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ABSTRACT

The solution of optimum rendezvous trajectories with bounded thrust magnitude and interim coasting arcs has been obtained. The Lagrange multipliers, determined from the impulsive solution, are used as starting values. The three-dimensional equations have been derived for the assumptions of a mass particle moving in an inverse square force field in a vacuum. A practical solution time was obtained by the use of a convergence technique which employs the impulsive solution to generate corrections to the initial multipliers.

THE UTILIZATION OF AN IMPULSIVE SOLUTION IN SOLVING OPTIMUM TRAJECTORY PROBLEMS WITH BOUNDED THRUST MAGNITUDE

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SUMMARY

A method is presented for obtaining solutions to nonlinear optimal trajectory problems with bounded thrust magnitude by use of the corresponding impulsive solution to generate the starting values of the Lagrange multipliers. The equations were derived in three dimensions for a mass particle moving in an inverse square force field in a vacuum. The convergence scheme which makes use of the impulsive multipliers has an efficient solution time as compared to a first-order perturbation technique.

An example problem of a transfer from Earth to Mars for fixed initial thrust accelerations has been used to demonstrate the solution technique. The number of iterations and the time required for a solution increased rapidly as the assumptions of the impulsive solution were violated.

INTRODUCTION

The solution of optimal trajectories with bounded control variables, which uses the indirect method of the calculus of variations, has been treated by many investigators (refs. 1 to 5). This formulation results in a two-point boundary-value problem which usually has unknown initial or final values of the Lagrange multipliers. The procedure has simply been to make an intuitive guess of these values. For special problems in particular coordinate frames, an experienced investigator can attach some physical meaning to these multipliers (ref. 6). The extreme sensitivity of the terminal boundary conditions to variations in the initial or final values of these variables is well known. This deterrent to the indirect solution method gives rise to other techniques such as the gradient or steepest-ascent method (ref. 7). Difficulties are also inherent in this method; for example, the choice of proper step size and the manner in which it is varied (ref. 8). The tendency of investigators is to look to other methods of solution. One interesting possibility, a hybrid optimization technique, consists of the method of steepest ascent for the initial phase and one of the indirect methods for the latter phase.

Pines (ref. 9) suggested that the Lagrange multipliers obtained by assuming an impulsive solution be used as starting values. He reasoned that, "We can look at the impulsive solution as a limiting point in a simply connected region in the space of the initial conditions of the adjoint variables." This indicates that, at worst, a solution could be obtained by stepping the initial thrust acceleration from the impulsive case to the desired finite level.

Handelsman (ref. 4) solved two-dimensional Earth-to-Mars trajectory problems by use of the impulsive multipliers as starting values. The author's work is an extension of and improvement on Handelsman's solution. The extension involves adding the third dimension to the solution. The improvement derived is the increased efficiency, as defined by the number and time for an iteration, in obtaining a solution. This efficiency is accomplished not only by using the impulsive multipliers as starting values, but also in the convergence scheme of a finite thrust-acceleration problem. The method of using an approximate closed-form solution in a convergence scheme (feedback loop) of a nonlinear problem is exemplified in Battin's work (ref. 10) and in Jezewski's work (ref. 6) for the circumlunar and time-optimal trajectory problems.

SYMBOLS

A_{ij}	matrix defined in equation (78)
A_z	azimuth angle measured from 1_θ, deg
$\underline{B}$	terminal boundary vector
$C_1, C_2, \ldots, C_6$	constants defined in equation (78)
C_o	constant defined in equation (19)
e	eccentricity of the coasting ellipse
$f_1, f_2, \ldots, f_6$	functions defined in equations (51) to (56)
I_{sp}	specific impulse, sec
k	logical control switch
m	vehicle mass, slugs
P	period of circular orbit whose radius is r_R, sec
q	function defined in equation (57)
r	radius to mass particle from center of attracting body, ft

S	transformation matrix defined in equation (59)
T	thrust, lb
t	time, sec
u, v, w	components of total velocity in directions in 1_ϕ, 1_ρ, and 1_θ directions, respectively, ft/sec
V	total velocity, ft/sec
V_e	exhaust velocity, ft/sec
v_e	nondimensional exhaust velocity
x, y, z	inertial reference frame, ft
Z	nondimensional velocity
1_R	unit vector in the ν direction
1_T	unit vector in the direction normal to the plane of motion
$1_\rho, 1_\phi, 1_\theta$	unit vector in ρ, ϕ, and θ directions, respectively
α	included angle between thrust and velocity vectors, deg
β	variable defined by equation (103)
γ	flightpath angle, deg
Δ	change of
δ	nondimensional time rate of change of latitude
ϵ	included angle between 1_R and 1_ϕ unit vectors, deg
θ	latitude angle, deg
κ	switch function defined in equation (17)
$\underline{\Lambda}$	primer vector
$\lambda_1, \lambda_2, \ldots, \lambda_6, \lambda_\mu$	Lagrange multipliers
μ	mass fraction

μ_G	gravitational constant, ft^3/sec^2
ν	true anomaly angle, deg
π	3.1415927, rad
ρ	nondimensional radius
σ	nondimensional time rate of change of longitude
τ	nondimensional time
ϕ	longitude angle, deg
χ	thrust pitch angle measured from local horizontal plane, deg
ψ	thrust yaw angle measured from 1_ϕ, deg
Ω	time rate of change of ϵ, sec^{-1}

Subscripts:

C	refers to a corrected solution
I	refers to an integrated solution
i, j	indices
max	maximum value of
R	reference quantity
u, v, w	refer to components in 1_ϕ, 1_ρ, and 1_θ directions, respectively
0,1	refer to conditions before and after first impulse
2, f	refer to conditions before and after second impulse
ρ	in the direction of the unit vector 1_ρ
_	below a variable indicates a vector quantity

Superscripts:

a, b	value of a quantity after and before an event
R	with respect to the 1_ρ, 1_R, and 1_T coordinate frame
S	with respect to the 1_ρ, 1_ϕ, and 1_θ coordinate frame
-1	inverse of a matrix
^	approximate value of variables

Operators:

$(\dot{\ })$	derivative with respect to time t
$(\ )'$	derivative with respect to time τ

OPTIMUM TRAJECTORY EQUATIONS

The nonlinear, nondimensional equations of motion of a mass particle moving in an inverse square force field in a vacuum and acted upon by a thrust acceleration can be written in the following manner (see appendix):

$$Z_v' - \rho\left(\sigma^2\cos^2\theta + \delta^2\right) + \frac{1}{\rho^2} + \frac{\mu'}{\mu} v_e \sin\chi = 0 \tag{1}$$

$$\rho\sigma'\cos\theta + 2\sigma\left(Z_v \cos\theta - \rho\delta\sin\theta\right) + \frac{\mu'}{\mu} v_e \cos\chi\cos\psi = 0 \tag{2}$$

$$\rho\delta' + 2\delta Z_v + \rho\sigma^2\sin\theta\cos\theta + \frac{\mu'}{\mu} v_e \cos\chi\sin\psi = 0 \tag{3}$$

$$\rho' - Z_v = 0 \tag{4}$$

$$\phi' - \sigma = 0 \tag{5}$$

$$\theta' - \delta = 0 \tag{6}$$

$$\mu' + \beta = 0 \tag{7}$$

The prime refers to the nondimensional time $\tau = tV_R/r_R$, and β is proportional to the thrust T. The state variables are the position and velocity coordinates ρ, ϕ, θ, Z_v, σ, δ, and the mass μ. The control variables are the magnitude and direction of the thrust βv_e, χ, and ψ, respectively. The exhaust velocity v_e is assumed constant. The axis system and the associated notation is illustrated in figure 1.

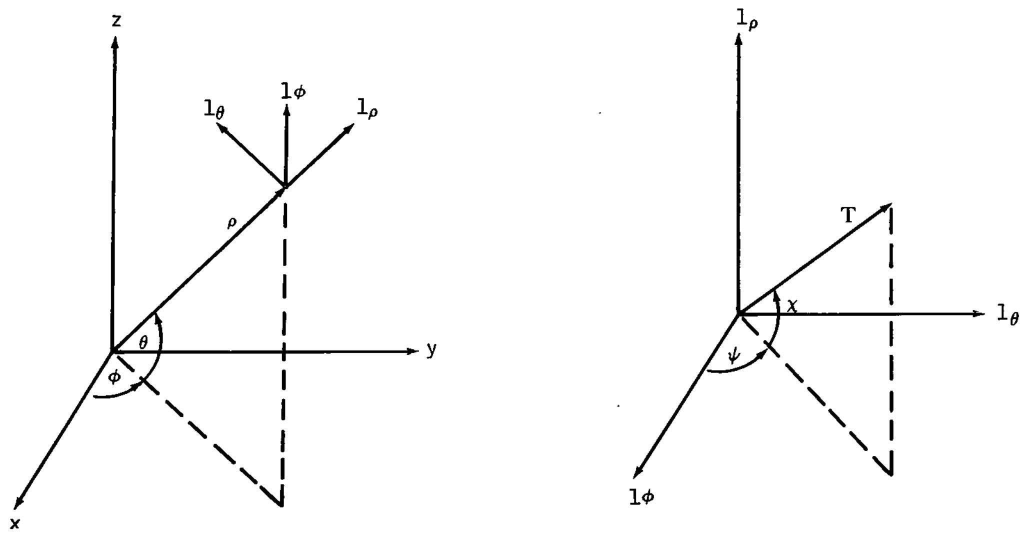

Figure 1. - Coordinate system and angle definition.

For this system of constraint equations, it is desirable to determine the thrust-vector program (bounded in magnitude) for a trajectory between two fixed positions, velocities, and times which minimizes the propellant consumption. The necessary and sufficient conditions for an optimum trajectory may be found in reference 1. It will suffice to state the variational equations and indicate their functions.

$$\lambda_1' = 2\left(\lambda_2\sigma \cos\theta + \lambda_3\delta\right) - \lambda_4 \tag{8}$$

$$\lambda_2' = -2\sigma\left(\lambda_1 \cos\theta - \lambda_3 \sin\theta\right) + \lambda_2\left(\frac{Z_v}{\rho} - \delta \tan\theta\right) - \frac{\lambda_5}{\rho\cos\theta} \tag{9}$$

$$\lambda_3' = -2\left(\lambda_1\delta + \lambda_2\sigma \sin\theta\right) + \lambda_3 \frac{Z_v}{\rho} - \frac{\lambda_6}{\rho} \tag{10}$$

$$\lambda_4' = -\lambda_1\left(\delta^2 + \sigma^2\cos^2\theta + \frac{2}{\rho^3}\right) + \lambda_2(\sigma'\cos\theta - 2\sigma\delta\sin\theta)$$

$$+ \lambda_3\left(\delta' + \sigma^2\sin\theta\cos\theta\right) \tag{11}$$

$$\lambda_5' = 0 \tag{12}$$

$$\lambda_6' = 2\lambda_1\rho\sigma^2\sin\theta\cos\theta + \lambda_3\rho\sigma^2\cos 2\theta$$

$$-\lambda_2\left[\rho\sigma'\sin\theta + 2\sigma\left(Z_v\sin\theta + \rho\delta\cos\theta\right)\right] \tag{13}$$

$$\lambda_\mu' = \beta\frac{v_e}{\mu^2}|\underline{\Lambda}| \tag{14}$$

$$\tan\chi = \frac{\lambda_1}{\lambda_2}\cos\psi, \quad \tan\psi = \frac{\lambda_3}{\lambda_2} \tag{15}$$

$$|\underline{\Lambda}| = \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^{1/2} \tag{16}$$

$$\kappa = \frac{v_e}{\mu}|\underline{\Lambda}| - \lambda_\mu \tag{17}$$

$$\beta = \begin{cases}\beta_{max}, & \kappa > 0 \\ 0 \quad , & \kappa < 0\end{cases} \tag{18}$$

$$C_o = -\lambda_1 Z_v' - \lambda_2\rho\sigma'\cos\theta - \lambda_3\rho\delta' - \lambda_4\rho'$$

$$-\lambda_5\phi' - \lambda_6\theta' - \lambda_\mu\mu' \tag{19}$$

Equations (8) to (14), the Euler-Lagrange equations, are the first necessary conditions for an optimum trajectory. The optimum conditions on the thrust vector are given by equations (15) to (18). These equations show the thrust direction to be alined with the primer vector $\underline{\Lambda}$ (Lawden's notation, ref. 2) and indicate the condition for the change in magnitude of the bounded thrust (the switching function $\kappa(\tau)$). The condition for $\kappa \equiv 0$ (intermediate thrust arcs) will not be considered in this study. The primer vector $\underline{\Lambda}$ has components λ_1, λ_2, and λ_3 in the directions of the unit vectors 1_ρ, 1_ϕ, and 1_θ, respectively. The direction cosines of the thrust vector can be written as

$$\sin \chi = \frac{\lambda_1}{|\underline{\Lambda}|} \tag{20}$$

$$\cos \chi \cos \psi = \frac{\lambda_2}{|\underline{\Lambda}|} \tag{21}$$

$$\cos \chi \sin \psi = \frac{\lambda_3}{|\underline{\Lambda}|} \tag{22}$$

This allows a replacement of the angles χ and ψ in the formulation by the primer-vector magnitude and its components. Equation (19) results from a first integral; C_o is a constant if the external accelerations are explicitly independent of the variable τ.

IMPULSIVE RELATIONSHIPS

An impulsive solution is assumed to occur in two possible manners: the external force acceleration approaches infinity or the independent variable τ goes to zero. The form of equations (1) to (14) is unsuitable for determination of the relationships between the variables when an impulse is applied. A change in variables is made from τ to μ because the mass ratio has a finite variation across the impulse. Performing this change in the independent variables and taking the limit of equations (1) to (14) as β approaches infinity, the following equations result.

$$\Delta Z_V = |\Delta \underline{V}| \sin \chi \tag{23}$$

$$\rho \, \Delta\sigma \cos \theta = |\Delta \underline{V}| \cos \chi \cos \psi \tag{24}$$

$$\rho \, \Delta\delta = |\Delta \underline{V}| \cos \chi \sin \psi \tag{25}$$

$$\Delta\rho = 0 \tag{26}$$

$$\Delta\phi = 0 \tag{27}$$

$$\Delta\theta = 0 \tag{28}$$

$$\Delta\lambda_1 = 0 \tag{29}$$

$$\Delta\lambda_2 = 0 \tag{30}$$

$$\Delta\lambda_3 = 0 \tag{31}$$

$$\Delta\lambda_4 = \lambda_2\ \Delta\sigma \cos\theta + \lambda_3\ \Delta\delta \tag{32}$$

$$\Delta\lambda_5 = 0 \tag{33}$$

$$\Delta\lambda_6 = -\rho\lambda_2\ \Delta\sigma \sin\theta \tag{34}$$

$$\Delta\lambda_\mu = v_e |\underline{\Lambda}| \left(\frac{1}{\mu^a} - \frac{1}{\mu^b}\right) \tag{35}$$

The quantity $|\Delta\underline{V}|$ is

$$|\Delta\underline{V}| = -v_e \int_{\mu^b}^{\mu^a} \frac{d\mu}{\mu} = -v_e \ln \frac{\mu^a}{\mu^b} \tag{36}$$

This equation is not actually used to determine the magnitude of the impulse because the change in the mass ratios is not known a priori. The ΔV is determined by differencing the velocity vectors before and after the impulse.

$$\Delta\underline{V} = \underline{V}_1 - \underline{V}_0 \tag{37}$$

Because the impulse is alined with this vector, the thrust direction angles χ and ψ are also specified.

In equations (23) to (35), the quantities ρ, ϕ, θ, λ_1, λ_2, λ_3, and λ_5 do not change, which indicates that the position and primer vector are invariant over the impulse. The variable λ_5 is a constant of motion of the solution and is unaffected by the impulse. The relationships between the variables that change across the impulse are

$$Z_v{}^a = Z_v{}^b + |\Delta \underline{V}| \sin \chi \tag{38}$$

$$\sigma^a = \sigma^b + \frac{|\Delta \underline{V}|}{\rho \cos \theta} \cos \chi \cos \psi \tag{39}$$

$$\delta^a = \delta^b + \frac{|\Delta \underline{V}|}{\rho} \cos \chi \sin \psi \tag{40}$$

$$\lambda_4{}^a = \lambda_4{}^b + \lambda_2 \frac{|\Delta \underline{V}|}{\rho} \cos \chi \cos \psi + \lambda_3 \frac{|\Delta \underline{V}|}{\rho} \cos \chi \sin \psi \tag{41}$$

$$\lambda_6{}^a = \lambda_6{}^b - \lambda_2 |\Delta \underline{V}| \cos \chi \cos \psi \tan \theta \tag{42}$$

$$\lambda_\mu{}^a = \lambda_\mu{}^b + v_e |\underline{\Delta}| \left(\frac{1}{\mu^a} - \frac{1}{\mu^b} \right) \tag{43}$$

where equations (39) and (40) were used in equations (41) and (42).

PRIMER VECTOR ON A COASTING ARC

Lawden's equations for the Lagrange multipliers and their time rate of change on a coasting arc (ref. 2) are

$$\lambda_1 = C_1 \cos \nu + C_2 e \sin \nu + C_3 f_2(\nu) \tag{44}$$

$$\lambda_2 = C_1 f_5(\nu) + C_2 f_1(\nu) + C_3 f_3(\nu) + \frac{C_4}{f_1(\nu)} \tag{45}$$

$$\lambda_3 = \frac{\left(C_5 \cos \nu + C_6 \sin \nu\right)}{f_1(\nu)} \tag{46}$$

$$\dot{\lambda}_1 = \dot{\nu}\left[-C_1 \sin \nu + C_2 e \cos \nu + C_3 f_4(\nu)\right] \tag{47}$$

$$\dot{\lambda}_2 = \dot{\nu}\left\{-C_1\left[\frac{e + \cos \nu}{f_1^2(\nu)} + \cos \nu\right] - C_2 e \sin \nu + C_3\left[\frac{\cos \nu}{f_1^2(\nu)} - f_2(\nu)\right] + \frac{C_4 e \sin \nu}{f_1^2(\nu)}\right\} \tag{48}$$

$$\dot{\lambda}_3 = -\frac{\dot{\nu}\left[-C_5 \sin \nu + C_6(e + \cos \nu)\right]}{f_1^2(\nu)} \tag{49}$$

where

$$\dot{\nu} = \left(\sigma^2 \cos^2\theta + \delta^2\right)^{1/2} \tag{50}$$

and

$$f_1(\nu) = 1 + e \cos \nu \tag{51}$$

$$f_2(\nu) = \frac{-\cos \nu}{f_1^2(\nu)} + \frac{e f_6(\nu) \sin \nu}{\left(1 - e^2\right)} \tag{52}$$

$$f_3(\nu) = \frac{f_1(\nu)\, f_6(\nu)}{\left(1 - e^2\right)} \tag{53}$$

$$f_4(\nu) = \frac{\sin\nu}{f_1^2(\nu)} + \frac{e f_6(\nu)\cos\nu}{\left(1 - e^2\right)} \tag{54}$$

$$f_5(\nu) = -\left[1 + \frac{1}{f_1(\nu)}\right]\sin\nu \tag{55}$$

$$f_6(\nu) = \frac{\sin\nu}{f_1^2(\nu)} + \frac{\left(1 + 2e^2\right)\sin\nu}{\left(1 - e^2\right) f_1(\nu)}$$

$$- \frac{6e \tan^{-1}\left[q \tan\left(\frac{\nu}{2}\right)\right]}{\left(1 - e^2\right)^{3/2}} \tag{56}$$

$$q = \left[\frac{(1 - e)}{(1 + e)}\right]^{1/2} \tag{57}$$

This set of equations is a function of only three quantities: the constants $C_1, C_2, \ldots, C_6$ defined by the initial conditions on the multipliers and their derivatives, the true anomaly ν, and the eccentricity e of the coasting arc. The components $\left(\lambda_1, \lambda_2, \text{ and } \lambda_3\right)$ of the primer vector as defined above are with respect to an altitude, range, and track coordinate frame. The transformation to the spherical coordinate frame is given as

$$\underline{\Lambda}^S = S\underline{\Lambda}^R \tag{58}$$

where the superscripts R and S refer to the altitude, range, and track coordinate frame, and the spherical coordinate frame, respectively. The transformation S is given as

$$S = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{vmatrix} \tag{59}$$

where ϵ (fig. 2) is a negative rotation about the unit ρ-axis.

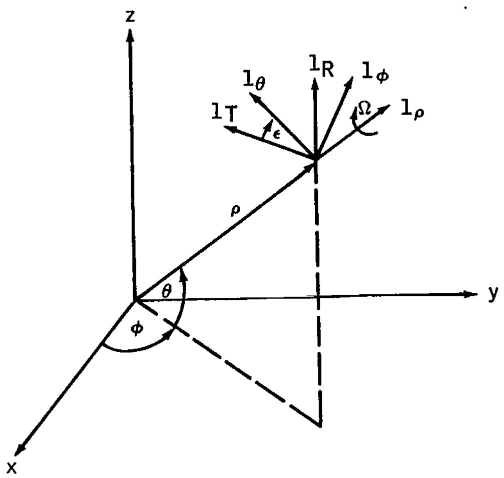

Figure 2. - Angular relationship between an altitude, range, and track frame and a spherical coordinate frame.

The time rate of change of this vector is given as

$$\dot{\underline{\Lambda}}^S = S\dot{\underline{\Lambda}}^R + \underline{\Omega} \times \underline{\Lambda}^S \tag{60}$$

where the vector $\underline{\Omega}$ is the angular rate between the two coordinate frames. This vector is directed along the ρ-axis and may be readily verified to have a magnitude given as

$$\left|\underline{\Omega}\right| = \Omega_\rho = -\sigma \sin \theta \tag{61}$$

INITIAL MULTIPLIER DETERMINATION

It may be noted from equations (29) to (31) that the components $\left(\lambda_1, \ \lambda_2, \text{ and } \lambda_3\right)$ of the primer vector are continuous across an impulse. Therefore,

$$\lambda_1{}^b = \lambda_1{}^a \tag{62}$$

$$\lambda_2{}^b = \lambda_2{}^a \tag{63}$$

$$\lambda_3^{\,b} = \lambda_3^{\,a} \tag{64}$$

From the previous section the primer vector and its derivative on the coasting arc are known by equations (44) to (49). This vector is, therefore, also known before the impulse.

The Euler-Lagrange equations are a homogeneous set. This condition indicates that only their ratios are significant and that one of the multipliers may be chosen arbitrarily. Since the multiplier λ_4 will remain nearly constant over the solution, its initial value will be chosen on the following basis:

$$\lambda_4^{\,b} = \begin{cases} +1, & -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} \leq \alpha \leq 3\frac{\pi}{2} \end{cases} \tag{65}$$

where

$$\alpha = \cos^{-1}\left[\frac{\underline{T} \cdot \underline{V}}{|\underline{T}|\ |\underline{V}|}\right] \tag{66}$$

This choice of sign for the initial value of λ_4 is based on whether the impulse is a posigrade or retrograde maneuver. Using equations (9) and (33), the constant λ_5 may be determined as

$$\lambda_5^{\,b} = \lambda_5^{\,a} = \rho \cos\theta \left[\lambda_2\left(\frac{Z_v}{\rho} - \delta \tan\theta\right) - \lambda_2' - 2\sigma\left(\lambda_1 \cos\theta - \lambda_3 \sin\theta\right)\right] \tag{67}$$

Similarly, from equations (10) and (34)

$$\lambda_6^{\,a} = \lambda_3 Z_v - 2\rho\left(\lambda_1 \delta + \lambda_2 \sigma \sin\theta\right) - \rho\lambda_3' \tag{68}$$

The initial value of the mass multipliers λ_μ must be determined before a solution can be generated. Recalling equation (19)

$$C_o = -\lambda_1 Z_v' - \lambda_2 \rho \sigma' \cos\theta - \lambda_3 \rho \delta' - \lambda_4 \rho'$$

$$-\lambda_5 \phi' - \lambda_6 \theta' - \lambda_\mu \mu'$$

and using the equations of motion, this constant may be written as

$$C_o = \lambda_1 \left[-\rho\left(\sigma^2 \cos^2\theta + \delta^2\right) + \frac{1}{\rho^2} \right] + 2\lambda_2 \sigma \left(Z_v \cos\theta - \rho\delta \sin\theta \right)$$

$$+ \lambda_3 \left(2\delta Z_v + \rho\sigma^2 \sin\theta \cos\theta \right) - \lambda_4 Z_v - \lambda_5 \sigma - \lambda_6 \delta - \beta\kappa \tag{69}$$

where equation (17), the relationship for κ, has also been used. During the coasting portion of the solution, $\kappa < 0$ and $\beta = 0$, which facilitates a solution for the constant C_o in terms of known functions evaluated after the impulse. Since C_o has now been uniquely defined, the initial value of κ, and hence of λ_μ, may be determined from an evaluation of equation (69) before the impulse. Solving for λ_μ

$$\lambda_\mu{}^b = \frac{1}{\beta} \Bigg\{ C_o + \frac{\beta v_e}{\mu} |\underline{\Delta}| + \lambda_1 \left[\rho\left(\sigma^2 \cos^2\theta + \delta^2\right) - \frac{1}{\rho^2} \right]$$

$$-2\sigma\lambda_2 \left(Z_v \cos\theta - \rho\delta \sin\theta \right) - \lambda_3 \left(2\delta Z_v + \rho\sigma^2 \sin\theta \cos\theta \right)$$

$$+ \lambda_4 Z_v + \lambda_5 \sigma + \lambda_6 \delta \Bigg\} \tag{70}$$

The initial values of the Lagrange multipliers have now been determined as functions of the state variables before and after the impulse, of the magnitude of the impulse, and of the primer vector and its derivative after the impulse. The determination of this vector and its derivative requires a solution of the constants $C_1, C_2, \ldots, C_6$ in Lawden's equations. A solution for these constants requires six independent equations which govern the relationship of the multipliers on the coasting arc. The simplest set of equations available is

$$\lambda_{10} \cos \psi_0 - \lambda_{20} \tan \chi_0 = 0 \tag{71}$$

$$\lambda_{1f} \cos \psi_f - \lambda_{2f} \tan \chi_f = 0 \tag{72}$$

$$\lambda_{30} - \lambda_{20} \tan \psi_0 = 0 \tag{73}$$

$$\lambda_{3f} - \lambda_{2f} \tan \psi_f = 0 \tag{74}$$

$$\frac{\lambda_{10}}{\sin \chi_0} = \frac{\lambda_{1f}}{\sin \chi_f} \tag{75}$$

$$\left(\lambda_{10}'\right)^a \pm 1 = \lambda_{20}\left(2\sigma^b \cos \theta + \frac{|\Delta \underline{V}|}{\rho} \cos \chi \cos \psi\right)$$

$$+ \lambda_{30}\left(2\delta^b + \frac{|\Delta \underline{V}|}{\rho} \cos \chi \sin \psi\right) \tag{76}$$

where the following notation has been used

$$\lambda_{10} = \lambda_1\left(t_0\right) \tag{77a}$$

$$\lambda_{1f} = \lambda_1\left(t_f\right) \tag{77b}$$

Equations (71) to (74) result from evaluation of equation (15) at the initial and terminal values of the coasting arc. Equation (75) was obtained by evaluating equation (20) at both impulsive times and by eliminating the magnitude of the primer vector between the two relationships. This is possible because the magnitude of the primer vector must take on identical values at the two impulsive times, as is indicated by examination of equations (14) and (17). Equation (76) is equation (8) evaluated after the first impulse, with λ_4 having been eliminated by use of equations (41), (65), and (66).

After transformation of this set of equations into the altitude, range, and track coordinate frame by means of equations (58) to (61), the constants $C_1, C_2, \ldots, C_6$ can then be evaluated by inversion of the matrix of coefficients A_{ij} and by solution of the following equation:

$$C_i = A_{ij}^{-1} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \pm 1 \end{vmatrix} \quad i, j = 1, 2, \ldots, 6 \tag{78}$$

The proper sign is determined from the relationships established in equations (65) and (66). It is noted that the values of the λ_4 and λ_6 multipliers after the first impulse were required for a solution of the initial values of the Lagrange multipliers.

CONVERGENCE TECHNIQUE

With a technique for generating the initial values of the Lagrange multipliers from an approximate closed-form solution, a convergence process may be established for the exact solution. The basis for convergence is that the impulsive (approximate) and nonimpulsive (exact) trajectories have solution curves which are similar, such that an identical change in the initial values of each will produce approximately the same change in the terminal boundary conditions. These terminal boundary errors can be nullified by generation of corrections to the initial values from the impulsive solution. Convergence will occur if assumptions of the impulsive solution are not violated excessively.

Figure 3 is a logical flow chart of the iterative technique used in the convergence process. A switch k governs the various phases of the solution.

With the desired terminal boundary conditions and the switch k equal to 0, the impulsive solution is solved for the initial values of the Lagrange multipliers $\underline{\hat{\lambda}}_R$ and is denoted as the reference solution. If the nonimpulsive system of equations is integrated to a terminal cutoff using the initial values $\underline{\hat{\lambda}}_R$, the resulting terminal vector $\underline{B}_I$ should resemble the desired terminal boundary conditions. The error in these conditions is indicative of the accuracy of the approximate solution. After replacement of the desired boundary conditions by those obtained from the nonimpulsive solution $\underline{B} = \underline{B}_I$ the impulsive solution is resolved for the initial vector $\underline{\hat{\lambda}}$. The switch k is set equal to +1. In the impulsive solution, a change in the initial vector of

$$\Delta\underline{\hat{\lambda}} = \underline{\hat{\lambda}} - \underline{\hat{\lambda}}_R \tag{79}$$

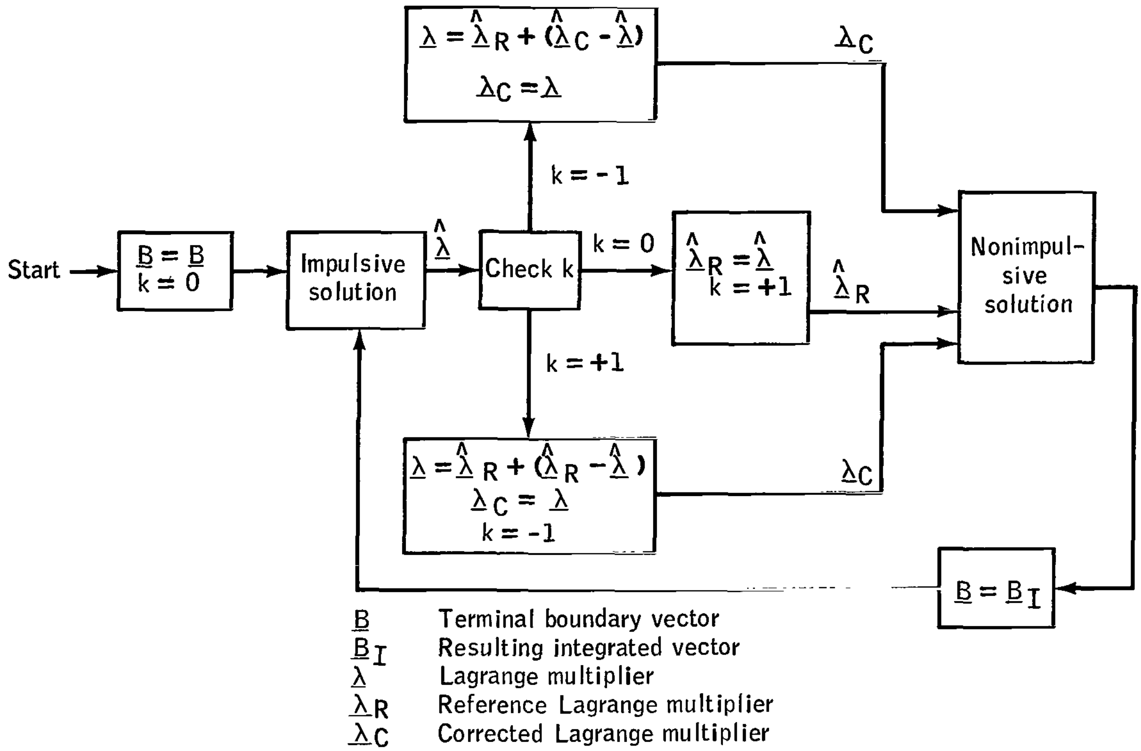

Figure 3. - The use of an impulsive solution in a convergence technique.

has produced a change in the terminal conditions of

$$\Delta \underline{B} = \underline{B}_I - \underline{B} \tag{80}$$

Because the impulsive solution is an approximation to the exact system of equations, it is assumed that the negative of this correction should produce an approximate change in the terminal boundary conditions of the nonimpulsive solution. Therefore, the corrected initial values of the Lagrange multipliers are

$$\underline{\lambda}_C = \hat{\underline{\lambda}}_R - \Delta\hat{\underline{\lambda}} \tag{81}$$

The switch k for the next and all future passes is set equal to -1. The procedure is repeated with one exception. The correction to the reference vector $\hat{\underline{\lambda}}_R$ is made with respect to the previous integrated solution $\underline{\lambda}_C$. This correction is given by

$$\Delta\hat{\underline{\lambda}} = \hat{\underline{\lambda}} - \hat{\underline{\lambda}}_C \tag{82}$$

The solution is continued in this manner until the terminal boundary conditions approach the desired conditions within some arbitrary tolerance.

EXAMPLE PROBLEM

To adequately demonstrate this solution technique, a fuel-optimum Earth-to-Mars transfer problem is solved for a wide range of initial thrust accelerations. The solution is initiated on Julian date 2 442 680 with a 213-day transfer time. The exhaust velocity is fixed at 1.0231545 EMOS (Earth mean orbital speed). Convergence was rapidly obtained for thrust accelerations ranging from the impulsive case to a value of 5.0×10^{-4}g $\left(1\text{g} = 32.2 \text{ ft/sec}^2\right)$. Solutions for accelerations less than this value could not be obtained without modifying the covergence technique. The variation in the initial values of the Lagrange multipliers with initial thrust acceleration for this transfer problem is illustrated in figure 4. The multipliers can be noted to diverge from their impulsive values as the initial thrust acceleration is decreased. It may appear that the initial values of the Lagrange multipliers are insensitive to the initial thrust acceleration. In this respect, this figure is misleading. The sensitivity of the multipliers to changes in the initial conditions is obscured by the nondimensionalization of the problem variables. Because, for this solution technique, only the mass multiplier λ_μ is a function of T/m_0 and the remaining lambdas use the impulsive solution multipliers as starting values, convergence will be limited to thrust-acceleration levels which do not violate excessively the assumptions of the analysis. For example, the impulsive solution is based on a thrusting logic cycle of thrust-coast-thrust. If, for a finite acceleration level, the starting values produce a solution which does not complete this cycle before reaching a terminal cutoff condition, convergence will not occur.

The significance of figure 4 should be reemphasized concerning the problems to which this solution technique is applicable. It will be recalled that the equations developed in this analysis were formulated in a nondimensional form. In this formulation, the particular example problem is immaterial with respect to the developed equations. The characteristics of the Lagrange multipliers for the investigated range of the initial thrust acceleration are shown in figure 4.

A measure of the efficiency of the solution technique is the number of iterations and the time required for a convergence to a desired set of boundary conditions. An iteration is defined as the number of corrections that are made to the multipliers such that the terminal constraints are satisfied on the succeeding solution within an arbitrary tolerance. For this example problem, a uniform tolerance of 2.0×10^{-4} was selected because all variables are nondimensionalized. The data for each iteration of the Earth-to-Mars transfer with an initial thrust acceleration of 9.0×10^{-4}g are

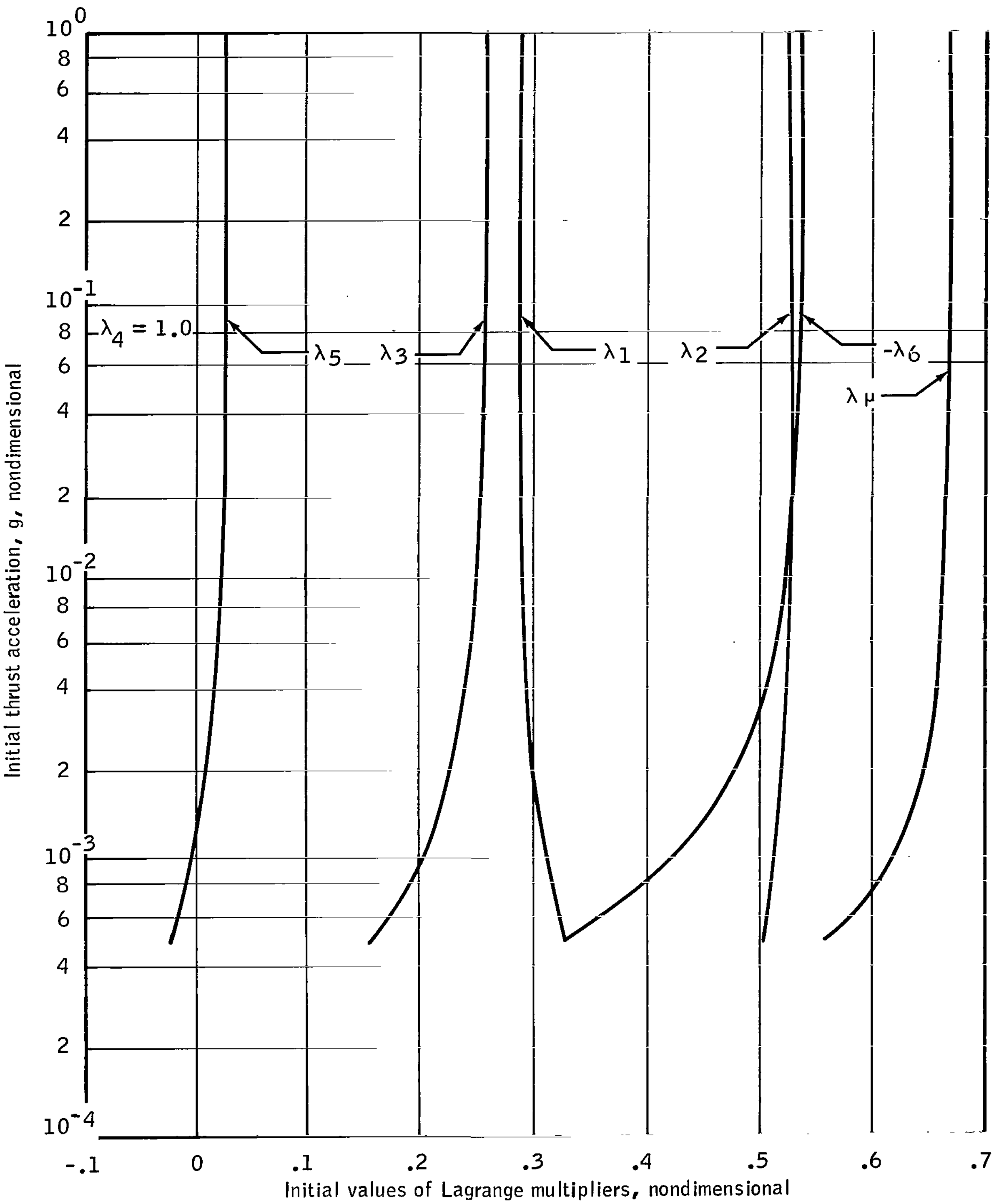

Figure 4. - Variation of initial values of Lagrange multipliers with thrust acceleration for a 213-day Earth-to-Mars transfer; Julian date 2 442 680, v_e = 1.0231545 Earth mean orbital speed.

TABLE I. - EARTH-TO-MARS TRANSFER (213-DAY)[a]

Variables	Iteration no.						
	0	1	2	3	4	5	6
V_F, EMOS	0.70447625	0.70764348	0.73613807	0.73759156	0.7394565	0.7394178	0.7395386
γ_F, deg	3.5489735	2.6832596	1.4632210	1.3113635	1.236200	1.233388	1.2288496
A_{ZF}, deg	90.521965	90.098216	90.118342	90.123595	90.12412	90.12409	90.12396
r_F, AU	1.7477922	1.7384044	1.6700365	1.6666388	1.662387	1.6624722	1.6621979
θ_F, deg	2.5647125	1.8270476	1.9067470	1.8395678	1.849077	1.844119	1.8451757
ϕ_F, deg	143.07035	142.06986	143.11270	143.15160	143.2138	143.2086	143.21273
$\lambda_{10} \cdot 10^1$	2.8987639	3.2015208	3.1592476	3.1478997	3.145361	3.144586	3.144495
$\lambda_{20} \cdot 10^1$	5.2501955	5.1380339	5.1221021	5.1420125	5.1414238	5.1427927	5.142652
$\lambda_{30} \cdot 10^1$	2.5655474	1.7708472	2.0219506	1.9684969	1.990323	1.985170	1.987014
$\lambda_{50} \cdot 10^1$	.24761119	-.09333626	.00145934	.00513479	.01298817	.01273934	.01327756
$\lambda_{60} \cdot 10^1$	-5.3477658	-3.6911911	-4.2379409	-4.092365	-4.1393818	-4.126159	-4.130388
$\lambda\mu_o \cdot 10^1$	6.4003868	6.0136106	6.1206711	6.1079195	6.117931	6.1162759	6.1170768
t, days	219.61825	219.15351	213.76202	213.37727	213.02954	213.02164	213.00003

[a]Julian date 2 442 680.

listed in table I. It is noted that convergence to the desired boundary conditions is rapid and uniform. Computer time for this solution on a CDC 3600 electronic data processing machine was approximately 2 seconds.

An initial thrust acceleration of 5.0×10^{-4}g required 20 iterations and a solution time of approximately 7 seconds. This indicates that, as the assumptions of the impulsive solution are violated, the number of iterations and the time required for a solution rapidly increase.

For a comparison with other convergence techniques, an example problem with an initial thrust acceleration of 3.1×10^{-1}g was solved using a first-order perturbation technique. This convergence method required six times as much computer time as the present technique.

CONCLUSIONS

A practical technique has been presented for obtaining solutions to optimum trajectory problems with bounded thrust magnitude by use of an impulsive solution to generate the starting values of the Lagrange multipliers.

This simplifies the problem by eliminating the necessity of choosing the initial values of the Lagrange multipliers. The convergence scheme employs the impulsive solution to generate corrections to the initial multipliers. The technique is demonstrated by solving an Earth-to-Mars transfer problem for a wide range of initial thrust accelerations. Computer (CDC 3600) solution time for the specific case of the thrust-to-mass ratio $T/m_o = 9.0 \times 10^{-4}$g was 2 seconds. The solution time was shown to increase rapidly as the assumptions of the impulsive solution are violated. In a comparison case with a $T/m_o = 3.1 \times 10^{-1}$g, the convergence scheme was six times more efficient than a first-order perturbation technique.

Lawden's equations for the Lagrange multipliers were used on all nonthrusting arcs. This facilitated a rapid solution of the Euler-Lagrange equations and eliminated the need for integrating on coasting arcs.

For a problem in which the variables have been nondimensionalized, the initial value of the Lagrange multipliers appear to be constant for a wide range of value of T/m_o.

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National Aeronautics and Space Administration
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APPENDIX

DERIVATION OF NONDIMENSIONAL EQUATIONS

The equations of motion in spherical coordinates of a mass particle acted upon by the accelerations of gravity and thrust T/m are

$$\dot{v} - \frac{\left(u^2 + w^2\right)}{r} + \frac{\mu_G}{r^2} - \frac{T}{m} \sin \chi = 0 \tag{83}$$

$$\dot{u} + \frac{u(v - w \tan \theta)}{r} - \frac{T}{m} \cos \chi \cos \psi = 0 \tag{84}$$

$$\dot{w} + \frac{\left(vw + u^2 \tan \theta\right)}{r} - \frac{T}{m} \cos \chi \sin \psi = 0 \tag{85}$$

$$\dot{r} - v = 0 \tag{86}$$

$$r\dot{\phi} \cos \theta - u = 0 \tag{87}$$

$$r\dot{\theta} - w = 0 \tag{88}$$

$$\dot{m} + \frac{T}{V_e} = 0 \tag{89}$$

where χ and ψ are arbitrarily chosen control variables and V_e is the exhaust velocity, assumed constant. Choosing a reference position r_R and a reference velocity V_R, defined as circular velocity at the position r_R,

$$V_R = \left(\frac{\mu_G}{r_R}\right)^{1/2} \tag{90}$$

the nondimensional equations of motion are

$$Z_v' - \frac{\left(Z_u^2 + Z_w^2\right)}{\rho} + \frac{1}{\rho^2} + \frac{\mu'}{\mu} v_e \sin \chi = 0 \tag{91}$$

$$Z_u' + Z_u \frac{\left(Z_v - Z_w \tan \theta\right)}{\rho} + \frac{\mu'}{\mu} v_e \cos \chi \cos \psi = 0 \tag{92}$$

$$Z_w' + \frac{\left(Z_v Z_w + Z_u^2 \tan \theta\right)}{\rho} + \frac{\mu'}{\mu} v_e \cos \chi \sin \psi = 0 \tag{93}$$

$$\rho' - Z_v = 0 \tag{94}$$

$$\rho\phi' \cos \theta - Z_u = 0 \tag{95}$$

$$\rho\theta' - Z_w = 0 \tag{96}$$

$$\mu' + \beta = 0 \tag{97}$$

The superscript prime refers to the nondimensional time

$$\tau = \frac{2\pi t}{P} = \frac{V_R t}{r_R} \tag{98}$$

where P is the period of the circular orbit. The new variables are defined by the following equations.

$$v_e = \frac{V_e}{V_R} \tag{99}$$

$$\rho = \frac{r}{r_R} \tag{100}$$

$$Z_v = \frac{v}{V_R} \tag{101a}$$

$$Z_u = \frac{u}{V_R} \tag{101b}$$

$$Z_w = \frac{w}{V_R} \tag{101c}$$

$$\mu = \frac{m}{m_o} \tag{102}$$

$$\beta = - \frac{\dot{m}}{m_o} \frac{r_R}{V_R} \tag{103}$$

The equations of motion are further transformed by letting

$$\sigma = \phi' = \frac{Z_u}{\rho \cos \theta} \tag{104}$$

$$\delta = \theta' = \frac{Z_w}{\rho} \tag{105}$$

This change of variables results in the following nondimensional equations in terms of the variables Z_v, σ, δ, ρ, ϕ, θ, and μ.

$$Z_v' - \rho\left(\sigma^2 \cos^2 \theta + \delta^2\right) + \frac{1}{\rho^2} + \frac{\mu'}{\mu} v_e \sin \chi = 0 \tag{106}$$

$$\rho\sigma' \cos \theta + 2\sigma\left(Z_v \cos \theta - \rho\delta \sin \theta\right) + \frac{\mu'}{\mu} v_e \cos \chi \cos \psi = 0 \tag{107}$$

$$\rho\delta' + 2\delta Z_v + \rho\sigma^2 \sin \theta \cos \theta + \frac{\mu'}{\mu} v_e \cos \chi \sin \psi = 0 \tag{108}$$

$$\rho' - Z_V = 0 \tag{109}$$

$$\phi' - \sigma = 0 \tag{110}$$

$$\theta' - \delta = 0 \tag{111}$$

$$\mu' + \beta = 0 \tag{112}$$

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