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CLOSED-LOOP DOPPLER CORRECTION SCHEME

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CLOSED-LOOP DOPPLER CORRECTION SCHEME

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Communications Research Branch

June 1967

Goddard Space Flight Center
Greenbelt, Maryland

FOREWORD

This document is based on a thesis submitted by the author to the faculty of the George Washington University School of Engineering and Applied Science.

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CLOSE-LOOP DOPPLER CORRECTION SCHEME

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ABSTRACT

Doppler frequency shift occurs when there is relative velocity between a satellite and ground station. The normalized frequency error is v/c, where v is the relative velocity between the satellite and the ground station and c is the speed of light.

This document describes a technique for reducing the effective Doppler frequency shift. This technique can reduce the Doppler effect to approximately second order, $(v/c)^2$. Experimental data have verified the technique, differing from theoretical predictions by 3 percent. A ground-controlled satellite clock with a stability of 5 parts in 10^{10} was established in Relay II. Without the Doppler correction scheme, the normalized frequency error can exceed 1 part in 10^5.

This document presents what is believed to be a unique analysis of the Doppler correction scheme in the variable velocity case with associated experimental verification.

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CLOSED-LOOP DOPPLER CORRECTION SCHEME

INTRODUCTION

When there is relative motion between a transmitter (T) and a receiver (R), the frequency of the signal detected at the receiver will not be the same as that sent from the transmitter. This change in frequency is called the Doppler frequency shift.

By using closed-loop techniques, it is possible to correct for Doppler frequency shift. The signal at R is retransmitted back to T. The frequency difference between this signal and the originally transmitted signal is a measure of the two-way Doppler frequency shift. Using the standard technique,[1] it can be shown that:

$$f_R = \frac{c - v}{c} f_T , \tag{1}$$

$$f'_R = \frac{c}{c + v} f_R , \tag{2}$$

$$= \frac{c}{c + v} \frac{c - v}{c} f_T ,$$

$$= \frac{c - v}{c + v} f_T . \tag{3}$$

where f_R is the frequency received at R,

f_T is the frequency transmitted at T,

f'_R is the frequency received at T.

[1]Brown, R. C.: A Textbook of Physics. Longmans, London, 1961, p. 1013

Because $v \ll c$,

$$f'_R \approx \frac{c - 2v}{c} f_T. \tag{4}$$

For small values of v/c, the frequency shift of the signal received at R will be approximately one-half the frequency shift of the signal received at T.

A servomechanism was devised which forced

$$f'_R + f_T = 2 f_{REF}, \tag{5}$$

where f_{REF} is the desired signal frequency in the absence of Doppler. Substituting equation (4) into equation (5) and solving for f_T yields:

$$\frac{c - 2v}{c} f_T + f_T = 2 f_{REF},$$

$$f_T = \frac{c}{c - v} f_{REF}. \tag{6}$$

By using equation (1), it is now possible to determine the frequency which will be received at R.

$$f_R = \frac{c - v}{c} f_T,$$

$$= \frac{c - v}{c} \frac{c}{c - v} f_{REF},$$

$$f_R = f_{REF}. \tag{7}$$

Appendix A contains a description and block diagram of the servomechanism.

It has been shown that the Doppler frequency shift can be eliminated from a signal under the following special circumstances:

1. $v \ll c$
2. $v \neq v(t)$

If $v = v(t)$, then the two v's in equation (3) are not equal, because they are evaluated at different times.

The preceding development is common in the literature on this subject. The case where $v = v(t)$ has not been thoroughly analyzed. The case where $v = v(t)$ is treated both theoretically and experimentally in this document.

ANALYSIS OF THE CONSTANT VELOCITY CASE

In order to keep the analysis as general as possible, the constant velocity case will be rederived without the constraint that $v \ll c$. Repeating equation (1) and equation (3),

$$f_R = \frac{c - v}{c} f_T, \tag{1}$$

$$f'_R = \frac{c - v}{c + v} f_T, \tag{3}$$

where f_R is the frequency received at the satellite,

f_T is the frequency transmitted from the ground,

f'_R is the frequency received on the ground.

The servomechanism forced

$$f'_R + f_T = 2 f_{REF}, \tag{8}$$

where f_{REF} is the desired signal frequency in the absence of Doppler. Substituting equation (3) into equation (8) yields:

$$\frac{c-v}{c+v} f_T + f_T = 2 f_{REF}, \tag{9}$$

$$f_T = \frac{c+v}{c} f_{REF}. \tag{10}$$

By substituting this into equation (1), it is possible to find the signal which will be received at the satellite.

$$f_R = \frac{c-v}{c} f_T, \tag{1}$$

$$= \frac{c-v}{c} \frac{c+v}{c} f_{REF},$$

$$= \left(1 - \frac{v^2}{c^2}\right) f_{REF}. \tag{11}$$

Note that the Doppler is not reduced to zero as before, but is reduced to second order in v/c.

A normalized frequency error is defined for use in subsequent sections.

$$E_N = \frac{f_R - f_{REF}}{f_{REF}}, \tag{12}$$

$$= -(v/c)^2. \tag{13}$$

ANALYSIS OF THE VARIABLE VELOCITY CASE

Classical Approach

This section describes a still more general analysis of the closed-loop Doppler correction scheme in which v/c is significant and velocity is a function of time. Let

$$v = v(t),$$

$$= v + at, \tag{14}$$

where v is the velocity at time $t = 0$,

a is the acceleration at time $t = 0$.

For short periods of time ($t \leq 100$ sec), the relative velocity between a ground station and satellite can generally be expressed as the summation of an initial velocity and a constant acceleration.

The frequency received at the satellite at time t, $f_R(t)$, is a function of the relative velocity at time t, and the frequency transmitted at time $t - T$, $f_T(t - T)$. T is the finite propagation time of the signal from the ground to the satellite, and is also a function of time. Substituting equation (14) into equation (1) yields:

$$f_R(t) = \frac{c - v - at}{c} f_T(t - T). \tag{15}$$

The frequency received at the ground at time t, $f'_R(t)$, is a function of the relative velocity at time $t - T$ and the frequency transmitted from the satellite at time $t - T$, $f_R(t - T)$. T is the finite propagation delay and is equal to T in equation (15) because the turnaround time of the signal in the satellite is negligible relative to the propagation time delay. Substituting equation (14) into equation (2) yields:

$$f'_R(t) = \frac{c}{c + v + a(t - T)} f_R(t - T). \tag{16}$$

Replacing t with $t - T$ in equation (15) and substituting into equation (16) yields:

$$f'_R(t) = \frac{c - v - a(t - T)}{c + v + a(t - T)} f_T(t - 2T). \tag{17}$$

Equation (17) provides a relationship between the frequency received on the ground at time t, $f'_R(t)$, and the frequency transmitted from the ground at time $t - 2T$, $f_T(t - 2T)$. To describe the servomechanism, however, it is necessary to have a relationship between $f_T(t)$ and $f_R(t)$. Because there was no obvious way to establish this relationship, this approach was abandoned.

Series Approach

The equations of motion of most macroscopic objects (e.g., a satellite) generate relatively smooth, continuous curves. Short segments of smooth continuous curves can be accurately represented by a truncated power series. The range to a satellite, for example, can be defined as:

$$R(t) = r + vt + \frac{1}{2}at^2, \tag{18}$$

where $R(t)$ is the range at any time,

r is the range at time $t = 0$,

v is the velocity at time $t = 0$,

a is the acceleration at time $t = 0$.

This equation is valid only over a finite time interval because the satellite is still at a finite range after infinite time. Over a proper time interval, however, the errors introduced by the use of equation (18) are insignificant.

Because the Doppler effect is linearly related to the rate of change of range, the Doppler frequency perturbation can also be adequately represented by a truncated power series. The phase at the satellite, at any time t, can be expressed as:

$$\phi(t) = b_0 + (b_1 + w)t + b_2 t^2, \tag{19}$$

where $\phi(t)$ is the phase at any time t,

b_0 is the initial phase at time $t = 0$,

$(b_1 + w)$ is the initial frequency at time $t = 0$,

b_2 is the initial frequency rate at time $t = 0$.

In equation (19), b_1 is the frequency error in radians per second caused by Doppler, and w is the frequency in radians per second in the absence of Doppler $(2\pi f_{REF})$. For simplicity, assume that b_2 is small relative to $b_1 + w$ (this assumption is justified by the experimental data) and therefore has negligible effect on $\phi(t)$ for small values of t. Hence,

$$\phi(t) = b_0 + (b_1 + w)\, t, \tag{20}$$

where b_1 is a function of both the initial velocity and acceleration.

The phase of a signal being transmitted from the ground at time t is the identical phase which will be received at the satellite at time $t + T$, if T is the propagation time from the ground to the satellite of a signal arriving at the satellite at time $t + T$. Therefore,

$$\phi_T(t) = b_0 + (b_1 + w)(t + T), \tag{21}$$

where $\phi_T(t)$ is the phase of the signal transmitted from the ground at any time t in terms of the phase at the satellite.

The phase of a signal being received on the ground at time t is the identical phase transmitted from the satellite at time $t - \tau$, if τ is the propagation time from the satellite to the ground of a signal transmitted from the satellite at time $t - \tau$. Therefore,

$$\phi_R'(t) = b_0' + (b_1 + w)(t - \tau), \tag{22}$$

where $\phi_R'(t)$ is the phase of the signal received on the ground at time t in terms of the phase at the satellite.

The servomechanism causes the sum of $\phi_T(t)$ and $\phi'_R(t)$ to equal a reference, or

$$\phi_T(t) + \phi'_R(t) = 2wt + \psi, \tag{23}$$

where ψ is an arbitrary initial phase angle. Substituting equation (21) and equation (22) into equation (23) yields:

$$b_0 + (b_1 + w)(t + T) + b_0 + (b_1 + w)(t - \tau) = 2wt + \psi, \tag{24}$$

$$2b_0 + 2(b_1 + w)t + (b_1 + w)(T - \tau) = 2wt + \psi. \tag{25}$$

Both T and τ can be expanded into a Maclaurin series so that

$$T(t) = T(0) + \dot{T}(0)t + \ddot{T}(0)\frac{t^2}{2!}, \tag{26}$$

$$\tau(t) = \tau(0) + \dot{\tau}(0)t + \ddot{\tau}(0)\frac{t^2}{2!}. \tag{27}$$

Substituting this into equation (25) yields:

$$\begin{aligned} 2b_0 + 2(b_1 + w)t + (b_1 + w) \Big[T(0) + \dot{T}(0)t \\ + \ddot{T}(0)\frac{t^2}{2!} - \tau(0) - \dot{\tau}(0)t - \ddot{\tau}(0)\frac{t^2}{2!} \Big] \\ = 2wt + \psi. \end{aligned} \tag{28}$$

Equating powers of t yields:

$$2b_0 + (b_1 + w)\,[T(0) - \tau(0)] = \psi, \tag{29}$$

$$(b_1 + w)\,[2 + \dot{T}(0) - \dot{\tau}(0)] = 2w, \tag{30}$$

$$(b_1 + w)\,[\ddot{T}(0) - \ddot{\tau}(0)] = 0. \tag{31}$$

Solving equation (30) for b_1 yields

$$b_1 = \frac{-w\,[\dot{T}(0) - \dot{\tau}(0)]}{2 + \dot{T}(0) - \dot{\tau}(0)}, \tag{32}$$

where b_1 is the frequency error in radians per second due to Doppler in the variable velocity case. The general expressions for $T(t)$ and $\tau(t)$, as well as their series expansions, are developed in Appendix B. Substituting the expressions for $\dot{T}(t)$ and $\dot{\tau}(t)$ into equation (32) yields:

$$b_1 = -w \left[\frac{\dfrac{(c-v)v + ra}{(c-v)^2} - \dfrac{(c+v)v - ra}{(c+v)^2}}{2 + \dfrac{(c-v)v + ra}{(c-v)^2} - \dfrac{(c+v)v - ra}{(c+v)^2}} \right]. \tag{33}$$

In Appendix C, a first-order approximation is made for equation (33):

$$b_1 \approx \frac{-w}{c^2}\,(v^2 + ra). \tag{34}$$

The normalized theoretical error, E_{NT}, is obtained by dividing equation (34) by w.

$$E_{NT} = \frac{-(v^2 + ra)}{c^2}. \tag{35}$$

This is in complete agreement with the constant velocity case. If the acceleration, a, in equation (35) is set equal to zero, then the normalized error calculated by using equation (35) will equal the normalized error found by using equation (13).

EXPERIMENTAL TEST SYSTEM

General Description

A comparison of the corrected frequency at the satellite with the reference frequency was required to experimentally determine the quality of the Doppler correction scheme. However, the satellite was only a simple repeater and contained no reference clocks. It was therefore impossible to make the required measurements in the satellite.

To overcome this difficulty, the system was modified so that the signal looped through the satellite twice. This placed the center of the system, a virtual satellite, on the ground. A description and block diagram of the servomechanism of the test system is contained in Appendix D.

Because both the corrected signal at the virtual satellite and the reference frequency were available on the ground, it was possible to measure the quality of the Doppler correction scheme.

Appendix D also contains a description and functional block diagram of the method used to compare the frequency at the virtual satellite, f_{vs}, with the reference frequency, f_{REF}.

Test System Analysis

The analysis of the test system is similar to that of the operational system. Assume that, as in equation (20), the phase at the satellite, $\phi(t)$, can be expressed as

$$\phi(t) = b_0 + (b_1 + w)t, \tag{36}$$

where $\phi(t)$ is the phase at any time t,

b_0 is the initial phase at time $t = 0$,

$(b_1 + w)$ is the initial frequency at time $t = 0$.

As defined in the section on the analysis of the variable velocity case, b_1 is the frequency error in radians per second caused by Doppler and w is the frequency in radians per second in the absence of Doppler ($2\pi f_{REF}$).

The transit time of a signal from the ground to the satellite is exactly the transit time of the same signal from the satellite to the ground if it is retransmitted immediately. This is intuitively justified by noting that the range up is equal to the range down if the signal is retransmitted immediately. The phase of a signal being transmitted from the ground at time t is the identical phase which will be received at the virtual satellite at time $t + 2T$, if T is the propagation time from the ground to the actual satellite of a signal arriving at the actual satellite at time $t + T$. Therefore,

$$\phi_T(t) = b_0 + (b_1 + w)(t + 2T), \tag{37}$$

where $\phi_T(t)$ is the phase of the signal transmitted from the ground at any time t in terms of the phase at the virtual satellite.

The phase of a signal being received on the ground at time t is the identical phase which was transmitted from the virtual satellite at time $t - 2\tau$, if τ is the propagation time from the actual satellite to the ground of a signal transmitted from the actual satellite at time $t - \tau$. Therefore,

$$\phi_R'(t) = b_0 + (b_1 + w)(t - 2\tau), \tag{38}$$

where $\phi_R'(t)$ is the phase of the signal received on the ground at time t in terms of the phase at the virtual satellite.

The servomechanism causes the sum of $\phi_T(t)$ and $\phi_R'(t)$ to equal a reference, or

$$\phi_T(t) + \phi_R'(t) = 2wt + \psi, \tag{39}$$

where ψ is the arbitrary initial phase angle. Substituting equation (37) and equation (38) into equation (39) yields:

$$b_0 + (b_1 + w)\ (t + 2T) + b_0 + (b_1 + w)\ (t - 2\tau) = 2wt + \psi, \tag{40}$$

$$2b_0 + 2\,(b_1 + w)\,t + 2\,(b_1 + w)\ (T - \tau) = 2wt + \psi. \tag{41}$$

As in the preceding section, T and τ can be expanded into a Maclaurin series so that

$$T(t) = T(0) + \dot{T}(0)\,t + \ddot{T}(0)\,\frac{t^2}{2!}, \tag{42}$$

$$\tau(t) = \tau(0) + \dot{\tau}(0)\,t + \ddot{\tau}(0)\,\frac{t^2}{2!}. \tag{43}$$

Substituting this into equation (41),

$$2b_0 + 2\,(b_1 + w)\,t + 2\,(b_1 + w)\left[T(0) + \dot{T}(0)\,t\right.$$

$$\left. + \dot{T}(0)\,\frac{t^2}{2!} - \tau(0) - \dot{\tau}(0)\,t - \ddot{\tau}(0)\,\frac{t^2}{2!}\right]$$

$$= 2wt + \psi. \tag{44}$$

Equating powers of t yields:

$$2b_0 + 2\,(b_1 + w)\ [T(0) - \tau(0)] = \psi, \tag{45}$$

$$(b_1 + w)\ [1 + \dot{T}(0) - \dot{\tau}(0)] = w, \tag{46}$$

$$(b_1 + w)\ [\ddot{T}(0) - \ddot{\tau}(0)] = 0. \tag{47}$$

Solving equation (46) for b_1 yields:

$$b_1 = \frac{-w\,[\dot{T}(0) - \dot{\tau}(0)]}{1 + \dot{T}(0) - \dot{\tau}(0)}, \tag{48}$$

where b_1 is the frequency error in radians per second due to Doppler. Appendix B contains the general expressions for $T(0)$ and $\tau(0)$. Substitution yields:

$$b_1 = -w \left[\frac{\dfrac{(c-v)v + ra}{(c-v)^2} - \dfrac{(c+v)v - ra}{(c+v)^2}}{1 + \dfrac{(c-v)v + ra}{(c-v)^2} - \dfrac{(c+v)v - ra}{(c+v)^2}} \right]. \tag{49}$$

In Appendix E, a first-order approximation is made for equation (49):

$$b_1 \approx \frac{-2w\,(v^2 + ra)}{c^2}. \tag{50}$$

The normalized theoretical error of the test system, E'_{NT}, is obtained by dividing equation (50) by w.

$$E'_{NT} = \frac{-2\,(v^2 + ra)}{c^2}. \tag{51}$$

A comparison of equation (51) and equation (35) indicates that the error at the virtual satellite in the test system will be exactly twice the error at the actual satellite in the operational system.

THE EXPERIMENT

The Relay satellite was used to obtain experimental data. Figures 1, 2, and 3 show the range, velocity, and acceleration of Relay with respect to the Mojave ground station as a function of time after acquisition for orbit 4472, September 17, 1965.

The first experimental run was initiated at time $t = 2.75$ minutes with the following initial conditions:

$$r = 6.8 \times 10^6 \text{ meters,}$$

$$v = 4.2 \times 10^3 \text{ meters per second,}$$

$$a = -.25 \text{ meters per second squared.}$$

The second experiment was started at time $t = 25.25$ minutes with the following initial conditions:

$$r = 3.3 \times 10^6 \text{ meters,}$$

$$v = 2.0 \times 10^3 \text{ meters per second,}$$

$$a = 4.3 \text{ meters per second squared.}$$

Note that $v^2 \gg ra$ for the first part of the experiment and $ra > v^2$ for the second part, thus providing a reliable test of the theoretical prediction.

The recorded data are in the general form of a scatter diagram as shown in Figure 4. A least-squares polynomial approximation is made for the data as discussed in Appendix F.[2]

Table 1 lists the normalized measured errors, E_{NM}, as determined by the computer and the normalized theoretical errors for the test system, E'_{NT}, as calculated in Appendix G. Because of the close agreement between the measured and predicted results, it is assumed that the theoretical analysis is valid.

Table 1
Comparison of E_{NM*} and E'_{NT**}

Test No.	E_{NM}	E'_{NT}	Difference Between E_{NM} and E'_{NT} (%)
1	-3.59 parts in 10^{10}	-3.54 parts in 10^{10}	1.4
2	-7.40 parts in 10^{10}	-7.16 parts in 10^{10}	3.0

*Normalized measured error of the test system.
**Normalized theoretical error of the test system.

[2]Hilderbrand, F. B.: Introduction to Numerical Analysis. ch. vii, McGraw-Hill Book Co., Inc., New York, 1965

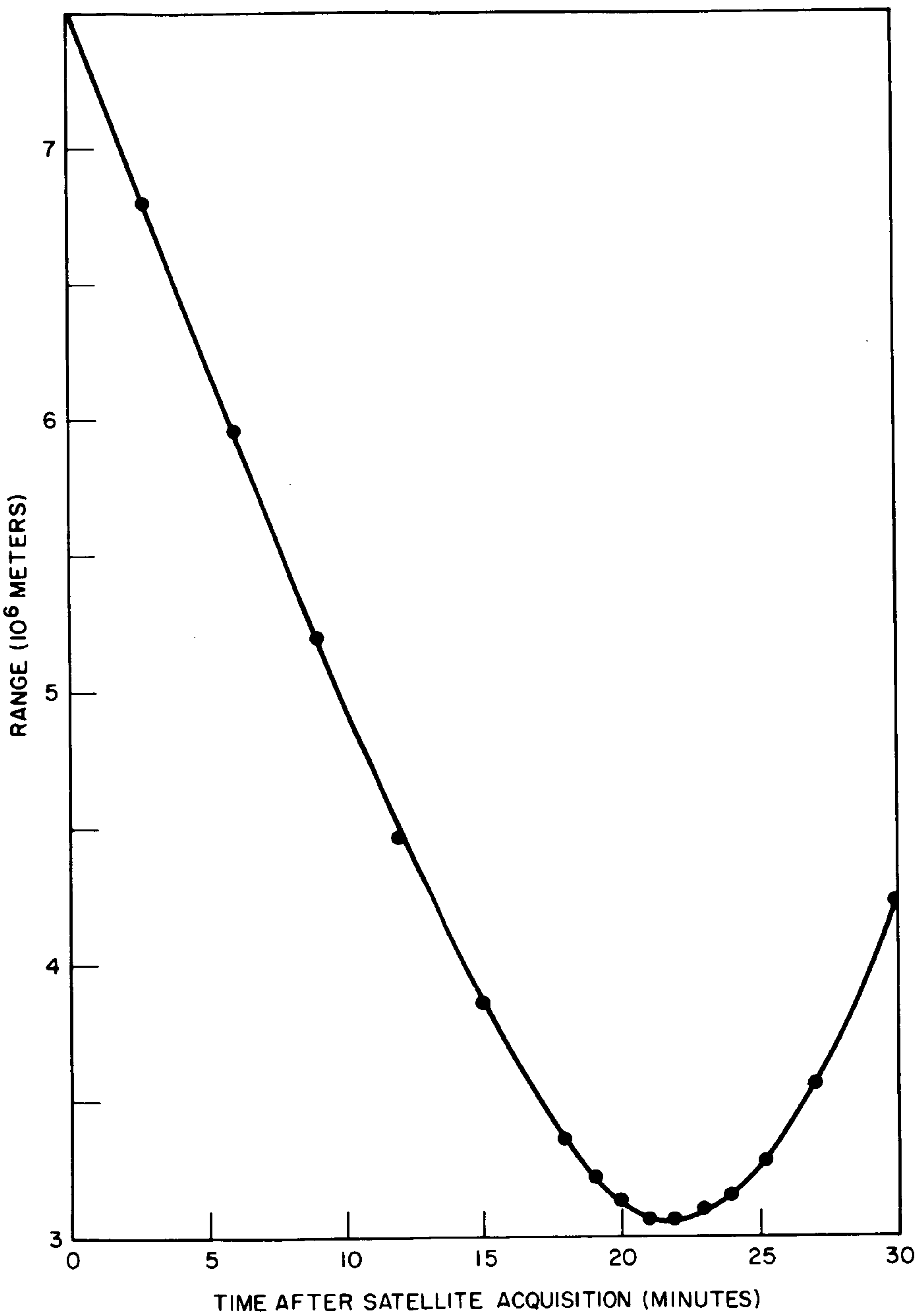

Figure 1–Satellite Range vs Time

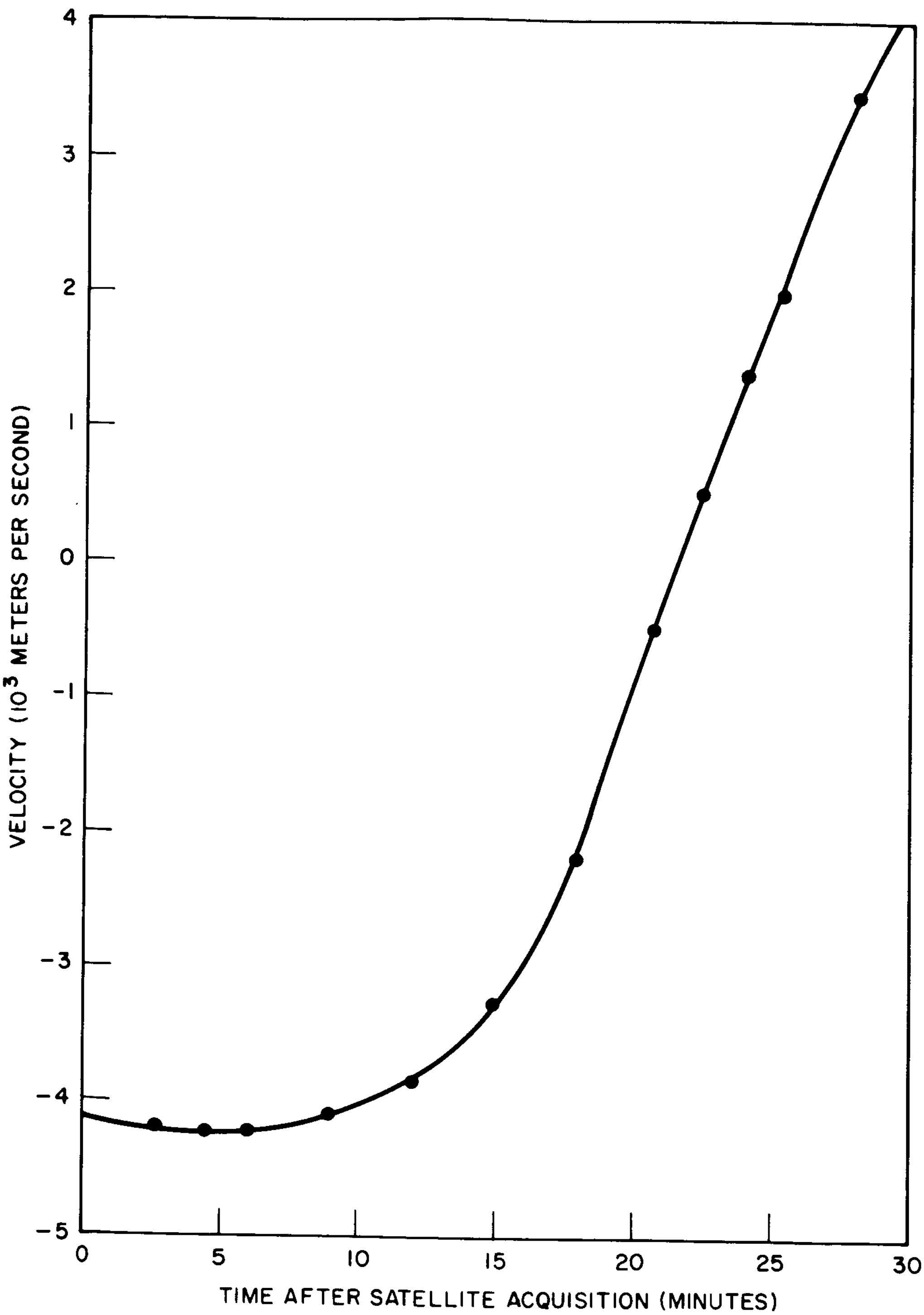

Figure 2–Satellite Velocity vs Time

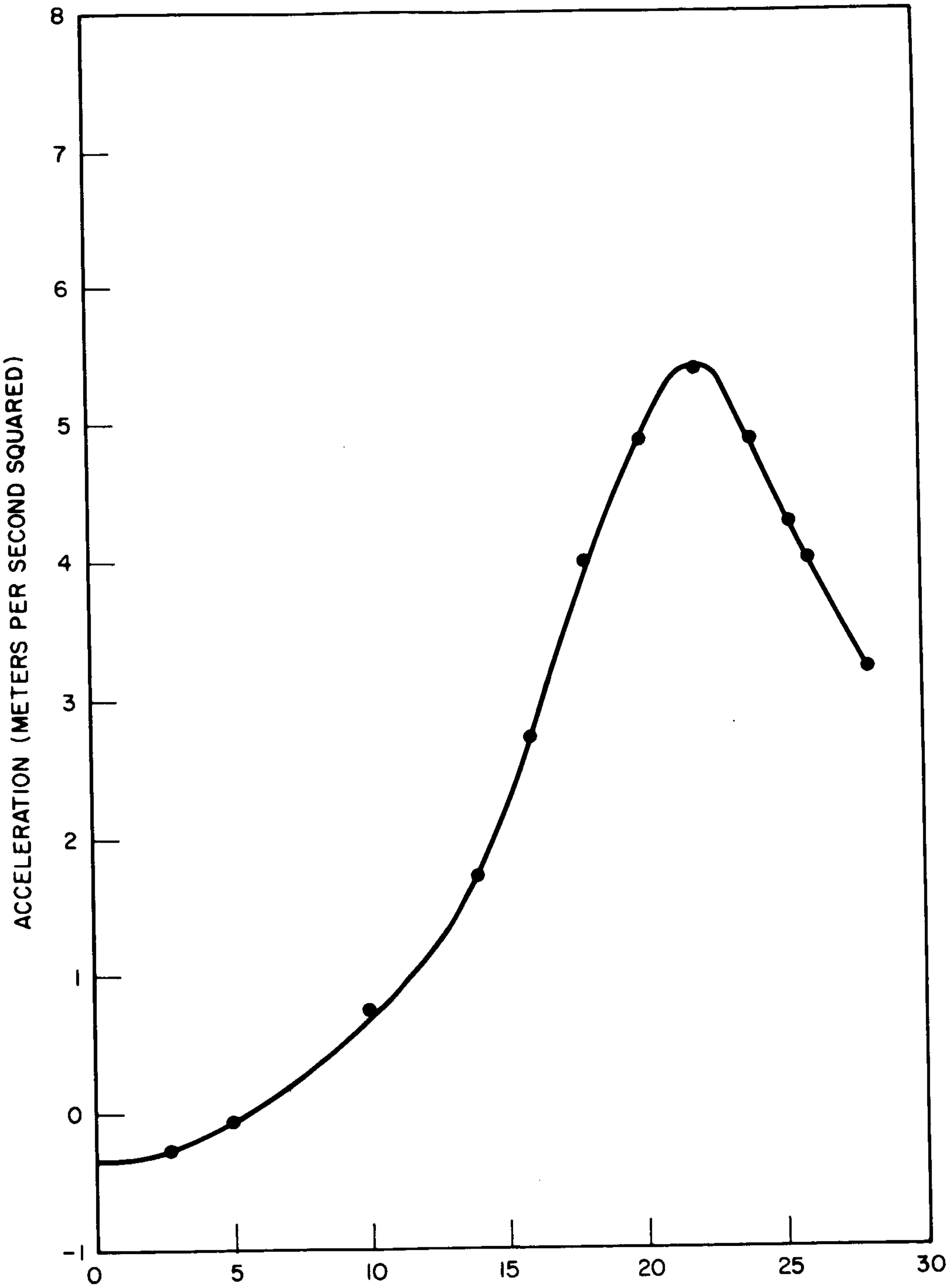

Figure 3–Satellite Acceleration vs Time

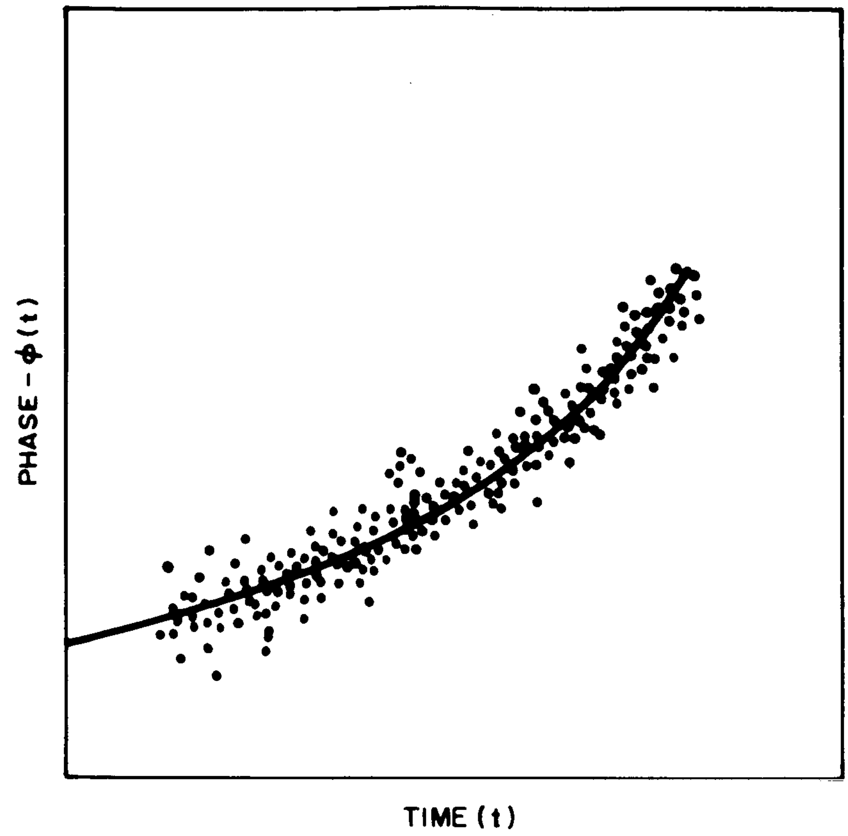

Figure 4–Scatter Diagram of Recorded Data

WORST-CASE ANALYSIS FOR RELAY II

Relay II, launched in 1964, is in an elliptical orbit with an apogee of 7500 km and a perigee of 2100 km. The worst-case relative velocity and acceleration occurs when perigee is approximately overhead. For a simplified worst-case analysis, assume that the orbit is circular at the altitude of perigee, 2100 km, and that the orbital angular velocity of the satellite is constant and equal to the angular velocity at perigee, 7×10^{-4} rad/sec.

Let r_e (Figure 5) be the radius of the earth, 6371 km. Let d represent the distance from the center of the earth to the satellite. If a ground station is located at g on the surface of the earth, then the range to the satellite is:[3]

[3]Handbook of Mathematical Tables. 2d ed., Chemical Rubber Publishing Co., Cleveland, 1964, p. 570

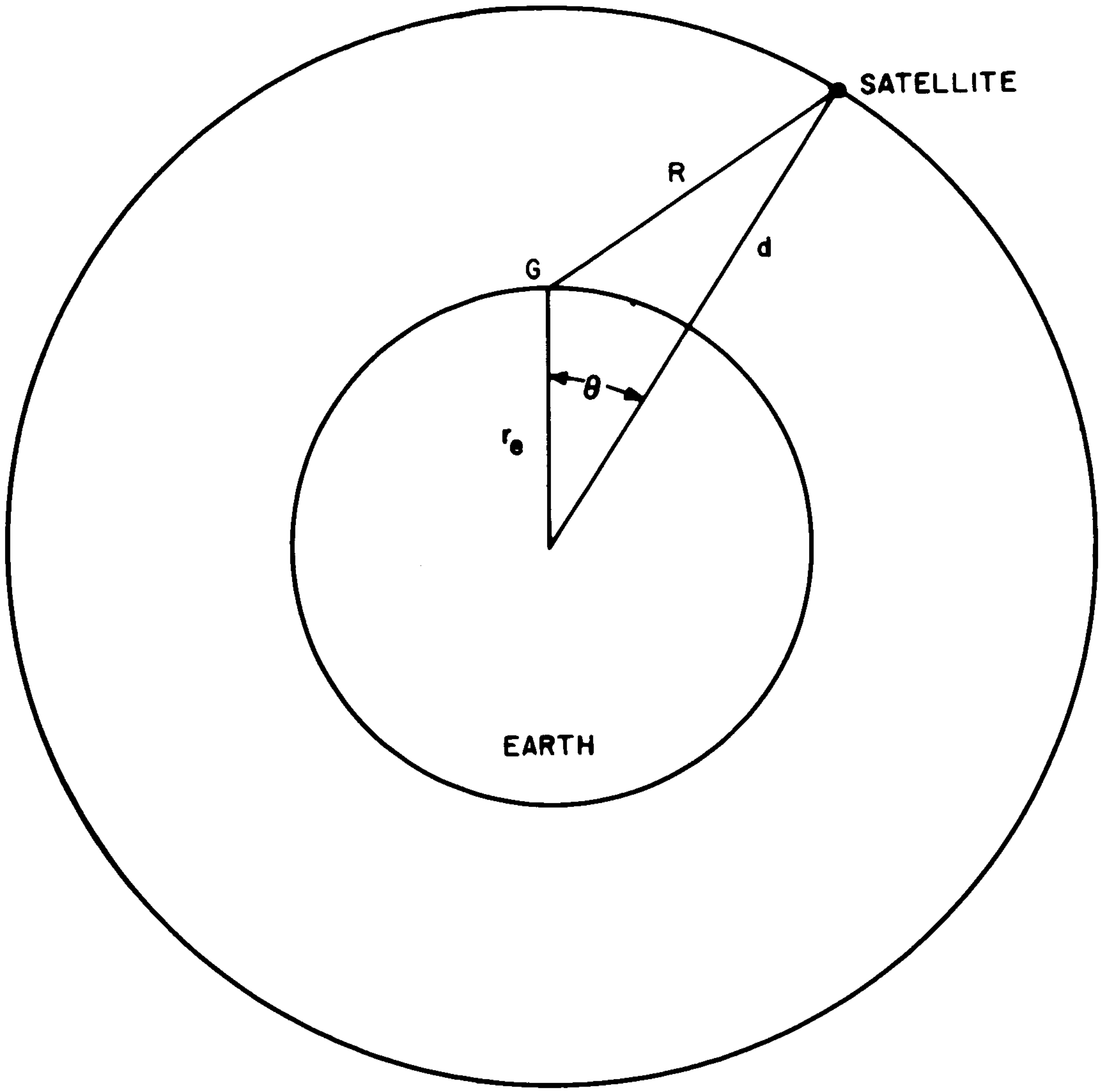

Figure 5–Worst-Case Analysis for Relay II

$$r = (d^2 + r_e^2 - 2dr_e \cos \theta)^{1/2}. \tag{52}$$

But,

$$\theta = \theta(t), \tag{53}$$

$$\approx \alpha t,$$

where α = angular velocity of satellite referenced to the center of the earth at perigee.

Therefore,

$$r = (d^2 + r_e^2 - 2dr_e \cos \alpha t)^{1/2}, \tag{54}$$

$$v = \frac{dr}{dt} = dr_e \alpha \sin \alpha t \ (d^2 + r_e^2 - 2dr_e \cos \alpha t)^{-1/2}, \tag{55}$$

$$a = \frac{d^2 r}{dt^2} = dr_e \alpha^2 \cos \alpha t \ (d^2 + r_e^2 - 2dr_e \cos \alpha t)^{-1/2}$$

$$-d^2 r_e^2 \alpha^2 \sin^2 \alpha t \ (d^2 + r_e^2 - 2dr_e \cos \alpha t)^{-3/2}. \tag{56}$$

In the operational system the normalized error was, from equation (35):

$$E_{NT} = \frac{-(v^2 + ra)}{c^2}.$$

Substituting,

$$E_{NT} = \left[\frac{d^2 r_e^2 \alpha^2 \sin^2 \alpha t}{d^2 + r_e^2 - 2dr_e \cos \alpha t} \right.$$

$$+ dr_e \alpha^2 \cos \alpha t$$

$$\left. \frac{-d^2 r_e^2 \alpha^2 \sin^2 \alpha t}{d^2 + r_e^2 - 2dr_e \cos \alpha t} \right] \Big/ c^2$$

$$= -(dr_e \alpha^2 \cos \alpha t)/c^2. \tag{57}$$

When $\theta = \alpha t = 0$, $E_{NT} = E_{NT}(\max)$,

$$E_{NT}(\max) = -d r_e \alpha^2/c^2, \tag{58}$$

where $d = 7500 \text{ km} + 6371 \text{ km}$,

$= 13{,}871 \text{ km}$,

$r_e = 6371 \text{ km}$,

$\alpha = 7 \times 10^{-4} \text{ rad/sec}$,

$c = 3 \times 10^5 \text{ km/sec}$,

$$E_{NT}(\max) = 4.7 \times 10^{-10}. \tag{59}$$

In the operational system the worst-case frequency error due to Doppler for Relay II is 4.7 parts in 10^{10}. Without the Doppler correction scheme, the maximum normalized Doppler error would be approximately 1.25 parts in 10^5. With the Doppler correction scheme, the maximum error will occur directly overhead ($\theta = 0$); without it, the maximum Doppler occurs at the maximum range.

CONCLUSIONS

It has been demonstrated that the stability of ground-controlled satellite clocks can be greatly increased by the use of the Doppler correction scheme. It has been shown both theoretically and experimentally that the correction scheme is useful in the variable velocity case. The close agreement between the theoretical and measured results (Table 1) indicates that the theoretical analysis is valid.

Maximum error with the Doppler correction scheme for Relay II occurs when the satellite is directly overhead. Because the relative velocity of the satellite becomes zero when the satellite is directly overhead, the maximum error is a function of the product of range and acceleration, equation (35). The true significance of this variable velocity analysis is now obvious because the constant velocity analysis predicts zero error when the satellite is directly overhead, at which time the relative velocity is zero, equation (13).

The worst-case normalized error for Relay II was 4.7 parts in 10^{10}. Without the Doppler correction scheme, this normalized error would be greater than 1 part in 10^5.

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APPENDIX A

SERVOMECHANISM OF THE OPERATIONAL SYSTEM

A comparison of equation (1) and equation (4) indicates that, for small values of v/c, the Doppler frequency shift of the signal received at the satellite is approximately one-half the Doppler frequency shift of the signal received on the ground.

$$f_R = \left(\frac{c - v}{c}\right) f_T , \tag{1}$$

$$f'_R \approx \left(\frac{c - 2v}{c}\right) f_T . \tag{4}$$

where f_T is the frequency transmitted from the ground,

f_R is the frequency received at the satellite,

f'_R is the frequency received on the ground.

Therefore, the frequency of the signal at the satellite is the average of the frequencies of the signals transmitted and received on the ground. If the average of the frequencies of the signals transmitted and received on the ground, f_T and f'_R, respectively, is maintained at f_{REF}, then f_{REF} will always be present at the satellite.

The servomechanism must therefore perform the following function:

$$\frac{f_T + f'_R}{2} = f_{REF} , \tag{A1}$$

or

$$f_T + f'_R = 2 f_{REF} , \tag{A2}$$

$$f_T + f'_R - 2 f_{REF} = 0 . \tag{A3}$$

Figure A-1 is a simplified block diagram of the servomechanism. The sum, f_T and f'_R, is formed in mixer 1 and is passed through a selective filter to mixer 2. $2\,f_{REF}$ is also an imput to mixer 2. The difference output of mixer 2 is

$$f'_R + f_T - 2f_{REF}.$$

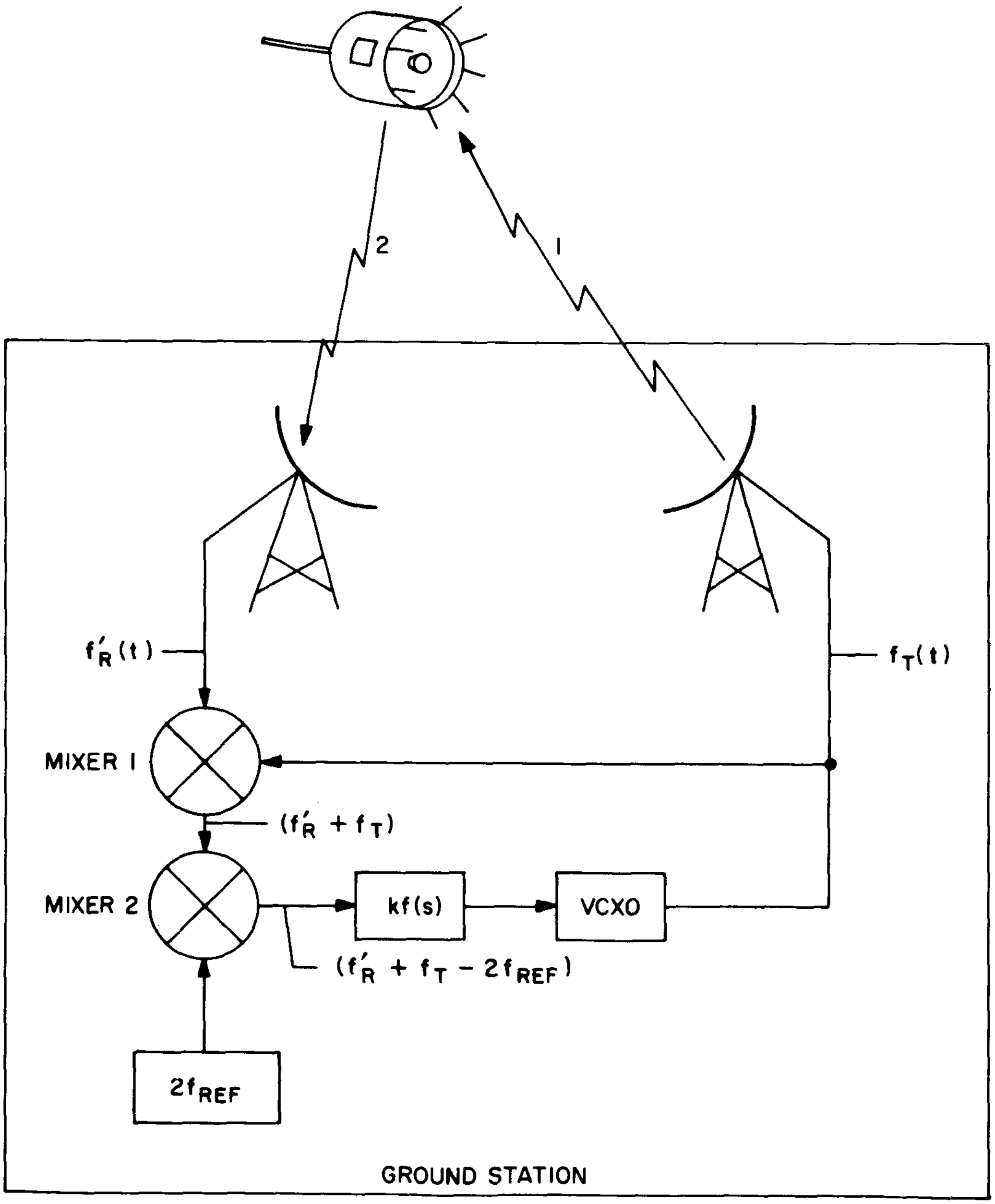

Figure A-1–Servomechanism of Operational System

This output is passed through the low-pass loop filter, $kf(s)$, and is applied as a correction voltage to the voltage-controlled crystal oscillator (VCXO). The output of the VCXO is directly f_T. The correction voltage will be zero only when

$$f'_R + f_T - 2f_{REF} = 0.$$

When $f'_R(t)$ is instantaneously perturbed by Doppler,

$$f'_R(t) + f_T(t) \neq 2f_{REF}. \tag{A4}$$

A correction voltage is instantaneously developed at the output of the low-pass filter, $kf(s)$, which adjusts the VCXO so that the sum of $f'_R(t)$ and $f_T(t)$ is again $2f_{REF}$.

Because the average of f'_R and f_T is maintained at f_{REF}, f_{REF} will always be present at the satellite.

APPENDIX B

CALCULATION OF TIME DELAYS

As discussed in the section on the variable velocity case, the range can be expressed as:

$$R(t) = r + vt + \frac{1}{2}at^2, \tag{B1}$$

where $R(t)$ is the range at any time t,

r is the range at time $t = 0$,

v is the velocity at time $t = 0$,

a is the acceleration at time $t = 0$.

The range at some other time $t + T$ can be expressed as:

$$R(t + T) = R(t) + \frac{dR}{dt} T, \tag{B2}$$

where

$$\frac{dR}{dt} = v + at. \tag{B3}$$

Substituting equation (B3) and (B1) into (B2),

$$R(t + T) = r + vt + \frac{1}{2}at^2 + vT + at\,T. \tag{B4}$$

The time, T, required for a signal to reach the satellite is equal to the range when the signal reaches the satellite, $R(t + T)$, divided by the speed of light, c.

$$T = \frac{R(t + T)}{c}. \tag{B5}$$

Therefore,

$$R(t + T) = cT. \tag{B6}$$

Substitution of equation (B6) into equation (B4) yields:

$$cT = r + vt + \frac{1}{2}at^2 + vT + at\ T. \tag{B7}$$

Gathering terms in T,

$$(c - v - at)\,T = r + vt + \frac{1}{2}at^2, \tag{B8}$$

or

$$T = \frac{r + vt + \frac{1}{2}at^2}{c - v - at}. \tag{B9}$$

It is useful to expand this into a Maclaurin series of the form:

$$f(t) = f(0) + \dot{f}(0)\,t. \tag{B10}$$

Therefore,

$$T(t) = T(0) + \dot{T}(0)\,t, \tag{B11}$$

where

$$T(0) = \frac{r}{c - v}, \tag{B12}$$

$$\dot{T}(0) = \frac{(c - v)v + ar}{(c - v)^2}. \qquad \text{(B13)}$$

The range at some other time $t - \tau$ can be expressed as:

$$R(t - \tau) = R(t) - \frac{dR}{dt}\tau, \qquad \text{(B14)}$$

where

$$\frac{dR}{dt} = v + at. \qquad \text{(B15)}$$

Substituting equation (B1) and (B15) into (B14),

$$R(t - \tau) = r + vt + \frac{1}{2}at^2 - v\tau - at\,\tau. \qquad \text{(B16)}$$

The time, τ, required for a signal to arrive from the satellite is equal to the range when the signal was transmitted from the satellite, $R(t - \tau)$, divided by the speed of light, c.

$$\tau = \frac{R(t - \tau)}{c}. \qquad \text{(B17)}$$

Therefore,

$$R(t - \tau) = c. \qquad \text{(B18)}$$

Substitution of equation (B18) into equation (B16),

$$c\tau = r + vt + \frac{1}{2}at^2 - v\tau - at\,\tau. \qquad \text{(B19)}$$

Gathering terms in τ,

$$(c + v + at)\,\tau = r + vt + \frac{1}{2} at, \tag{B20}$$

or

$$\tau = \frac{r + vt + \frac{1}{2} at^2}{c + v + at}. \tag{B21}$$

Expansion into a Maclaurin series yields:

$$\tau(t) = \tau(0) + \dot{\tau}(0)\,t, \tag{B22}$$

where

$$\tau(0) = \frac{r}{c + v}, \tag{B23}$$

$$\dot{\tau}(0) = \frac{(c + v)v - ar}{(c + v)^2}. \tag{B24}$$

APPENDIX C

REDUCTION OF EQUATION (33) TO A FIRST-ORDER APPROXIMATION

$$b_1 = -w \left[\frac{\dfrac{(c-v)\,v + ra}{(c-v)^2} - \dfrac{(c+v)\,v - ra}{(c+v)^2}}{2 + \dfrac{(c-v)\,v + ra}{(c-v)^2} - \dfrac{(c+v)\,v - ra}{(c+v)^2}} \right]. \tag{33}$$

To determine the first-order approximation, the following maximum anticipated values must be assigned to the variables:

$r < 7 \times 10^6$ meters

$v < 4 \times 10^3$ meters per second

$a < 5$ meters per second squared

$c = 3 \times 10^8$ meters per second

When these values are substituted into the denominator, it is obvious that:

$$2 \gg \frac{(c-v)v + ra}{(c-v)^2} - \frac{(c-v)v - ra}{(c+v)^2}. \tag{C1}$$

Therefore,

$$b_1 = -\frac{w}{2}\left[\frac{(c-v)\,v + ra}{(c-v)^2} - \frac{(c+v)\,v - ra}{(c+v)^2}\right], \tag{C2}$$

$$= -\frac{w}{2}\left[\frac{(c-v)\,v + ra}{(c-v)^2}\,\frac{(c+v)^2}{(c+v)^2} - \frac{(c+v)\,v - ra}{(c+v)^2}\,\frac{(c-v)^2}{(c-v)^2}\right], \tag{C3}$$

$$= \frac{-w}{2(c^2 - v^2)^2} [(cv - v^2 + ra)(c^2 + 2cv + v^2)$$

$$- (cv + v^2 - ra)(c^2 - 2cv + v^2)]. \tag{C4}$$

But

$$c^2 \gg v^2. \tag{C5}$$

Therefore,

$$b_1 = \frac{-w}{2c^4} [c^3 v + 2c^2 v^2 + cv^3 - c^2 v^2 - 2cv^3 - v^4$$

$$+ rac^2 + 2crav + v^2 ra - c^3 v + 2c^2 v^2 - cv^3$$

$$- c^2 v^2 + 2cv^3 - v^4 + c^2 ra - 2cvra + rav^2], \tag{C6}$$

$$= \frac{-w}{2c^4} [2c^2 v^2 + 2c^2 ra - 2v^4 + 2v^2 ra], \tag{C7}$$

$$= \frac{-w}{c^2} \left[v^2 + ra - \frac{1}{c^2} (v^4 - v^2 ra)\right], \tag{C8}$$

where

$$v^2 + ra \gg \frac{v^4}{c^2}, \tag{C9}$$

and

$$v^2 + ra \gg \frac{v^2}{c^2} ra. \tag{C10}$$

Therefore,

$$b_1 \approx \frac{-w}{c^2} (v^2 + ra). \tag{C11}$$

APPENDIX D

SERVOMECHANISM OF THE TEST SYSTEM

As discussed in the experimental test system section, it was impossible to measure the quality of the Doppler correction scheme in the satellite. The signal was therefore looped through the satellite twice, placing a virtual satellite (Point A, Figure D-1) on the ground. The only additional circuit functions required were two frequency translators to prevent interference of the two transmitted signals and of the two received signals, respectively. Appendix A describes the operation of the basic servomechanism.

In the test system, f_{REF}, is established at the virtual satellite. Figure D-2 is a functional block diagram of the error measurement circuitry. Both signals were passed through hard-limiters, converting the sinusoidal waveforms to pulse waveforms. The positive-going edge of f_{vs} started the time interval counter; the positive-going edge of f_{REF} stopped the time interval counter. A printout recorded the time interval counts at a rate of six samples per second.

The initial time interval represents an initial relative phase difference. Any subsequent change in interval represents a change in relative phase, or frequency difference.

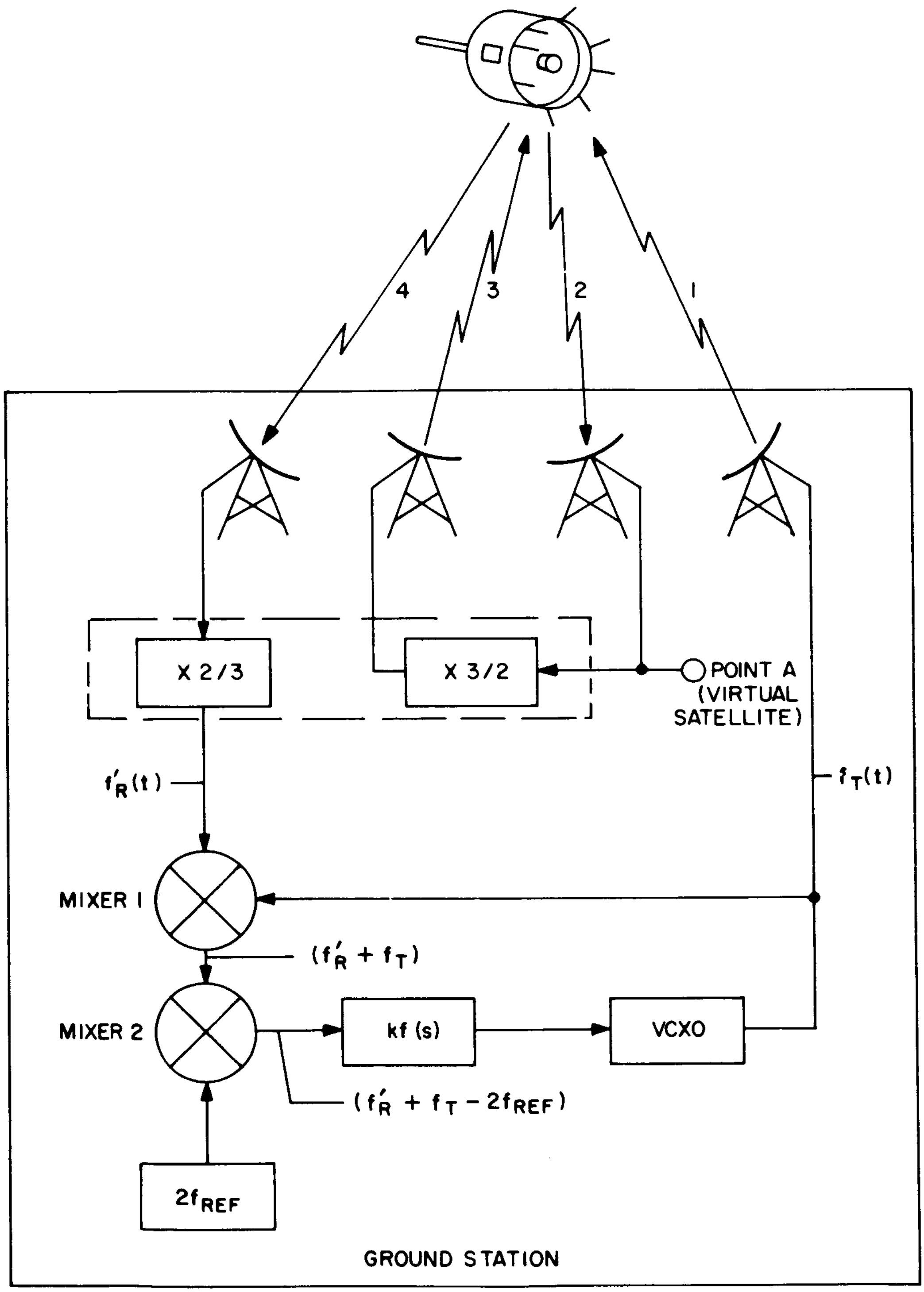

Figure D-1--Servomechanism of Test System

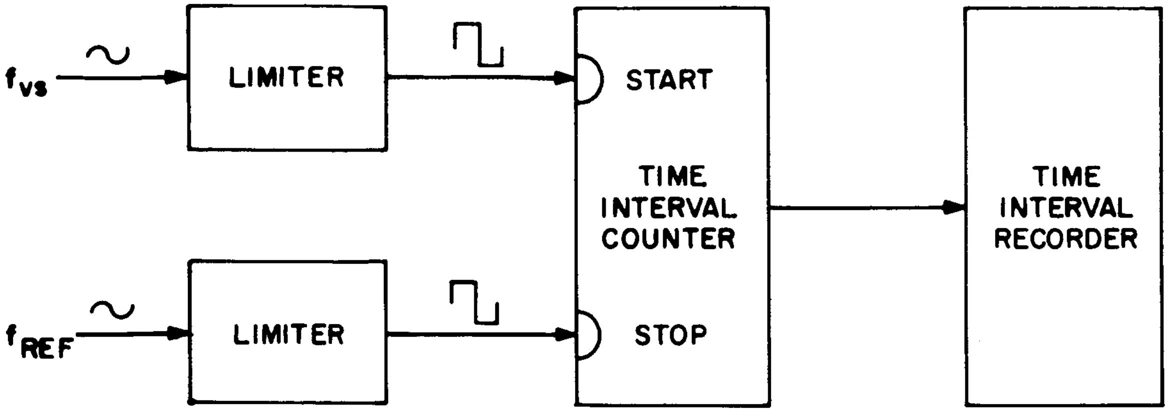

Figure D-2–Instrumentation of Test System

APPENDIX E

REDUCTION OF EQUATION (49) TO A FIRST-ORDER APPROXIMATION

$$b_1 = -w \left[\frac{\dfrac{(c-v)v + ra}{(c-v)^2} - \dfrac{(c+v)v - ra}{(c+v)^2}}{1 + \dfrac{(c-v)v + ra}{(c-v)^2} - \dfrac{(c+v)v - ra}{(c+v)^2}} \right]. \tag{49}$$

To determine the first-order approximation, the following maximum anticipated values must be assigned to the variables:

$r < 7 \times 10^6$ meters

$v < 4 \times 10^3$ meters per second

$a < 5$ meters per second squared

$c = 3 \times 10^8$ meters per second

When these values are substituted into the denominator, it is again obvious that:

$$1 \gg \frac{(c-v)v + ra}{(c-v)^2} - \frac{(c+v)v - ra}{(c+v)^2}. \tag{E1}$$

Therefore,

$$b_1 \approx -w \left[\frac{(c-v)v + ra}{(c-v)^2} - \frac{(c+v)v - ra}{(c+v)^2} \right]. \tag{E2}$$

A comparison of equation (C2) and equation (E2) indicates that b_1 in the test system is exactly twice b_1 in the operational system, equation (C11).

Therefore, in the test system:

$$b_1 \approx \frac{-2w}{c^2} (v^2 + ra). \tag{E3}$$

APPENDIX F

REDUCTION OF A SCATTER DIAGRAM TO A QUADRATIC CURVE BY THE CRITERIA OF LEAST-MEAN-SQUARED ERRORS

Assume that a set of observations, ϕ_1, ϕ_2, . . . , ϕ_n, which belong to an unknown function, $\phi = f(t)$, at some prescribed points of the independent variable

$$t = t_1, t_2, \ldots, t_n \tag{F1}$$

can be fitted by a polynomial of order n:

$$\phi = b_0 + b_1 t + \ldots + b_n t^n. \tag{F2}$$

The order, n, of the polynomial is chosen a priori; the coefficients, b_0, b_1, b_2, . . . , b_n, are determined by the measurements.

The unknown coefficients, b_i's, of the polynomial are determined as follows. At each point of observation, i, form a "residual" R, so that:

$$R_i = b_0 + b_1 t_i + \ldots + b_n t_i^n - \phi_i, \tag{F3}$$

where ϕ_i is the observed value at t_i.

Take the sum of the square of all residuals:

$$Q = \sum_{i=1}^{m} R_i^2, \tag{F4}$$

$$Q = \sum_{i=1}^{m} (b_0 + b_1 t_i + \ldots + b_n t_i^n - \phi_i)^2, \tag{F5}$$

The quantity, Q, is by nature positive or in the limit zero. The zero value is possible only if all of the residuals vanish independently. This implies that the measurements fit the n^{th} order polynomial exactly. This is not the usual case. There is, however, a unique value of the b_i's for which the sum, Q, becomes a minimum. The polynomial associated with these b_i's is the "best fit" of the measurements.

This minimization principle has a unique solution in the form of a linear set of equations with a nonvanishing determinant. According to the principles of calculus, the condition of minimization requires that the partial derivative of Q with respect to any and all b_i's vanishes. Repeating equation (F5),

$$Q = \sum_{i=1}^{m} (b_0 + b_1 t_i + \ldots + b_n t_i^n - \phi_i)^2. \tag{F5}$$

Taking the derivative with respect to b_j yields:

$$\frac{dQ}{db_j} = 0 = 2 \sum_{i=1}^{m} (b_0 + b_i t_i + \ldots + b_n t_i^n - \phi_i)\, t_i^j. \tag{F6}$$

Expanding,

$$\sum_{i=1}^{m} \phi_i t_i^j = b_0 \sum_{i=1}^{m} t_i^j + b_1 \sum_{i=1}^{m} t_i^{j+1} + \ldots + b_n \sum_{i=1}^{m} t_i^{j+n}, \tag{F7}$$

where

$$j = 0, 1, 2, \ldots, n.$$

In this case $n = 2$. Equation (F7) can now be expanded as follows for $j = 0, 1,$ and 2, respectively:

$$\sum_{i=1}^{m} \phi_i = b_0 m + b_1 \sum_{i=1}^{m} t_i + b_2 \sum_{i=1}^{m} t_i^2, \tag{F8}$$

$$\sum_{i=1}^{m} \phi_i \, t_i = b_0 \sum_{i=1}^{m} t_i + b_1 \sum_{i=1}^{m} t_i^2 + b_2 \sum_{i=1}^{m} t_i^3, \tag{F9}$$

$$\sum_{i=1}^{m} \phi_i \, t_i^2 = b_0 \sum_{i=1}^{m} t_i^2 + b_1 \sum_{i=1}^{m} t_i^3 + b_2 \sum_{i=1}^{m} t_i^4. \tag{F10}$$

Because all measurements of ϕ_i were made at regular intervals of t_i,

$$t_i = i \, \Delta t. \tag{F11}$$

The three preceding equations may now be rewritten as:

$$\sum_{i=1}^{m} \phi_i = b_0 \, m + b_1 \, \Delta t \sum_{i=1}^{m} i + b_2 \, \Delta t^2 \sum_{i=1}^{m} i^2, \tag{F12}$$

$$\Delta t \sum_{i=1}^{m} i \, \phi_i = b_0 \, \Delta t \sum_{i=1}^{m} i + b_1 \, \Delta t^2 \sum_{i=1}^{m} i^2 + b_2 \, \Delta t^3, \sum_{i=1}^{m} i^3 \tag{F13}$$

$$\Delta t^2 \sum_{i=1}^{m} i^2 \, \phi_i = b_0 \, \Delta t^2 \sum_{i=1}^{m} i^2 + b_1 \, \Delta t^3 \sum_{i=1}^{m} i^3 + b_2 \, \Delta t^4 \sum_{i=1}^{m} i^4. \tag{F14}$$

These equations, called the "normal equations" of the least-square problem, belong to a group of linear equations which have not only a symmetric but also a recurrent matrix, characterized by the property

$$A_{jk} = A_{j-1,\,k+1}. \tag{F15}$$

In the preceding set of equations, the b_i's are the unknowns; all other terms are constants. The constants involve simple summations which were computed on a digital computer. After the constants were determined, a matrix-solving routine was used to evaluate the values of b_0, b_1, and b_2.

Table 1 lists the normalized measured errors, E_{NM}, for the two experimental runs.

APPENDIX G

CALCULATION OF ERRORS

The theoretical errors of the test system, E'_{NT}, are calculated below. The errors are determined by using equation (51) and the initial condition listed in the experimental section.

$$E'_{NT} = \frac{-2\,(v^2 + r\,a)}{c^2}. \tag{51}$$

Experiment 1

$$E'_{NT} = -2\ \frac{(4.2)^2 \times 10^6 - 6.8 \times 10^6 \times .25}{3^2 \times 10^{16}},$$

$$= -3.54 \times 10^{-10}.$$

Experiment 2

$$E'_{NT} = -2\ \frac{(2.0)^2 \times 10^6 + 3.3 \times 10^6 \times 4.3}{3^2 \times 10^{16}},$$

$$= -7.16 \times 10^{-10}.$$