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GRAVITATIONAL AND PHASE CHANGE SOURCES OF ENERGY IN JUPITER

R. Smoluchowski

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UNIVERSITY OF MINNESOTA
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ABSTRACT

The energy release in giant planets is calculated using a simple solid hydrogen model without taking into account nuclear or other effects. The amount of energy being stored in the planet in the form of elastic strain depends upon the past history of the planet. For planets of the size of Jupiter the rate of release of gravitational energy is greater than, though comparable to, the rate of increase of the stored energy if an existing strain of 10 percent is assumed. It vanishes for sufficiently high pre-strains. If progressive change from molecular to metallic solid hydrogen is permitted then the release of energy is greater than without phase change. It appears that a phase change is a more likely source of energy in Jupiter than a gravitational contraction unless very low pre-strains are assumed. The rate controlling processes such as thermal conduction and helium diffusion are discussed.

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INTRODUCTION

Recent observations suggest that a large fraction of energy radiated by Jupiter, in the whole range of wavelengths, is of internal origin (ref. 1). It has been proposed that, among the various possible sources of this energy, gravitational contraction of about 1 mm per year (ref. 1) may account for the observed emission of about 10^{33} ergs per year. Apart from the question whether this mechanism actually operates in Jupiter a general investigation of the problem is of interest since other giant planets (Saturn, dark companion of Bernard's star, etc.) may belong to the same category.

Many theoretical investigations(ref. 2) of the interior of giant planets have been made assuming equilibrium configurations at absolute zero to which corrections for higher temperatures are made. Clearly, in order to obtain an internal source of energy, a non-equilibrium situation has to be considered apart from radioactivity and other nuclear reactions. Since the purpose of this note is only to estimate the magnitude of the various phenomena, a very simple model resembling numerically the known equilibrium conditions will be used. Thus no detailed assumptions are made about the past history of the planet except where this affects the presently observable phenomena. Also no attempt is made to discuss quantitatively the rate controlling processes such as diffusion or heat transfer in the interior of the solid core or through the planetary atmosphere (ref. 3). Similarly all effects caused by rotation or change of gravitational constant are omitted (ref. 4. The problem is thus limited to the question how much gravitational energy can be released and what is the role of possible solid state phase changes.

The author is indebted to his colleagues Professors W. B. Daniels and R. H. Dicke for stimulating discussions.

ONE-PHASE SYSTEM

The assumed model is a non-rotating sphere of a cold solid of uniform density. The gravitational potential energy is thus

$$\Omega = -\frac{3}{5}\frac{GM^2}{R} \tag{1}$$

where G is the gravitational constant, M is the total mass and R is the radius of the sphere. A decrease of the radius by δR releases

$$\delta\omega = \frac{3}{5}\frac{GM^2}{R^2}\delta R \tag{2}$$

energy. It follows that

$$\frac{\delta R}{R} = \frac{\delta\Omega}{\Omega} \tag{3}$$

The strain energy or work done on a solid depends on the strain already present. Thus if a strain $\frac{\Delta R}{R}$, from an initially strain-free condition, is followed by an additional strain $\frac{\delta R}{R}$ the additional work done per unit volume is

$$\delta\omega = \frac{9}{2}\frac{\Delta R}{R}\frac{\delta R}{R}K \tag{4}$$

where K is the modulus of compressibility. The increase of the strain energy in a sphere is then

$$\delta W = \frac{4}{3}\pi R^3 \; \delta\omega = 6\pi RK \; \Delta R \; \delta R \tag{5}$$

For simplicity let us assume that the compressibility is small so that a uniform average density $\rho(R)$ where R is the radius of the sphere can be used in the calculations. Then the net energy release caused by gravitational shrinkage δR is given by

$$\delta E = \delta\Omega - \delta W = 2\pi R \left(\frac{8}{15} \pi G R^3 \rho^2 - 3K\Delta R \right) \delta R \tag{6}$$

which vanishes for

$$\Delta R_{max} = \frac{8}{45} \pi G R^3 \rho^2 / K = \frac{2}{15} G \frac{\rho M}{K} \tag{7}$$

giving a maximum value of the total released gravitational energy

$$\Delta\Omega_{max} = \frac{128}{675} \pi^3 G^2 R^7 \rho^4 / K = \frac{8}{45} \pi G^2 R \frac{\rho^2 M^2}{K} \tag{8}$$

If we put $\Delta R < 0.5 R$ it follows from (7) that for R greater than

$$R_c = \left(\frac{45 K}{16\pi G \rho^2} \right)^{1/2} \tag{9}$$

δE is always positive. On the other hand, for smaller radii a stage could be reached when no further gravitational contraction is possible. It is interesting to note that for most solids at high pressures K is proportional to ρ^2 so that R_c is essentially independent of pressure.

In order to apply the various formulae to a sphere of pure hydrogen one has to consider the fact that at high pressures molecular solid hydrogen transforms into a metal. This limits the possible mass of a molecular sphere to a value beyond which a central metallic core is formed. This two-phase model will be considered later. Best equation of state data (ref. 2) indicate that for molecular solid hydrogen in the range of pressure p from 0.1 to 2×10^{12} dynes/cm^2 and for metallic hydrogen up to 3×10^{13} dynes/cm^2 the equation

$$p = A(\rho^2 - \rho'^2) \tag{10}$$

is very well obeyed. For the molecular and metallic forms the parameter A has the values 4.26 and 3.30×10^{12} dynes/cm^2 while ρ', which should not be identified with the real density at zero pressure, is 0.25 and 0.68 g/cm^3 respectively. Equation (10) leads to an expression for the modulus of compressibility

$$K = 2A\rho^2 \tag{11}$$

and to the Laplace form of the radial dependence of density

$$\rho(r) = \rho_c \frac{L}{r} \sin \frac{r}{L} \tag{12}$$

where ρ_c is the density at $r = 0$ and

$$L^2 = A/2\pi G \tag{13}$$

For the molecular and metallic forms L is 3.18 and 2.8×10^9 cm respectively.

The above equations determine uniquely the average density $\bar{\rho}$, and the corresponding modulus of compressibility K for molecular hydrogen spheres of various radii. Table 1 shows ΔR_{max} and $\Delta\Omega_{max}$ with R expressed in cm and Ω in ergs.

TABLE 1

ΔR_{max} and $\Delta\Omega_{max}$ with R Expressed in cm and Ω in Ergs

R	$\Delta R_{max}/R$		$\Delta\Omega_{max}$	
	mol.	met.	mol.	met.
10^7	4.4×10^{-7}	5.7×10^{-7}	4.6×10^{21}	1.7×10^{23}
10^8	4.4×10^{-5}	5.7×10^{-5}	4.6×10^{23}	1.7×10^{30}
10^9	4.4×10^{-3}	5.7×10^{-3}	4.6×10^{35}	1.7×10^{37}
5×10^9	0.11	0.14	5.4×10^{40}	2.3×10^{41}
7×10^9	0.21	0.28	9.8×10^{41}	3.7×10^{42}

For radii smaller than 10^9 cm the radial gradient of pressure and of density plays little role from the point of view of gravitational and elastic energy.

The radii of the planets Jupiter (7×10^9 cm) and Saturn (6×10^9 cm) are known to lie just beyond the upper limit of the applicability of the one-phase solid hydrogen model. They are also close to the range where the stored elastic energy begins to be always smaller than the available gravitational energy. If we use, however, the molecular model and assume that the present strain is about 10 percent, then nearly 40 percent of the evolved gravitational energy is being used for compressing the interior of the planet. The net rate is $\frac{\delta E}{\delta R} \sim 5\times10^{31}$ ergs/cm.

Another consequence of the gravitational contraction is an increase of pressure P at the center of the planet. The relative increase is approximately given by

$$\frac{\Delta P}{P} = -4 \frac{\Delta R}{R} \tag{14}$$

and, as Table I indicates, for large radii it can be considerable. In particular for a planet which is close to the maximum size compatible with solid molecular hydrogen the 1 mm per year shrinkage corresponds to an annual increase of central pressure by 140 cynes/cm^2 or to a 7 per cent increase in a billion years.

For hydrogen the limiting radius R_c as given by Eq. (9) is about 10^{10} cm which is close to Russell's upper limit of cold bodies 0.78×10^{10} cm as obtained by De Marcus (ref. 2) from detailed calculations. At extremely high pressures Eq. (10) is not applicable and the solid is better described as a Fermi gas (ref. 5) for which $p \sim \rho^{5/3}$. According to Alder (ref. 6) this is reached for hydrogen near 5×10^{13} dynes/cm^2 which is comparable to the central pressure of Jupiter.

TWO-PHASE SYSTEMS

As mentioned above a two-phase solid hydrogen system is required for an analysis of heavy planets such as Jupiter and Saturn. Unfortunately, the transformation pressure for the phase change in solid hydrogen, and especially the ratio of densities of the two phases at that pressure is poorly known. Using extrapolated equations of state De Marcus (ref. 2) arrived at a transformation pressure of about 2×10^{12} dynes/cm^2. Alder (ref. 6), on the other hand, questioned the validity of the extrapolated equation for the molecular form and, through other considerations, arrived at a pressure of 18×10^{12} dynes/cm^2. So far all studies of the giant planets were made using the lower of these two pressures. It should be pointed out, however, that if the higher value were true then rather drastic changes both in the radial dependence of pressure and of density for Jupiter and for Saturn will have to be made. Also serious consequences for the rate of internal heat transfer may follow since one expects the metallic phase to be a better heat conductor than the molecular phase.

In view of the commonly used transition pressure 2×10^{12} dynes/cm^2 this value will be used here too. This limits the solid molecular sphere to a mass of 4×10^{29} g compared to a mass of 5.69^{29} g for Saturn and 19.02×10^{29} g for Jupiter.

The pertinent equations for the two-phase model are:

$$\Omega = -\frac{3}{5}\frac{GM_c^2}{R_1}\left\{1 - \frac{5}{2\gamma} + \frac{3}{2\gamma^2} + \frac{1}{\gamma^2}\left[\frac{5(\gamma-1)}{2\beta^2} + \frac{1}{\beta^5}\right]\right\} = -\frac{3}{5}\frac{GM_c^2}{R_1} P \qquad (15)$$

$$\frac{\delta\Omega}{\delta R_2} = -\frac{3}{5}\frac{GM_c^2}{R_1^2}\left(R_1 \frac{\delta P}{\delta R_2} - P\frac{\delta R_1}{\delta R_2}\right) \qquad (16)$$

where $\beta = R_1/R_2$, $\gamma = \overline{\rho_1}/\overline{\rho_2}$ and M_c is the mass of the metallic core; further

$$\delta W = \delta W_1 + \delta W_2 \qquad (17)$$

where

$$\delta W_1 = 6\pi R_1 K_1 \Delta R_1 \delta R_1$$

$$\delta W_2 = \frac{6\pi K_2}{R_2^3 - R_1^3}\left(R_2^2 \Delta R_2 - R_1^2 \Delta R_1\right)\left(R_2^2 \delta R_2 - R_1^2 \delta R_1\right)$$

with subscripts 1 and 2 referring to the metalic core and to the molecular mantle respectively. These equations are based on the assumption that no phase change is induced by the shrinkage i.e., that the masses of the core M_c and of the mantle M_m remain unchanged. The requirement that the pressure is the same on both sides of the boundary between the core and the mantle leads to

$$\frac{\Delta R_1}{R_1}\left(K_1 + \frac{R_1^3}{R_2^3 - R_1^3}\right) = \frac{\Delta R_2}{R_2}\frac{R_2^3}{R_2^3 - R_1^3} K_2 \qquad (18)$$

a similar relation holding for δR_1 and δR_2.

De Marcus (ref. 2) has shown that in order to account for the known moment of inertia and the external potential parameters of Jupiter it is necessary to conclude that the planet consists of hydrogen and some helium. The latter is nearly all concentrated in the metallic hydrogen core and, in his model, up to one eighth of the radius of the core is essentially pure solid helium. There is nearly no helium in the mantle but about 20 weight per cent (or six atomic per cent) in the core. For simplicity we shall assume first that there is no helium either in the mantle or in the core but use the actual radii $R_1 = 5.6$ and $R_2 = 7 \times 10^9$ cm. The known pressure at the core boundary and the Laplace relationship give the average densities for the core 1.5 and 0.45 for the mantle. The corresponding values of K are 13.8 and 1.7×10^{12} dynes/cm^2. With these values and a total strain of 10 per cent one obtains $\frac{\delta E}{\delta R_2} \sim 2.9 \times 10^{33}$ ergs/cm with only 10 per cent of the available gravitational energy being stored as elastic strain.

The inclusion of helium in the model can be done easily since only the contribution of the core needs to be altered. A mixture of helium and hydrogen as given by De Marcus also obeys Eq. (10) with $A = 2.5 \times 10^{12}$ leading to an average density of the core 2.13 and a modulus of compressibility K of nearly 23×10^{12} dynes/cm^2. For a 10 per cent pre-strain one obtains $\frac{\delta E}{\delta R_2} \sim 3.5 \times 10^{33}$ ergs/cm with about 1/8th of the energy being stored in the elastic strain. It should be noted that the calculated pressure (ref. 2) at the center of the planet is higher than the limit of applicability of Eq. (10) to hydrogen but is lower than the corresponding limit for helium (ref. 5). It is doubtful whether, at this stage of approximation, a distinction need be made between a solid solution of hydrogen and helium and a two-phase mixture. Solid helium is close-packed-hexagonal while the lattice of metallic hydrogen is perhaps body-centered cubic or has a layer structure (ref. 7). Thus the possibility of a wide solid solubility range and continuous transition from nearly pure helium at the center of the planet to a nearly pure metallic hydrogen at larger radii, assumed by De Marcus, seems remote. It is not possible, at present, to evaluate this situation in a quantitative manner.

It is interesting to note that while the one-phase models gave a maximum obtainable stress ($\Delta R_{max}/R$ in Table I), even for a planet of the size of Jupiter, this is not necessarily the case for the two-phase model. The calculated ΔR_{max} turns out to be comparable to R_2 although the actual value is very sensitive to β and to γ which in turn depend upon the assumed transition pressure between the two phases. Since no calculations of the size of the metallic core for a higher transition pressure were made it is not possible to evaluate the maximum strain for that situation. Nevertheless, a rough consideration indicates that a higher transition pressure would lead to a lower maximum strain.

It may be pointed out too that a compression of a two-phase sphere in which the compressibilities of the two phases are appreciably different requires a net plastic flow in the tangential direction in the mantle. Thus part of the evolved energy is being used for this purpose but it reappears eventually as heat and thus is not lost from the energy balance considered above.

PHASE CHANGES

In the previous section it was assumed that the masses of the metallic core and of the molecular mantle remain unchanged during shrinkage. If this restriction is removed then additional terms in the energy balance should appear. In particular a change of entropy (or of specific heat) should be taken into account. Unfortunately, the highly approximate knowledge of the equations of state of the two phase does not permit a reasonable estimate of these effects. Thus the calculations are made for absolute zero hoping that they have significance at higher temperatures (ref. 2). The customary argument is that at high temperatures (of the order of 10^4 degrees) and high pressures the specific heats (and probably the entropies) do not differ much. Clearly this is far from a satisfactory situation but any attempts at making reasonable corrections at the present moment seem hopeless.

Let us assume first that the radius of the metallic core R_1 is increased by δR_1 at the expense of the molecular mantle without adjustment of the gravitational shrinkage. We have then

$$\delta R_2 = (1 - \rho_1/\rho_2) \frac{R_1^2}{R_2^2} \delta R_1 = (1-\gamma)\, \beta^2\, \delta R_1$$

and from Eq. (15)

$$\frac{\delta \Omega}{\delta R_1} = \frac{3}{5} \frac{G M_c^2}{R_1^2} \left(5P + R_1 \frac{dP}{dR_1}\right) \tag{19}$$

which for the numerical values appropriate for Jupiter gives $\frac{\delta E}{\delta R_1} \sim 7.5 \times 10^{33}$ ergs/cm. There is also an additional term PdV caused by the change of density which is an order of magnitude smaller.

If we now permit the enlarged metallic sphere to shrink so as to reach the proper Laplace density distribution (Eq. 12) then additional gravitational energy $\delta\Omega'$ will be released but part of it will be used for increasing the stored elastic energy. The shrinkage is equal to

$$-(1 - \frac{1}{f})\, \delta R_1$$

where

$$f = \frac{1}{x^2} \left(1 - 2\, x \operatorname{ctg} x + \frac{x^2}{\sin^2 x}\right) \tag{20}$$

with $x = R_1/L$. This shrinkage increases $\delta\Omega$, the amount calculated from Eq. (19), by about 30 per cent. Thus the net available energy is about 10^{34} ergs/cm with less then five per cent going into strain assuming as before a pre-strain of 10 per cent.

The rate of the progressive increase of the metallic core at the expense of the molecular mantle is important from the point of view of Ramsey's small core instability. Both Saturn and Jupiter are far from the critical limits.

DISCUSSION

Availability of gravitational energy in a solid hydrogen planet depends upon its size and upon its state of strain i.e. upon its past history. For a planet of the size of Jupiter and a 10 per cent pre-strain the largest amount of energy becomes available if a gradual change from a molecular to a metallic phase is accompanied by a gravitational contraction. Without phase change the effect is smaller and it vanishes for sufficiently large pre-strains.

The problem of the rate-controlling processes is quite difficult. If no phase change is taking place then the most obvious impediment to contraction is thermal expansion induced by heating which accompanies an almost adiabatic compression. In turn the loss of internal heat is a function of the thermal conductivity through the solid hydrogen and through the atmosphere which surrounds the real planet. The thermal conductivity of metallic hydrogen is known to be very high (ref. 8), that of solid hydrogen is probably lower because of lower electronic mobility. Radiativc heat transfer plays unquestionably an important role too.

If a phase change does take place then the rate-controlling process can be either the change of the internal temperature or chemical diffusion. One might expect the transformation pressure to be higher at higher temperatures and thus the actual rate-controlling process would be the gradual drop of temperature at the interphase interface, which in turn depends on thermal conductivity as discussed above.

It is well known that polymorphic phase changes are sensitive to composition. The addition of a solute may either increase or lower the transformation temperature or transformation pressure of a solvent. It is quite possible that for the phase transformation of hydrogen to proceed a solute, perhaps helium, has to change its local concentration. The rate controlling process is then diffusion which at the high pressures is probably quite slow in spite of high temperatures. Peebles (ref. 3) estimates that depending upon the necessary diffusion distance the required times may be as high as 10^9 years for the motion of helium in a solid hydrogen planet. Normally one assumes that the heavy helium diffuses towards the center of a hydrogen planet. One cannot exclude however the possibility that the solubility of helium in molecular solid hydrogen at the transformation pressure is higher than in the metallic phase. In that case a progressive growth of the metallic core would necessitate an outward diffusion of the excess helium. If this were true then the helium content of the core would depend only upon the solubility of helium in solid metallic hydrogen at the transformation pressure. This point warrants a further careful investigation.

One concludes that a progressive, diffusion controlled, phase change within the solid hydrogen is a more likely source of energy in Jupiter than a gravitational contraction. The latter would become important only if unreasonably small pre-strains were assumed.

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