TG #29

MEASUREMENT OF THE EINSTEIN EFFECT AND OTHER MEASUREMENTS REQUIRING EXTREMELY ACCURATE DETERMINATION OF STELLAR POSITIONS OR MOTIONS

H. Suhl

August, 1966

Contract No. NSR-24-005-047

Prepared by

UNIVERSITY OF MINNESOTA
Minneapolis, Minnesota

For

HEADQUARTERS, NATIONAL AERONAUTICS & SPACE ADMINISTRATION
Washington, D. C. 20546

MEASUREMENT OF THE EINSTEIN EFFECT AND OTHER MEASUREMENTS REQUIRING EXTREMELY ACCURATE DETERMINATION OF STELLAR POSITIONS OR MOTIONS

ABSTRACT

It appears that the main factor limiting the accuracy of measurement of the Einstein light bending carried out from a spacecraft will be the rotational Brownian motion induced by micrometeorite impacts. The attainable accuracy should be about one half of one percent, using reasonable restoring torques of the order of 200 kg cm/radian to correct the deviation.

PRECEDING PAGE BLANK NOT FILMED.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

MEASUREMENT OF THE EINSTEIN EFFECT AND OTHER MEASUREMENTS REQUIRING EXTREMELY ACCURATE DETERMINATION OF STELLAR POSITIONS OR MOTIONS

H. Suhl

August, 1966

INTRODUCTION

The bending of light rays passing by a massive object is predicted by general relativity to be of amount

$$\frac{4\,GM}{c^2 r} \text{ radians}$$

where M is the mass of the object, G the gravitational constant, and r the perpendicular distance from the center of M to one of the asymptotes of the ray path (which is a hyperbola). For light grazing the edge of the sun, this angle is 1.75 seconds of arc.

Attempts to measure this small angle have so far been carried out only during solar eclipse and they indicate that a deflection of this magnitude probably occurs. The accuracy of these measurements is optimistically put at 10 to 20% by some, and questioned altogether by others. Obviously one would like to build up much more convincing statistics than is possible with solar eclipses. The obvious way to do this is to obscure the sun with an occulting disc that is part of the instrument. Hitherto the problem with this method has been light scattering and refraction by the atmosphere, which makes the seeing and accurate localization of stars very difficult. One would, therefore, like to avoid all or part of the atmosphere, perhaps by placing the instrument aboard a spacecraft.

A very ambitious attempt to overcome atmospheric problems and perform the experiment from the ground with an occulting disc is currently being undertaken by Dr. Henry A. Hill, Wesleyan University and University of

Arizona. Using photodetectors, and sophisticated optics, he hopes to be able to follow sixth magnitude stars, to within about one and a half solar radii of the sun. This would enable him to measure about two stars a month. He hopes to attain about 1% accuracy.

If no unforeseen problems arise and Hill succeeds in this and related measurement within the next few years, there will obviously be little reason to perform this experiment in space. The fact remains, however, that considerably cruder apparatus aboard a spacecraft can attain a somewhat higher accuracy, probably about three parts in a thousand, backed by some impressive statistics.

LIGHT DEFLECTION MEASUREMENTS MADE FROM A SPACECRAFT

In the following analysis we shall assume that in a spacecraft the method of measurement will be exactly the same as proposed by Hill, except that the optical system will not require the elaborate provisions needed for compensating atmospheric dispersion, and that the whole system including the occulting disc can be considerably smaller, because of the lower signal discrimination requirements in the absence of an atmosphere. We will concentrate on the specific problems arising from the space environment. The main problem requiring attention is the question of pointing accuracy. For a spacecraft in solar orbit, servoed to a certain orientation, errors in pointing accuracy could arise from:

1) Signal to noise ratio of the electro-optical sensor
2) Rotation and vibration due to micrometeorite impacts (translation does not cause pointing error)
3) Flexure of the instrument due to gravitational gradients
4) Torques on the spacecraft due to differential radiation pressure

In Earth orbit an additional problem might arise from variation of temperature gradient due to thermal radiation from the Earth.

In connection with Point 1 it turns out that even with simplest possible pointing system based on an occulting disc in front of the Sun, one can easily discriminate angles of one millisecond of arc, this limitation coming from background due to diffraction around the disc and from corona.

That Point 2 requires careful examination follows from the fact that micrometeorites in the .01 to .02 cm range, of which there are about eight to ten a day, will induce angular rotation velocities of one or two milliseconds of arc per second in a spaceship weighing 200 kg, and with dimensions of the order of one meter. Thus, Point 2 imposes minimal requirements on the order of one meter. Thus, Point 2 imposes minimal requirements on the servo system. Elastic vibrations due to meteorite impact, even if totally unchecked by special mounting of the instrument, etc., turn out to be negligible.

Points 3 and 4 turn out to have completely negligible effect on the pointing accuracy.

We now consider these points in detail.

FACTORS DETERMINING POINTING ACCURACY

Signal to Noise Ratio of the Electro-Optical Sensor

For an order of magnitude estimate it is sufficient to consider the arrangement shown in Figure 1. It consists of a strip detector normal to the plane of the paper and parallel to it a semi-infinite screen rigidly attached to the detector. The detector area is assumed negligibly small. The angle between the normal from the detector to the screen, and a line from the detector to the edge of the screen is just slightly in excess of half the angle $2\eta_{\odot}$ subtended by the Sun at the detector. In zero error position, the normal from the detector to the screen is assumed to pass through the center of the Sun's disc.

The actual arrangement will differ from this model by having a circular disc of angle $2\eta_\odot$ in front of the detector, and by having the detector divided into four or more sectors whose outputs are added up. Any rotation of the instrument, say through angle φ about an axis normal to the line of center will induce a signal in one of the sectors, which will activate the servoes. However, we use the simplified model to calculate the minimum detectable φ. Let x be the distance from the point 0 (Fig. 1) to a point P in the plane of the screen, but off the screen. The signal at this point produced by an element of solar disc of angular width 2η is proportional to

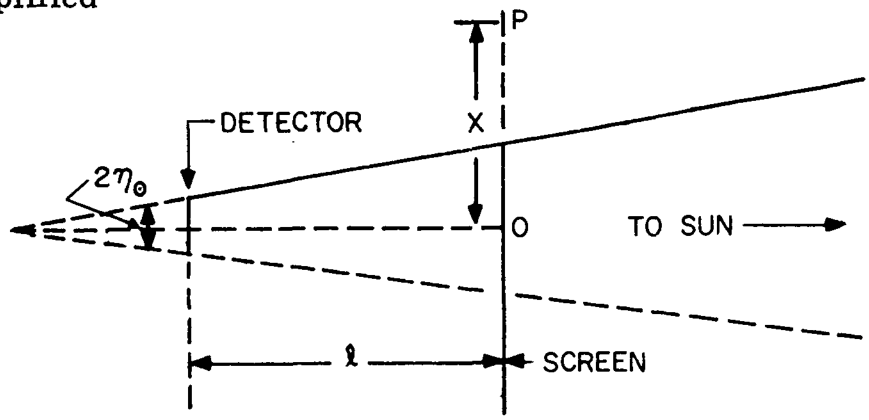

Figure 1. Occulting Geometry

$$\exp \frac{2\pi i x}{\lambda}$$

where for convenience we measure phases relative to the phase of the grazing ray. This signal in turn produces a signal at the detector with a further phase delay

$$\frac{2\pi}{\lambda}\sqrt{\ell^2 + x^2} - \frac{2\pi}{\lambda}\sqrt{\ell^2 + d^2} \doteq \frac{\pi}{\lambda}\frac{x^2}{\ell} - \frac{\pi}{\lambda}\ell\eta_\odot^2 \quad \text{if } x \ll \ell$$

Thus the total signal from x is proportional to $\exp \frac{2\pi i}{\lambda}\left(x\eta + \frac{x^2}{2e}\right)$ aside from an x-independent phasefactor. If we were to consider a rotationally symmetric geometry, this result would have to be multiplied by $2\pi x dx$, and integrated. In our present simplified geometry we integrate

$\exp \frac{2\pi i}{\lambda} \left(x\eta + \frac{x^2}{2e} \right)$ from $\ell(\eta_\odot - \varphi)$, the edge of the screen in error position φ, up to the width X of the aperture, and multiply by the length of the aperture normal to the plane of the paper. Thus we find a total signal proportional to

$$\int_{\ell(\eta_\odot - \varphi)}^{X} e^{\frac{2\pi i}{\lambda}\left(x\eta + \frac{x^2}{2\ell}\right)}\, dx \doteq \int_{\sqrt{\frac{\pi\ell}{\lambda}}(\eta + \eta_\odot - \varphi)}^{\infty} e^{i\xi^2}\, d\xi \qquad (1)$$

provided $x \gg \sqrt{2\ell\lambda}$. Those rays from the Sun that do not lie in the plane of the paper have to the lowest order in η the same phase delay with respect to the reference ray as do their projections onto the plane of the paper. Therefore to add up the incoherent contributions from all elements of the Sun, we square (1), integrate η from $-\eta_\odot$ to $\eta_\odot$, and multiply the result by a quantity of order $\eta_\odot$. Finally the result is normalized to the power received in the absence of the screen ($\varphi = \infty$). Thus we receive at the detector a fraction

$$\int_{-\eta_\odot}^{+\eta_\odot} \left| \int_{\sqrt{\frac{\pi\ell}{\lambda}}(\eta + \eta_\odot - \varphi)}^{\infty} e^{i\xi^2}\, d\xi \right|^2 d\eta \Bigg/ 2\eta_\odot \int_{-\infty}^{+\infty} e^{i\xi^2}\, d\xi \qquad (2)$$

of the "wide open" flux.

Let $\ell = 10^2$ cms, $\lambda = 10^{-4}$ cms. Then $\sqrt{\pi\ell/\lambda} \sim 1.7 \times 10^3$. Also $\eta_\odot = 4 \times 10^{-3}$ rad so that for $\eta \sim 0$, $\varphi \sim 0$, the inner integral is

$$\int_{6\cdot 8}^{\infty} e^{i\xi^2}\, d\xi$$

whose absolute value is quite small. In other words, because $\sqrt{\frac{\pi \ell}{\lambda}}$ is large most of the contribution to the integral comes from the range $\sqrt{\frac{\pi \ell}{\lambda}}(\eta + \eta_\odot - \varphi) < 1$ i.e. $-\eta_\odot < \eta < \sqrt{\frac{\lambda}{\pi \ell}} + \varphi - \eta_\odot$, and its value in that region is approximately

$$1/4 \left| \int_{-\infty}^{+\infty} e^{i\xi^2} \, d\xi \right|^2$$

Thus we get

$$\left(\sqrt{\frac{\lambda}{\pi \ell}} + \varphi \right) \Big/ 8\eta_\odot$$

for the ratio of powers at error position φ.

The flux from the Sun is 4.1×10^{10} photons of visible light per second per arc second^2 of Sun per cm^2 of receiving area. Taking a detector of area A cm^2, the "wide open" flux from the whole Sun (10^6 arc sec^2) is

$$4 A \times 10^{16} \text{ photons/sec.}$$

and so, in error position φ we have an incoming flux

$$n(\varphi) = 4 \times A \times 10^{16} \, \frac{\varphi + \sqrt{\frac{\lambda}{\pi \ell}}}{8\eta_\odot} + n_{cor} \quad .$$

where n_{cor} is the flux from the corona, of order $4 \times A \times 10^{10}$ photons/sec. If the detector has an integration time T, we require for easy detectability

$$\left\{ n(\varphi) - n(0) \right\} T \gg \sqrt{n(0)\, T}$$

since we must discriminate against the fluctuation in background flux $\sqrt{n(0)T}$.

Then we find

$$\frac{\varphi}{8\eta_\odot} \gg \frac{1}{2} \times 10^{-8} \sqrt{\frac{\sqrt{\lambda/\pi\ell}}{8\eta_\odot} + 10^{-6}} \Big/ \sqrt{AT}$$

or, since $\sqrt{\lambda\pi\ell} \gg 8\eta_\odot 10^{-6}$,

$$\varphi \gg \frac{4 \times 10^{-8}}{\sqrt{8}} \sqrt{\eta_\odot \frac{\lambda}{\pi e}} \Big/ AT$$

For $A = 1\ cm^2$, $T = 1$ sec, this number is of order 4×10^{-11} = 4 microseconds of arc. However, we have assumed an infinitely sharply defined Sun matched perfectly by the occulting disc. If instead we assume that in a small angular distance φ, the solar flux increases from essentially 0 to $\alpha\,\varphi$, where $\alpha = 4 \times 10^{16}$/arc sec $\div\ 10^{20}$/radian, then we require

$$\frac{\alpha\varphi^2}{8\eta_\odot} \gg 2 \times 10^8 \quad \frac{(\lambda/\pi\ell)^{1/4}}{8\eta_\odot\,(AT)^{1/2}}$$

or

$$\varphi \gg 10^{-7}/\sqrt{AT}$$

In that case, then, an integration time of 100 seconds and an area $A = 1\ cm^2$ will give an accuracy of order two milliseconds of arc. Of course, by more elaborate arrangements, to remove part of the diffracted light, this figure may be improved, probably two or three orders of magnitude. Thus, it is evident that pointing accuracy is no problem so far as the detecting system is concerned.

To estimate the root mean square pointing error $\sqrt{\langle \Phi^2 \rangle}$ due to micrometeorites, we consider a simple model of a spacecraft with all three principal moments of inertia equal. Then the equation of motion is

$$I \ddot{\vec{\Phi}} = \sum_{t_i < t} \vec{r}_i \times \vec{F}_i(t) = \vec{C}(t), \text{ say} \tag{3}$$

where I is the moment of inertia, and

$$\vec{F}_i(t) = m_i \vec{V}^{(i)} \delta(t - t_i)$$

is the force at time t due to a meteor of mass m_i impinging at time t_i with velocity $\vec{V}^{(i)}$ at position $\vec{r}_i$ on the craft. As already noted in the introduction, it is necessary to correct the drift in Φ that would result from equation (3). This we assume is done by a control system which exerts a restoring torque whose magnitude is proportional to the average of Φ over a time t to t - T, where T is the integration time already mentioned.

Then we must solve

$$I \ddot{\vec{\Phi}}(t) + \frac{K}{T} \int_{t-T}^{t} \Phi(t')\, dt' = \vec{C}(t)$$

Taking Fourier transforms of both sides, we find

$$\left(\text{with } \Phi(\omega) = \int \frac{d\omega}{\sqrt{2\pi}} e^{i\omega t} \Phi(t)\, dt \right), \text{ that}$$

$$\Phi(\omega) = - \frac{C(\omega)}{I\omega^2 + \frac{K}{iT\omega}(1 - e^{i\omega T})}$$

The standard theory of random processes may now be used to evaluate $\langle \Phi^2(t) \rangle$. This requires evaluation of the correlation function

$$\gamma_{\alpha\beta} = \sum_{\substack{\alpha'\alpha'' \\ \beta'\beta''}} \epsilon_{\alpha\alpha'\alpha''} \epsilon_{\beta\beta'\beta''} \sum_{ij} \left\langle m_i m_j V_{\alpha''}^{(i)} V_{\beta''}^{j} r_{\alpha'}(t_i) r_{\beta'}(t_j) \right\rangle \delta(t'-t_i)\delta(t''-t_j)$$

where $\mathcal{E}_{\alpha\alpha'\alpha''}$ is the usual antisymmetric symbol, and the subscripts indicate cartesian components. Assuming no correlation between any of the random quantities, we immediately find

$$\gamma_{\alpha\beta} = \gamma \cdot \delta_{\alpha\beta}$$

with $\gamma = 2\,\delta\,(t-t')\,\langle X^2 \rangle\, S \int N(mV)\, m^2 V_X^2\, dm d\vec{v}$

where

$$\langle X^2 \rangle = \int_{\text{surface of craft}} X^2 dS \Big/ \int dS$$

(X being the x coordinate of a surface element dS), and where N(mV) is the number of micrometeorites in mass and velocity ranges (m, m + dm) and (V, V + dV) impinging per second per cm^2. Then we have

$$\langle \Phi_\alpha^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\left| I\omega^2 + \frac{K}{i\omega T}\left(1-e^{i\omega t}\right)\right|^2} \cdot 2\,\langle X^2 \rangle\, S \int N(m,v) m^2 v^2 dm d\vec{v} \tag{4}$$

To a good approximation, the meteors may be considered to have velocities all very nearly equal to 3×10^5 cms/sec. Also, their mass distribution falls off rapidly with increasing mass. For meteors in the size range $1\mu < r < 5$ cms the flux density is about

$$N(m)dm = \frac{6.2 \times 10^{-17}}{m^{2.4}} \, dm$$

per cm^2 per second. Clearly, impacts less frequent than the typical period needed for following one star in traversing, say, six solar radii should not be considered on a statistical basis at all. This will take about three days, or 2.4×10^5 seconds. The number of meteors of size bigger than m hitting the craft (of 100 cm dimension) in this period is about

$$\frac{6.2 \times 2.4 \times 10^{-17} \times 10^5}{1.4 \; m^{1.4}} \times 10^4$$

and m_{max} is determined by equating this to unity. This gives $m_{max} = 10^{-7.15}$ gms. Also, the velocity distribution is (with $V_o = 3 \times 10^5$ cms/sec)

$$\frac{1}{4\pi} \; \frac{\delta(|v| - v_o)}{|v|^2} \; dv_x dv_y dv_z$$

and so

$$\int N(m, v) \, m^2 v_x^2 \, dm d\vec{v} = 1/3 \int N(m, v) \, m^2 V^4 dv$$

$$= 1/3 \; V_o^2 \int_o^{m_{max}} \frac{6.2 \times 10^{-17}}{m^{0.4}}$$

$$= 1/2 \; V_o^2 \times \frac{6.2 \times 10^{-17}}{.6} \times 10^{-6.6/1.4}$$

$$= 5 \times 10^{-7.7}$$

Also, for a sphere of radius R

$$\langle x^2 \rangle = R\frac{2}{3}$$

so that

$$\langle x^2 \rangle S = \frac{4\pi}{3} R^4 = 2.5 \times 10^7$$

with R = 50 cms. Thus we obtain, from equation(4)

$$\langle \Phi^2 \rangle \sim 3 \times 10^{-.7} \int_{-\infty}^{+\infty} \frac{d\omega}{\left| I\omega^2 + \frac{K}{i\omega t} (1-e^{i\omega t}) \right|^2}$$

To avoid numerical evaluation of the integral we consider the limiting cases $T << \sqrt{I/K}$, i.e. $\frac{1}{T}$ much greater than the natural period of oscillation $\sqrt{K/I}$ of the craft due to elastic binding by the servo), and $T >> \sqrt{I/K}$. In the former case the integral is

$$\int_{+\infty}^{+\infty} \left[(I\omega^2 - K)^2 + \frac{\omega^2 T^2 K^2}{4} \right]^{-1} d\omega = \frac{\pi}{2K^2 T}$$

if $T << \sqrt{I/K}$. Thus in this case

$$\langle \Phi^2 \rangle \doteqdot 3 \times 10^{-.7} \frac{\pi}{4K^2 T}$$

Setting $T = \frac{1}{z} \sqrt{I/K}$, where $z >> 1$ we get

$$\langle \Phi^2 \rangle \sim 3 \times 10^{-.7} \frac{\pi z}{2K\sqrt{KI}}$$

With $I = 2 \times 10^9$ gm cm^2 (200 kg with dimensions of one meter), and $K = 5 \times 10^7$ dynes cm/radian = 50 kg cm/radian, we find

$$\langle \Phi^2 \rangle \sim \frac{3 \times nz \times 10^{-15.7}}{2 \times 3 \times 5}$$

$$\sqrt{\langle \Phi^2 \rangle} \sim \frac{\sqrt{z}}{5} \times 10^{-7}$$

For $z \approx 4$, this gives 15 milliseconds of arc. Correspondingly $T \sim \frac{10}{1.5z}$ seconds. Hence in this regime the servo must supply a fairly substantial restoring torque; in fact 200 kg cm/radian are needed to hold $\sqrt{\Phi^2}$ to four milliseconds of arc.

On the other hand, for $T \gg \sqrt{I/K}$, it is evident that $K(1 - e^{i\omega t})/i\omega T$ is appreciably different from zero only when $|\omega| < \frac{1}{T}$. In that range, the supposedly large K makes the integrand vanish. The approximate value of the integral is then

$$2 \int_{\frac{1}{T}}^{\infty} \frac{d\omega}{I^2 \omega^4} = \frac{2}{3} \frac{T^3}{I^2}$$

Thus

$$\langle \Phi^2 \rangle = 1.6 \times 10^{-.7} \times \frac{T^3}{I^2}$$

Setting

$$T = z \sqrt{\frac{I}{K}}$$

where $z \gg 1$, we get

$$\langle \Phi^2 \rangle = 1.6 \times 10^{-.7} z^3 \frac{1}{K} \frac{1}{\sqrt{KI}}$$

with $I = 2 \times 10^9$ gm cm^2 and $K \quad 10^9$ dyne cm/radian $\sim$ 1000 kg cm/radian, we get

$$\sqrt{\langle \Phi^2 \rangle} \sim 1.6 z^3 \times 10^{-9.35}$$

for $z \sim 4$, this gives

$$\sqrt{\Phi^2} \sim 15 \text{ milliseconds of arc}$$

Evidently this regime requires even higher restoring torques than the short time constant regime.

Vibrations Due to Meteor Impacts

Next we consider the value of $\langle \Phi^2 \rangle$ due to elastic vibrations set up by meteor impacts.

We confine the analysis to a linear structure. The equation is

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - c_s^2 \frac{\partial^2 u}{\partial x^2} = \frac{F(t, x)}{\rho}$$

where u is the lateral displacement at position x and time t, c_s is the sound velocity, α a damping parameter needed to limit resonances, ρ the mass per unit length, and $F(t, x)$ the force density per unit length, at time t. We have, in terms of meteors impinging at times t_i,

$$F(t,x) = \sum_{t_i<t} m_i V_i \delta(t - t_i) \delta(x - x_i) = \sum_i f_i(t) \delta(x - x_i), \text{ say}$$

where now V_i is the velocity component perpendicular to the linear structure.

Taking Fourier transforms in time, we have

$$(-\omega^2 - i\omega) u(x,\omega) - c_s^2 \frac{\partial^2 u(x,\omega)}{\partial x^2} = \sum_i \frac{f(\omega) \delta(x - x_i)}{\rho}$$

The ends of this structure are free, and therefore, this equation must be solved subject to $\frac{\partial u}{\partial x} = 0$ at $x = -\ell, +\ell$ where 2ℓ is the length of the structure. It is sufficient to solve

$$(-\omega^2 - i\alpha\omega) u_1(x,\omega) - c_s^2 \frac{\partial^2 u}{\partial x^2} = \frac{f_1(\omega)}{\rho} \delta(x - x_1) \qquad (5)$$

The complete answer follows by superposition. Actually, we must subtract out the center of gravity motion so that $F(t,x)$ should be replaced by

$F(t,x) - \frac{1}{2\ell} \int_{-\ell}^{+\ell} F(t,x)\, dx$, but the same result is achieved by ultimately calculating $u(x,\omega) - \frac{1}{2\ell} \int_{-\ell}^{+\ell} u(x,\omega)\, dx$. The solution of equation (5) which satisfies $u_1(\pm\ell,\omega) = 0$ is

$$u_1 = -f_1(\omega) \cos \frac{\Omega}{c_s}(x_1 + \ell) \cos \frac{\Omega}{c_s}(x - \ell) / \Omega c_s \sin(2\ell/c_s)$$

$$= -f_1(\omega) \cos \frac{\Omega}{c_s}(x_1 - \ell) \cos \frac{\Omega}{c_s}(x + \ell) / \Omega c_s \sin(2\ell\Omega/c_s)$$

$$-\ell < x < x_1$$

where $\Omega = \sqrt{\omega^2 + i\,\alpha\,\omega} \triangleq \omega + \frac{i\,\alpha}{2}$. We are interested only in the relative displacement of the occulting disc (at $x = \ell$) and the detector (at $x = -\ell$). This displacement is

$$\Delta_1(\omega) = u_1(\omega\ell) - u_1(\omega_1 - \ell) = \frac{f_1(\omega)}{\rho}\ \frac{\sin\frac{\Omega}{c_s}\ell \sin\frac{\Omega}{c_s} x_1}{\Omega\, c_s \sin 2\,\ell\,\Omega/c_s}$$

$$\triangleq \frac{f_1(\omega) \sin\left(\frac{\omega}{c_s}\ell\right) \sin\left(\frac{\omega}{c_s} x_1\right)}{\rho\,\omega\, c_s\left[\sin\frac{2\ell\omega}{c_s} + i\,\frac{2\ell}{c_s}\cos\frac{2\ell\omega}{c_s}\right]}$$

In the last formula we have retained α only where needed to prevent an infinite resonance. The mean square angle is now

$$\langle\Phi^2\rangle = \left\langle \sum_i \Delta_i(t)^2 \right\rangle \Big/ 4\ell^2$$

and proceeding exactly as before we find

$$\left\langle \Delta^2(t) = \left(\sum \Delta_i\right)^2 \right\rangle = \frac{S}{2\pi} \int N(mV)\, m^1 V^2\, dm\, dV$$

$$x \int_{-\infty}^{+\infty} \frac{d\omega\left(\sin\frac{2\omega}{c_s}\ell\right)\left(\frac{1}{2\ell}\int_{-\ell}^{+\ell} \sin^2\frac{\omega}{c_s} x_1\, dx_1\right)}{\rho^2\,\omega^2\, c_s^2\left[\sin^2\frac{2\ell\omega}{c_s} + \frac{4\ell^2\alpha^2}{c_s^2}\cos^2\frac{2\ell\omega}{c_s}\right]}$$

where S is the total surface area of the linear structure. After the integration over x_1 is performed, it is noted that for small α, most of the contribution to the frequency integral comes from $\omega < \alpha$, and so its approximate value is

$$\frac{1}{2}\ \frac{c_s^2}{4\ell^2\alpha^2}\int_{-\infty}^{+\infty}\frac{d\omega\,\sin^2\frac{\omega\ell}{c_s}}{\omega^2} = \frac{\pi}{16}\ \frac{c_s}{\ell\,\alpha^2}$$

and the result is

$$\langle\Phi^2\rangle = \frac{\ell}{32\alpha^2 M^2 c_s}\int N\,(mV)\,m^2V^2\,dm\,dv$$

where M is the total mass of the structure. For a $Q = (\omega\alpha)^{-1} \sim$ one in the lowest mode ($\omega = \pi\, c_s/2\ell$), for $M \sim 2 \times 10^5$ gms, $c_s = 10^5$ cms/sec, $\ell = 10^2$ cms, and the same upper limit on meteor sizes as was used before $\sqrt{\langle\Phi^2\rangle}$ is of the order of 10^{-8} radians $\sim$ 2 milliseconds, even if no special arrangements for protecting the apparatus are made.

Gravitational Gradient

The effect of a gravitational gradient is not a random effect and is easily calculated from elasticity theory to give an angular deflection:

$$\Phi = \frac{1}{6}\cdot\left(\frac{1+\sigma}{E}\right)\ell\,\rho\,\sin(2\,\theta)\ \frac{\partial^2 V}{\partial r^2}$$

where θ is the inclination of the linear structure to the radius vector drawn from the center of the source of the gravitational field V to the mass center of the craft. σ is Poissons ratio, E Young's modulus, and ρ is the mass p.u. length. For standard materials

$$\frac{E}{1+\sigma} \sim 10^{11}\ \text{dynes/cm}^2$$

In the case of solar orbit, with same radius as Earth

$$\frac{\partial^2 V}{\partial r^2} - \frac{GM_\odot}{r^3} \sim 10^{-13}\ \mathrm{sec}^{-2}$$

For $\ell = 10^2\ \mathrm{cm}^2$, $\rho = 2\ \mathrm{kg/cm} = 2 \times 10^3\ \mathrm{gm/cm}$,

$$\Phi \sim 10^{-17},$$

completely negligible.

Radiation Pressure

To assess the torque exerted by radiation pressure (likewise not a random quantity), we make the pessimistic assumption that the craft has the geometry of a paddle wheel. The solar flux is 4.1×10^{16} photons/cm^2/sec. The momentum per photon is $\hbar\gamma_{\mathrm{visible}}/c = \hbar/\lambda_{\mathrm{visible}} \sim 2 \times 10^{-23}$ gm cm/sec, and so the total momentum destroyed per second is 8.2×10^{-7} dynes/cm^2. Roughly the maximum torque is of order $8.2 \times 10^{-7} \cdot A \cdot \ell$ where A is the area, ℓ the linear dimension of the spacecraft. For $\ell = 10^2$cm, $A = 10^4$cm^2, the maximum torque exerted by the radiation pressure is .82 dynes cm. With a servo torque of order 10^4 gms cm/radian $= 10^7$ dynes cm/radian, we would thus get a steady state angular deflection of $.41 \times 10^{-7}$ radians or 10 milliseconds of arc.

This is the most extreme result to be expected from radiation pressure. Some very elementary precautions would reduce this deflection by a very large factor.

Stability Against Rolling

So far we have considered only pointing stability towards the Sun. Rolling about an axis through the center of the Sun's disc would produce no error signal. If we seek three permil accuracy it is obviously necessary to prevent rolling to this degree of accuracy. Thus, we can tolerate rolling through about 1/3 of a degree. But this stability is readily achieved by using one additional star as control.

Orbital Stability Under Meteor Impact

In Hill's proposed system, a star is tracked into the Sun, the apparent distance of its image from that of the Sun being measured interferometrically. From the apparent variation of its rate of motion, the value of the Einstein shift can be calculated, if the velocity of the spacecraft is known. This velocity fluctuates under meteor impact. However, the fluctuation is totally negligible. Only about one meteor with mass as high as 10^{-4} gms hits a spacecraft of area 10^4 cm^2 with a speed of 3×10^5 cm/sec in one day. A spacecraft of 2×10^5 gms thus acquires a velocity increment of only $\sim 10^{-4}$ cms/sec. In solar orbit the craft moves with a speed of about 1/4 miles/sec, of the order of 4×10^4 cms/sec. The error introduced in this way is therefore only one part in 10^{-8}.

CONCLUSION

It appears that the main factor limiting the accuracy of measurement of the Einstein light bending carried out from a spacecraft will be the rotational Brownian motion induced by micrometeorite impacts. The attainable accuracy should be about one half of one percent, using reasonable restoring torques of the order of 200 kg cm/radian to correct the deviation.