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DAVID STERN

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by
David Stern
Laboratory for Theoretical Studies

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ABSTRACT

The equation of drift shells, traced by the guiding center motion of charged particles moving in a magnetic field, is discussed in terms of Euler potentials α and β. Particular attention is given to fields deviating to a limited extent from a dipole configuration, for which it is shown that the result is related to the drift-shell parameter L, with an added "shell splitting function" G_1. A perturbation method approximately deriving G_1 is described: it leads to results similar to those found by Pennington in his perturbation derivation of drift shells. The use by Pennington of a divergent expansion and the $I = 0$ limit of the equations obtained are also discussed.

EULER POTENTIALS AND GEOMAGNETIC DRIFT SHELLS

INTRODUCTION

A drift shell (sometimes termed "magnetic shell") is defined as the surface traced by the guiding center of a magnetically confined charged particle, as derived from the guiding center approximation of its motion. By this approximation, a particle generally follows magnetic field lines, and therefore such lines will be tangential to drift shells.

Magnetic field lines may be compactly described by the use of Euler potentials α and β

$$\underline{\mathbf{B}} = \nabla\alpha \times \nabla\beta \tag{1}$$

and therefore it may be surmised that drift shells are also best expressed by such potentials. It is the purpose of this work to approximately derive this relationship for drift shells in the geomagnetic field. For that field, approximate Euler potentials may be derived by perturbation (Stern, 1967; the notation and results of that work will be freely used here) from the spherical harmonic expansion of the geomagnetic scalar potential γ. A similar derivation allowing for external sources of the geomagnetic field is also possible; its various properties, including the form of drift shells in that case, will be discussed in a separate article.

With one exception, previous derivations of drift shells in the geomagnetic field have involved extensive numerical calculation of line integrals, by means of large digital computers. The exception is the perturbation derivation by R. Pennington (1967; performed in 1960 for the Argus experiment), of which unfortunately only a short summary (Pennington, 1961) has appeared in the periodical literature. The results obtained here are fully equivalent to those of Pennington; their derivation, however, is through the use of Euler potentials, which represents the more natural approach to problems of this sort. Certain mathematical aspects, not treated by Pennington, are also clarified.

FIRST-ORDER EULER POTENTIALS

Let the geomagnetic field $\underline{B}$ be given through its scalar potential γ

$$\underline{B} = -\nabla\gamma \tag{2}$$

and let γ be split (assuming tilted dipole coordinates) into a dipole component γ_0 and a sum of higher harmonics γ_1, assumed to be of the order of $\epsilon\gamma_0$, with $\epsilon \ll 1$:

$$\gamma_0 = a\, g_1^o\, (a/r)^2 \cos\theta \tag{3}$$

$$\gamma_1 = a \sum_{n=2} \sum_{m=0}^{m=n} (a/r)^{n+1} P_n^m(\theta) \left\{ g_n^m \cos m\varphi + h_n^m \sin m\varphi \right\} \tag{4}$$

In the summation of γ_1, the $n = 1$, $m = 1$ terms are absent, because tilted-dipole coordinates are used; some of the terms there are actually small enough to be ignored at this level and could be relegated to a second-order component γ_2 (Stern, 1967), but this point will not be stressed. We denote by α and β the first-order Euler potentials, as derived from the dipole potentials by first order perturbation (Stern, 1967)

$$\alpha = \alpha_0 + \alpha_1$$

$$\beta = \beta_0 + \beta_1$$

with

$$\alpha_0 = a\, g_1^{\circ}\, (a/r)\, \sin^2\theta \tag{5}$$

$$\alpha_1 = a \sum_{n=2} \sum_{m=0}^{m=n} (a/r)^n \sin^{2n}\theta \left[V_n^m(\theta) + \xi_n^m\right] \left\{ g_n^m \cos m\varphi + h_n^m \sin m\varphi \right\} \tag{6}$$

$$\beta_0 = a\varphi \tag{7}$$

$$\beta_1 = a/g_1^0 \sum_{n=2} \sum_{m=0}^{m=n} (a/r)^{n-1} \sin^{2n-2}\theta\; t_n^m(\theta) \left\{ h_n^m \cos m\varphi - g_n^m \sin m\varphi \right\} \tag{8}$$

Here $V_n^m(\theta)$ and $t_n^m(\theta)$ are trigonometrical polynomials introduced by Pennington, available in tabulated form (Pennington, 1961; Stern, 1965, 1967), and ξ_n^m are additive constants (Stern, 1967), chosen in such a manner that the analogous additive constants of $t_n^m(\theta)$ all vanish.

THE DRIFT SHELL EQUATION

Since field lines are tangential to drift shells, the equation of any such shell has the form

$$f(\alpha, \beta) = 0 \tag{9}$$

For a given particle, the associated drift shell and therefore also the function $f(\alpha,\beta)$ depends on the initial conditions of the particle's motion, e.g., its position and momentum at some given instant, or on suitable independent functions of these. By the adiabatic theory of guiding center motion it may be shown that in static cases no more than two such functions determine the shell — the magnetic moment of the particle and its longitudinal invariant. These, in turn, are functions only of the particle's mass and energy and of two quantities associated with the field, the "mirroring field intensity" B_m at which the particle is reflected in its motion along field lines and the integral

$$I = \int_{B<B_m} (1 - B/B_m)^{1/2}\, ds \tag{10}$$

evaluated along a field line. It is the property of a drift shell (by which, indeed, it is defined) that the same value of I is obtained no matter which of the field lines tangential to it is chosen for integration. The entire family of shells in the geomagnetic field may thus be characterised by an equation of the form

$$f(\alpha, \beta, I, B_m) = 0 \tag{11}$$

or

$$\alpha = G(\beta, I, B_m) \tag{12}$$

Since the geomagnetic field may be regarded as a perturbed dipole field, one may expect that its drift shells reduce to those of the dipole field as all higher harmonics tend to zero. In a dipole field the variable β, which is then given by Eq. (7), is absent from Eq. (11) due to the field's axial symmetry, so this equation may be written (subscript zero referring to the zeroth order, i.e., the dipole case)

$$\alpha_0 = G_o(I, B_m) \tag{13}$$

Inverting, one gets

$$I = I_o(\alpha_0, B_m) \tag{14}$$

The function I_o, defined here for later use, is none other than the integral of Eq. (10), expressed in (α_0, θ, B_m) variables for the dipole field (see Eqs. 23-25). If one allows the perturbation to shrink to zero, Eq. (12) reduces to (13); therefore, within first-order accuracy, Eq. (12) has the form

$$\alpha_0 + \alpha_1 = G_o(I, B_m) + G_1'(\beta, I, B_m) \tag{15}$$

The function G_1' appearing here is of the first order, and in evaluating it one may use zero-order relationships. Now, of all variables entering the last equation, the integral I is the most difficult to handle; it is therefore advantageous to eliminate it from G_1' by use of (14), even though this means reintroducing α to the right-hand side (although Eq. (14) is only correct to zeroth order, such a substitution is permissible in a first-order correction term). In the next section we shall discuss the deeper significance of this elimination — namely, that it allows labeling a shell by two parameters, the effects of which are of different orders and can be considered separately. We shall also replace α and β in G_1' by their zero-order parts α_0 and β_0; for practical applications, when we want the equation of a shell in (r, θ, φ) variables rather than in terms of α and β, one may substitute for them from Eqs. (5) and (7). The function obtained after all these changes will be denoted by unprimed G_1, and the equation becomes

$$\alpha = G_o(I, B_m) + G_1(\alpha_0, \varphi, B_m) \tag{16}$$

THE DRIFT-SHELL PARAMETER L

The preceding derivation bears a strong relation to the parameter L, introduced by McIlwain (1961) for labeling drift shells. There has existed some misunderstanding concerning this parameter, and in what follows we shall try to resolve it, at the same time defining the relationship between L and the present calculation.

Much of the above misunderstanding can be traced to the fact that there exist two different functions which are commonly denoted by L. On the one hand, there is the function $L(I, B_m)$ of the adiabatic invariants, which is used to label drift shells; on the other hand there is also a function of position ("L at a point"), denoted here by $L(r, \theta, \varphi)$, which is obtained, for a given position in space, by substituting in $L(I, B_m)$ the values of I and B_m appropriate to particles mirroring at that point. The purpose of this section is to show that

(1) $L(I, B_m)$ is a function naturally arising when classifying drift shells in a dipole field or in a perturbed dipole field, and

(2) $L(r, \theta, \varphi)$ is an approximation to a certain choice of the Euler potential α, obtained by averaging Eq. (16) over B_m.

To demonstrate the first point, consider again the labeling of drift shells: as was noted, such labeling may be accomplished by means of the two parameters I and B_m. This, however, is not the only possible choice, and two independent functions of I and B_m may serve equally well.

In particular, if the field is that of a magnetic dipole, it appears advantageous to choose as one of the labeling parameters the function $G_o(I, B_m)$, retaining, say B_m for the other one. By Eq. (13) the shape of the magnetic drift shell then depends only on one characterizing parameter, and the entire collection of shells reduces to a one-parameter family, rather than a two-parameter one.

There still remains some freedom of choice left — instead of labeling with $G_o(I, B_m)$, one may use some function of it. In particular, one may introduce the function

$$L(I, B_m) = a\, g_1^0 / G_o(I, B_m) \tag{17}$$

so that Eq. (13) for a shell in a dipole field takes the form

$$L(I, B_m) = a\, g_1^0 / \alpha_0 \tag{18}$$

By Eq. (5), a dipole shell crosses the equatorial plane at a fixed distance, and it can be seen from the last equation that the constant L associated with such a shell has the useful intuitive property of equaling this distance, as measured in earth radii. It is therefore to be identified with the function $L(I, B_m)$ introduced by McIlwain (1961).

Of more interest is the case in which the field is not strictly a dipole field but has a perturbation added to it. To the first order the equation of a drift shell is then given by (16), and replacing there G_o by L we get

$$\alpha = a\, g_1^0 / L + G_1(\alpha_0, \beta_0, B_m) \tag{19}$$

In this case the equation of a shell depends on both L and B_m. Since, however, this is a perturbed version of Eq. (18), the dependence is unequal: L appears in a zero-order term, whereas B_m enters only through the first-order correction G_1. The function G_1 represents the variation between shells sharing the same value of L, and it therefore appears appropriate to call it the (first-order) shell splitting function. Much of the present work is devoted to its explicit derivation.

In this connection it should also be realized that no simple expression is available for either $G_o(I, B_m)$ or $L(I, B_m)$. The only way to derive these functions is through the inverse function $I_o(\alpha_0, B_m)$, obtained by extracting I from $G_o(I, B_m)$ and shown in Eq. (14); this function is explicitely defined in equations (23)-(25) and is by no means a simple one. Because of this difficulty, an analytical approximation to $L(I, B_m)$, introduced by McIlwain (1961, 1966), is nowadays generally used.

Turning now to the second definition of L, by calculating "L at a point" one is actually deriving a function which to first order can be approximated by

$$L(r, \theta, \varphi) = \psi(\alpha, \beta, B) = a\, g_1^0 / \left[\alpha - G_1(\alpha, \beta, B)\right] . \tag{20}$$

On the right-hand side, α, β and B are all to be evaluated at the point (r, θ, φ). Since B enters only through a first-order correction term, the dependence on it is weak and may be averaged out

$$L(r, \theta, \varphi) \cong \langle \psi(\alpha, \beta, B) \rangle_{\text{aver. over B}} = \chi(\alpha, \beta) .$$

Since χ is function of α and β only, it may be introduced as an Euler potential to replace α (Stern, 1967). In fact, if the correction term G_1 is altogether ignored, we find that $L(r, \theta, \varphi)$ closely approximates $(a\, g_1^0/\alpha)$, with α defined as in Eq. (6). One may even improve such an approximation by an appropriate choice of the constants ξ_n^m appearing in that equation, but this point will not be discussed.

In summary, then, the use of $L(I, B_m)$ for labeling drift shells naturally enters into the present treatment through the function G_o, and is further augmented by the addition of a "shell-splitting" correction term G_1. If the perturbation is mild — as is the case near earth — the correction term is small and it is a good approximation to assume that the same value of $L(I, B_m)$ characterises all particles attached to a given field line. This value can then be derived as $L(r, \theta, \varphi)$ for some arbitrary point on the line — e.g., the point at which it meets the earth's surface, which is the choice used by most experimenters.

On the other hand, the use of $L(r, \theta, \varphi)$ as an Euler potential for labeling geomagnetic field lines (as has been practiced by many workers in the field) is not particularly encouraged. As has been shown, it is not entirely accurate, and although the inaccuracy (for the internal magnetosphere) is no greater than that introduced by the first-order approximation of Eq. (6), it is a basic one and is independent of mathematical precision. Moreover, the Euler potentials of a field depend only on its sources and structure, and it should not be necessary to involve adiabatic invariants of particles trapped in it for their derivation.

THE FIRST-ORDER SHELL SPLITTING FUNCTION

In order to derive an approximate expression for G_1, Eq. (16) is rewritten

$$\alpha - G_1 = G_o(I, B_m) . \tag{21}$$

This resembles (13), except that α_0 is replaced by $(\alpha - G_1)$. We therefore get for the perturbed field, in analogy with (14)

$$I = I_o(\alpha - G_1, B_m)$$

$$= I_o(\alpha, B_m) - G_1 \, \partial I_o / \partial \alpha + O(\epsilon^2) .$$

No first-order errors are committed by replacing $\partial I_o(\alpha, B_m)/\partial\alpha$ by $\partial I_o(\alpha_0, B_m)/\partial\alpha_0$, so that one finds, to first order

$$G_1(\alpha_0, \varphi, B_m) = - \frac{I(\alpha, \beta, B_m) - I_o(\alpha, B_m)}{\partial I_o(\alpha_0, B_m)/\partial\alpha_0} . \tag{22}$$

To evaluate the denominator, one has to differentiate

$$I_o(\alpha_0, B_m) = \int_{B_o<B_m} \left[1 - B_o(\alpha_0, \theta)/B_m\right]^{1/2} \chi_0(\alpha_0, \theta) \, d\theta \tag{23}$$

with

$$B_o = |\nabla\gamma_0| \tag{24}$$

$$= g_1^0 (\alpha_0/a \, g_1^0)^3 \sin^{-6}\theta \, (1 + 3\cos^2\theta)^{1/2}$$

$$\chi_0 = (\partial s/\partial\theta)_{\text{dipole}} = r^2 B_o/(\partial\gamma_0/\partial\theta)\,|_r$$

$$= - a\,(\alpha_0/a\; g_1^0)^{-1} \sin\theta\; (1 + 3\cos^2\theta)^{1/2}. \tag{25}$$

Using

$$\partial B_o/\partial\alpha_0 = 3\, B_o/\alpha_0 \tag{26}$$

$$\partial\chi_0/\partial\alpha_0 = -\chi_0/\alpha_0 \tag{27}$$

and denoting by θ_m ("mirroring θ") the value of θ corresponding to B_m in a dipole field, one gets

$$\partial I_o/\partial\alpha_0 = -\alpha_0^{-1}\int_{\pi-\theta_m}^{\theta_m}(1 + B_o/2B_m)\,(1 - B_o/B_m)^{-1/2}\,\chi_0\, d\theta$$

$$- (a^2 g_1^0/2\alpha_0^2)\int_{\pi-\theta_m}^{\theta_m}(2 + B_o/B_m)\,(1 - B_o/B_m)^{-1/2}\sin\theta\;(1 + 3\cos^2\theta)^{1/2}\, d\theta\,. \tag{28}$$

The integral derived here is the same as the integral K_4 defined by Pennington (1961, Eq. 11). It has an integrable singularity at the limits of integration.

EVALUATION OF THE NUMERATOR

The numerator in Eq. (22) consists of the first-order difference between two integrals. The first of these is

$$I(\alpha, \beta, B_m) = \int_{\substack{\pi-\theta_s \\ \alpha,\beta = \text{const.}}}^{\theta_n} (1 - B/B_m)^{1/2} \chi \, d\theta \tag{29}$$

where $B = |\nabla\gamma|$ is the magnitude of the perturbed field, integration is performed with constant (perturbed) α and β, and where

$$\chi = \partial s/\partial\theta|_{\alpha,\beta} = r^2 B/(\partial\gamma/\partial\theta)|_{r,\varphi} \, . \tag{30}$$

The integrand of I vanishes at $\theta = \theta_n$ and at $\theta = \pi - \theta_s$ (subscripts for "θ-north" and "θ-south"); because the perturbed field is not necessarily symmetric, θ_n and θ_s will in general be unequal, and may depend on φ.

The second integral in the numerator is $I_o(\alpha, B_m)$, already defined in Eq. (23). It should be noted that here its first argument is α, not α_0 as in Eq. (23), so that θ_m is defined by $B_o(\alpha, \theta_m)$, <u>not</u> $B_o(\alpha_0, \theta_m)$, being equal to B_m.

The two integrals represent two functions $I(\alpha, \beta, B_m)$ and $I_o(\alpha, B_m)$ that differ only slightly, and it is therefore natural to try to expand I, in some way, around I_o; when such an expansion is then substituted in the enumerator, its zero-order terms cancel, leaving the first-order difference explicitely stated. In what follows the procedure for such an expansion, which is somewhat tricky, will be derived; the actual calculation, which is merely tedious, will not be given.

Three factors have to be considered in deriving the difference between I and I_o. First of all, there is the difference in limits of integration: this will be dealt with later. Secondly, I contains the perturbed variables B and γ, whereas only their dipole components B_o and γ_0 appear in I_o. Finally, the form in which B and γ are available is as functions of the spatial coordinates (r, θ, φ). By means of Eqs. (5) and (7), this dependence is easily transformed into a dependence on $(\alpha_0, \beta_0, \theta)$, but since it is α and β, not α_0 and β_0, that stay constant during integration, one has then to substitute in the above dependence

$$\begin{aligned} \alpha_0 &= \alpha - \alpha_1 \\ \beta_0 &= \beta - \beta_1 \end{aligned} \tag{31}$$

and expand, thus expressing the variables in terms of (α, β, θ). Because the calculation is to be accurate only to the first order of perturbation, the substitution (31) is only required in zero-order terms, while in first-order terms the difference between (α_0, β_0) and (α, β) may be ignored.

As an example, consider the field intensity B (given for the dipole field in Eq. 24)

$$\begin{aligned} B(\alpha, \beta, \theta) &= |\nabla\gamma| \\ &\cong B_o(\alpha_0, \theta) + (\nabla\gamma_0 \cdot \nabla\gamma_1)/B_o \\ &= B_o(\alpha_0, \theta) + B_1(\alpha_0, \beta_0, \theta) \quad \text{(defining } B_1\text{)} \\ &\cong B_o(\alpha - \alpha_1, \theta) + B_1(\alpha, \beta, \theta) \\ &\cong B_o(\alpha, \theta) - \alpha_1(\partial B_o/\partial\alpha) + B_1(\alpha, \beta, \theta). \end{aligned} \tag{32}$$

For convenience, we introduce a special notation for the first-order component

$$\Delta B(\alpha, \beta, \theta) = -\alpha_1(\partial B_o/\partial\alpha) + B_1(\alpha, \beta, \theta). \tag{33}$$

Similarly, from the dipole relation

$$r = r_o(\alpha_0, \theta) \tag{34}$$

one get

$$\begin{aligned} r(\alpha, \beta, \theta) &\cong r_o(\alpha, \theta) - \alpha_1(\partial r_o/\partial\alpha) \\ &= r_o + \Delta r. \end{aligned} \tag{35}$$

Finally, denoting for brevity

$$\partial\gamma_0/\partial\theta = \Gamma_0(\alpha_0, \theta)$$

one finds

$$\partial\gamma/\partial\theta = \Gamma_0(\alpha, \theta) + \Delta\Gamma \tag{36}$$

where

$$\Delta\Gamma = \partial\gamma_1/\partial\theta - \alpha_1(\partial\Gamma_0/\partial\alpha). \tag{37}$$

For reasons which will become clear later, it is advisable to leave the difference arising from the square root terms for separate consideration. Expansion of all other terms to the first order yields

$$I(\alpha, \beta, B_m) = \int_{\substack{\pi-\theta_s \\ \alpha,\beta=\text{const.}}}^{\theta_n} \left\{1 - \frac{B_o + \Delta B}{B_m}\right\}^{1/2} \left\{1 + \frac{\Delta B}{B_o} + \frac{2\Delta r}{r_o} - \frac{\Delta\Gamma}{\Gamma_0}\right\} \chi_0(\alpha, \theta)\, d\theta. \tag{38}$$

Let us write

$$I - I_o = \int_{\pi-\theta_s}^{\theta_n} \{1 - (B_o + \Delta B)/B_m\}^{1/2} \chi_0 \, d\theta$$

$$- \int_{\pi-\theta_m}^{\theta_m} (1 - B_o/B_m)^{1/2} \chi_0 \, d\theta + \Delta I_1 . \tag{39}$$

The quantity ΔI_1 is of the first order and may therefore be evaluated, to the present order of approximation, between the limits θ_m and $\pi-\theta_m$. Making other allowable approximations, and replacing (α, β) by (α_0, β_0) during integration, we get

$$\Delta I_1 \cong \int_{\pi-\theta_m}^{\theta_m} \left(1 - \frac{B_o}{B_m}\right)^{1/2} \left\{\frac{\Delta B}{B_o} + \frac{2\Delta r}{r_o} - \frac{\Delta\Gamma}{\Gamma_0}\right\} \chi_0 \, d\theta . \tag{40}$$

At the corresponding point in Pennington's calculation, the square root in the first part of (39) was expanded by the binomial theorem:

$$\{1 - (B_o + \Delta B/B_m)\}^{1/2} = (1 - B_o/B_m)^{1/2} \{ 1 - \Delta B/(B_m - B_o)\}^{1/2}$$

$$\cong (1 - B_o/B_m)^{1/2} - (\Delta B/2\, B_m)\ (1 - B_o/B_m)^{-1/2} . \tag{41}$$

In addition, the difference between the limits of the two integrals in (39) was ignored, leading to

$$I - I_o = (-1/2B_m) \int_{\pi-\theta_m}^{\theta_m} \Delta B\, (1 - B_o/B_m)^{-1/2} \chi_0 \, d\theta + \Delta I_1 + o(\epsilon^2) . \tag{42}$$

Actually the binomial expansion is divergent when $(B_m - B_o)$ is less than ΔB and as a result, the integrand in (42) has a singularity at its limits, where the quantity it tries to approximate really tends to zero. The singularity is integrable, however, and it is shown in the appendix that Eq. (42) is in fact correct to the first order in ϵ.

The explicit derivation of $(I - I_o)$ from here on will not be described: as was noted before, it is lengthy, though not too difficult. Since the first order quantities $\Delta B, \Delta\Gamma$ and Δr all involve α, or γ, in a linear fashion, the final expressions for $(I - I_o)$, and consequently also for G_1, split up into a sum of terms, each of which is proportional to one of the harmonic coefficients g_n^m or h_n^m. The calculation of these terms leads to a series of integrals, first given by Pennington (1961), which combine to form a set of functions α_n^m (Pennington's notation, retained here; not related to the Euler potential α) of the mirroring angle θ_m. The functions $\alpha_n^m(\theta_m)$ have been evaluated numerically and tabulated by Pennington (1961) and more extensively and accurately by Stern (1965); using them one gets

$$G_1 = (1/g_1^0) \sum_{n=2} \sum_{m=0}^{m=n} (a_0/a g_1^0)^n \left[\alpha_n^m(\theta_m) + \xi_n^m\right] \left\{g_n^m \cos m\varphi + h_n^m \sin m\varphi\right\} . \tag{43}$$

Combining this with Eqs. (4), (5) and (19) gives the shell's first-order equation as

$$1/L = (a/r)\sin^2\theta$$

$$+ (1/g_1^0)\sum_{n=2}\sum_{m=0}^{m=n}(a/r)^n \sin^{2n}\theta\left[V_n^m(\theta) + \alpha_n^m(\theta_m)\right]\left\{g_n^m \cos m\varphi + h_n^m \sin m\varphi\right\}. \tag{44}$$

It should be noted that the integration constants ξ_n^m cancel out in the final result. This was to be expected, since in characterizing drift shells, as in Eq.(9) and the ones following it, the only requirement for α and β was that they be conserved along field lines, and it was not necessary that they "match" (Stern, 1967).

THE $I = 0$ LIMIT

One may test Eq. (44) by considering the limit $I = 0$, for which the shape of a drift shell (which then contracts to a line) may be separately derived from first principles. Such a test also checks whether the ratio of expanded integrals tends to the correct limit when the integration ranges become very short.

In a dipole field, shells with $I = 0$ describe particles confined to the equatorial plane. Since (aL) then equals their (constant) distance from the dipole, one finds in that case the limit of the function $L(I, B_m)$ as

$$L(0, B_m) = (g_1^0/B_m)^{1/3}. \tag{45}$$

Substituting this into (44) and inserting there $\pi/2$ for the mirror angle θ_m gives the first-order equation of the $(0, B_m)$ shell, by the present perturbation scheme, as

$$B_m^2 = (g_1^0)^2\,(\alpha/a\,g_1^0)^6$$

$$+\ 6\,g_1^0 \sum_{n,m} (\alpha_0/a\,g_1^0)^{n+5}\left\{\xi_n^m - \alpha_n^m(\pi/2)\right\}(g_n^m \cos m\varphi + h_n^m \sin m\varphi)\,. \tag{46}$$

For an alternative derivation of the shell's equation, we note that a particle with $I = 0$ will always be located at points at which the field intensity is B_m. This gives to first order the condition

$$B_m^2 = B^2 = B_o^2 + 2\nabla\gamma_0 \cdot \nabla\gamma_1$$

$$= (g_1^0)^2\,(\alpha/a\,g_1^0)^6\,F(\theta) - g_1^0 \sum_{n,m} (\alpha_0/a\,g_1^0)^{n+5}\,Q_n^m(\theta)\left[\,g_n^m \cos m\varphi + h_n^m \sin m\varphi\right] \tag{47}$$

where

$$F(\theta) = (1 + 3\cos^2\theta)\,\sin^{-12}\theta$$

$$Q_n^m(\theta) = 6\,F(\theta)\left\{V_n^m(\theta) + \xi_n^m\right\}$$

$$-\ 2\,\sin^{-(2n+10)}\theta\left\{2\,(n+1)\cos\theta\,P_n^m(\theta) - \sin\theta\,(dP_n^m/d\theta)\right\}.$$

To obtain the shell's equation, one would now have to eliminate θ, using the additional condition that the position of the particle on any field line corresponds to the minimum of B there. However, it is sufficient to assume that the angle

$$\lambda = \pi/2 - \theta$$

between the particle's location and the equatorial plane, is of the first order in smallness. With this assumption one may substitute

$$\theta = \pi/2$$

$$F(\theta) = 1$$

not only in first order terms but also in zero-order ones, since

$$\partial F/\partial\theta|_{\pi/2} = 0$$

and thus

$$F(\theta) = F(\pi/2) + O(\epsilon^2) .$$

Comparing then (48) with (47) gives

$$6\left\{\xi_n^m - \alpha_n^m(\pi/2)\right\} = Q_n^m(\pi/2) = 6\xi_n^m + 2(dP_n^m/d\theta)|_{\pi/2} \tag{48}$$

from which one finds the requirement

$$\alpha_n^m(\pi/2) = -\frac{1}{3}(dP_n^m/d\theta)|_{\pi/2} \tag{49}$$

which indeed is met by the functions α_n^m (Stern, 1965, last equation).

APPENDIX

PROOF OF EQUATION (42)

Let the integration region be divided at $\theta = \pi/2$, and consider the contribution to the first two terms in (39) from angles θ smaller than this value

$$\Delta I_n = \int_{\pi/2}^{\theta_n} \left[1 - (B_o + \Delta B/B_m)\right]^{1/2} \chi_0 \, d\theta - \int_{\pi/2}^{\theta_m} (1 - B_o/B_m)^{1/2} \chi_0 \, d\theta . \quad \text{(A-1)}$$

Defining a constant $\Delta\theta$, which may be of either sign

$$\begin{aligned} \Delta\theta &= \theta_n - \theta_m \\ &\cong - \Delta B(\theta_m)/(\partial B_o/\partial\theta)|_{\theta_m} \end{aligned} \quad \text{(A-2)}$$

one changes variables in the first part of (39) to

$$\psi = \theta - \Delta\theta .$$

To the first order

$$\begin{aligned} B_o(\alpha, \theta) &= B_o(\alpha, \psi) + \Delta\theta\,(\partial B_o/\partial\psi) \\ \Delta B(\alpha, \beta, \theta) &= \Delta B(\alpha, \beta, \psi) \\ \chi_0(\alpha, \theta) &= \chi_0(\alpha, \psi) + \Delta\theta\,(\partial\chi_0/\partial\psi) \end{aligned}$$

giving, to the same order

$$\int_{\pi/2}^{\theta_n}\left[1 - (B_o + \Delta B)/B_m\right]^{1/2} \chi_0 \, d\theta =$$

$$= \int_{\pi/2-\Delta\theta}^{\theta_m} \left\{1 - B_o + \Delta B + \Delta\theta\,(\partial B_o/\partial\psi)\ /B_m\right\}^{1/2} (\chi_0 + \Delta\theta\,\partial\chi_0/\partial\psi)\, d\psi$$

$$= \Delta\theta\left\{\int_{\pi/2}^{\theta_m} (1 - B_o/B_m)^{1/2}\,(\partial\chi_0/\partial\psi)\, d\psi + \left[1 - B_o(\alpha,\pi/2)/B_m\right]^{1/2} \chi_0(\alpha,\pi/2)\right\}$$

$$+ \int_{\pi/2}^{\theta_m}\left\{1 - \left[B_o + \Delta B + \Delta\theta\,(\partial B_o/\partial\psi)\right]/B_m\right\}^{1/2} \chi_0\, d\psi\,. \tag{A-3}$$

The square root appearing in the last integral may be written

$$(1 - B_o/B_m)^{1/2}\left\{1 - \left[\Delta B + \Delta\theta\,(\partial B_o/\partial\theta)\right]/\left[B_m - B_o\right]\right\}^{1/2}. \tag{A-4}$$

Unlike in the case of Eq. (41), the second factor here may be expanded, for the ratio contained in it is less than unity at all points at which the integrand of (A-3) is real. There exists no problem at the end of the integration range, for both numerator and denominator vanish there. One may therefore write (reinstating θ as integration variable)

$$\Delta I_n \cong (-1/2B_m)\int_{\pi/2}^{\theta_m}\left[\Delta B + \Delta\theta\,(\partial B_o/\partial\theta)\right](1 - B_o/B_m)^{-1/2}\,\chi_0\, d\theta$$

$$+ \Delta\theta\int_{\pi/2}^{\theta_m}(1 - B_o/B_m)^{1/2}\,(\partial\chi_0/\partial\theta)\, d\theta + \Delta\theta\left[1 - B_o(\alpha,\pi/2)/B_m\right]^{1/2}\chi_0(\alpha,\pi/2)\,. \tag{A-5}$$

However, integration by parts shows that all terms involving $\Delta\theta$ cancel. A similar cancellation occurs in the remaining half of the integration range, and the only terms left are those in (42).

Figure 1 schematically illustrates the meaning of the preceding result. Graphs (1) and (2) give the integrands appearing in equation (A-1), and the curve bounding the hatched area describes their difference. Graph (3) describes the binomial approximation to this curve as used in Eq. (42), an approximation which evidently breaks down near $\theta = \theta_m$. In spite of this breakdown, the use of the approximation in evaluating integrals leads to correct results, because the area between graph (3) and the two orthogonal axes shown in the figure equals (to the order of approximation) the hatched area beneath the curve.

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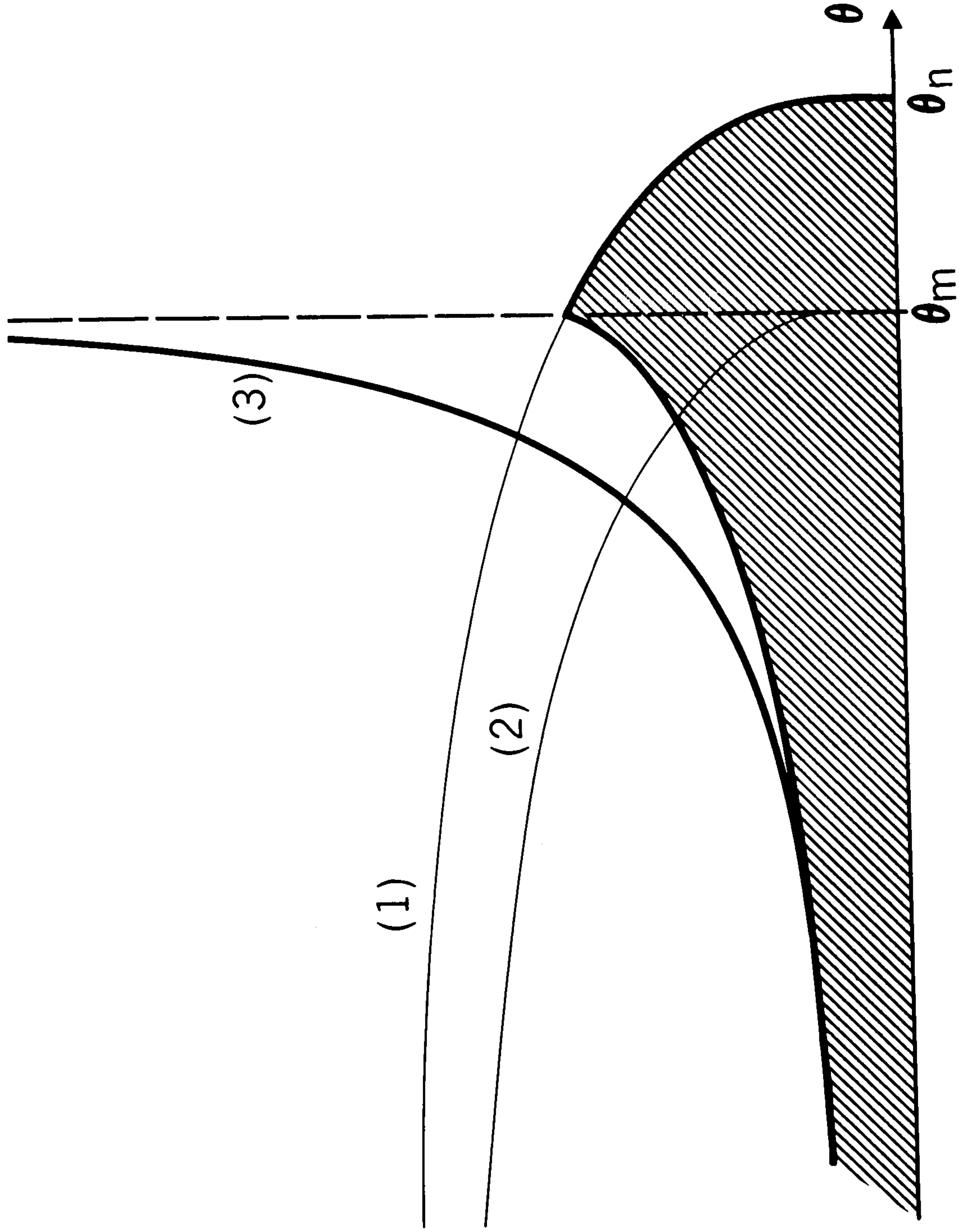

Figure 1.