X-507-67-43

NASA TM X- 55820

MAXIMUM POSSIBLE ERROR IN POSITION IN A LEAST SQUARE ORBIT

BY

R. G. LANGEBARTEL

FEBRUARY 1967

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

X-507-67-43

MAXIMUM POSSIBLE ERROR IN POSITION IN A LEAST SQUARES ORBIT

by

R. G. Langebartel

February, 1967

Goddard Space Flight Center
Greenbelt, Maryland

MAXIMUM POSSIBLE ERROR IN POSITION IN A LEAST SQUARES ORBIT

R. G. Langebartel

SUMMARY

A second order theory is developed for the problem of determining the maximum possible error in position if a Keplerian orbit is fitted by least squares to a set of observational data under the condition that the sum of the squares of the distances between the true and observed positions be held constant. A result of the first order theory is that the maximum error at any one position in the orbit occurs when the observed positions coincide with the least squares computed positions. This doesn't remain true in the second order theory.

Application of the second order theory is made in detail to the case of circular orbits.

CONTENTS

PRECEDING PAGE BLANK NOT FILMED.

LIST OF SYMBOLS

a	semi-major axis of true orbit
e	eccentricity of true orbit
p	angle of perigee of true orbit
z	instant of perigee passage in true orbit
α	semi-major axis of least squares orbit
ϵ	eccentricity of least squares orbit
π	angle of perigee of least squares orbit
ζ	instant of perigee passage in least squares orbit
(r, f)	polar coordinates of position in true orbit
(ρ, φ)	polar coordinates of position in least squares orbit
s	mean anomaly for true orbit
g	eccentric anomaly for true orbit
σ	mean anomaly for least squares orbit
ν	eccentric anomaly for least squares orbit
$(\hat{r}, \hat{f})$	polar coordinates of observed position
β_i, b_i	see (1.4)
W^2	see (2.1)
Q^2	see (2.2)
$\mathcal{E}$	see (3.28)

MAXIMUM POSSIBLE ERROR IN POSITION
IN A LEAST SQUARES ORBIT

R. G. Langebartel

INTRODUCTION

A pertinent problem in orbit determination theory is the gauging of the effect of observational data errors. One approach, suggested by Dr. B. Kruger, is to investigate the nature of the distribution of errors that gives rise to the maximum error in one predicted position. That is the problem considered here for the case of two-dimensional Keplerian motion where it is assumed the computed orbit is an ellipse fitted to the observational data by least squares. This least squares ellipse is, of course, a specialized curve fit in that it takes into account the special parameterization of the ellipse (with respect to time) peculiar to Newtonian two-body motion. As constraint in the maximum problem it is assumed that the sum of the squares of the distances between the observed and true positions is a constant. No other assumption on the errors is made.

THE LEAST SQUARES ORBIT

The elements of the true elliptic orbit in the plane are the semi-major axis a, the eccentricity e, the angle of perigee p, and the instant of perigee passage z. The elements for the least squares orbit are indicated by the corresponding Greek letters: α, ϵ, π, and ζ. Position in the true orbit is given by the polar coordinates r (distance) and f (angle), position in the least squares orbit by ρ and φ. The direction from one focus as the origin from which angles are measured is arbitrary but fixed throughout the discussion. The functional relations

$$\begin{cases} r = r\,(a, e, p, z) \\ f = f\,(a, e, p, z) \end{cases} \qquad \begin{cases} \rho = \rho\,(\alpha, \epsilon, \pi, \zeta) \\ \varphi = \varphi\,(\alpha, \epsilon, \pi, \zeta) \end{cases} \tag{1.1}$$

are those of the Newtonian theory:

$$
\begin{cases}
r = a(1 - e\cos g) \\
\cos(f - p) = a r^{-1} (\cos g - e) \\
s = \sqrt{\mu}\, a^{-3/2} (t - z) \\
s = g - e \sin g
\end{cases}
\qquad
\begin{cases}
\rho = \alpha(1 - \epsilon\cos\gamma) \\
\cos(\varphi - \pi) = \alpha \rho^{-1} (\cos\gamma - \epsilon) \\
\sigma = \sqrt{\mu}\, \alpha^{-3/2} (t - \zeta) \\
\sigma = \gamma - \epsilon \sin\gamma
\end{cases}
\tag{1.2}
$$

Thus, s and g are the mean and eccentric anomalies, respectively, for the true orbit, and σ and γ are those for the least squares orbit. The gravitational parameter μ is the product of the gravitation constant by the sum of the masses of the two bodies.

The least squares orbit is defined by requiring α, ϵ, π, and ζ to be so chosen as to minimize the sum of the squares of the distances between the observed and computed positions. Suppose that positions have been observed at $n + 1$ different instants of time and let these observed positions be denoted by $(\hat{r}_k, \hat{f}_k)$, $k = 0, 1, \ldots, n$. Consequently, the function to be minimized is

$$
P^2 \equiv \sum_{k=0}^{n} (\hat{r}_k^2 + \rho_k^2 - 2\hat{r}_k \rho_k \cos(\hat{f}_k - \varphi_k)). \tag{1.3}
$$

Introduce the notation

$$
\begin{cases}
b_1 \equiv a \\
b_2 \equiv e \\
b_3 \equiv p \\
b_4 \equiv z
\end{cases}
\quad
\begin{cases}
\beta_1 \equiv \alpha \\
\beta_2 \equiv \epsilon \\
\beta_3 \equiv \pi \\
\beta_4 \equiv \zeta
\end{cases}
\tag{1.4}
$$

The function P^2 is a function of the β_i through the variables ρ_k and φ_k. Hence the four equations of condition for the least squares orbit are

$$
\frac{1}{2}\frac{\partial P^2}{\partial \beta_i} = \sum_{k=0}^{n} \left\{ \left[\rho_k - \hat{r}_k \cos(\hat{f}_k - \varphi_k)\right] \frac{\partial \rho_k}{\partial \beta_i} - \hat{r}_k \rho_k \sin(\hat{f}_k - \varphi_k) \frac{\partial \varphi_k}{\partial \beta_i} \right\} = 0. \tag{1.5}
$$

The solving of these equations determines β_i in terms of the quantities $\hat{r}_k$ and $\hat{f}_k$.

2. THE MAXIMUM PROBLEM

We wish to ascertain what distribution of errors in the observational data will bring about the greatest discrepancy between the true and least squares computed positions at the k^{th} instant, t_k, if we impose the constraint that the sum of the squares of the distances between the true and the observed positions, i.e. the sum of the squares of the errors, is held constant. That is to say, we are to maximize

$$W^2 \equiv r_k^2 + \rho_k^2 - 2\, r_k \rho_k \cos\,(f_k - \varphi_k) \tag{2.1}$$

subject to the side condition

$$Q^2 \equiv \sum_{k=0}^{n} \left\{ r_k^2 + \hat{r}_k^2 - 2\, r_k \hat{r}_k \cos\,(f_k - \hat{f}_k) \right\} = c^2 . \tag{2.2}$$

The problem is to determine the values of $\hat{r}_k$ and $\hat{f}_k$ that make W^2 a maximum. The variables ρ_k and φ_k in W^2 are to be regarded as functions of $\hat{r}_k$ and $\hat{f}_k$ by virtue of (1.5) and (1.2). It should be noted that s, g, σ, γ vary with t so we write

$$\left\{ \begin{aligned} r_k &= a\,(1 - e \cos g_k) \\ \cos\,(f_k - p) &= a\, r_k^{-1}\,(\cos g_k - e) \\ s_k &= \sqrt{\mu}\, a^{-3/2}\,(t_k - z) \\ s_k &= g_k - e \sin g_k \end{aligned} \right. \qquad \left\{ \begin{aligned} \rho_k &= \alpha\,(1 - \epsilon \cos \gamma_k) \\ \cos\,(\varphi_k - \pi) &= \alpha\, \rho^{-1}\,(\cos \gamma_k - \epsilon) \\ \sigma_k &= \sqrt{\mu}\, \alpha^{-3/2}\,(t_k - \zeta) \\ \sigma_k &= \gamma_k - \epsilon \sin \gamma_k . \end{aligned} \right. \tag{2.3}$$

The fundamental equations for the isoperimetric problem are

$$\left\{ \begin{aligned} &[\rho_k - r_k \cos\,(\varphi_k - f_k)]\, \frac{\partial \rho_k}{\partial \hat{r}_m} + r_k \rho_k \sin\,(\varphi_k - f_k)\, \frac{\partial \varphi_k}{\partial \hat{r}_m} + \lambda\, [\hat{r}_m - r_m \cos\,(\hat{f}_m - f_m)] = 0 \\ &[\rho_k - r_k \cos\,(\varphi_k - f_k)]\, \frac{\partial \rho_k}{\partial \hat{f}_m} + r_k \rho_k \sin\,(\varphi_k - f_k)\, \frac{\partial \varphi_k}{\partial \hat{f}_m} + \lambda\, [\hat{r}_m\, r_m \sin\,(\hat{f}_m - f_m)] = 0 \end{aligned} \right.$$

where λ is the Lagrange multiplier.

A more compact form for the equations involved results if we introduce the notation

$$
\begin{cases}
\Theta_k \equiv \rho_k - r_k \cos(\varphi_k - f_k) \\
\Phi_k \equiv r_k \, \rho_k \sin(\varphi_k - f_k)
\end{cases}
$$

$$
\begin{cases}
\tilde{\Theta}_k \equiv \hat{r}_k - r_k \cos(\hat{f}_k - f_k) \\
\tilde{\Phi}_k \equiv r_k \, \hat{r}_k \sin(\hat{f}_k - f_k)
\end{cases}
\tag{2.4}
$$

$$
\begin{cases}
\hat{\Theta}_k \equiv \rho_k - \hat{r}_k \cos(\varphi_k - \hat{f}_k) \\
\hat{\Phi}_k \equiv \hat{r}_k \, \rho_k \sin(\varphi_k - \hat{f}_k).
\end{cases}
$$

The necessary conditions for a maximum thus have the form

$$
\begin{cases}
\Theta_k \dfrac{\partial \rho_k}{\partial \hat{r}_m} + \Phi_k \dfrac{\partial \varphi_k}{\partial \hat{r}_m} + \lambda \tilde{\Theta}_m = 0, \\
\qquad\qquad\qquad\qquad\qquad m = 0, 1, \cdots, n. \\
\Theta_k \dfrac{\partial \rho_k}{\partial \hat{f}_m} + \Phi_k \dfrac{\partial \varphi_k}{\partial \hat{f}_m} + \lambda \tilde{\Phi}_m = 0,
\end{cases}
\tag{2.5}
$$

The dependence of ρ_k and φ_k on $\hat{r}_m$ and $\hat{f}_m$ has been specified by the least squares equations of condition (1.5). Use can be made of (1.5) if we write (2.5) in the form

$$
\begin{cases}
\displaystyle\sum_{i=1}^{4} \left[\Theta_k \frac{\partial \rho_k}{\partial \beta_i} + \Phi_k \frac{\partial \varphi_k}{\partial \beta_i} \right] \frac{\partial \beta_i}{\partial \hat{r}_m} + \lambda \tilde{\Theta}_m = 0 \\
\displaystyle\sum_{i=1}^{4} \left[\Theta_k \frac{\partial \rho_k}{\partial \beta_i} + \Phi_k \frac{\partial \varphi_k}{\partial \beta_i} \right] \frac{\partial \beta_i}{\partial \hat{f}_m} + \lambda \tilde{\Phi}_m = 0
\end{cases}
\tag{2.6}
$$

and then use (1.5) to eliminate $\partial \beta_i / \partial \hat{r}_m$ and $\partial \beta_i / \partial \hat{f}_m$.

The equation (1.5) is

$$\sum_{k=0}^{n} \left[\hat{\Theta}_k \frac{\partial \rho_k}{\partial \beta_i} + \hat{\Phi}_k \frac{\partial \varphi_k}{\partial \beta_i} \right] = 0 \tag{2.7}$$

and upon differentiation with respect to $\hat{r}_m$ and with respect to $\hat{f}_m$ this becomes

$$\left\{ \begin{array}{l} \sum_{i=1}^{4} G_{ji} \dfrac{\partial \beta_i}{\partial \hat{r}_m} = H_{jm}, \\ \qquad\qquad\qquad\qquad j = 1,2,3,4 \\ \sum_{i=1}^{4} G_{ji} \dfrac{\partial \beta_i}{\partial \hat{f}_m} = K_{jm} \end{array} \right. \tag{2.8}$$

where

$$\left\{ \begin{array}{l} G_{ji} \equiv \sum_{k=0}^{n} \left\{ \hat{\Theta}_k \dfrac{\partial^2 \rho_k}{\partial \beta_i \, \partial \beta_j} + \hat{\Phi}_k \dfrac{\partial^2 \varphi_k}{\partial \beta_i \, \partial \beta_j} + \dfrac{\partial \hat{\Theta}_k}{\partial \beta_i} \dfrac{\partial \rho_k}{\partial \beta_j} + \dfrac{\partial \hat{\Phi}_k}{\partial \beta_i} \dfrac{\partial \varphi_k}{\partial \beta_j} \right\} \\ H_{jm} \equiv \cos\,(\hat{f}_m - \varphi_m) \dfrac{\partial \rho_m}{\partial \beta_j} + \rho_m \sin\,(\hat{f}_m - \varphi_m) \dfrac{\partial \varphi_m}{\partial \beta_j} \\ K_{jm} \equiv - \hat{r}_m \sin\,(\hat{f}_m - \varphi_m) \dfrac{\partial \rho_m}{\partial \beta_j} + \hat{r}_m \rho_m \cos\,(\hat{f}_m - \varphi_m) \dfrac{\partial \varphi_m}{\partial \beta_j}. \end{array} \right. \tag{2.9}$$

Upon introduction of the matrices

$$\frac{\partial \rho}{\partial \beta} \equiv \begin{pmatrix} \dfrac{\partial \rho_0}{\partial \beta_1} & \cdots & \dfrac{\partial \rho_n}{\partial \beta_1} \\ \vdots & & \vdots \\ \dfrac{\partial \rho_0}{\partial \beta_4} & \cdots & \dfrac{\partial \rho_n}{\partial \beta_4} \end{pmatrix}, \qquad \frac{\partial \varphi}{\partial \beta} \equiv \begin{pmatrix} \dfrac{\partial \varphi_0}{\partial \beta_1} & \cdots & \dfrac{\partial \varphi_n}{\partial \beta_1} \\ \vdots & & \vdots \\ \dfrac{\partial \varphi_0}{\partial \beta_4} & \cdots & \dfrac{\partial \varphi_n}{\partial \beta_4} \end{pmatrix},$$

$$\frac{\partial \rho_k}{\partial \beta} \equiv \begin{pmatrix} \frac{\partial \rho_k}{\partial \beta_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \rho_k}{\partial \beta_4} \end{pmatrix}, \qquad \frac{\partial \varphi_k}{\partial \beta} \equiv \begin{pmatrix} \frac{\partial \varphi_k}{\partial \beta_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial \varphi_k}{\partial \beta_4} \end{pmatrix},$$

$$\frac{\partial \beta}{\partial \hat{r}} \equiv \begin{pmatrix} \frac{\partial \beta_1}{\partial \hat{r}_0} & \cdots & \frac{\partial \beta_1}{\partial \hat{r}_n} \\ \vdots & & \vdots \\ \frac{\partial \beta_4}{\partial \hat{r}_0} & \cdots & \frac{\partial \beta_4}{\partial \hat{r}_n} \end{pmatrix}, \qquad \frac{\partial \beta}{\partial \hat{f}} \equiv \begin{pmatrix} \frac{\partial \beta_1}{\partial \hat{f}_0} & \cdots & \frac{\partial \beta_1}{\partial \hat{f}_n} \\ \vdots & & \vdots \\ \frac{\partial \beta_4}{\partial \hat{f}_0} & \cdots & \frac{\partial \beta_4}{\partial \hat{f}_n} \end{pmatrix},$$

$$\widetilde{\Theta} \equiv (\widetilde{\Theta}_0, \widetilde{\Theta}_1, \cdots, \widetilde{\Theta}_n), \ \widetilde{\Phi} \equiv (\widetilde{\Phi}_0, \cdots, \widetilde{\Phi}_n), \tag{2.10}$$

$$P_k \equiv \Theta_k \frac{\partial \rho_k^T}{\partial \beta} + \Phi_k \frac{\partial \varphi_k^T}{\partial \beta} = \left(\Theta_k \frac{\partial \rho_k}{\partial \beta_1} + \Phi_k \frac{\partial \varphi_k}{\partial \beta_1}, \cdots, \Theta_k \frac{\partial \rho_k}{\partial \beta_4} + \Phi_k \frac{\partial \varphi_k}{\partial \beta_4}\right),$$

$$G \equiv \begin{pmatrix} G_{11} & \cdots & G_{14} \\ \vdots & & \vdots \\ G_{41} & \cdots & G_{44} \end{pmatrix},$$

$$H \equiv \begin{pmatrix} H_{10} & \cdots & H_{1n} \\ \vdots & & \vdots \\ H_{40} & \cdots & H_{4n} \end{pmatrix},$$

$$K \equiv \begin{pmatrix} K_{10} & \cdots & K_{1n} \\ \vdots & & \vdots \\ K_{40} & \cdots & K_{4n} \end{pmatrix},$$

we have for (2.6) and (2.8)

$$\begin{cases} P_k \dfrac{\partial \beta}{\partial \hat{r}} + \lambda \tilde{\Theta} = 0 \\ \\ P_k \dfrac{\partial \beta}{\partial \hat{f}} + \lambda \tilde{\Phi} = 0 \end{cases} \tag{2.11}$$

$$\begin{cases} G \dfrac{\partial \beta}{\partial \hat{r}} = H \\ \\ G \dfrac{\partial \beta}{\partial \hat{f}} = K. \end{cases} \tag{2.12}$$

Evidently the necessary condition for a maximum now has the form

$$\begin{cases} P_k \, G^{-1} \, H + \lambda \tilde{\Theta} = 0 \\ P_k \, G^{-1} \, K + \lambda \tilde{\Phi} = 0 \end{cases} \tag{2.13}$$

These equations are highly non-linear in the $\hat{r}_m$ and $\hat{f}_m$ and recourse must be had to some approximation procedure. Suppose that the functions involved are expandable in powers of a variable ε:

$$
\begin{cases}
P_k = P_k^{(1)}\,\mathcal{E} + P_k^{(2)}\,\mathcal{E}^2 + \cdots \\
\widetilde{\Theta} = \widetilde{\Theta}^{(1)}\,\mathcal{E} + \widetilde{\Theta}^{(2)}\,\mathcal{E}^2 + \cdots \\
\widetilde{\Phi} = \widetilde{\Phi}^{(1)}\,\mathcal{E} + \widetilde{\Phi}^{(2)}\,\mathcal{E}^2 + \cdots \\
G = G^{(0)} + G^{(1)}\,\mathcal{E} + G^{(2)}\,\mathcal{E}^2 + \cdots \\
H = H^{(0)} + H^{(1)}\,\mathcal{E} + H^{(2)}\,\mathcal{E}^2 + \cdots \\
K = K^{(0)} + K^{(1)}\,\mathcal{E} + K^{(2)}\,\mathcal{E}^2 + \cdots \\
\lambda = \lambda^{(0)} + \lambda^{(1)}\,\mathcal{E} + \lambda^{(2)}\,\mathcal{E}^2 + \cdots .
\end{cases} \tag{2.14}
$$

Note from (2.4) and (2.10) that if the observed positions coincide with the true positions, P_k , $\widetilde{\Theta}$, and $\widetilde{\Phi}$ all vanish (which is, of course, in conflict with the constraint $Q^2 = c^2$ if $c > 0$). If $\mathcal{E}$ is a parameter such that the observed configuration becomes the true configuration when $\mathcal{E}$ vanishes then the constant terms in the expansions of P_k, $\widetilde{\Theta}$, and $\widetilde{\Phi}$ must all be zero as is indicated in (2.14). In view of the parenthetical statement above we must also have c and $\mathcal{E}$ vanishing simultaneously. Actually, we shall ultimately take $\mathcal{E}$ to be a multiple of c but this specialization is not necessary now. It is only required at this point that the vanishing of $\mathcal{E}$ bring about the coincidence of the observed positions with the true positions.

We need the expansion of the matrix G^{-1}.

$$G^{-1} = F^{(0)} + F^{(1)}\,\mathcal{E} + F^{(2)}\,\mathcal{E}^2 + \cdots$$

$$\therefore GG^{-1} = I = (G^{(0)} + G^{(1)}\,\mathcal{E} + G^{(2)}\,\mathcal{E}^2 + \cdots)\,(F^{(0)} + F^{(1)}\,\mathcal{E} + F^{(2)}\,\mathcal{E}^2 + \cdots).$$

$$= G^{(0)}F^{(0)} + (G^{(0)}F^{(1)} + G^{(1)}F^{(0)})\,\mathcal{E} + \cdots .$$

This shows that

$$
\begin{cases}
G^{-1} = F^{(0)} - F^{(0)}G^{(1)}F^{(0)}\,\mathcal{E} + \cdots, \\
F^{(0)} = {G^{(0)}}^{-1}
\end{cases} \tag{2.15}
$$

The substitution of (2.14) and (2.15) into (2.13) leads after some matrix algebra to the infinite system of equations

$$\begin{cases} P_k^{(1)} F^{(0)} H^{(0)} + \lambda^{(0)} \tilde{\Theta}^{(1)} = 0 \\ P_k^{(1)} F^{(0)} K^{(0)} + \lambda^{(0)} \tilde{\Phi}^{(1)} = 0 \end{cases} \tag{2.16}$$

$$\begin{cases} P_k^{(1)} F^{(0)} H^{(1)} - P_k^{(1)} F^{(0)} G^{(1)} F^{(0)} H^{(0)} + P_k^{(2)} F^{(0)} H^{(0)} + \lambda^{(0)} \tilde{\Theta}^{(2)} + \lambda^{(1)} \tilde{\Theta}^{(1)} = 0 \\ P_k^{(1)} F^{(0)} K^{(1)} - P_k^{(1)} F^{(0)} G^{(1)} F^{(0)} K^{(0)} + P_k^{(2)} F^{(0)} K^{(0)} + \lambda^{(0)} \tilde{\Phi}^{(2)} + \lambda^{(1)} \tilde{\Phi}^{(1)} = 0 \end{cases} \tag{2.17}$$

. . . .

3. THE FIRST ORDER THEORY

The particular character of the first order equations (2.16) rests on the manner in which the low order terms in the expansions of $\hat{r}_m$, $\hat{f}_m$, ρ_m, and φ_m are involved. Since $\mathcal{E}$ has been specialized to the extent that $\hat{r}_k \to r_k$, $\rho_k \to r_k$, $\hat{f}_k \to f_k$, $\varphi_k \to f_k$, as $\mathcal{E} \to 0$ and $k = 0, 1, \ldots, n$, the expansions are written as

$$\begin{cases} \hat{r}_k = r_k + \hat{r}_k^{(1)} \mathcal{E} + \hat{r}_k^{(2)} \mathcal{E}^2 + \cdots \\ \hat{f}_k = f_k + \hat{f}_k^{(1)} \mathcal{E} + \hat{f}_k^{(2)} \mathcal{E}^2 + \cdots \end{cases} \tag{3.1}$$

$$\begin{cases} \rho_k = r_k + \rho_k^{(1)} \mathcal{E} + \rho_k^{(2)} \mathcal{E}^2 + \cdots \\ \varphi_k = f_k + \varphi_k^{(1)} \mathcal{E} + \varphi_k^{(2)} \mathcal{E}^2 + \cdots \end{cases} \tag{3.2}$$

Further,

$$\begin{cases} \dfrac{\partial \rho_k}{\partial \beta} = \dfrac{\partial r_k}{\partial b} + \left[\dfrac{\partial \rho_k}{\partial \beta}\right]^{(1)} \mathcal{E} + \left[\dfrac{\partial \rho_k}{\partial \beta}\right]^{(2)} \mathcal{E}^2 + \cdots \\ \dfrac{\partial \varphi_k}{\partial \beta} = \dfrac{\partial f_k}{\partial b} + \left[\dfrac{\partial \varphi_k}{\partial \beta}\right]^{(1)} \mathcal{E} + \left[\dfrac{\partial \varphi_k}{\partial \beta}\right]^{(2)} \mathcal{E}^2 + \cdots \end{cases} \tag{3.3}$$

where

$$\frac{\partial r_k}{\partial b} = \begin{pmatrix} \frac{\partial r_k}{\partial b_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial r_k}{\partial b_4} \end{pmatrix}, \qquad \frac{\partial f_k}{\partial b} = \begin{pmatrix} \frac{\partial f_k}{\partial b_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f_k}{\partial b_4} \end{pmatrix}$$

and

$$\begin{cases} \Theta_k = \Theta_k^{(1)} \mathcal{E} + \Theta_k^{(2)} \mathcal{E}^2 + \cdots \\ \Phi_k = \Phi_k^{(1)} \mathcal{E} + \Phi_k^{(2)} \mathcal{E}^2 + \cdots . \end{cases} \tag{3.4}$$

The first few terms in the expansions of the quantities appearing explicitly in (2.13) are easily obtainable and in particular we find

$$P_k^{(1)} = \Theta_k^{(1)} \frac{\partial r_k^T}{\partial b} + \Phi_k^{(1)} \frac{\partial f_k^T}{\partial b} = \rho_k^{(1)} \frac{\partial r_k^T}{\partial b} + r_k^2 \varphi_k^{(1)} \frac{\partial f_k^T}{\partial b}$$

$$F^{(0)} = \left(\sum_{k=0}^{n} \left[\frac{\partial r_k}{\partial b} \frac{\partial r_k^T}{\partial b} + r_k^2 \frac{\partial f_k}{\partial b} \frac{\partial f_k^T}{\partial b} \right] \right)^{-1}$$

$$H^{(0)} = \frac{\partial r}{\partial b} \equiv \begin{pmatrix} \frac{\partial r_0}{\partial b_1} & \cdots & \frac{\partial r_n}{\partial b_1} \\ \vdots & & \vdots \\ \frac{\partial r_0}{\partial b_4} & \cdots & \frac{\partial r_n}{\partial b_4} \end{pmatrix}$$

$$K^{(0)} = \left(r^2 \frac{\partial f}{\partial b}\right) \equiv \begin{pmatrix} r_0^2 \dfrac{\partial f_0}{\partial b_1} & \cdots\cdots & r_n^2 \dfrac{\partial f_n}{\partial b_1} \\ \vdots & & \vdots \\ r_0^2 \dfrac{\partial f_0}{\partial b_4} & \cdots\cdots & r_n^2 \dfrac{\partial f_n}{\partial b_4} \end{pmatrix} \tag{3.5}$$

$$\widetilde{\Theta}^{(1)} = \hat{X}_1^{(1)} \equiv (\hat{r}_0^{(1)}, \cdots, \hat{r}_n^{(1)})$$

$$\widetilde{\Phi}^{(1)} = \hat{X}_2^{(1)} R \equiv (r_0^2 \hat{f}_0^{(1)}, \cdots, r_n^2 \hat{f}_n^{(1)})$$

where

$$X_2^{(1)} \equiv (r_0 \hat{f}_0^{(1)}, \cdots, r_n \hat{f}_n^{(1)})$$

$$R \equiv \begin{pmatrix} r_0 & & & \\ & r_1 & & 0 \\ & & \ddots & \\ 0 & & & r_n \end{pmatrix} .$$

There will also be needed the expansions of the least square elliptic orbit parameters. Upon writing

$$\beta \equiv \begin{pmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_4 \end{pmatrix}, \qquad b \equiv \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_4 \end{pmatrix}, \qquad \beta^{(1)} \equiv \begin{pmatrix} \beta_1^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \beta_4^{(1)} \end{pmatrix}, \qquad \cdots \tag{3.6}$$

we have

$$\beta = b + \beta^{(1)} \mathcal{E} + \beta^{(2)} \mathcal{E}^2 + \cdots . \tag{3.7}$$

Now

$$\left\{\begin{aligned}
\rho_k^{(1)} &= \left[\frac{d\rho_k}{d\mathcal{E}}\right]_{\mathcal{E}=0} = \left[\sum_{i=1}^{4} \frac{\partial \rho_k}{\partial \beta_i}\frac{d\beta_i}{d\mathcal{E}}\right]_{\mathcal{E}=0} = \sum_{i=1}^{4} \frac{\partial r_k}{\partial b_i}\beta_i^{(1)} = \beta^{(1)^T}\frac{\partial r_k}{\partial b} \\
\varphi_k^{(1)} &= \left[\frac{d\varphi_k}{d\mathcal{E}}\right]_{\mathcal{E}=0} = \left[\sum_{i=1}^{4} \frac{\partial \varphi_k}{\partial \beta_i}\frac{d\beta_i}{d\mathcal{E}}\right]_{\mathcal{E}=0} = \sum_{i=1}^{4} \frac{\partial f_k}{\partial b_i}\beta_i^{(1)} = \beta^{(1)^T}\frac{\partial f_k}{\partial b}
\end{aligned}\right. \tag{3.8}$$

Consequently, there is the alternative expression for $P_k^{(1)}$:

$$P_k^{(1)} = \beta^{(1)^T}\left(\frac{\partial r_k}{\partial b}\frac{\partial r_k^T}{\partial b} + r_k^2\frac{\partial f_k}{\partial b}\frac{\partial f_k^T}{\partial b}\right). \tag{3.9}$$

It is apparent that there is still needed the relation between $\beta^{(1)}$ and $\hat{r}^{(1)}$, $\hat{f}^{(1)}$. This relation results from the lead off term in the expansion of (1.5), the equation of condition for the least squares orbit, in conjunction with (3.8). The coefficient of $\mathcal{E}$ in (1.5) when set to zero gives an equation which when rearranged has the form

$$\sum_{k=0}^{n}\left(\rho_k^{(1)}\frac{\partial r_k^T}{\partial b} + r_k^2\varphi_k^{(1)}\frac{\partial f_k^T}{\partial b}\right) = \sum_{k=0}^{n}\left(\hat{r}_k^{(1)}\frac{\partial r_k^T}{\partial b} + r_k^2\hat{f}_k^{(1)}\frac{\partial f_k^T}{\partial b}\right). \tag{3.10}$$

And so the required relation between $\beta^{(1)}$ and $\hat{r}^{(1)}$, $\hat{f}^{(1)}$ is afforded by

$$\beta^{(1)^T}\sum_{k=0}^{n}\left(\frac{\partial r_k}{\partial b}\frac{\partial r_k^T}{\partial b} + r_k^2\frac{\partial f_k}{\partial b}\frac{\partial f_k^T}{\partial b}\right) = \sum_{k=0}^{n}\left(\hat{r}_k^{(1)}\frac{\partial r_k^T}{\partial b} + r_k^2\hat{f}_k^{(1)}\frac{\partial f_k^T}{\partial b}\right). \tag{3.11}$$

The matrix R that appears in $\tilde{\Phi}^{(1)}$ is an encumbrance that can be made to disappear from the equations (2.16) by the introduction of a new matrix $J^{(0)}$ in place of $K^{(0)}$ where

$$J^{(0)} \equiv \begin{pmatrix} r_0 \dfrac{\partial f_0}{\partial b_1} \cdots r_n \dfrac{\partial f_n}{\partial b_1} \\ \vdots \qquad\qquad \vdots \\ r_0 \dfrac{\partial f_0}{\partial b_4} \cdots r_n \dfrac{\partial f_n}{\partial b_4} \end{pmatrix}$$

Note that $K^{(0)} = J^{(0)}R$ with R non-singular, so that the equations (2.16) have the form

$$\begin{cases} P_k^{(1)}\, F^{(0)}\, H^{(0)} + \lambda^{(0)}\, \hat{X}_1^{(1)} = 0 \\ P_k^{(1)}\, F^{(0)}\, J^{(0)} + \lambda^{(0)}\, \hat{X}_2^{(1)} = 0. \end{cases} \tag{3.12}$$

It may be noted that $\partial r_k/\partial b$ and $r_k\, \partial f_k/\partial b$ are the $k+1$ st columns of the matrices $H^{(0)}$ and $J^{(0)}$, respectively, whereupon there is suggested the notation

$$H_k^{(0)} \equiv \frac{\partial r_k}{\partial b} = \begin{pmatrix} \dfrac{\partial r_k}{\partial b_1} \\ \vdots \\ \dfrac{\partial r_k}{\partial b_4} \end{pmatrix}, \qquad J_k^{(0)} \equiv r_k \frac{\partial f_k}{\partial b} = \begin{pmatrix} r_k \dfrac{\partial f_k}{\partial b_1} \\ \vdots \\ r_k \dfrac{\partial f_k}{\partial b_4} \end{pmatrix}. \tag{3.13}$$

And it is convenient to combine these into one matrix:

$$\frac{\partial X_k}{\partial b} \equiv \begin{pmatrix} \dfrac{\partial r_k}{\partial b_1} & r_k \dfrac{\partial f_k}{\partial b_1} \\ \vdots & \vdots \\ \dfrac{\partial r_k}{\partial b_4} & r_k \dfrac{\partial f_k}{\partial b_4} \end{pmatrix}. \tag{3.14}$$

The matrix of these matrices likewise occurs and so we define

$$\frac{\partial X}{\partial b} \equiv \begin{pmatrix} \frac{\partial r_0}{\partial b_1} & r_0 \frac{\partial f_0}{\partial b_1} & \frac{\partial r_1}{\partial b_1} & r_1 \frac{\partial f_1}{\partial b_1} \cdots & \frac{\partial r_n}{\partial b_1} & r_n \frac{\partial f_n}{\partial b_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial r_0}{\partial b_4} & r_0 \frac{\partial f_0}{\partial b_4} & \frac{\partial r_1}{\partial b_4} & r_1 \frac{\partial f_1}{\partial b_4} \cdots & \frac{\partial r_n}{\partial b_4} & r_n \frac{\partial f_n}{\partial b_4} \end{pmatrix} . \tag{3.15}$$

Then

$$G^{(0)} = \sum_{k=0}^{n} \left(H_k^{(0)} H_k^{(0)^T} + J_k^{(0)} J_k^{(0)^T}\right) = H^{(0)} H^{(0)^T} + J^{(0)} J^{(0)^T} \tag{3.16}$$

$$= \sum_{k=0}^{n} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} = \frac{\partial X}{\partial b} \frac{\partial X^T}{\partial b} ,$$

$$P_k^{(1)} = \beta^{(1)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} , \tag{3.17}$$

$$\beta^{(1)^T} = \left(X_1^{(1)} H^{(0)^T} + \hat{X}_2^{(1)} J^{(0)^T}\right) F^{(0)} \tag{3.18}$$

Thus the observed position coordinates as represented by the matrices $\hat{X}_1^{(1)}$ and $\hat{X}_2^{(1)}$ must satisfy

$$\begin{cases} \left(\hat{X}_1^{(1)} H^{(0)^T} + \hat{X}_2^{(1)} J^{(0)^T}\right) F^{(0)} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} F^{(0)} H^{(0)} + \lambda^{(0)} \hat{X}_1^{(1)} = 0 \\ \\ \left(\hat{X}_1^{(1)} H^{(0)^T} + \hat{X}_2^{(1)} J^{(0)^T}\right) F^{(0)} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} F^{(0)} J^{(0)} + \lambda^{(0)} \hat{X}_2^{(1)} = 0 \end{cases} \tag{3.19}$$

This is an eigenvalue problem for the eigenvectors $\hat{X}_1^{(1)}, \hat{X}_2^{(1)}$ and the eigenvalue $\lambda^{(0)}$. Some information about the eigenvectors is readily obtained by postmultiplying the first equation by $H^{(0)^T}$, the second by $J^{(0)^T}$, and subtracting since these operations, if $\lambda^{(0)} \neq 0$, result in

$$\hat{X}_1^{(1)} H^{(0)^T} \left(H^{(0)} H^{(0)^T}\right)^{-1} - \hat{X}_2^{(1)} J^{(0)^T} \left(J^{(0)} J^{(0)^T}\right)^{-1} = 0. \tag{3.20}$$

This cannot be used by itself to solve for $\hat{X}_1^{(1)}$ in terms of $\hat{X}_2^{(1)}$ or vice versa if $n > 3$ because $H^{(0)}$ and $J^{(0)}$ are non-square and thus have no inverses.

The two matrix equations (3.19) represent $2n + 2$ equations in the unknowns $\hat{r}_k^{(1)}$, $\hat{f}_k^{(1)}$ and so considerable complexity in the spectrum could be expected. Fortunately, this is not the case as only two of the eigenvalues can be different from zero. This circumstance arises essentially from the fact that $\partial X_k/\partial b$ is a non-square matrix. Let $X \equiv (\hat{X}_1^{(1)}, \hat{X}_2^{(1)})$, i.e. a row matrix with $2n + 2$ elements, and let L be the $2n + 2 \times 4$ matrix

$$\begin{pmatrix} H^{(0)} \\ J^{(0)} \end{pmatrix}.$$

Then the matrix $A \equiv L F^{(0)} \partial X_k/\partial b$ is a $2n+2 \times 2$ matrix and the eigenvalue equation becomes simply

$$X A A^T + \lambda^{(0)} X = 0.$$

The two columns of A may be regarded as the components of two vectors $\vec{p}$ and $\vec{q}$ in a vector space of $2n + 2$ dimensions. The vectors orthogonal to both $\vec{p}$ and $\vec{q}$ fill out a $2n$-dimensional vector space. Let X_0 be any non-zero vector in this $2n$-dimensional space. Then $X_0 AA^T = 0A^T = 0$ and so every vector in this $2n$-dimensional subspace is annihilated by AA^T and is therefore an eigenvector of this matrix with eigenvalue $\lambda^{(0)} = 0$. Since the $2n$-dimensional space is spanned by $2n$ independent vectors the multiplicity of the zero eigenvalue is $2n$. In the unlikely event that $\vec{p}$ and $\vec{q}$ are not independent but are non-zero the vectors orthogonal to both $\vec{p}$ and $\vec{q}$ will fill out a space of $2n+1$ dimensions and the multiplicity of the zero eigenvalue in this case will be $2n+1$.

Thus, in general, there are only two non-zero eigenvalues. The search for these two eigenvalues is considerably simplified by reducing the eigenvalue equation in $\hat{X}_1^{(1)}$, $\hat{X}_2^{(1)}$ to one in $\beta^{(1)}$. This is accomplished by postmultiplying the first equation in (3.19) by $H^{(0)^T}$, the second by $J^{(0)^T}$, and then adding and using (3.18) and (3.16). This eigenvalue equation for $\beta^{(1)}$,

$$\left(\frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} + \lambda^{(0)} G^{(0)}\right) \beta^{(1)} = 0, \tag{3.21}$$

can be analyzed in the manner above to obtain the result that in this fourth order system two of the eigenvalues are, in general, zero. The zero eigenvalues correspond to the eigenvectors $\beta^{(1)}$ that are orthogonal to both columns of $\partial X_k/\partial b$, i.e., $\partial r_k^T/\partial b\ \beta^{(1)} = r_k\ \partial f_k^T/\partial b\ \beta^{(1)} = 0$. But this implies by (3.8) that $\rho_k^{(1)} = r_k\ \varphi_k^{(1)} = 0$ which in turn implies, as we shall see later, that at least to the first and second order $W^2 = 0$, indicating a minimum rather than a maximum.

The eigenvalues are, of course, the roots of the determinantal equation

$$\left| \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} + \lambda^{(0)} \frac{\partial X}{\partial b} \frac{\partial X^T}{\partial b} \right| = 0. \tag{3.22}$$

The eigenvalues are all real since $\partial X_k/\partial b\ \partial X_k^T/\partial b$ and $\partial X/\partial b\ \partial X^T/\partial b$ are both symmetric with real elements [ref. 1, p. 306].

We now show that the original eigenvectors, $\hat{X}_1^{(1)}$, $\hat{X}_2^{(1)}$, have a very simple relationship to the $\rho_m^{(1)}$, $\varphi_m^{(1)}$. If (3.20) is used in (3.18) there is obtained

$$\begin{aligned} \beta^{(1)^T} &= [\hat{X}_1^{(1)} H^{(0)^T} + \hat{X}_1^{(1)} H^{(0)^T} (H^{(0)} H^{(0)^T})^{-1} (J^{(0)} J^{(0)^T})]\ F^{(0)} \\ &= \hat{X}_1^{(1)} H^{(0)^T} \left(H^{(0)} H^{(0)^T}\right)^{-1} [H^{(0)} H^{(0)^T} + J^{(0)} J^{(0)^T}]\ F^{(0)} \\ &= \hat{X}_1^{(1)} H^{(0)^T} \left(H^{(0)} H^{(0)^T}\right)^{-1} G^{(0)} G^{(0)^{-1}} \\ &= \hat{X}_1^{(1)} H^{(0)^T} \left(H^{(0)} H^{(0)^T}\right)^{-1}. \end{aligned} \tag{3.23}$$

Similarly,

$$\beta^{(1)^T} = \hat{X}_2^{(1)} J^{(0)^T} \left(J^{(0)} J^{(0)^T}\right)^{-1}. \tag{3.24}$$

Hence,

$$\begin{cases} \hat{X}_1^{(1)} H^{(0)^T} = \beta^{(1)^T} H^{(0)} H^{(0)^T}, \\ \hat{X}_2^{(1)} J^{(0)^T} = \beta^{(1)^T} J^{(0)} J^{(0)^T} \end{cases} \tag{3.25}$$

We cannot conclude from this alone that $\hat{X}_1^{(1)} = \beta^{(1)^T} H^{(0)}$ and $\hat{X}_2^{(1)} = \beta^{(1)^T} J^{(0)}$ since $H^{(0)}$ and $J^{(0)}$, being non-square for $n > 3$, have no inverses. However, these formulae are indeed true for eigenvectors corresponding to non-zero eigenvalues. We can conclude from (3.25) that $\hat{X}_1^{(1)}$ is $\beta^{(1)^T} H^{(0)}$ except possibly for an additive zero divisor of $H^{(0)^T}$. That is, we can write

$$\begin{cases} \hat{X}_1^{(1)} = \beta^{(1)^T} H^{(0)} + Y_1, \\ \hat{X}_2^{(1)} = \beta^{(1)^T} J^{(0)} + Y_2, \end{cases} \quad \text{where} \quad \begin{cases} Y_1 \hat{H}^{0^T} = 0, \\ Y_2 \hat{J}^{0^T} = 0. \end{cases}$$

Substitute these into the eigenvalue equations (3.19) to give

$$\begin{cases} \beta^{(1)^T} \left(\dfrac{\partial X_k}{\partial b} \dfrac{\partial X_k^T}{\partial b} F^{(0)} + \lambda^{(0)} I \right) H^{(0)} + \lambda^{(0)} Y_1 = 0, \\ \\ \beta^{(1)^T} \left(\dfrac{\partial X_k}{\partial b} \dfrac{\partial X_k^T}{\partial b} F^{(0)} + \lambda^{(0)} I \right) J^{(0)} + \lambda^{(0)} Y_2 = 0. \end{cases}$$

But if $\hat{X}_1^{(1)}, \hat{X}_2^{(1)}$ are eigenvectors then by (3.21)

$$\beta^{(1)^T} \left(\frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} F^{(0)} + \lambda^{(0)} I \right) = 0.$$

Therefore, if $\lambda^{(0)} \neq 0$, we must have $Y_1 = Y_2 = 0$ and, consequently,

$$\begin{cases} \hat{X}_1^{(1)} = \beta^{(1)^T} H^{(0)}, \\ \hat{X}_2^{(1)} = \beta^{(1)^T} J^{(0)}. \end{cases} \tag{3.26}$$

This joins up with (3.8) to give the result that if $\lambda^{(0)} \neq 0$,

$$\begin{cases} \hat{r}_k^{(1)} = \rho_k^{(1)}, \\ \hat{f}_k^{(1)} = \varphi_k^{(1)}, \end{cases} \tag{3.27}$$

that is to say, for a maximum to occur the observed positions must coincide with the least squares computed positions and therefore must fall along a Keplerian ellipse (though not, in general, the true orbit). This property, to be sure, is a consequence of the first order theory. As we shall discover later, it does not carry over to the second order theory.

Nothing has been said so far about the normalization of the eigenvectors. The size of the eigenvectors is regulated by the side condition $Q^2 = c^2$. Before we can determine its effect we must decide on the variation of c with $\mathcal{E}$. We choose

$$\mathcal{E} = \frac{c}{a\sqrt{n+1}}. \tag{3.28}$$

Since the lead off term in the expansion of Q^2 is

$$\sum_{k=0}^{n} \left(\hat{r}_k^{(1)^2} + r_k^2\, \hat{f}_k^{(1)^2}\right) \mathcal{E}^2,$$

the eigenvectors must satisfy

$$\hat{X}_1^{(1)}\, \hat{X}_1^{(1)^T} + \hat{X}_2^{(1)}\hat{X}_2^{(1)^T} = a^2\,(n+1). \tag{3.29}$$

The vector $\beta^{(1)}$is governed accordingly. Working with (3.26) we find

$$\begin{aligned}
\hat{X}_1^{(1)}\, \hat{X}_1^{(1)^T} + \hat{X}_2^{(1)}\, \hat{X}_2^{(1)^T} &= \beta^{(1)^T} H^{(0)} \left(\beta^{(1)^T} H^{(0)}\right)^T + \beta^{(1)^T} J^{(0)} \left(\beta^{(1)^T} J^{(0)}\right)^T \\
&= \beta^{(1)^T} H^{(0)}\, H^{(0)^T} \beta^{(1)} + \beta^{(1)^T} J^{(0)}\, J^{(0)^T} \beta^{(1)} \\
&= \beta^{(1)^T} G^{(0)}\, \beta^{(1)}.
\end{aligned}$$

Consequently, $\beta^{(1)}$ is normalized by

$$\beta^{(1)^T} \frac{\partial X}{\partial b} \frac{\partial X^T}{\partial b} \beta^{(1)} = a^2\,(n+1). \tag{3.30}$$

The maximum squared distance between the k + 1 st true point and the k + 1 st least squares computed point is

$$\begin{aligned}
W^2 &\equiv r_k^2 + \rho_k^2 - 2r_k\,\rho_k \cos\,(f_k - \varphi_k) \\
&= \left(\rho_k^{(1)^2} + r_k^2\,\varphi_k^{(1)^2}\right) \mathcal{E}^2 + \ldots \\
&= \left[\left(\beta^{(1)^T} H_k^{(0)}\right)\left(\beta^{(1)^T} H_k^{(0)}\right)^T + \left(\beta^{(1)^T} J_k^{(0)}\right)\left(\beta^{(1)^T} J_k^{(0)}\right)^T\right] \mathcal{E}^2 + \ldots
\end{aligned}$$

$$= \beta^{(1)^T} \left(H_k^{(0)} H_k^{(0)^T} + J^{(0)} J_k^{(0)^T} \right) \beta^{(1)} \varepsilon^2 + \ldots$$

$$= \beta^{(1)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k}{\partial b} \beta^{(1)} \varepsilon^2 + \ldots . \tag{3.31}$$

This can be expressed in terms of the eigenvalue $\lambda^{(0)}$ by seeing from (3.21) that

$$\beta^{(1)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} \beta^{(1)} = - \lambda^{(0)} \beta^{(1)^T} G^{(0)} \beta^{(1)}$$

and this in turn is $-\lambda^{(0)} a^2 (n+1)$ by (3.30). Hence, an alternative form for W^2 is

$$W^2 = - \lambda^{(0)} c^2 + \ldots . \tag{3.32}$$

4. THE SECOND ORDER THEORY

Pushing the analysis to the second order consists essentially in treating the set of equations (2.17). The following matrices will be needed in the work:

$$\hat{X}_1^{(2)} \equiv (\hat{r}_0^{(2)}, \ldots , \hat{r}_n^{(2)})$$

$$\hat{X}_2^{(2)} \equiv (r_0 \hat{f}_0^{(2)}, \ldots , r_n \hat{f}_n^{(2)})$$

$$\hat{V}^{(1)} \equiv \begin{pmatrix} \hat{f}_0^{(1)} & & & \\ & \hat{f}_1^{(1)} & & 0 \\ & & \ddots & \\ 0 & & & \hat{f}_n^{(1)} \end{pmatrix}, \qquad \hat{R}^{(1)} \equiv \begin{pmatrix} \hat{r}_0^{(1)} & & & \\ & \hat{r}_1^{(1)} & & 0 \\ & & \ddots & \\ 0 & & & \hat{r}_n^{(1)} \end{pmatrix}, \tag{4.1}$$

$$\left[\frac{\partial \Xi}{\partial \beta}\right]^{(1)} \equiv \begin{pmatrix} \left[\frac{\partial \rho_0}{\partial \beta_1}\right]^{(1)} & \left[r_0 \frac{\partial \varphi_0}{\partial \beta_1}\right]^{(1)} & \left[\frac{\partial \rho_1}{\partial \beta_1}\right]^{(1)} & \left[r_1 \frac{\partial \varphi_1}{\partial \beta_1}\right]^{(1)} & \cdots & \left[\frac{\partial \rho_n}{\partial \beta_1}\right]^{(1)} & \left[r_n \frac{\partial \varphi_n}{\partial \beta_1}\right]^{(1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \left[\frac{\partial \rho_0}{\partial \beta_4}\right]^{(1)} & \left[r_0 \frac{\partial \varphi_0}{\partial \beta_4}\right]^{(1)} & \left[\frac{\partial \rho_1}{\partial \beta_4}\right]^{(1)} & \left[r_1 \frac{\partial \varphi_1}{\partial \beta_4}\right]^{(1)} & \cdots & \left[\frac{\partial \rho_n}{\partial \beta_4}\right]^{(1)} & \left[r_n \frac{\partial \varphi_n}{\partial \beta_4}\right]^{(1)} \end{pmatrix}$$

$$\left[\frac{\partial \Xi_k}{\partial \beta}\right]^{(1)} \equiv \begin{pmatrix} \left[\frac{\partial \rho_k}{\partial \beta_1}\right]^{(1)} & \left[r_k \frac{\partial \varphi_k}{\partial \beta_1}\right]^{(1)} \\ \vdots & \vdots \\ \left[\frac{\partial \rho_k}{\partial \beta_4}\right]^{(1)} & \left[r_k \frac{\partial \varphi_k}{\partial \beta_4}\right]^{(1)} \end{pmatrix}$$

$$\left[\frac{\partial \rho}{\partial \beta}\right]^{(1)} \equiv \begin{pmatrix} \left[\frac{\partial \rho_0}{\partial \beta_1}\right]^{(1)} & \cdots & \left[\frac{\partial \rho_n}{\partial \beta_1}\right]^{(1)} \\ \vdots & & \vdots \\ \left[\frac{\partial \rho_0}{\partial \beta_4}\right]^{(1)} & \cdots & \left[\frac{\partial \rho_n}{\partial \beta_4}\right]^{(1)} \end{pmatrix}, \qquad \left[\frac{\partial \rho_k}{\partial \beta}\right]^{(1)} \equiv \begin{pmatrix} \left[\frac{\partial \rho_k}{\partial \beta_1}\right]^{(1)} \\ \vdots \\ \left[\frac{\partial \rho_k}{\partial \beta_4}\right]^{(1)} \end{pmatrix},$$

$$\left[\frac{\partial \varphi}{\partial \beta}\right]^{(1)} \equiv \begin{pmatrix} \left[\frac{\partial \varphi_0}{\partial \beta_1}\right]^{(1)} & \cdots & \left[\frac{\partial \varphi_n}{\partial \beta_1}\right]^{(1)} \\ \vdots & & \vdots \\ \left[\frac{\partial \varphi_0}{\partial \beta_4}\right]^{(1)} & \cdots & \left[\frac{\partial \varphi_n}{\partial \beta_4}\right]^{(1)} \end{pmatrix}, \qquad \left[\frac{\partial \varphi_k}{\partial \beta}\right]^{(1)} \equiv \begin{pmatrix} \left[\frac{\partial \varphi_k}{\partial \beta_1}\right]^{(1)} \\ \vdots \\ \left[\frac{\partial \varphi_k}{\partial \beta_4}\right]^{(1)} \end{pmatrix}.$$

Then

$$\begin{cases} \tilde{\Theta}^{(2)} = \hat{X}_1^{(2)} + \frac{1}{2} \hat{X}_2^{(1)} \hat{V}^{(1)} \\ \tilde{\Phi}^{(2)} R^{-1} = \hat{X}_2^{(2)} + \hat{X}_1^{(1)} \hat{V}^{(1)} \end{cases} \tag{4.2}$$

$$\begin{cases} H^{(1)} = \left[\frac{\partial \rho}{\partial \beta} \right]^{(1)} \\ K^{(1)} \equiv J^{(1)} R = \left(\left[\frac{\partial \varphi}{\partial \beta} \right]^{(1)} R + 2 \frac{\partial f}{\partial b} \hat{R}^{(1)} \right) R \end{cases}$$

$$\begin{cases} G^{(1)} = H^{(1)} H^{(0)^T} + J^{(1)} J^{(0)^T} + \frac{\partial X}{\partial b} \left[\frac{\partial \Xi}{\partial \beta} \right]^{(1)^T} \\ P_k^{(2)} = \Lambda_k^{(1)} + \rho_k^{(2)} H_k^{(0)^T} + r_k \varphi_k^{(2)} J_k^{(0)^T} \\ \Lambda_k^{(1)} \equiv \rho_k^{(1)} \left[\frac{\partial \rho_k}{\partial \beta} \right]^{(1)^T} + r_k^2 \varphi_k^{(1)} \left[\frac{\partial \varphi_k}{\partial \beta} \right]^{(1)^T} + \frac{1}{2} r_k \varphi_k^{(1)^2} \frac{\partial r_k^T}{\partial b} + r_k \rho_k^{(1)} \varphi_k^{(1)} \frac{\partial f_k^T}{\partial b} . \end{cases} \tag{4.3}$$

Also, from (3.17) and (3.21) we get

$$P_k^{(1)} = - \lambda^{(0)} \beta^{(1)^T} G^{(0)} . \tag{4.4}$$

The fundamental second order equations (2.17) thus have the form

$$\begin{cases} - \lambda^{(0)} \beta^{(1)^T} H^{(1)} + \lambda^{(0)} \beta^{(1)^T} G^{(1)} F^{(0)} H^{(0)} + \Lambda_k^{(1)} F^{(0)} H^{(0)} + (H_k^{(0)^T} \rho_k^{(2)} + J_k^{(0)^T} r_k \varphi_k^{(2)}) F^{(0)} H^{(0)} \\ + \lambda^{(0)} \left(\hat{X}_1^{(2)} + \frac{1}{2} \hat{X}_2^{(1)} \hat{V}^{(1)} \right) + \lambda^{(1)} \beta^{(1)^T} H^{(0)} = 0 \\ - \lambda^{(0)} \beta^{(1)^T} J^{(1)} + \lambda^{(0)} \beta^{(1)^T} G^{(1)} F^{(0)} J^{(0)} + \Lambda_k^{(1)} F^{(0)} J^{(0)} + (H_k^{(0)^T} \rho_k^{(2)} + J_k^{(0)^T} r_k \varphi_k^{(2)}) F^{(0)} J^{(0)} \\ + \lambda^{(0)} (\hat{X}_2^{(2)} + \hat{X}_1^{(1)} \hat{V}^{(1)}) + \lambda^{(1)} \beta^{(1)^T} J^{(0)} = 0 \end{cases} \tag{4.5}$$

These equations involve two sets of second order unknowns, $\rho^{(2)}, \varphi^{(2)}$ and $\hat{X}_1^{(2)}$, $\hat{X}_2^{(2)}$, besides the eigenvalue perturbation $\lambda^{(1)}$. In order to obtain the necessary further relations among these we note first that

$$\rho_k^{(2)} = \frac{1}{2}\left[\frac{d^2\rho_k}{d\mathcal{E}^2}\right]_{\mathcal{E}=0} = \frac{1}{2}\left[\frac{d}{d\mathcal{E}}\sum_{i=1}^{4}\frac{\partial\rho_k}{\partial\beta_i}\frac{d\beta_i}{d\mathcal{E}}\right]_{\mathcal{E}=0}$$

$$= \sum_{i=1}^{4}\frac{\partial r_k}{\partial b_i}\beta_i^{(2)} + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial^2 r_k}{\partial b_i\,\partial b_j}\beta_i^{(1)}\beta_j^{(1)} ,$$

with a similar formula for $\varphi_k^{(2)}$, and that

$$\frac{\partial\rho_k}{\partial\beta_i} = \frac{\partial r_k}{\partial b_i} + \sum_{j=1}^{4}\frac{\partial^2 r_k}{\partial b_j\,\partial b_i}\beta_j^{(1)}\,\mathcal{E} + \cdots ,$$

so that

$$\sum_{j=1}^{4}\frac{\partial^2 r_k}{\partial b_j\,\partial b_i}\beta_j^{(1)} = \left[\frac{\partial\rho_k}{\partial\beta_i}\right]^{(1)} .$$

The connection between $\rho_k^{(2)}$, $\varphi_k^{(2)}$ and the orbital elements is thereupon seen to be, in matrix form,

$$\left\{\begin{aligned} \rho_k^{(2)} &= \beta^{(2)^T} H_k^{(0)} + \frac{1}{2}\beta^{(1)^T}\left[\frac{\partial\rho_k}{\partial\beta}\right]^{(1)} \\ r_k\,\varphi_k^{(2)} &= \beta^{(2)^T} J_k^{(0)} + \frac{1}{2}\beta^{(1)^T} r_k\left[\frac{\partial\varphi_k}{\partial\beta}\right]^{(1)} . \end{aligned}\right. \qquad (4.6)$$

To complete the connection of $\rho_k^{(2)}, \varphi_k^{(2)}$ with $\hat{X}_1^{(2)}$, $\hat{X}_2^{(2)}$ we need now the relation between $\beta^{(2)}$ and $\hat{X}_1^{(2)}, \hat{X}_2^{(2)}$. This relation resides in the least squares equation of condition (1.5) which, when expanded to the second order, gives

$$\left\{\begin{aligned} &\beta^{(2)^T} G^{(0)} = \hat{X}_1^{(2)} H^{(0)^T} + \hat{X}_2^{(2)} J^{(0)^T} + \Omega, \\ &\Omega \equiv -\frac{1}{2}\beta^{(1)^T}\left[\frac{\partial\Xi}{\partial\beta}\right]^{(1)}\frac{\partial X^T}{\partial b} , \end{aligned}\right. \qquad (4.7)$$

the counterpart to (3.18). We note further from (4.6) that

$$
\left\{
\begin{aligned}
&\rho_k^{(2)} H_k^{(0)^T} + r_k\, \varphi_k^{(2)} J_k^{(0)^T} = \beta^{(2)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} - \Omega_k \\
&\Omega_k \equiv -\frac{1}{2} \beta^{(1)^T} \left[\frac{\partial \Xi_k}{\partial \beta}\right]^{(1)} \frac{\partial X_k}{\partial b} .
\end{aligned}
\right.
\tag{4.8}
$$

This enables us to remove $\rho_k^{(2)}, \varphi_k^{(2)}$ from (4.5) in favor of $\beta^{(2)}$, thereby giving

$$
\left\{
\begin{aligned}
&\beta^{(2)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} F^{(0)} H^{(0)} + \lambda^{(0)} \hat{X}_1^{(2)} \\
&= \lambda^{(0)} \beta^{(1)^T} H^{(1)} - \lambda^{(0)} \beta^{(1)^T} G^{(1)} F^{(0)} H^{(0)} - \Lambda_k^{(1)} F^{(0)} H^{(0)} + \Omega_k F^{(0)} H^{(0)} \\
&- \lambda^{(0)} \frac{1}{2} \hat{X}_2^{(1)} \hat{V}^{(1)} - \lambda^{(1)} \beta^{(1)^T} H^{(0)} \equiv \Pi_1 \\
&\beta^{(2)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} F^{(0)} J^{(0)} + \lambda^{(0)} \hat{X}_2^{(2)} \\
&= \lambda^{(0)} \beta^{(1)^T} J^{(1)} - \lambda^{(0)} \beta^{(1)^T} G^{(1)} F^{(0)} J^{(0)} - \Lambda_k^{(1)} F^{(0)} J^{(0)} + \Omega_k F^{(0)} J^{(0)} \\
&- \lambda^{(0)} \hat{X}_1^{(1)} \hat{V}^{(1)} - \lambda^{(1)} \beta^{(1)^T} J^{(0)} \equiv \Pi_2
\end{aligned}
\right.
\tag{4.9}
$$

The unknowns $\hat{X}_1^{(2)}, \hat{X}_2^{(2)}$, can be eliminated from these equations to leave an equation in $\beta^{(2)}$ alone by postmultiplying these two equations by $H^{(0)^T}$ and $J^{(0)^T}$, respectively, adding, and using (4.7). The resulting equation is

$$
\beta^{(2)^T} \left(\frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} + \lambda^{(0)} G^{(0)} \right) = \Pi_1 H^{(0)^T} + \Pi_2 J^{(0)^T} + \lambda^{(0)} \Omega , \tag{4.10}
$$

the counterpart to (3.21). Since $\lambda^{(0)}$ has already been chosen so as to make the determinant of the coefficient of $\beta^{(2)^T}$ vanish, the number $\lambda^{(1)}$, which is present in Π_1 and Π_2, must be chosen so as to make the rank of the augmented matrix equal to that of the coefficient matrix. After this has been done, (4.10) can be solved for $\beta^{(2)}$. If, on the other hand, $\beta^{(2)}$ is removed from (4.9) by the use of (4.7) there are displayed equations from which $\hat{X}_1^{(2)}$, $\hat{X}_2^{(2)}$ can be obtained.

A simple relation exists between $\hat{X}_1^{(2)}$ and $\hat{X}_2^{(2)}$, a relation derivable by multiplying the first equation of (4.9) by $H^{(0)T}\ (H^{(0)}\ H^{(0)T})^{-1}$, the second by $J^{(0)T}\ (J^{(0)}\ J^{(0)T})^{-1}$, and subtracting. It is

$$\hat{X}_1^{(2)}\, H^{(0)^T}\, (H^{(0)}\, H^{(0)^T})^{-1} - \hat{X}_2^{(2)}\, J^{(0)^T}\, (J^{(0)}\, J^{(0)^T})^{-1}$$

$$= \beta^{(1)^T}\, [(H^{(1)} - \frac{1}{2} J^{(0)}\, \hat{V}^{(1)})\, H^{(0)^T}\, (H^{(0)}\, H^{(0)^T})^{-1} - (J^{(1)} - H^{(0)}\, V^{(1)})\, J^{(0)^T}\, (J^{(0)}\, J^{(0)^T})^{-1}]\,. \tag{4.11}$$

This is evidently the counterpart of (3.20).

The coefficient of ε^3 in the expansion representing the constraining equation (2.2) is

$$2 \sum_{k=0}^{n} (\hat{r}_k^{(1)}\, \hat{r}_k^{(2)} + r_k^2\, \hat{v}_k^{(1)}\, \hat{v}_k^{(2)} + \frac{1}{2}\, r_k\, \hat{r}_k^{(1)}\, \hat{v}_k^{(1)^2})$$

so that we must have

$$\hat{X}_1^{(1)}\, \hat{X}_1^{(2)^T} + \hat{X}_2^{(1)}\, \hat{X}_2^{(2)^T} + \frac{1}{2} \hat{X}_1^{(1)}\, (\hat{X}_2^{(1)}\, \hat{V}^{(1)})^T = 0\,. \tag{4.12}$$

which is for the second order theory what (3.29) is for the first. The corresponding equation for $\beta^{(2)}$ results from the application of (3.26) and (4.7) to (4.12). We find that

$$\beta^{(2)^T} \frac{\partial X}{\partial b} \frac{\partial X^T}{\partial b} \beta^{(1)} + \frac{1}{2} \beta^{(1)^T} \frac{\partial X}{\partial b} \left[\frac{\partial \Xi}{\partial \beta}\right]^{(1)} \beta^{(1)} + \frac{1}{2} \beta^{(1)^T}\, J^{(0)}\, \hat{V}^{(1)}\, H^{(0)^T}\, \beta^{(1)} = 0\,, \tag{4.13}$$

the counterpart to (3.30).

A similar computation leads to the expression for W^2,

$$W^2 = (\rho_k^{(1)^2} + r_k^2\, \varphi_k^{(1)^2})\, \mathcal{E}^2 + (2\rho_k^{(1)}\, \rho_k^{(2)} + 2\, r_k^2\, \varphi_k^{(1)}\, \varphi_k^{(2)} + r_k\, \rho_k^{(1)}\, \varphi_k^{(1)^2})\, \mathcal{E}^3 + \cdots$$

$$= \beta^{(1)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} \beta^{(1)}\, \mathcal{E}^2$$

$$+ \left\{ 2\beta^{(2)^T} \frac{\partial X_k}{\partial b} \frac{\partial X_k^T}{\partial b} \beta^{(1)} + \beta^{(1)^T} \frac{\partial X_k}{\partial b} \left[\frac{\partial \Xi_k}{\partial \beta}\right]^{(1)^T} \beta^{(1)} \right.$$

$$\left. + \beta^{(1)^T} J_k^{(0)}\, \hat{f}_k^{(1)}\, H_k^{(0)^T}\, \beta^{(1)} \right\} \mathcal{E}^3 + \cdots. \tag{4.14}$$

Note that the coefficient of $\mathcal{E}^3$ (as well as that of $\mathcal{E}^2$) vanishes if $\beta^{(1)^T}\, \partial X_k/\partial b = 0$ which occurs for $\lambda^{(0)} = 0$. Thus the vanishing of the eigenvalue causes the vanishing of W^2 through terms in $\mathcal{E}^3$ at least.

5. KEPLERIAN FORMULAE

The work of the last two sections is independent of the type of orbit involved. All that is required is that the orbit depend in a well-behaved fashion on four parameters (the extension to any number of parameters is immediate). In the case of two-dimensional Newtonian two-body motion the parameter dependence is given by (1.2). From these the fundamental quantities appearing in the formulae of the preceding sections are easily computed. The column matrices $H_k^{(0)}$ and $J_k^{(0)}$ are

$$H_k^{(0)} = \begin{pmatrix} a^{-1}\, r_k - \frac{3}{2} a\, r_k^{-1}\, e\, s_k \sin g_k \\ - a \cos (f_k - p) \\ 0 \\ - \sqrt{\mu}\, a^{1/2}\, r_k^{-1}\, e \sin g_k \end{pmatrix}, \tag{5.1}$$

$$J_k^{(0)} = \begin{pmatrix} -\frac{3}{2} a\, r_k^{-1} \sqrt{1-e^2}\, s_k \\ [a + r_k (1-e^2)^{-1}] \sin (f_k - p) \\ r_k \\ -\sqrt{\mu}\, a^{1/2}\, r_k^{-1} \sqrt{1-e^2} \end{pmatrix} \tag{5.2}$$

The matrices $H^{(0)}$ and $J^{(0)}$ are $4 \times n + 1$ and have the elements of $H_k^{(0)}$ and $J_k^{(0)}$, respectively, as their $k + 1$ columns.

Let $G_{ij}^{(0)}$ represent the element in the i^{th} row and j^{th} column of the 4×4 matrix $G^{(0)}$. Then

$$G_{11}^{(0)} = \sum_{k=0}^{n} \left[a^{-2} r_k^2 - 3\, e\, s_k \sin g_k + \frac{9}{4} a\, r_k^{-1} (2 - a^{-1} r_k)\, s_k^2 \right],$$

$$G_{12}^{(0)} = \sum_{k=0}^{n} [-3\, a^2\, r_k^{-1}\, s_k \sin g_k - r_k \cos (f_k - p)],$$

$$G_{13}^{(0)} = \sum_{k=0}^{n} \left[-\frac{3}{2} a \sqrt{1-e^2}\; s_k \right],$$

$$G_{14}^{(0)} = \sum_{k=0}^{n} \left[-\sqrt{\mu}\, a^{-1/2}\, e \sin g_k + \frac{3}{2} \sqrt{\mu}\, a^{1/2}\, r_k^{-1} (2 - a^{-1} r_k)\, s_k \right],$$

$$G_{22}^{(0)} = \sum_{k=0}^{n} [a^2 + a^2 \{2 a r_k^{-1} + (1-e^2)^{-1}\} \sin^2 g_k],$$

$$G_{23}^{(0)} = \sum_{k=0}^{n} [a r_k + r_k^2 (1-e^2)^{-1}] \sin (f_k - p),$$

$$G_{24}^{(0)} = \sum_{k=0}^{n} [-2\sqrt{\mu}\, a^{3/2}\, r_k^{-1} \sin g_k],$$

$$G_{33}^{(0)} = \sum_{k=0}^{n} r_k^2,$$

$$G_{34}^{(0)} = \sum_{k=0}^{n} [-\sqrt{\mu}\, a^{1/2} \sqrt{1 - e^2}],$$

$$G_{44}^{(0)} = \sum_{k=0}^{n} [\mu a^{-1}\, r_k^{-1} (2a - r_k)]. \qquad (5.3)$$

This determines all the elements of $G^{(0)}$ since $G^{(0)}$ is symmetric.

If we suppose that the observations are made at equal intervals in time then the summations with respect to k that are present in the formulae for $G_{ij}^{(0)}$ can be replaced by summations of a different sort. This is accomplished by employing the Fourier series expansions in s for the various orbital quantities, since then the original summation index k will occur only on the variable s and will appear only as a simple factor. The sum over k is then accomplished through the use of the following series which represent the complex geometric series and its first two derivatives:

$$\begin{cases} \sum_{k=0}^{n} 1 = n + 1 \\ \sum_{k=0}^{n} k = \frac{1}{2} n(n+1) \\ \sum_{k=0}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1) \end{cases}$$

$$
\left\{
\begin{aligned}
&\sum_{k=0}^{n} \cos pk\theta = \csc\frac{1}{2}p\theta \,\sin\frac{n+1}{2}p\theta \,\cos\frac{n}{2}p\theta \\
&\sum_{k=0}^{n} k\cos pk\theta = n\csc\frac{1}{2}p\theta \,\sin\frac{n+1}{2}p\theta \,\cos\frac{n}{2}p\theta \\
&\qquad\qquad -\frac{1}{4}\csc^2\frac{1}{2}p\theta \,[(n+1) - n\cos p\theta - \cos np\theta] \\
&\sum_{k=0}^{n} k^2\cos pk\theta = n^2\csc\frac{1}{2}p\theta \,\sin\frac{n+1}{2}p\theta \,\cos\frac{n}{2}p\theta \\
&\qquad\qquad -\frac{n}{4}\csc^2\frac{1}{2}p\theta \,[(n+1) - n\cos p\theta - \cos np\theta] \\
&\qquad\qquad +\frac{1}{4}\csc^3\frac{1}{2}p\theta \left[n\sin\frac{1}{2}p\theta\cdot(1+\cos np\theta) - \sin np\theta\cos\frac{1}{2}p\theta\right]
\end{aligned}
\right. \tag{5.4}
$$

$$
\left\{
\begin{aligned}
&\sum_{k=0}^{n} \sin pk\theta = \csc\frac{1}{2}p\theta \,\sin\frac{n+1}{2}p\theta \,\sin\frac{n}{2}p\theta \\
&\sum_{k=0}^{n} k\sin pk\theta = n\csc\frac{1}{2}p\theta \,\sin\frac{n+1}{2}p\theta \,\sin\frac{n}{2}p\theta \\
&\qquad\qquad -\frac{1}{4}\csc^2\frac{1}{2}p\theta \,[n\sin p\theta - \sin np\theta] \\
&\sum_{k=0}^{n} k^2\sin pk\theta = n^2\csc\frac{1}{2}p\theta \,\sin\frac{n+1}{2}p\theta \,\sin\frac{n}{2}p\theta \\
&\qquad\qquad -\frac{n}{4}\csc^2\frac{1}{2}p\theta \,[n\sin p\theta - \sin np\theta] \\
&\qquad\qquad +\frac{1}{4}\csc^3\frac{1}{2}p\theta \left[n\sin\frac{1}{2}p\theta\sin np\theta + \cos\frac{1}{2}p\theta\cdot(\cos np\theta - 1)\right].
\end{aligned}
\right.
$$

Let X_k be one of the quantities connected with the orbit. Its Fourier expansion in terms of the mean anomaly s is of the form

$$X_k = A + \sum_{p=1}^{\infty} (a_p \cos ps_k + b_p \sin ps_k) \tag{5.5}$$

where s_k is the value of s at the k +1st instant: $s_k = \sqrt{\mu}\, a^{3/2} (t_k - z)$. The constants A, a_p, b_p are all independent of k. We are assuming the observations are made at equal intervals in time and so we write

$$s_k = k\theta - \Theta \tag{5.6}$$

where $\theta \equiv \sqrt{\mu}\, a^{-3/2}\tau$, $\Theta \equiv \sqrt{\mu} a^{-3/2} z$, and τ is the interval (in time) between successive observations. Then the Fourier expansion of X_k becomes

$$X_k = A + \sum_{p=1}^{\infty} \Big\{ [a_p \cos p\Theta - b_p \sin p\Theta] \cos pk\theta$$

$$+ [a_p \sin p\Theta + b_p \cos p\Theta] \sin pk\theta \Big\}. \tag{5.7}$$

For the elements of $G^{(0)}$ we require the sum of X_k over k together with its first two moments. In forming this finite sum we add the Fourier series term by term, a process justified bv the fact that the Fourier series converge for all values of s. [ref. 2, p. 210]. The result is, upon using (5.4),

$$\sum_{k=0}^{n} X_k = (n+1)A + \sum_{p=1}^{\infty} \csc\frac{1}{2}p\theta \sin\frac{n+1}{2}p\theta\, (a_p \cos ps_{n/2} + b_p \sin ps_{n/2})$$

where $s_{n/2}$ is the value of the mean anomaly at the instant $t=(n/2)\tau$. In a similar fashion we can compute the first two moments. The formulae are

$$
\left\{
\begin{aligned}
\sum_{k=0}^{n} X_k &= (n+1)\,A + \sum_{p=1}^{\infty} D_p^{(0)}\,(a_p \cos ps_{n/2} + b_p \sin ps_{n/2}),\\
\sum_{k=0}^{n} s_k X_k &= s_{n/2} \sum_{k=0}^{n} X_k + \theta \sum_{p=1}^{\infty} D_p^{(1)}\,(a_p \sin ps_{n/2} - b_p \cos ps_{n/2}),\\
\sum_{k=0}^{n} s_k^2\, X_k &= (\hat{s}^2 - 2s_{n/2}^2) \sum_{k=0}^{n} X_k + 2s_{n/2} \sum_{k=0}^{n} s_k X_k\\
&\quad + \theta^2 \cdot \sum_{p=1}^{\infty} D_p^{(2)} \cdot (a_p \cos ps_{n/2} + b_p \sin ps_{n/2})
\end{aligned}
\right. \tag{5.8}
$$

where

$$
\left\{
\begin{aligned}
D_p^{(0)} &\equiv \csc \frac{1}{2} p\theta \,\sin \frac{n+1}{2}\, p\theta,\\
D_p^{(1)} &\equiv \frac{1}{2} \csc^2 \frac{1}{2}\, p\theta \cdot \left(n \sin \frac{1}{2}\, p\theta \cos \frac{n+1}{2} p\theta - \sin \frac{n}{2} p\theta\right),\\
D_p^{(2)} &\equiv \frac{1}{6} \csc^3 \frac{1}{2}\, p\theta \cdot \left[n^2 \sin^2 \frac{1}{2} p\theta \sin \frac{n+1}{2}\, p\theta + n \sin \frac{1}{2} p\theta \cdot \left(\cos \frac{1}{2} p\theta \cos \frac{n+1}{2} p\theta\right.\right.\\
&\qquad \left.\left. + 2 \cos \frac{n}{2} p\theta\right) - 3 \cos \frac{1}{2} p\theta \sin \frac{n}{2} p\theta\right],
\end{aligned}
\right. \tag{5.9}
$$

$$
\left\{
\begin{aligned}
s_{n/2} &\equiv \frac{n}{2}\,\theta - \Theta,\\
\hat{s}^2 &\equiv \frac{n(2n+1)}{6}\,\theta^2 - n\theta\,\Theta + \Theta^2.
\end{aligned}
\right. \tag{5.10}
$$

In order to use these formulae to obtain representations for the elements of $G^{(0)}$ we need to know A, a_p, b_p, the coefficients in the Fourier expansions for the various quantities connected with the true orbit. These expansions are known [ref. 2, p. 205] and the particular ones we require are given in the following table:

X	A	a_p	b_p
$\sin g$	0	0	$\frac{2}{ep} J_p(pe)$
$\cos g$	$-\frac{1}{2}e$	$\frac{2}{p} J_p'(pe)$	0
r	$a\left(1+\frac{1}{2}e^2\right)$	$-2\frac{ae}{p} J_p'(pe)$	0
r^{-1}	a^{-1}	$2a^{-1} J_p(pe)$	0
r^2	$a^2\left(1+\frac{3}{2}e^2\right)$	$-\frac{4a^2}{p^2} J_p(pe)$	0
$\cos(f-p)$	$-e$	$\frac{2(1-e^2)}{e} J_p(pe)$	0
$\sin(f-p)$	0	0	$2\sqrt{1-e^2}\, J_p'(pe)$
$r\cos(f-p)$	$-\frac{3}{2}ae$	$\frac{2a}{p} J_p'(pe)$	0
$r\sin(f-p)$	0	0	$\frac{2a\sqrt{1-e^2}}{ep} J_p(pe)$
$\sin^2 g$	$\frac{1}{2}$	$-4p\frac{d}{d(pe)}\left[\frac{1}{pe} J_p(pe)\right]$	0
$r^{-1}\sin g$	0	0	$2a^{-1} J_p'(pe)$
$r^2\sin(f-p)$	0	0	$a^2\sqrt{1-e^2}\left[\frac{-2+2e^2}{ep} J_p(pe)+\frac{4}{p^2} J_p'(pe)\right]$
$r^{-1}\sin^2 g$	$\frac{1}{2}a^{-1}$	$2a^{-1}\left[\frac{e^2-1}{e^2} J_p(pe)+\frac{1}{pe} J_p'(pe)\right]$	0

In these formulae $J_p(z)$ is the Bessel function of the first kind and pth order, and $J_p'(z)$ is its derivative with respect to the argument, z. Then, as alternatives to (5.3) we have

$$G_{11}^{(0)} = (n+1)\left(1+\frac{3}{2}e^2+\frac{9}{4}\hat{s}^2\right) - 4\sum_{p=1}^{\infty} D_p^{(0)} p^{-2} J_p(pe)\cos ps_{n/2}$$

$$-6s_{n/2}\sum_{p=1}^{\infty} D_p^{(0)} p^{-1} J_p(pe)\sin ps_{n/2} + 9\hat{s}^2\sum_{p=1}^{\infty} D_p^{(0)} J_p(pe)\cos ps_{n/2}$$

$$+6\theta \sum_{p=1}^{\infty} D_p^{(1)} p^{-1} J_p(pe) \cos ps_{n/2} + 18 s_{n/2} \theta \sum_{p=1}^{\infty} D_p^{(1)} J_p(pe) \sin ps_{n/2}$$

$$+9 \sum_{p=1}^{\infty} D_p^{(2)} J_p(pe) \cos ps_{n/2},$$

$$G_{12}^{(0)} = \frac{3}{2}(n+1)\, ae - \sum_{p=1}^{\infty} D_p^{(0)}\, 2ap^{-1} J_p'(pe) \cos ps_{n/2} - 6as_{n/2} \sum_{p=1}^{\infty} D_p^{(0)} J_p'(pe) \sin ps_{n/2}$$

$$+6a\theta \sum_{p=1}^{\infty} D_p^{(1)} J_p'(pe) \cos ps_{n/2},$$

$$G_{13}^{(0)} = -\frac{3}{2} a \sqrt{1-e^2}\,(n+1)\, s_{n/2},$$

$$G_{14}^{(0)} = \frac{3}{2}\sqrt{\mu}\, a^{-1/2}(n+1)\, s_{n/2} - 2\sqrt{\mu}\, a^{-1/2} \sum_{p=1}^{\infty} D_p^{(0)} p^{-1} J_p(pe) \sin ps_{n/2}$$

$$+6\sqrt{\mu}\, a^{-1/2} s_{n/2} \sum_{p=1}^{\infty} D_p^{(0)} J_p(pe) \cos ps_{n/2} + 6\sqrt{\mu}\, a^{-1/2}\theta \sum_{p=1}^{\infty} D_p^{(1)} J_p(pe) \sin ps_{n/2},$$

$$G_{22}^{(0)} = (n+1)\, a^2 \frac{5-4e^2}{2(1-e^2)} + 4a^2 \sum_{p=1}^{\infty} D_p^{(0)} \left[\frac{-p(1-e^2)^2+1}{e^2(1-e^2)p} J_p(pe) + \frac{1-e^2-p}{e(1-e^2)p} J_p'(pe)\right] \cos ps_{n/2},$$

$$G_{23}^{(0)} = \frac{4a^2}{\sqrt{1-e^2}} \sum_{p=1}^{\infty} D_p^{(0)} p^{-2} J_p'(pe) \sin ps_{n/2}, \qquad (5.11)$$

$$G_{24}^{(0)} = -4\sqrt{\mu}\, a^{1/2} \sum_{p=1}^{\infty} D_p^{(0)} J_p'(pe) \sin ps_{n/2},$$

$$G_{33}^{(0)} = (n+1)\, a^2 \left(1 + \frac{3}{2} e^2\right) - 4a^2 \sum_{p=1}^{\infty} D_p^{(0)}\, p^{-2}\, J_p(pe) \cos ps_{n/2},$$

$$G_{34}^{(0)} = -\sqrt{\mu}\, a^{1/2} \sqrt{1-e^2}\,(n+1),$$

$$G_{44}^{(0)} = (n+1)\,\mu a^{-1} + 4\mu a^{-1} \sum_{p=1}^{\infty} D_p^{(0)}\, J_p(pe) \cos ps_{n/2}.$$

The factors $J_p(pe)$ and $J_p'(pe)$ in the summands in these series bring about quite rapid convergence. In fact, these factors are both positive decreasing functions of p [ref. 3, p. 254] and, moreover,

$$\begin{cases} J_p(pe) \leq (2\pi p)^{-1/2}(1-e^2)^{-1/4} \exp\left[p\left(\sqrt{1-e^2} - \log \dfrac{1+\sqrt{1-e^2}}{e}\right)\right], \\ J_p'(pe) \leq (2\pi p e^2)^{-1/2}(1+e^2)^{1/4} \exp\left[p\left(\sqrt{1-e^2} - \log \dfrac{1+\sqrt{1-e^2}}{e}\right)\right], \end{cases} \tag{5.12}$$

which shows that the convergence is exponential inasmuch as the coefficient of p in the argument of the exponential functions in (5.12) is negative for $0 < e < 1$. If e is very small the first term in the series for $G_{ij}^{(0)}$ serves as a good approximation for the series. Besides this only a small extra error will be introduced if $J_1(e)$ and $J_1'(e)$ are replaced by $(1/2)e$ and $1/2$, respectively, since the error terms are $0(e^3)$ and $0(e^2)$, respectively.

The quantities occurring in the second order theory for the maximum problem depend to a great extent on the second derivatives of r and f with respect to the orbital parameters. In a straightforward computation these are found to be

$$\frac{\partial^2 r}{\partial b_1^2} = \frac{\partial^2 r}{\partial a^2} = \frac{3}{4} a^{-1} e(1-e^2)^{-1/2} s \sin(f-p) + \frac{9}{4} a r^{-2} e s^2 \cos(f-p),$$

$$\frac{\partial^2 r}{\partial b_2\, \partial b_1} = \frac{\partial^2 r}{\partial e\, \partial a} = -\cos(f-p) - \frac{3}{2} a^2 r^{-2} s \sqrt{1-e^2} \sin(f-p),$$

$$\frac{\partial^2 r}{\partial b_3\, \partial b_1} = \frac{\partial^2 r}{\partial p\, \partial a} = 0,$$

$$\frac{\partial^2 r}{\partial b_4\, \partial b_1} = \frac{\partial^2 r}{\partial z\, \partial a} = \frac{1}{2}\sqrt{\mu}\, a^{-3/2}\, e(1 - e^2)^{-1/2} \sin(f - p) + \frac{3}{2}\sqrt{\mu}\, a^{1/2}\, r^{-2}\, es \cos(f - p),$$

$$\frac{\partial^2 r}{\partial b_2^2} = \frac{\partial^2 r}{\partial e^2} = [a^2 r^{-1} + a(1 - e^2)^{-1}] \sin^2(f - p),$$

$$\frac{\partial^2 r}{\partial b_3\, \partial b_2} = \frac{\partial^2 r}{\partial p\, \partial e} = 0,$$

$$\frac{\partial^2 r}{\partial b_4\, \partial b_2} = \frac{\partial^2 r}{\partial z\, \partial e} = -\sqrt{\mu}\, a^{3/2}\, r^{-2} \sqrt{1 - e^2} \sin(f - p),$$

$$\frac{\partial^2 r}{\partial b_3^2} = \frac{\partial^2 r}{\partial p^2} = 0,$$

$$\frac{\partial^2 r}{\partial b_4\, \partial b_3} = \frac{\partial^2 r}{\partial z\, \partial p} = 0,$$

$$\frac{\partial^2 r}{\partial b_4^2} = \mu r^{-2}\, e \cos(f - p),$$

$$\frac{\partial^2 f}{\partial b_1^2} = \frac{15}{4} r^{-2}\, s\sqrt{1 - e^2} - \frac{9}{2} a r^{-3}\, es^2 \sin(f - p),$$

$$\frac{\partial^2 f}{\partial b_2\, \partial b_1} = \frac{3}{2} a r^{-3}\, s(1 - e^2)^{-1/2}\, [re - 2a(1 - e^2)\cos(f - p)],$$

$$\frac{\partial^2 f}{\partial b_3\, \partial b_1} = 0,$$

$$\frac{\partial^2 f}{\partial b_4\, \partial b_1} = \frac{3}{2}\sqrt{\mu}\, a^{-1/2}\, r^{-3}\, [r\sqrt{1 - e^2} - 2aes\sin(f - p)],$$

$$\frac{\partial^2 f}{\partial b_2^2} = \left\{a^2 r^{-2}\cos(f-p) + [ar^{-1} + (1-e^2)^{-1}]^2\cos(f-p) + 2e(1-e^2)^{-2}\right\}\sin(f-p),$$

$$\frac{\partial^2 f}{\partial b_3\, \partial b_2} = 0,$$

$$\frac{\partial^2 f}{\partial b_4\, \partial b_2} = \sqrt{\mu}\, a^{1/2}\, r^{-3}\, (1 - e^2)^{-1/2}\, [re - 2a(1 - e^2)\cos(f - p)],$$

$$\frac{\partial^2 f}{\partial b_3^2} = 0,$$

$$\frac{\partial^2 f}{\partial b_4\, \partial b_3} = 0,$$

$$\frac{\partial^2 f}{\partial b_4^2} = -2\mu\, r^{-3}\, e\sin(f - p).$$

In the work leading to the expressions (4.6) for $\rho^{(2)}$ and $r\varphi^{(2)}$ there are the relations

$$\begin{cases} \left[\dfrac{\partial \rho}{\partial \beta_i}\right]^{(1)} = \displaystyle\sum_{j=1}^{4} \frac{\partial^2 r}{\partial b_j\, \partial b_i} \beta_j^{(1)}, \\ \\ \left[\dfrac{\partial \varphi}{\partial \beta_i}\right]^{(1)} = \displaystyle\sum_{j=1}^{4} \frac{\partial^2 f}{\partial b_j\, \partial b_i} \beta_j^{(1)}, \end{cases} \tag{5.13}$$

showing the connection of the second derivatives of r and f with the matrices occurring in section 4. Indeed, with $\beta^{(1)}$ determined in a particular case from the first order theory and with $[\partial\rho/\partial\beta_i]^{(1)}$ and $[\partial\varphi/\partial\beta_i]^{(1)}$ computed with the aid of (5.13) and the above second order derivative formulae all matrices are computable that are needed in (4.10) so that this equation may be solved for $\beta^{(2)}$ and $\lambda^{(1)}$.

6. CIRCULAR ORBIT

In the case of a circular orbit the eccentricity is zero and the angle of perigee and instant of perigee passage become indeterminate. Thus, in dealing with circular orbits we have only to consider the radius, a. Consequently, such matrices as $H^{(0)}$ and $G^{(0)}$ which are $4 \times (n+1)$ and 4×4 in the two-dimensional elliptic orbit case become $1 \times (n+1)$ and 1×1 in the two-dimensional circular orbit case.

The true circular orbit is defined by

$$\begin{cases} r = a \\ f = \sqrt{\mu}\, a^{-3/2}\, t \end{cases} \tag{6.1}$$

and it is on these formulae that the matrices occurring in the maximum problem are based. In particular,

$$
\begin{cases}
H^{(0)} = (1, 1, 1, \dots, 1) \\
J^{(0)} = -\frac{3}{2}\sqrt{\mu}\, a^{-3/2}\, \tau\, (0, 1, 2, \dots, n) \\
\frac{\partial X_k}{\partial b} = \left(1, -\frac{3}{2}\sqrt{\mu}\, a^{-3/2}\, k\tau\right) \\
\frac{\partial X}{\partial b} = \left(1, 0, 1, -\frac{3}{2}\sqrt{\mu}\, a^{-3/2}\, \tau, \dots, 1, -\frac{3}{2}\sqrt{\mu}\, a^{-3/2}\, n\tau\right) \\
G^{(0)} = (n+1)\left[1 + \frac{3}{8}\mu\, a^{-3}\, n(2n+1)\,\tau^2\right]
\end{cases}
\tag{6.2}
$$

The position at the $k+1$st instant in the least squares circular orbit is determined by

$$
\begin{cases}
\rho_k^{(1)} = H_k^{(0)}\,\alpha^{(1)} = \alpha^{(1)} \\
\varphi_k^{(1)} = -\frac{3}{2}\sqrt{\mu}\,\alpha^{(1)}\, a^{-5/2}\, k\tau
\end{cases}
\tag{6.3}
$$

which were computed from (3.8) and (6.2). The first order Langrange multiplier, $\lambda^{(0)}$, is found from (3.22) and (6.2) to be

$$
\lambda^{(0)} = -\left(1 + \frac{9}{4}\mu a^{-3}\, k^2\, \tau^2\right)(n+1)^{-1}\left[1 + \frac{3}{8}\mu a^{-3}\, n(2n+1)\,\tau^2\right]^{-1}, \tag{6.4}
$$

but the one-dimensional eigenvector $\beta^{(1)} = \alpha^{(1)}$ is not determined by the equation (3.21) precisely because the matrices involved are 1×1. That is to say, the only freedom for the direction of $\beta^{(1)}$ is that afforded by the sign of $\beta^{(1)}$ and this is left unspecified by (3.21). The magnitude of $\beta^{(1)}$ is, however, another matter and it is found from (3.30):

$$
\beta^{(1)} = \alpha^{(1)} = \pm\, a\left[1 + \frac{3}{8}\mu a^{-3}\, n(2n+1)\,\tau^2\right]^{-1/2}
$$

Probably the most satisfactory way of representing the maximum value of W^2 is in terms of the true anomaly, f. By (3.32), (6.4), and (6.1) we obtain for the maximum value of the squared distance between the $k+1$st true point and the $k+1$st least squares computed point (to the first order)

$$W^2 = \left(1 + \frac{9}{4} f_k^2\right) \left(1 + \frac{3}{8} \frac{2n+1}{n} f_n^2\right)^{-1} c^2/(n+1). \tag{6.5}$$

To get an idea of the size of this we note that if $k = n$ and $f_n = 1$ radian with n regarded as large W^2 is approximately $2c^2/(n+1)$.

In extending the work to include the second order terms we need the following basic results:

$$\left\{\begin{aligned}
&\left[\frac{\partial \rho_k}{\partial \beta}\right]^{(1)} = 0 \\
&\left[\frac{\partial \varphi_k}{\partial \beta}\right]^{(1)} = \frac{15}{4} \sqrt{\mu}\, a^{-7/2}\, \alpha^{(1)}\, k\tau \\
&H^{(1)} = (0, 0, 0, \ldots, 0) = 0 \cdot H^{(0)} \\
&J^{(1)} = \frac{3}{4} \sqrt{\mu}\, \alpha^{(1)}\, a^{-5/2}\, \tau\, (0, 1, 2, \ldots, n) = -\frac{1}{2} \alpha^{(1)} a^{-1} J^{(0)} \\
&G^{(1)} = -\frac{9}{8} n(n+1)(2n+1)\, \mu a^{-4}\, \alpha^{(1)}\, \tau^2 \\
&\Lambda_n^{(1)} = -\frac{9}{4} \mu {\alpha^{(1)}}^2\, a^{-4}\, n^2 \tau^2
\end{aligned}\right. \tag{6.6}$$

The second order terms in the least squares computed position at the $k + 1$ st instant are then available from (4.6) as expressions in terms of $\alpha^{(2)}$ and these are

$$\left\{\begin{aligned}
&\rho_k^{(2)} = \alpha^{(2)}, \\
&\varphi_k^{(2)} = \sqrt{\mu}\, a^{-7/2}\, k\tau \left[-\frac{3}{2} a\, \alpha^{(2)} + \frac{15}{8} {\alpha^{(1)}}^2\right]
\end{aligned}\right. \tag{6.7}$$

The remaining second order quantities present no problems and we obtain

$$\begin{cases} \lambda^{(1)} = \dfrac{27}{32}\,\mu\alpha^{(1)}\, a^{-4}\, n(n+1)\,(4n-1)\,\tau^2\, {G^{(0)}}^{-2} \\[2ex] \alpha^{(2)} = \dfrac{3}{32}\,\mu{\alpha^{(1)}}^2\, a^{-4}\, n(n+1)\,(2n+1)\,\tau^2\, {G^{(0)}}^{-1} \\[2ex] \hat{X}_1^{(2)} = \dfrac{3}{8}\mu{\alpha^{(1)}}^2\, a^{-4}\,\tau^2\, {G^{(0)}}^{-1}\,[n(n+1)(2n+1)\cdot(1,1,1,\cdots,1) - 3G^{(0)}\cdot(0,1,4,\cdots,n^2)] \\[2ex] \hat{X}_2^{(2)} = \dfrac{9}{4}\sqrt{\mu}\,{\alpha^{(1)}}^2\, a^{-5/2}\,(n+1)\,\tau\, {G^{(0)}}^{-1}\left[1+\dfrac{1}{8}\mu a^{-3}\, n(2n+1)\,\tau^2\right]\cdot(0,1,2,\cdots,n). \end{cases} \tag{6.8}$$

This formula for $\lambda^{(1)}$ is for the case $k = n$, i.e., for the maximization of the error occurring at the end of the run. Formulae (6.7) and (6.8) show that the coincidence of the least squares computed positions with the observed positions (for maximum error) that held in the first order theory (cf. (3.27)) does not carry over, in general, to the second order. And, finally, the maximum squared error at the end of the run ($k=n$) for a circular orbit is

$$\begin{aligned} W^2 = {G^{(0)}}^{-1}\Big\{ & a^2\,(n+1)\left(1+\frac{9}{4}\mu a^{-3}\, n^2\,\tau^2\right)\mathcal{E}^2 \\ & -\frac{3}{16}\mu{\alpha^{(1)}}^3\, a^{-4}\, n(n+1)\,\tau^2\left[(16n-1)-\frac{9}{2}\mu a^{-3}\, n^2\,(n-1)\,\tau^2\right]\mathcal{E}^3+\cdots\Big\}, \\ = (n+1)^{-1}\left[1+\frac{3}{8}\frac{2n+1}{n}\, f_n^2\right]^{-1} \Big\{ & \left(1+\frac{9}{4}\, f_n^2\right) c^2 \\ & -\frac{3}{16}\, a^{-1}\, n^{-1}\,(n+1)^{-1/2}\, f_n^2\left(1+\frac{3}{8}\frac{2n+1}{n}\, f_n^2\right)^{-3/2}\left[(16n-1)-\frac{9}{2}(n-1)\, f_n^2\right] c^3+\cdots\Big\}. \end{aligned}$$

The ratio of the second order term to the first order term for a sweep of one radian ($f_n = 1$) and large n is, approximately,

$$0.287 \frac{c}{a\sqrt{n+1}}$$

in absolute value. Since in practice c can be expected to be very much smaller than $a\sqrt{n+1}$, perhaps around 1/500th its size, the indication is that the first order theory will be adequate in the case of a single pass of moderate length.

REFERENCES

1. Birkhoff and MacLane, Survey of Modern Algebra, Macmillan, New York, 1946.

2. Wintner, Analytical Foundations of Celestial Mechanics, Princeton, New Jersey, 1941.

3. Watson, Treatise on Bessel Functions, Macmillan, New York, 1943.