = \pi/2$, $\beta = \omega$), one has

$$|\underline{F}|_{\beta=\pi/2} = p_o R^2 \operatorname{cotg} \omega$$

$$|\underline{F}|_{\beta=\omega} = \pi p_o R^2 \cos \omega.$$

It is seen that if $\sin \omega < 1/\pi$ the force at $\beta = \pi/2$ is stronger than that at $\beta = \omega$; that is, with sufficiently slender cones the decrease of the force with increasing β will not continue indefinitely up to $\beta = \pi/2$. It can be shown that in such circumstances, the force goes through a minimum followed by a maximum and ends up decreasing at $\beta = \pi/2$. In the special case where the force magnitude here is identical with that at $\beta = \omega$ ($\sin \omega = 1/\pi$, $\omega \approx 17.5°$), the minimum was found at $\beta \approx 19.5°$, the maximum at $\beta \approx 58°$.

With $\omega = 33°$, the magnitude of the force on a non-reflecting cone decreases through the entire β-interval from $\beta = 0$ to $\beta = \pi/2$. From unity at $\beta = 0$ it is down to 0.84 at $\beta = \omega$, and to 0.49 at $\beta = \pi/2$.

V. CONCLUDING REMARKS

Diffuse re-emission was thought in the foregoing to be governed by Lambert's cosine law. This law furnished the model from which to calculate the overall diffusion force acting on an elemental surface. From the intensities re-emitted in the several directions, corresponding light forces were determined and added up vectorially.

Other approaches to the problem of diffuse reflectivity can be imagined.

As an example, one may assume that an infinitesimal half-sphere erected over the elemental surface is uniformly illuminated by the diffusely reflected light. Contrary to Lambert's rule, its intensity then would be the same in all directions. This entails that a greater part than before of the re-emitted energy is caught up in cancelling "stresses" parallel to the elemental surface. Computation shows that the factor 2/3 in expressions (4) and (9) is to be replaced by 1/2, so that instead of $(1 - q)/3$ in the fundamental expression (10) one has to write $(1 - q)/4$.

In both these approaches fictitious model forces are treated as if they were actual forces; the factors 2/3 and 1/2 are uncertain to this extent. From the random distribution of the surface irregularities one would conclude that the integrated force actually exerted on the surface element in diffuse reflection should be essentially in the direction of the interior surface normal (as in fact it turned out to be with the two models). Without going into any more detail, one might argue that the amount of energy not reflected specularly is reflected diffusely, thus converting the factor 2/3 in expression (4) into unity and the term $(1 - q)/3$ into $(1 - q)/2$.

From all this it would seem that the "diffusion" portion of the force differential (10) is likely to be proportional to a factor between $(1 - q)/2$ and $(1 - q)/4$.

To illustrate the differences involved the forces arising with $\kappa = 1$ on the fully illuminated cone with half-angle $\omega = 33°$ were computed using the largest factor $(1 - q)/2$ in order to compare them with those previously found for $(1 - q)/3$. In the worst case ($q = 0$; exclusively diffuse reflection) the force magnitudes are multiplied by nearly the same factor 1.2 in the entire range $0 \leqq \beta \leqq \omega$, whereas the force inclinations toward the cone axis decrease by a rather constant factor of again ≈ 1.2. A tendency of this kind is in keeping with what an earlier result should suggest to expect from increasing diffusivity; the force on the cone becomes stronger and less inclined. Its moment about the origin, however, and thus the center of pressure, are independent of speculations on the magnitude of the diffusion part, because with full illumination the latter's line of attack is the cone axis. The diffusion torque therefore is zero irrespective of the choice of n in $(1 - q)/n$ and, by the way, of all other characteristic quantities such R, L, q, κ.

ON MIXED REFLECTION OF SUNLIGHT

by W. H. Heybey

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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