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NEWTON'S METHOD AND THE OPTIMIZATION OF NONLINEAR SYSTEMS

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Newton's Method for the solution of n simultaneous nonlinear equations in n unknowns is applied to the variational two-point boundary value problem arising in trajectory optimization. A 3-dimensional low-thrust Earth-Mars trajectory is used as an example problem to illustrate the computational algorithm.

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1. Introduction.

Newton's Method[1] is one of the most powerful methods for the solution of n simultaneous nonlinear equations in n unknowns. When applied to the equations

$$V_i(y_j) = 0 \qquad (i,j = 1,2,\ldots, n)$$

the method takes the form

$$y^{(\ell+1)} = y^{(\ell)} - A^{-1}(y^{(\ell)})V(y^{(\ell)}) \qquad \ell = 0,1,2,\ldots \qquad (1)$$

where y_j and V_i are components of the column vectors y and V respectively; $A(y)$ is an $n \times n$ matrix whose (i,j)-th element is $\partial V_i/\partial y_j$; and $y^{(\ell)}$ denotes the ℓ-th iterant of y. One of the characteristics of Newton's Method is its quadratic convergence rate, provided the initial guess $y^{(0)}$ is "sufficiently close" to the solution y^*. Further details are given in the Kantorovich Theorem[1].

Recently, a generalized form of Newton's Method, under the names of quasilinearization or generalized Newton-Raphson[2], has been applied in the solution of the two-point boundary value problems arising in trajectory optimization. The basis of the generalized Newton-Raphson procedure is the linearization of the differential equations describing the trajectory (state variables and Lagrange multipliers). Roberts and Shipman[3] show that the perturbation method of Goodman and Lance[4] is also a form of the general Newton's Method.

It is the purpose of this note to show how Newton's Method, Eq. (1), can be directly applied to the two-point boundary value problems arising in trajectory optimization.

2. Variational two-point boundary value problem.

Suppose that a trajectory optimization problem is given in the form of minimizing the functional

$$\Gamma = G(X(t_f),t_f) + \int_0^{t_f} Q(X,U,t)dt \tag{2}$$

subject to the conditions that

$$S_i(X(0)) = 0 \qquad (i = 1,2,\ldots, q \leq n) \tag{3}$$

$$R_j(X(t_f),t_f) = 0 \qquad (j = 1,2,\ldots, r \leq n+1) \tag{4}$$

$$\dot{X} = \frac{dX}{dt} = f(X,U,t) \tag{5}$$

where $X = (X_1, X_2,\ldots, X_n)^T$ is the state and $U = (U_1,U_2,\ldots, U_m)^T$ is the control. It is known that the solution of this problem is given in terms of the differential equations

$$\dot{X} = f(X,U,t) \tag{6a}$$

$$\dot{P} = -\left(\frac{\partial f}{\partial X}\right)^T P - \left(\frac{\partial Q}{\partial X}\right)^T = g(X,U,P,t) \tag{6b}$$

where U is determined from the equation

$$0 = P^T \frac{\partial f}{\partial U} + \frac{\partial Q}{\partial U} ,$$

and $P = (P_1, P_2, \ldots, P_n)^T$. Assume that the $m \times m$ matrix $\frac{\partial}{\partial U}(P^T \frac{\partial f}{\partial U} + \frac{\partial Q}{\partial U})$ is positive definite. Then one can solve for U in terms of X, P, and t, and then substitute for U in Eq. (6) to obtain

$$\dot{X} = \bar{f}(X,P,t) \tag{7a}$$

$$\dot{P} = \bar{g}(X,P,t) . \tag{7b}$$

The natural boundary conditions from the variational problem supply (1) n boundary conditions at $t = 0$, with n of the X_j and P_i unknown, (2) $n+1$ boundary conditions at t_f, with t_f and n of the X_i and P_j unknown. Therefore, the boundary conditions can be written as

$$\bar{S}_k(X(0),P(0)) = 0 \quad k = 1,2,\ldots, n$$

$$\bar{R}_j(X(t_f),P(t_f),t_f) = 0 \quad j = 1,2,\ldots, n+1 .$$

The problem could be solved if the correct values of the unknowns at $t = 0$ and the value of the terminal time t_f were known. Let these unknowns form the $(n+1)$-vector C, with $C_{n+1} = t_f$. A change of independent variable will be made using a device due to Long[5].

Let $t = C_{n+1}s$ $(0 \leq s \leq 1)$. Then the differential equations (7) can be written as

$$\frac{dX}{ds} = C_{n+1}\,\bar{f}(X,P,C_{n+1}s) \tag{8a}$$

$$\frac{dP}{ds} = C_{n+1}\bar{g}(X,P,C_{n+1}s) \tag{8b}$$

$$0 \leq s \leq 1$$

and the boundary conditions become

$$\left.\begin{aligned} X_i(0) &= J_i(C_1,C_2,\ldots,C_n) \\ P_i(0) &= J_{i+n}(C_1,C_2,\ldots,C_n) \end{aligned}\right\} \quad i = 1,2,\ldots,n$$

$$\bar{R}_j(X(1),P(1),C_{n+1}) = 0 \qquad (j = 1,2,\ldots,n+1)\ .$$

Introducing the notation $Y^T = (X^T,P^T)$ the two-point boundary value problem becomes

$$\frac{dY}{ds} = F(Y,C,s) \quad , \quad 0 \leq s \leq 1$$

$$Y(0) = J(C)$$

$$V(Y(1),C) = 0$$

which is in form that can be used for solution by Newton's Method. Notice that C and V have the same dimensionality.

3. Solution by Newton's Method.

A two-point boundary value problem can be stated as follows. Determine the constant vector B so that the vector $Z(t)$ satisfies the following equations on the interval $0 \leq t \leq 1$:

$$Z(0) = K(B) \tag{9}$$

$$\dot{Z} = \frac{dZ}{dt} = F(Z,B,t) \tag{10}$$

$$L(Z(1),B) = 0 \tag{11}$$

where K, Z and F are column vectors of dimension n; L and B are column vectors of dimension m.

Equation (11) represents m simultaneous equations in the m unknowns B_j, since $Z(1)$ can be expressed as a function of B from Eq. (9) and Eq. (10). Apply Newton's Method [Eq. (1)]

to Eq. (11) to obtain

$$B^{(\ell+1)} = B^{(\ell)} - A^{-1}(B^{(\ell)})L^{(\ell)} \tag{12}$$

where

$$L^{(\ell)} = L(Z(1),B^{(\ell)})$$

$$A(B) = \left(\frac{\partial L}{\partial Z}\frac{\partial Z}{\partial B} + \frac{\partial L}{\partial B}\right)_{t=1} \tag{13}$$

$\frac{\partial L}{\partial Z}$ is an $m \times n$ matrix whose (j,i)-th element is $\partial L_j/\partial Z_i$;
$\frac{\partial Z}{\partial B}$ is an $n \times m$ matrix whose (i,k)-th element is $\partial Z_i/\partial B_k$;
$\frac{\partial L}{\partial B}$ is an $m \times m$ matrix whose (j,k)-th element is $\partial L_j/\partial B_k$;
and $(j,k = 1,2,\ldots, m)$, $(i = 1,2,\ldots, n)$. The matrices $\partial L/\partial Z$ and $\partial L/\partial B$ can be obtained from Eq. (11) by taking the appropriate derivatives. The matrix $\partial Z/\partial B$ is obtained by writing Eq. (10) in integral form and then differentiating with respect to B. Thus

$$Z(t) = Z(0) + \int_0^t F(Z,B,s)ds \tag{14}$$

and

$$\frac{\partial Z(t)}{\partial B} = \frac{\partial Z(0)}{\partial B} + \int_0^t \left(\frac{\partial F}{\partial Z}\frac{\partial Z}{\partial B} + \frac{\partial F}{\partial B}\right)ds \ . \tag{15}$$

If Eq. (15) is differentiated with respect to t then the matrix $\partial Z/\partial B$ will satisfy the matrix differential equation

$$\frac{d}{dt}\frac{\partial Z}{\partial B} = \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial B} + \frac{\partial F}{\partial B} \tag{16}$$

$$\frac{\partial Z(0)}{\partial B} = \frac{\partial K}{\partial B} \tag{17}$$

where

$\frac{\partial F}{\partial Z}$ is an $n\times n$ matrix whose (i,j)-th element is $\partial F_i/\partial Z_j$;
$\frac{\partial F}{\partial B}$ is an $n\times m$ matrix whose (i,k)-th element is $\partial F_i/\partial B_k$;
$i,j = 1,2,\ldots, n$ and $k = 1,2,\ldots, m$.

Suppose that an initial guess at B is made, $Z(0)$ is computed from Eq. (9), and Eq. (10) is integrated from $t = 0$ to $t = 1$. The value of $L(Z(1),B)$ will generally not be zero. (A trajectory for which $L \neq 0$ will be called a nominal trajectory.) Equation (16) is integrated from $t = 0$ to $t = 1$, with initial conditions given by Eq. (17). In Eq. (16) the matrices $(\partial F/\partial Z)$ and $(\partial F/\partial B)$ are evaluated on the nominal trajectory. Then $(\partial Z/\partial B)$ will be the rate of change of Z due to a change in B, on the nominal trajectory. Thus $(\partial Z/\partial B)$ at $t = 1$ is obtained, $A(B)$ can be evaluated, and a new value of B can be obtained from Eq. (12) provided $A(B)$ is nonsingular.

If the differential equations (10) are highly nonlinear then the change in B, as given by Eq. (18), may be too large.

$$b = -A^{-1}(B^{(\ell)})L^{(\ell)} . \tag{18}$$

This occurs when the norm of $L^{(\ell+1)}$ is greater than the norm of $L^{(\ell)}$. In this case the following scheme is employed.

$$B^{(\ell+1)} = B^{(\ell)} + \alpha b \tag{19}$$

where $0 < \alpha \leq 1$. The "best" value of α must be determined empirically for each iteration. The use of α in the computational algorithm is a realization of Theorem 3 in the paper by Moore[6].

The computational algorithm is summarized as follows. Set $\ell = 0$ and choose $B^{(0)}$.

(I) Integrate Eq. (10) from $t = 0$ to $t = 1$ with initial conditions given by $Z(0) = K(B^{(\ell)})$.

(II) Determine $L(Z(1),B^{(\ell)}) = L^{(\ell)}$ and the norm of $L^{(\ell)}$, denoted by $\|L^{(\ell)}\|$. If $\|L^{(\ell)}\|$ is less than some preassigned ε then $B^{(\ell)}$ is the accepted solution. If $\|L^{(\ell)}\| > \varepsilon$ then

(1) if $\ell = 0$ go to step (IV)

(2) if $\ell > 0$ go to step (III) .

(III) Compare $\|L^{(\ell)}\|$ and the previous norm, W.

(1) $\|L^{(\ell)}\| \geq W$. Decrease α and form $B^{(\ell)} = C + \alpha b$. Go to step (I).

(2) $\|L^{(\ell)}\| < W$. Go to step (IV).

(IV) Set C equal to $B^{(\ell)}$, α equal to unity and W equal to $\|L^{(\ell)}\|$.

(V) Integrate Eq. (16) from $t = 0$ to $t = 1$ with initial condition given by $\frac{\partial Z(0)}{\partial B} = \frac{\partial K(B^{(\ell)})}{\partial B}$.

(VI) Form $A(B^{(\ell)})$ according to Eq. (13) and solve the linear system $A(B^{(\ell)})b = - L^{(\ell)}$ for b.

(VII) Form $B^{(\ell+1)} = B^{(\ell)} + \alpha b$. Add 1 to the value of ℓ. Go to step (I).

This computational algorithm was programmed in FORTRAN for the CDC 6600 Computer at the University of Texas. The problem given in the next section was used to test the effectiveness of the algorithm.

4. Example problem.

A low-thrust Earth-Mars trajectory is sought. The vehicle is assumed to travel in an inverse square gravitational field. The orbit of Mars is assumed to be an ellipse with an eccentricity of $e = 0.093393$, a semi-major axis of $a = 1.523691$ AU (astronomical units), lying in a plane which is inclined to the ecliptic at an angle of $i = 0.032289$ radians. The equations of motion which describe the transfer trajectory are expressed in a heliocentric rectangular cartesian coordinate system whose X_4-axis coincides with the line of ascending node for the Mars orbit. The X_5-axis lies in the Ecliptic plane and the X_6-axis coincides with the angular momentum vector of the earth with respect to the sun. Letting (X_1, X_2, X_3) and (X_4, X_5, X_6) be the velocity and position components respectively in the (X_4, X_5, X_6)-coordinate system, the equations of motion are

$$\frac{dX_1}{d\tau} = -\gamma R^{-3} X_4 + \Omega \cos U_1 \cos U_2$$

$$\frac{dX_2}{d\tau} = -\gamma R^{-3} X_5 + \Omega \cos U_1 \sin U_2$$

$$\frac{dX_3}{d\tau} = -\gamma R^{-3} X_6 + \Omega \sin U_1$$

$$\frac{dX_4}{d\tau} = X_1$$

$$\frac{dX_5}{d\tau} = X_2$$

$$\frac{dX_6}{d\tau} = X_3$$

where $R^2 = X_4^2 + X_5^2 + X_6^2$ and $\Omega = \beta c/(1-\beta\tau)$ on the interval $0 \leq \tau \leq \tau_f$.

For the units chosen in the problem:

time is days

position is in AU

velocity is in AU/day

mass is in vehicle mass

with

$\tau = 0 =$ (12:00 noon May 9, 1971)

$\beta = 0.00108$ vehicle mass/day

$c = 0.0453649854$ AU/day

$\gamma = 0.000296007536$ AU^3/day^2 .

The control angles are shown in Fig. 1. The initial conditions at $\tau = 0$ are

$X_1(0) = -0.0003455906$

$X_2(0) = -0.0171986836$

$X_3(0) = 0.0$

$X_4(0) = -0.9998$

$X_5(0) = 0.02009$

$X_6(0) = 0.0$.

The terminal conditions are $X_i(\tau_f) - Y_i(\tau_f) = 0 \quad i = 1,2,\ldots,6$ with $Y_i = \frac{dY_{i+3}}{d\tau} \quad i = 1,2,3$.

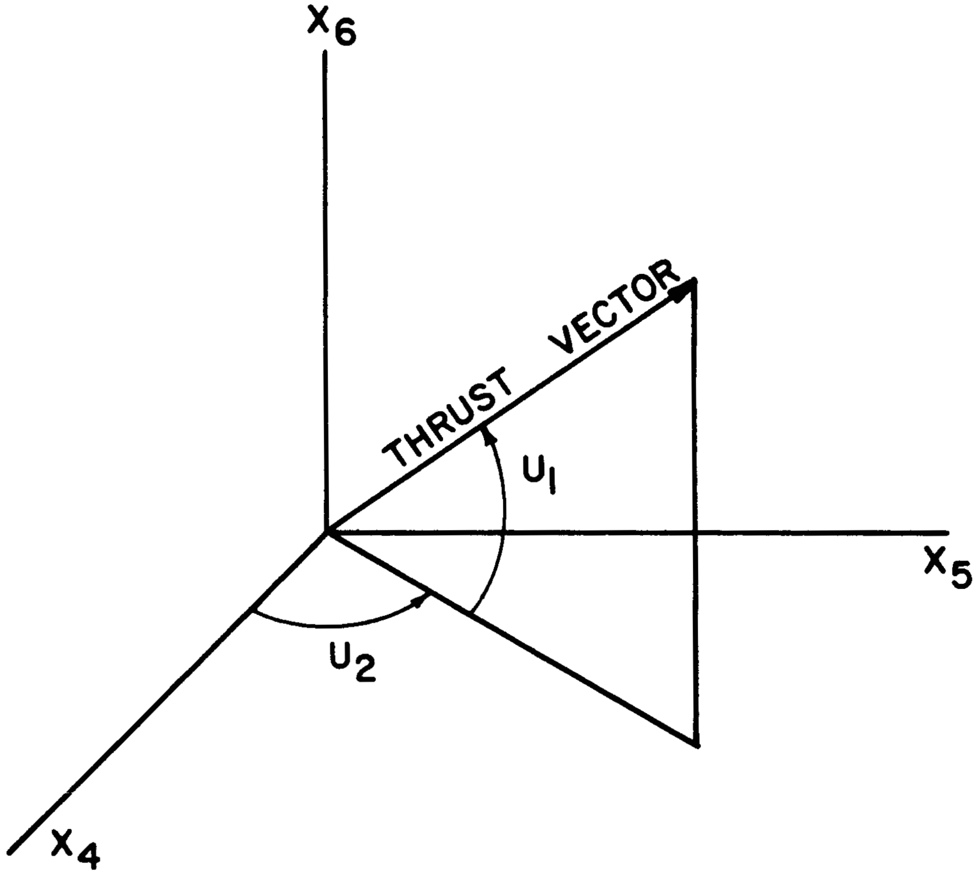

FIGURE 1: CONTROL ANGLES

The position of Mars at time τ is given by

$$Y_4(\tau) = k_{11}D_1 + k_{12}D_2$$
$$Y_5(\tau) = k_{21}D_1 + k_{22}D_2$$
$$Y_6(\tau) = k_{31}D_1 + k_{32}D_2$$

where
$$k_{11} = \cos\omega$$
$$k_{12} = -\sin\omega$$
$$k_{21} = (\cos i)(\sin\omega)$$
$$k_{22} = (\cos\omega)(\cos i)$$
$$k_{31} = (\sin i)(\sin\omega)$$
$$k_{32} = (\cos\omega)(\sin i)$$
$$D_1 = a(\cos E - e)$$
$$D_2 = a(\sin E)(1-e^2)^{1/2}$$
$\omega = 5.8541335$, the argument of perihelion of Mars at $\tau = 0$.

E is the eccentric anomaly of Mars. It satisfies Kepler's equation $(\tau \geq 0)$

$$E - e(\sin E) = \tau(\gamma/a^3)^{1/2} + E_0 - e(\sin E_0)$$
$$E_0 = 4.250885 .$$

It is required that the vehicle arrive at Mars with maximum mass. Therefore the quantity $\beta\tau_f - 1$ is to be minimized. The equations governing the Lagrange multipliers are

$$\frac{dP_i}{d\tau} = -P_{i+3}$$

$$i = 1,2,3$$

$$\frac{dP_{i+3}}{d\tau} = \gamma R^{-3} P_i + hX_{i+3}$$

$$h = -3\gamma R^{-5}[P_1 X_4 + P_2\ X_5 + P_3\ X_6]\ .$$

The controls U_1 and U_2 are given by

$$\sin(U_1) = -P_3/\Delta$$
$$\cos(U_1) = \delta/\Delta$$
$$\sin(U_2) = -P_2/\delta$$
$$\cos(U_2) = -P_1/\delta$$
$$\Delta^2 = \delta^2 + P_3^2$$
$$\delta^2 = P_1^2 + P_2^2\ .$$

Thus U_1 and U_2 can be eliminated from the differential equations in X. In addition to the 6 terminal boundary conditions given by $X_i(\tau_f) - Y_i(\tau_f) = 0$ the variational analysis gives the additional boundary condition (corresponding to τ_f being free)

$$\beta + \sum_{i=1}^{6} P_i\left(\frac{dX_i}{d\tau} - \frac{dY_i}{d\tau}\right) = 0 \quad \text{at} \quad \tau = \tau_f\ .$$

The unknowns are $P_i(0)$, $(i = 1,2,\ldots, 6)$, and τ_f. Make the change of variable $\tau = B_7 t$, $(0 \leq t \leq 1)$, and let $P_i(0) = B_i$ $(i = 1,2,\ldots, 6)$. In the notation of Section 3, with $Z_i = X_i$, $Z_{i+6} = P_i$ $(i = 1,2,\ldots, 6)$ the differential equations become $(\dot{Z} = \frac{dZ}{dt})$

$$\dot{Z}_i = -B_7[\gamma Z_{i+3} R^{-3} + \frac{\beta c Z_{i+6}}{(1-\beta B_7 t)v}] = F_i$$

$$\dot{Z}_{i+3} = B_7 Z_i = F_{i+3}$$

$$\dot{Z}_{i+6} = -B_7 Z_{i+9} = F_{i+6}$$

$$\dot{Z}_{i+9} = B_7[\gamma Z_{i+6} R^{-3} + h Z_{i+3}] = F_{i+9}$$

for $i = 1,2,3$

where

$$R^2 = Z_4^2 + Z_5^2 + Z_6^2$$
$$v^2 = Z_7^2 + Z_8^2 + Z_9^2$$
$$h = -3\gamma R^{-5}[Z_7 Z_4 + Z_8 Z_5 + Z_9 Z_6] .$$

The initial conditions are

$$\left.\begin{aligned} Z_i(0) &= X_i(0) \\ Z_{i+6}(0) &= B_i \end{aligned}\right\} \quad i = 1,2,\ldots, 6 .$$

The terminal conditions at $t = 1$ are

$$L_i = Z_i(1) - Y_i(B_7) = 0 \qquad i = 1,2,\ldots, 6$$

$$L_7 = \beta + \sum_{i=1}^{6} Z_{i+6}(F_i B_7^{-1} - \frac{dY_i}{dB_7}) = 0 \ .$$

In the numerical solution of this problem the following initial values were guessed

$$B_1 = 0.3455906$$
$$B_2 = 17.1986836$$
$$B_i = 0.0 \qquad (i = 3,4,5,6)$$
$$B_7 = 184.0 \ .$$

These values of B_1, B_2, B_3 aligned the initial thrust direction along the vehicle's velocity vector.

The factor α was determined by the following procedure. A number $r>1$ was chosen; then α was set equal to r^{-k} for $k = 0,1,2,\ldots, p$, where p is the least integer for which $\|L^{(\ell+1)}\| < \|L^{(\ell)}\|$. (Other procedures such as a Fibonaccian search are possible.)

Several different computer runs were made. In each run the process was terminated whenever $\|L^{(\ell)}\|$ became less than 10^{-11}. The error norm was chosen as

$$\|L\| = \left(\sum_{i=1}^{7} L_i^2\right)^{1/2} .$$

Convergence curves for four different values of r are shown in Fig. 2. Each of the four curves contains a plateau region in which the terminal error norm decreases very slowly. In this region some of the B_i $(1 \leq i \leq 6)$ were negative (note that the converged values, B_i^*, given below, are positive). At the end of each plateau region, all B_i were positive. Further data is given in Tables 1 and 2. Table 1 shows that the speed of computation is dependent on the choice of r.

The converged values B_j^* are

$$\begin{aligned}
B_1^* &= 14.065205938 \\
B_2^* &= 17.179740719 \\
B_3^* &= 1.7012931523 \\
B_4^* &= 0.39534671015 \\
B_5^* &= 0.18375894864 \\
B_6^* &= 0.0021643563219 \\
B_7^* &= 175.46074200 \quad .
\end{aligned}$$

The Earth-to-Mars transfer is accomplished in 175.460742 days.

Table 1: Convergence Data

r	Iterations	Computing Time in Seconds
2.0	41	308.502
3.0	46	289.495
3.5	23	138.029
4.0	31	187.668

Table 2: Last ten iterations for r = 3.5

Iteration Number	Terminal Error Norm
14	1.731×10^{-1}
15	1.627×10^{-1}
16	1.535×10^{-1}
17	1.323×10^{-1}
18	7.909×10^{-2}
19	1.425×10^{-2}
20	4.214×10^{-4}
21	7.765×10^{-7}
22	8.951×10^{-11}
23	1.286×10^{-14}

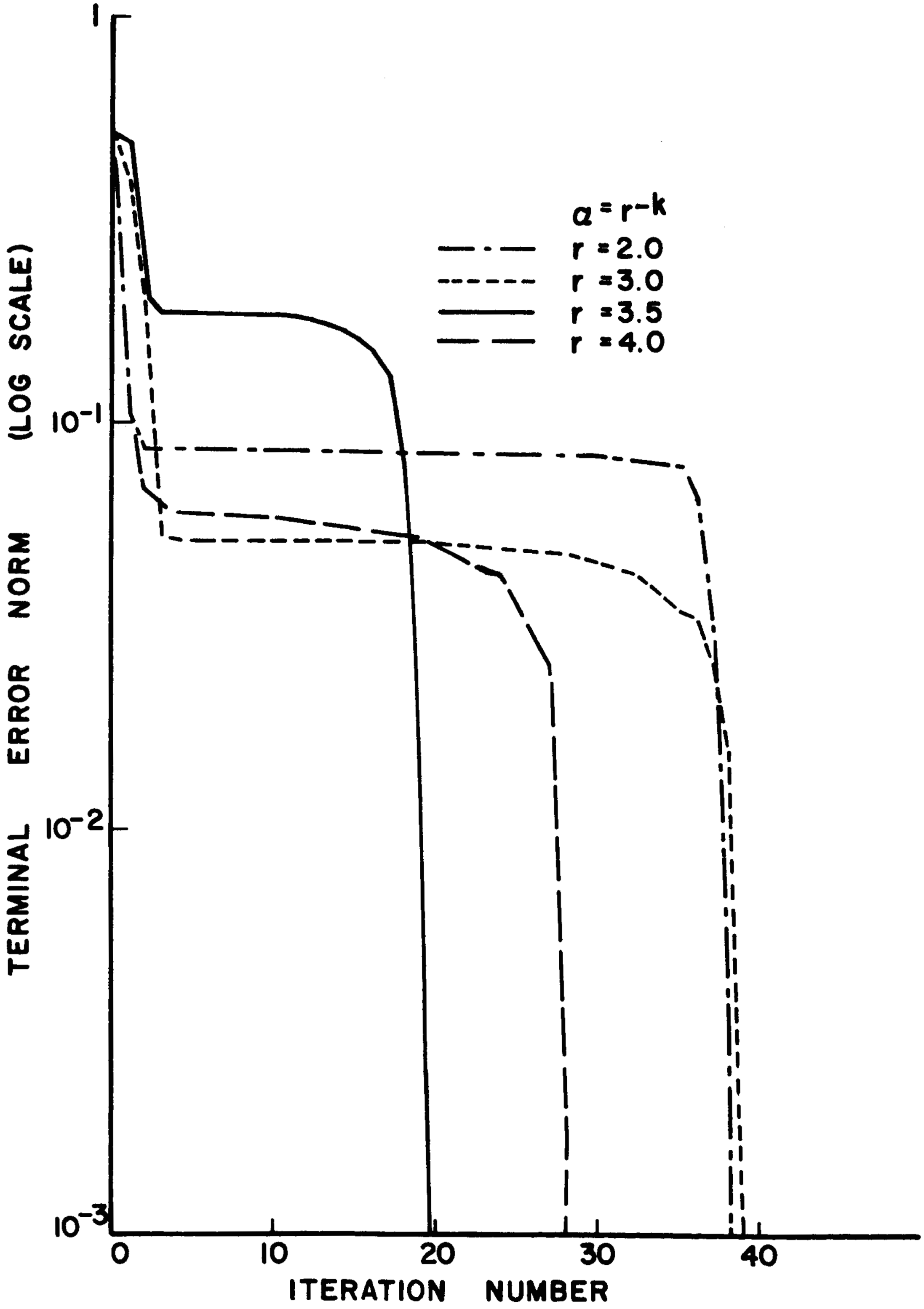

FIGURE 2 : CONVERGENCE CURVES

5. Conclusions.

Formulating a trajectory optimization problem in the notation of Section 3 allows one to make direct use of Newton's Method. The resulting trajectory optimization scheme is a rapidly converging computational procedure as was shown in the example problem. Further extensions of this method to problems with intermediate boundaries (stages) will be treated in a later report.

REFERENCES

1. Henrici, P.,
Discrete Variable Methods in Ordinary Differential Equations, (Wiley, New York, 1962), pp. 366-371.

2. McGill, R., and Kenneth, P.,
"Solution of Variational Problems by Means of a Generalized Newton-Raphson Operator", A.I.A.A. Journal 2, 1761 - 1766, (1964).

3. Roberts, S. M., and Shipman, J. S.,
"The Kantorovich Theorem and Two-Point Boundary Value Problems", I.B.M. Journal of Research and Development 10, 402-406, (1966).

4. Goodman, T. R., and Lance, G. N.,
"The Numerical Integration of Two-Point Boundary Value Problems", Math. Tables and Other Aids to Computation 10, 82-86, (1956).

5. Long, R. S.,
"Newton-Raphson Operator: Problems with Undetermined End Points", A.I.A.A. Journal 3, 1351-1352, (1965).

6. Moore, R. H.,
"Newton's Method and Variations", Nonlinear Integral Equations, edited by P. M. Anselone, (University of Wisconsin Press, 1964), pp. 65-98.