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ANALYSIS OF STRUCTURAL RESPONSE WITH DIFFERENT FORMS OF DAMPING

by R. R. Reed

George C. Marshall Space Flight Center
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DEFINITION OF SYMBOLS

Symbol	Definition
A	amplitude of the response
A_n	the n^{th} complex coefficient of the exponential series
A_0	steady state response amplitude
C_0, C_1, C_2, C_3	damping coefficients
C_{1r}	a reference value of C_1
K	linear spring stiffness coefficient
M	generalized mass
P	magnitude of step input force
R	energy loss ratio
R_r	referenced energy loss ratio
τ	the initial time reference for energy measurements
W	total energy content
a_n, b_n	real and imaginary parts of A_n
n, m	whole numbers
t	independent variable time
x	generalized mass displacement
x_0	displacement when $\tau = 0$
x_{10}, x_{20}	points about which the equation is expanded in a power series
ΔW	energy loss or change

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
ΔW_n	energy loss for a nonlinear damping function
ΔW_v	energy loss for viscous damping
α	damping coefficient, Case V
α_n	the n^{th} complex exponent of the exponential series
α_1, α_2	basic exponents of the approximate solution
δ	energy content, $\Delta W/2W$
ϵ	nonlinear spring coefficient
σ	exponent of e, exponential decay factor
τ	nondimensional time
ϕ	phase angle of displacement response
ω	frequency of the displacement response

ANALYSIS OF STRUCTURAL RESPONSE WITH DIFFERENT FORMS OF DAMPING

SUMMARY

Five different analytical representations of the damping force are investigated to provide a basis for deciding what type may exist in a given structural system. These representations of the damping force are incorporated along with a nonlinear spring force in a single-degree-of-freedom representation of the structural response.

Energy loss per cycle calculations provide one basis for comparison. The dependence of energy losses on amplitude and frequency for sustained and transient response is shown. For sustained oscillations, a ratio of the energy loss for each damping force to that for viscous damping is plotted as a function of amplitude and frequency. Spring plus damping force as a function of displacement yields hysteresis loops of different shapes for different damping forces. Also, by selecting reasonable numerical values, transient displacement versus time curves and hysteresis loops are obtained by numerical integration on the digital computer.

Approximate analytical solutions for each of the five cases are also shown, and equations for the displacement versus time response envelope are obtained. Considerable differences in the response envelopes are observed using the same numerical coefficients in each case.

Energy loss calculations along with the displacement versus time response envelope and hysteresis loops provide bases for deciding what type of damping may exist in a structural system when observing experimental results.

INTRODUCTION

In the present analysis and design of missiles, the damping is generally considered to be viscous damping. When considering the complex system as a series of lumped masses, the damping force on each generalized mass is considered proportional to velocity. When exciting the first mode of vibration, the modal damping is also taken as a function of velocity. As missile designs become more complicated, the multiple stage interfaces can produce slip damping, and large deflections can cause nonlinear stiffness and damping.

The investigation of different analytical representations of damping has lagged far behind efforts to obtain a better representation of stiffness. To improve this situation, five different forms of damping are investigated herein. These five functions are each separately included in a single-degree-of-freedom equation with a nonlinear cubic spring force. The step input as well as the free oscillation condition is studied.

Three different criteria are used to study the five different damping cases. Energy loss per cycle is shown to change from case to case with amplitude and frequency. Steady state as well as transient energy losses are calculated. Secondly, hysteresis loops can be used to compare each case. Finally, numerical results are obtained by numerical integration on the digital computer. The envelope of each displacement versus time curve is calculated by first obtaining an approximate solution for the nonlinear differential equation.

To obtain analytical representations of slip damping, it is necessary to match experimental hysteresis loops with those produced analytically. Thus, hysteresis loops for each analytical representation of the damping force shown herein can be correlated with current experimental efforts.

ANALYTICAL REPRESENTATION OF DAMPING

Consider the differential equation of motion,

$$M \frac{d^2x}{dt^2} + g\left(\frac{dx}{dt}, x\right) + h(x) = f(t) \quad , \tag{1}$$

where M is the generalized mass, x is the generalized mass displacement, g is the damping force function, h is the spring force function, and f is the input force. Only one type of spring force function, given by

$$h(x) = K(x + \epsilon x^3) \quad , \tag{2}$$

is considered. This function is plotted in Figure 1 and is a typical representation of a nonlinear spring [1]. In this instance, ϵ is negative. With $\epsilon = 0$,

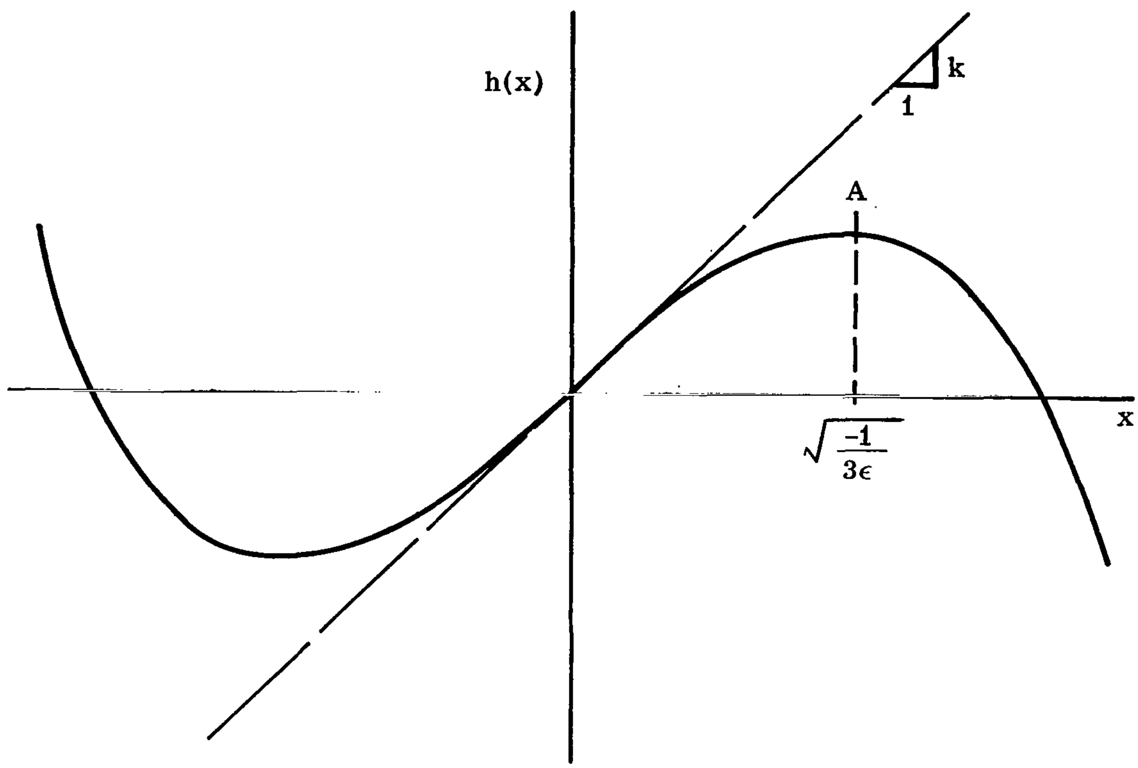


FIGURE 1. NONLINEAR SPRING FORCE

the spring force is linear with a spring stiffness K. Point A on Figure 1 may be thought of as the point of instability in the spring force and may be used to select the value of ϵ. For this study, the value $\epsilon = -1/3$ is used.

The free response and the response to a step input are considered. Thus,

$$\left.\begin{array}{ll} f(t) = 0 & \text{(free)} \\ f(t) = P & \text{(step)}\ . \end{array}\right\} \qquad (3)$$

The purpose of this study is to investigate different forms of the damping force function g(dx/dt, x). A study recently completed by this author [2], discusses different sources of damping and shows different analytical representations of damping. The five different damping functions considered in this report are listed in Table I. Also shown is the possible source of each damping function.

TABLE I. ANALYTICAL REPRESENTATION OF DAMPING

Case	$g\left(\frac{dx}{dt}, x\right)$	Source
I	$C_1 \frac{dx}{dt}$	Viscous or environmental
II	$C_0 \lvert x \rvert \left(\operatorname{sgn} \frac{dx}{dt}\right)$	Material or slip
III	$C_2 \left(\frac{dx}{dt}\right)^2 \left(\operatorname{sgn} \frac{dx}{dt}\right)$	Environmental
IV	$C_3 \lvert x \rvert \frac{dx}{dt}$	Material
V	$C_1(1 + \alpha x^2) \frac{dx}{dt}$	All

To study the systems which can be described by equation (1), it is desirable to normalize the independent variable "time." First substitute equation (2) into equation (1) and divide by mass M to obtain

$$\frac{d^2x}{dt^2} + \frac{1}{M} g\left(\frac{dx}{dt}, x\right) + \frac{K}{M}(x + \epsilon x^3) = \frac{P}{M}$$

for the step input. To normalize let

$$\tau = \sqrt{\frac{K}{M}}\, t \quad ,$$

where $\sqrt{K/M}$ is the linear circular frequency. Obtaining derivatives with respect to τ and dividing by K/M gives

$$\ddot{x} + \frac{1}{K}\, g(\dot{x}, x) + x + \epsilon x^3 = \frac{P}{K} \quad , \tag{4}$$

where

$$\dot{x} = \frac{dx}{d\tau} \quad \text{and} \quad \ddot{x} = \frac{d^2x}{d\tau^2} \quad .$$

The damping force terms shown in Table I are expressed in terms of the normalized time as follows

$$\frac{1}{K}\, g(\dot{x}, x) = \left\{ \begin{array}{lll} \dfrac{C_1}{\sqrt{KM}}\, \dot{x} & , & \mathrm{I} \\[2ex] \dfrac{C_0}{K}\, |x|\,(\mathrm{sgn}\ \dot{x}) & , & \mathrm{II} \\[2ex] \dfrac{C_2}{M}\, \dot{x}^2\,(\mathrm{sgn}\ \dot{x}) & , & \mathrm{III} \\[2ex] \dfrac{C_3}{\sqrt{KM}}\, |x|\,\dot{x} & , & \mathrm{IV} \\[2ex] \dfrac{C_1}{\sqrt{KM}}\, (1 + \alpha x^2)\,\dot{x} & , & \mathrm{V} \end{array} \right\} \quad . \tag{5}$$

The units of x are consistent with the other units. With mass in slugs and K in pounds per foot, the C's are in terms of pounds, feet, and seconds, and x is the displacement in feet. Any other set of units is equally applicable.

Three different but related approaches for studying similarities of, and differences between, these damping terms are included in the following sections.

ENERGY LOSSES AND HYSTERESIS CURVES

A useful technique in studying different analytical representations of damping is to make a comparison of energy losses per cycle, ΔW/cycle. The value, ΔW/cycle, is obtained by integrating the spring plus damper force times the displacement over a vibration cycle. The spring force integrated over a cycle is zero; thus,

$$\frac{\Delta W}{\text{cycle}} = \int_{\text{cycle}} \left[g\left(\frac{dx}{dt}, x\right) \right] dx \quad .$$

This equation is changed to an integral over time, or

$$\frac{\Delta W}{\text{cycle}} = \int_{\text{period}} \left[g\left(\frac{dx}{dt}, x\right) \right] \frac{dx}{dt}\, dt \quad . \tag{6}$$

Some authors prefer to use the energy content of the vibrating system, which is the energy loss per cycle divided by twice the vibration energy, 2W; that is,

$$\delta = \frac{\Delta W/\text{cycle}}{2W} = \frac{\Delta W/\text{cycle}}{M\left(\frac{dx}{dt}\right)^2_{max}} \quad . \tag{7}$$

First consider the input force f(t) to be a type which maintains a constant response amplitude. The response may be approximated by

$$x = A \sin(\omega t - \phi) \quad , \tag{8}$$

where A is the amplitude, ω is the frequency, and ϕ is the phase angle. The amplitude and frequency may be determined by using the method of Klotter [3]. The velocity becomes

$$\frac{dx}{dt} = A\omega \cos(\omega t - \phi) \quad . \tag{9}$$

Substituting any of the damping functions of Table I into equation (6), with displacement and velocity given by equations (8) and (9), the energy loss per cycle is obtained by integrating between the limits of ϕ/ω and $2\pi + \phi/\omega$. The energy content is similarly obtained from equation (7). The results for each damping function are shown in Table II.

TABLE II. ENERGY LOSSES - SINUSOIDAL RESPONSE

Case	ΔW/cycle	$\delta = \Delta W/2W$
I	$C_1 \pi A^2 \omega$	$\frac{C_1 \pi}{M\omega}$
II	$2C_0 A^2$	$\frac{2C_0}{M\omega^2}$
III	$\frac{8}{3} C_2 A^3 \omega^2$	$\frac{8C_2 A}{3M}$
IV	$\frac{4}{3} C_3 A^3 \omega$	$\frac{4C_3 A}{M\omega}$
V	$C_1 \pi A^2 \omega + \frac{\pi}{4} A^4 \omega C_1 \alpha$	$\frac{C_1 \pi}{M\omega} + \frac{C_1 \pi \alpha A^2}{4M\omega}$

As shown by this table, various damping functions in effect make the energy loss dependent on different powers of amplitude and frequency.

An interesting comparison of energy losses is obtained by defining an energy loss ratio

$$R = \frac{\Delta W_n}{\Delta W_v} \quad . \tag{10}$$

The numerator, ΔW_n, is the energy loss per cycle for any nonlinear damping function and the denominator, ΔW_v, is the energy loss per cycle for viscous damping, Case I. This ratio would be the same using energy content, δ. These ratios are shown in Table III.

TABLE III. ENERGY LOSS RATIOS

Case	$R = \Delta W_n / \Delta W_v$	$R_r = \Delta W_n / \Delta W_{vr}$
I	1.0 (reference)	$\frac{\text{measured } C_1}{\text{reference } C_1} = \frac{C_1}{C_{1r}}$
II	$\frac{2C_0}{C_1 \pi \omega}$	$\frac{2C_0}{C_{1r} \pi \omega}$
III	$\frac{8C_2 A\omega}{3C_1 \pi}$	$\frac{8C_2 A\omega}{3C_{1r} \pi}$
IV	$\frac{4C_3 A}{C_1 \pi}$	$\frac{4C_3 A}{C_{1r} \pi}$
V	$1 + \frac{\alpha A^2}{4}$	$1 + \frac{\alpha A^2}{4}$

The commonly employed method for experimental determination of missile damping presupposes viscous damping, Case I. The results show different values of C_1 as amplitude and/or frequency vary. This suggests that possibly some other form of damping would be more appropriate. Energy loss ratios provide a possible means of selecting a more appropriate damping term. Defining one viscous coefficient as the reference,

$$C_1(\text{reference}) = C_{1r} \quad ,$$

a reference energy loss ratio is obtained as

$$R_r = \frac{\Delta W_n}{\Delta W_{vr}} \tag{11}$$

where

$$\frac{\Delta W_v}{\Delta W_{vr}} = \frac{C_1}{C_{1r}} \quad .$$

These reference ratios are also shown in Table III.

The reference energy loss ratios that are a function of frequency appear in Figure 2; those that are a function of amplitude are shown in Figure 3. Selecting one experimentally determined value of the viscous damping coefficient as the reference value, a plot of C_1/C_{1r} is determined as a function of both amplitude and frequency. Comparing this plot with those of Figures 2 and 3 may indicate a better analytical representation of the damping coefficient. The coefficient would then be constant. This suggested method is to be used with existing experimental data and should be investigated in more detail.

Relatively little information is available on energy losses in transient systems. For these systems the response can be approximated by

$$x = A_0 + Ae^{\sigma t}\sin(\omega t - \phi) \ . \tag{12}$$

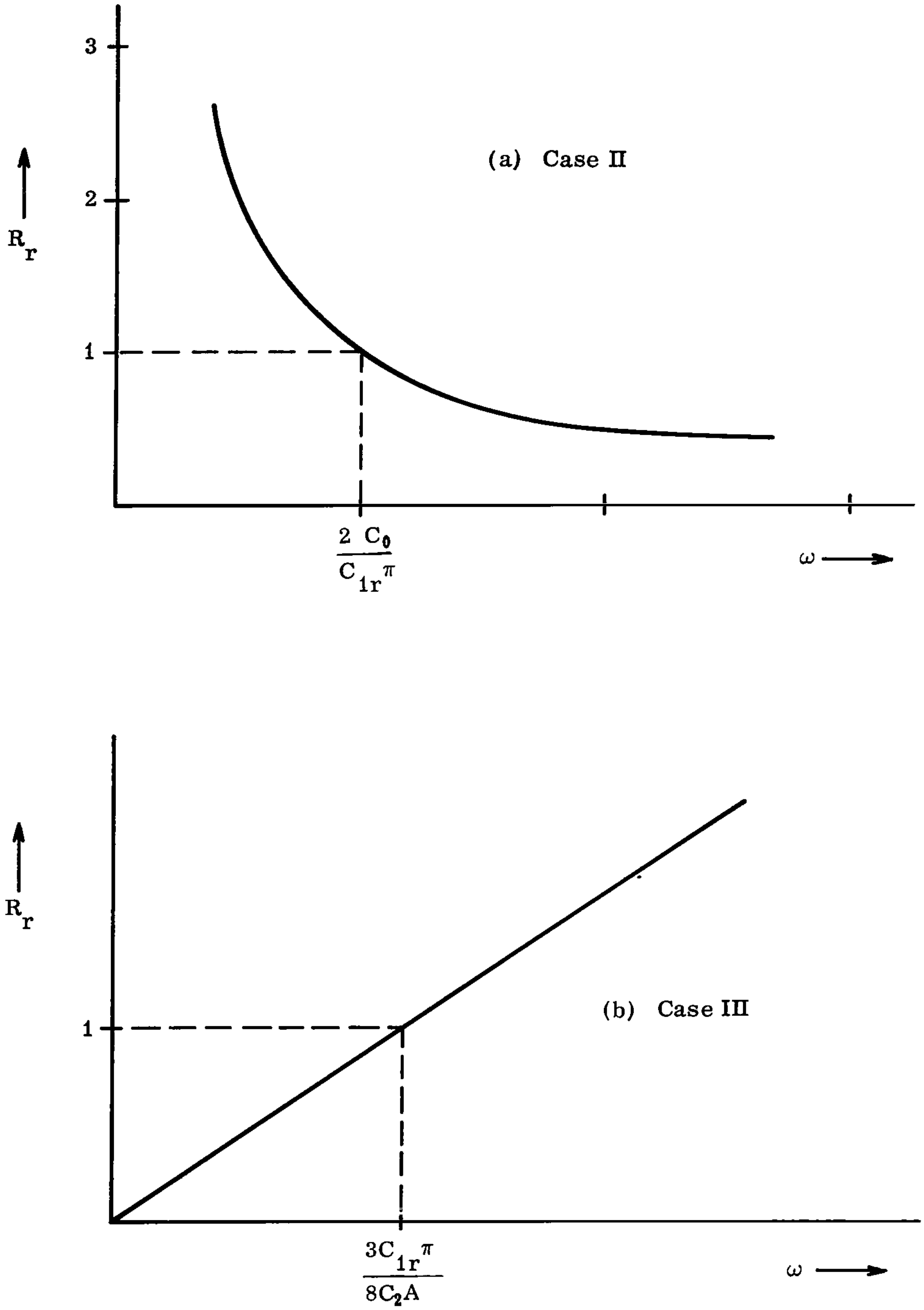
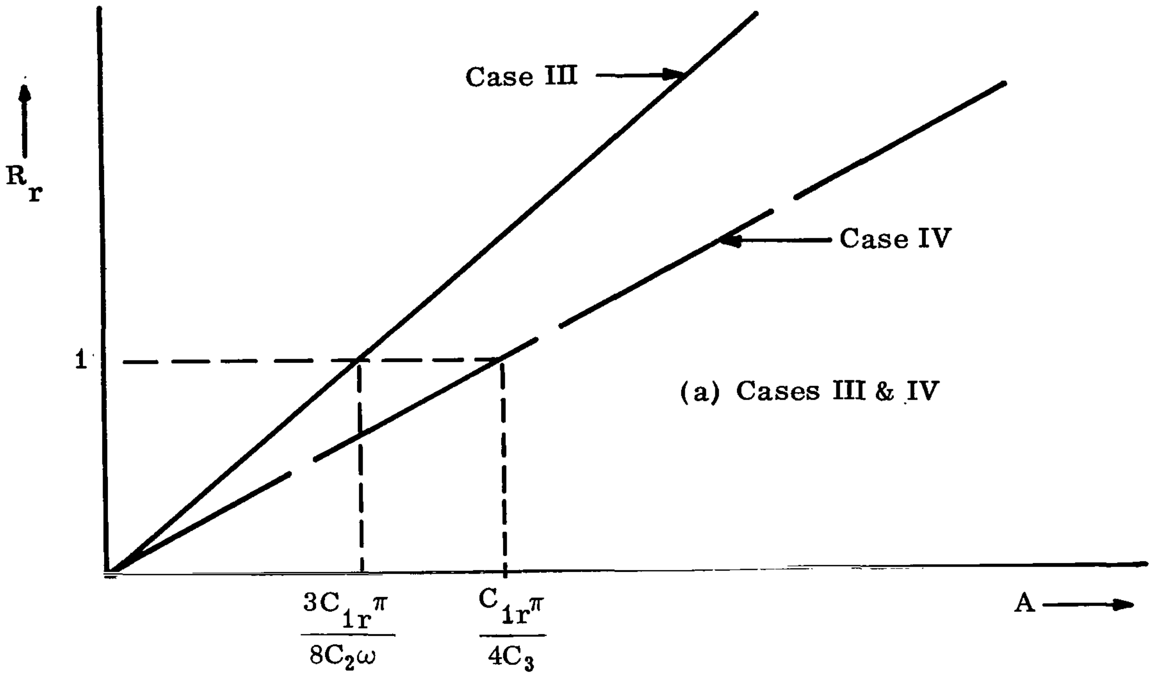


FIGURE 2. ENERGY LOSS RATIO VERSUS FREQUENCY



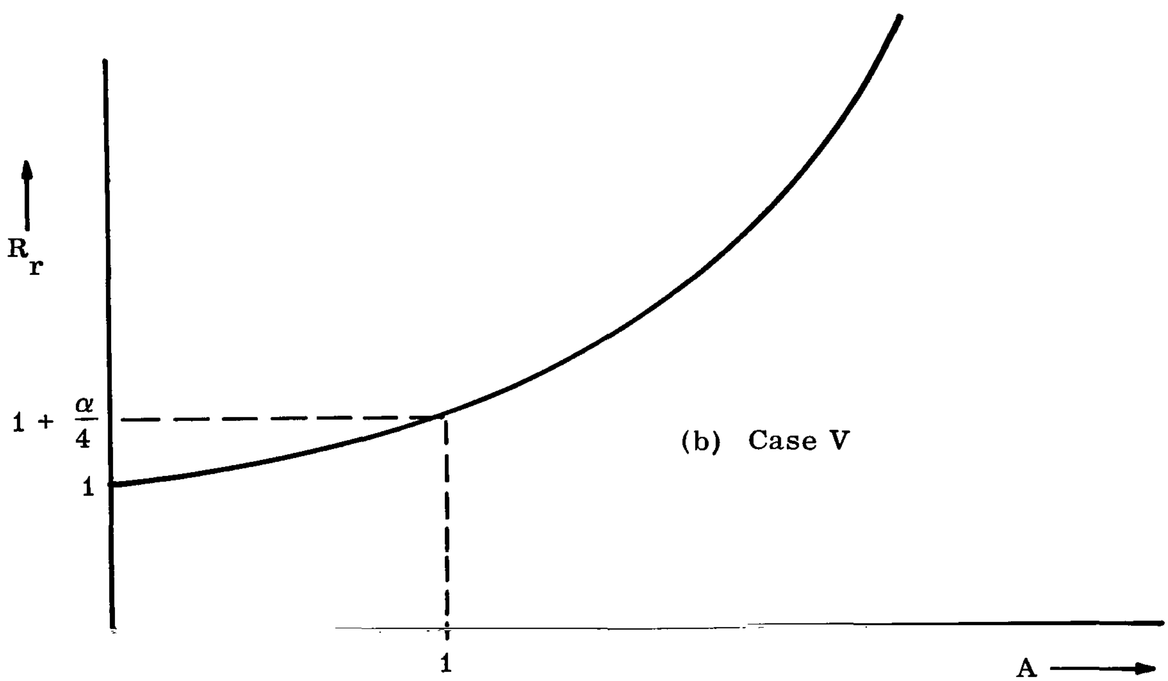


FIGURE 3. ENERGY LOSS RATIO VERSUS AMPLITUDE

The constant A_0 is introduced for those systems which do not vibrate about the null, $x = 0$, position. The exponential exponent, σ, must be negative for decaying vibrations and the final steady state ($t \rightarrow \infty$) amplitude is A_0. A method for determining A_0, A, σ, and ω in terms of the system parameters and initial conditions is shown in the Appendix and in the following section. The velocity is obtained from the derivative of equation (12) as

$$\frac{dx}{dt} = A\sigma e^{\sigma t}\sin(\omega t - \phi) + A\omega e^{\sigma t}\cos(\omega t - \phi) \; . \tag{13}$$

The study of transient energy losses includes the following refinements not previously considered:

1. The effect of a decay in the response

2. The effect of a constant nonzero steady state amplitude

The input forcing functions used are those given in equation (3). With the free vibration $A_0 = 0$, the system is considered to be given an initial displacement and released. With the step input $f(t) = P$, the constant amplitude is considered to be positive.

Substituting the various damping functions from Table I into equation (6), with displacement and velocity given by equations (12) and (13), respectively, the transient energy loss is obtained for each function. Care must be taken to integrate over the proper limits to ensure that absolute values and signs are correct. The results are separated into two parts. The transient energy losses for free response are shown in Table IV for one cycle starting at time T. The transient energy losses for step input are shown in Table V for one cycle starting at time T.

The transient energy losses in Table IV will reduce to those in Table II when $\sigma = 0$. In general, the transient energy losses become increasingly important the greater the ratio σ/ω. For most applications with slow decay and large frequency, the assumption of a steady vibration does not yield large energy loss errors. However, as missiles become more flexible and slip joints at stage interfaces increase the rate of vibration decay, the transient energy losses should be considered. The transient energy losses of Tables IV and V vary with amplitude and frequency in the same way as in Table II.

TABLE IV. TRANSIENT ENERGY LOSSES - FREE RESPONSE

Case	ΔW/Cycle
I	$C_1A^2e^{2\sigma T}\left(e^{\frac{2\sigma}{\omega}2\pi}-1\right)\dfrac{\omega}{4\left(\frac{\sigma}{\omega}\right)}$
II	$C_0A^2e^{2\sigma T}\left(e^{\frac{3\sigma}{\omega}\pi}+e^{\frac{\sigma}{\omega}\pi}\right)$
III	$C_2A^3e^{3\sigma T}\left(e^{\frac{3\sigma}{\omega}\pi}+1\right)^2\dfrac{\omega^2\left[2+7\left(\frac{\sigma}{\omega}\right)^2+9\left(\frac{\sigma}{\omega}\right)^4\right]}{3\left[1+\left(\frac{3\sigma}{\omega}\right)^2\right]}$
IV	$C_3A^3e^{3\sigma T}\left(e^{\frac{3\sigma}{\omega}\pi}+1\right)^2\dfrac{\omega}{3\left[1+\left(\frac{3\sigma}{\omega}\right)^2\right]}$
V	$C_1A^2e^{2\sigma T}\left(e^{\frac{2\sigma}{\omega}2\pi}-1\right)\dfrac{\omega}{4\left(\frac{\sigma}{\omega}\right)}$
	$+\,C_1\alpha A^4\left(e^{4\sigma T}-1\right)\dfrac{\omega}{32\frac{\sigma}{\omega}\left[1+\left(\frac{2\sigma}{\omega}\right)^2\right]}$

TABLE V. TRANSIENT ENERGY LOSSES - STEP RESPONSE

Case	
I	$C_1A^2e^{2\sigma T}\left(e^{\frac{2\sigma}{\omega}2\pi}-1\right)\frac{\omega}{4\left(\frac{\sigma}{\omega}\right)}$
II	$C_0A_0Ae^{\sigma T}\left(e^{\frac{\sigma}{\omega}\pi}+1\right)^2\frac{1}{\left[1+\left(\frac{\sigma}{\omega}\right)^2\right]^{1/2}}$
	$+\,C_0A^2e^{2\sigma T}\left(e^{\frac{2\sigma}{\omega}\pi}-1\right)\frac{3-\left(\frac{\sigma}{\omega}\right)^2}{4\left[1+\left(\frac{\sigma}{\omega}\right)^2\right]^2}$
III	$C_2A^3e^{3\sigma T}\left(e^{\frac{3\sigma}{\omega}\pi}+1\right)\frac{\omega^2\left[2+7\left(\frac{\sigma}{\omega}\right)^2+9\left(\frac{\sigma}{\omega}\right)^4\right]}{3\left[1+\left(\frac{3\sigma}{\omega}\right)^2\right]}$
IV	$C_3A_0A^2e^{2\sigma T}\left(e^{\frac{2\sigma}{\omega}2\pi}-1\right)\frac{\omega}{4\left(\frac{\sigma}{\omega}\right)}$
	$+\,C_2A^3e^{3\sigma T}\left(e^{\frac{3\sigma}{\omega}2\pi}-1\right)\frac{4\sigma}{3\left[1+\left(\frac{3\sigma}{\omega}\right)^2\right]\left[1+\left(\frac{\sigma}{\omega}\right)^2\right]^{1/2}}$
V	$(C_1+C_1\alpha A_0^2)A^2e^{2\sigma T}\left(e^{\frac{2\sigma}{\omega}2\pi}-1\right)\frac{\omega}{4\left(\frac{\sigma}{\omega}\right)}$
	$+\,C_1\alpha A_0A^3e^{3\sigma T}\left(e^{\frac{3\sigma}{\omega}2\pi}-1\right)\frac{8\sigma}{3\left[1+\left(\frac{3\sigma}{\omega}\right)^2\right]\left[1+\left(\frac{\sigma}{\omega}\right)^2\right]^{1/2}}$
	$+\,C_1\alpha A^4e^{4\sigma T}\left(e^{\frac{4\sigma}{\omega}2\pi}-1\right)\frac{\omega\left[1+13\left(\frac{\sigma}{\omega}\right)^2\right]}{32\frac{\sigma}{\omega}\left[1+\left(\frac{2\sigma}{\omega}\right)^2\right]\left[1+\left(\frac{\sigma}{\omega}\right)^2\right]}$

The transient energy losses of Table V vary considerably from those in Table IV when the damping term involves the displacement x. This shows the dependence of energy losses on the average displacement which can be related to average stress in many problems.

Related to the energy loss is the force versus displacement, hysteresis loop, curve. In fact, the energy loss per cycle is the area enclosed by the hysteresis loop in the same cycle. To obtain a better understanding of the different damping terms of Table I, these hysteresis loops are considered.

The force is the spring plus damping force from equations (1) and (2):

$$\text{force} = g\left(\frac{dx}{dt}, x\right) + K(x + \epsilon x^3) \quad . \tag{14}$$

The sinusoidal displacement and velocity of equations (8) and (9) are substituted into this equation, and force may be plotted as a function of displacement x. The result is either a single or double loop curve. Those cases that cause a single loop are sketched in Figure 4. Those which produce a double loop are sketched in Figure 5.

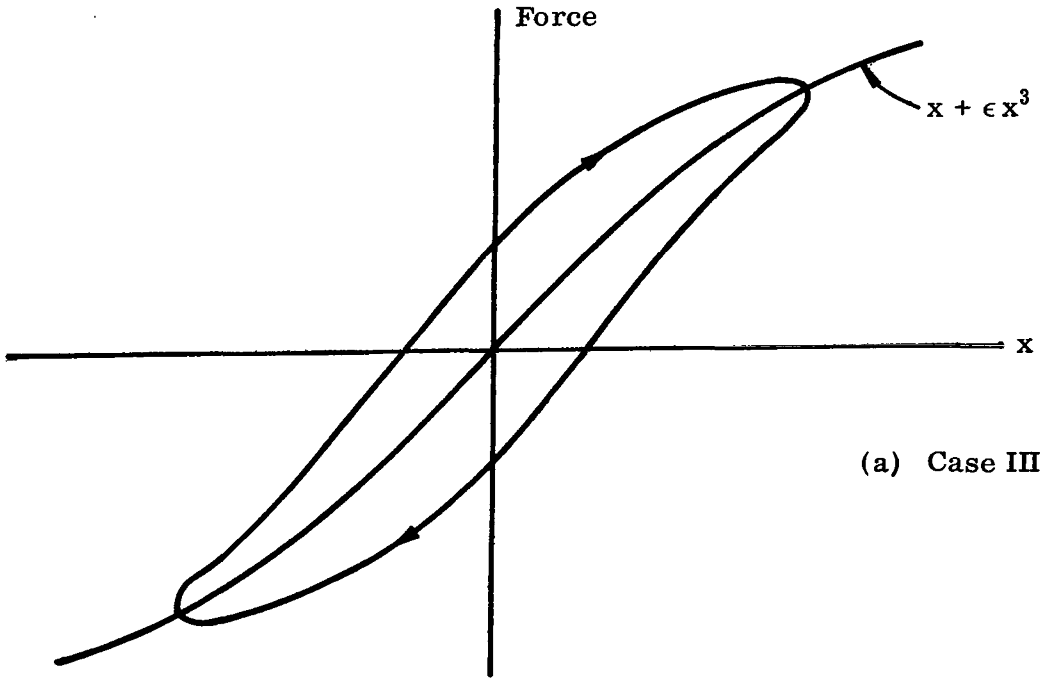
The effect of slip damping is commonly obtained as a force-displacement hysteresis loop. A combination of curves given in Figures 4 and 5 can be used to approximate the slip damping condition. This in turn will allow slip damping to be described analytically.

TRANSIENT RESPONSE

Another method for comparing various analytical representations of the damping force is to observe displacement versus time curves and in particular the envelope of these curves. This requires a solution of each equation obtained by substituting equations (5) into equation (4). An approximate solution is obtained in the form

$$x = A_0 + Ae^{\sigma\tau}\sin(\omega\tau - \phi) \tag{15}$$

which is the same form used to obtain transient energy losses (equation (12)). A numerical integration solution of each equation is also obtained.



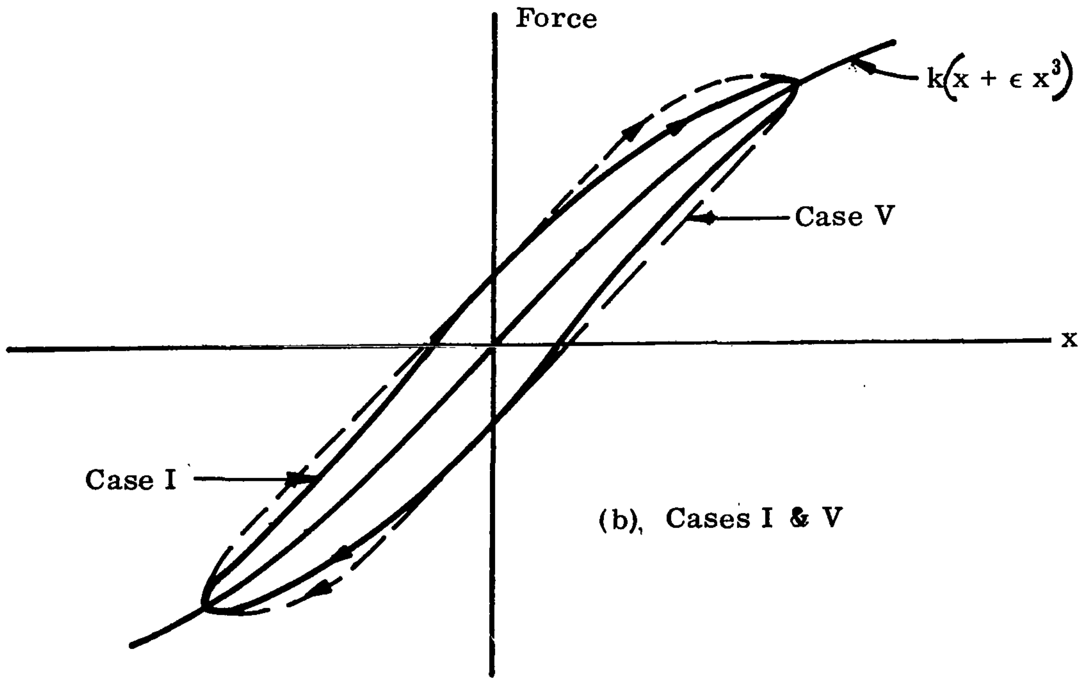


FIGURE 4. SINGLE HYSTERESIS LOOPS

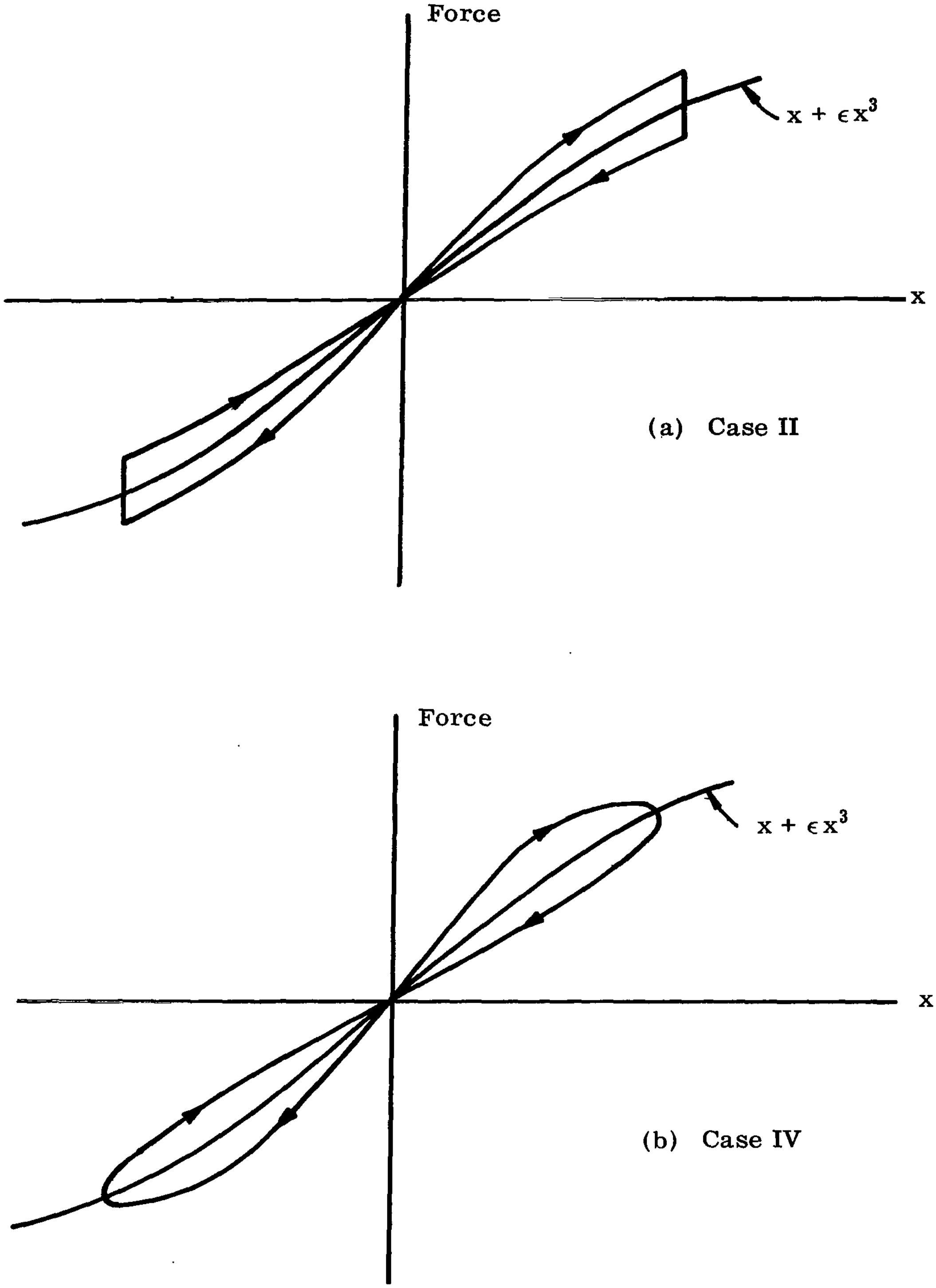


FIGURE 5. DOUBLE HYSTERESIS LOOPS

A method for obtaining the coefficients A_0 and A, the exponent σ, the normalized frequency ω, and phase angle ϕ is shown in the Appendix. Two inputs to each system are considered, as shown in equation (3).

For the step input, P/K, the exponent σ and the frequency ω are obtained from equations (A. 2), (A. 6) and (A. 7) with $x_{10} = 0$ and $x_{20} = A_0$. The constant amplitude A_0 is obtained from equation (A. 10). The amplitude A and phase angle ϕ are available from equation (A. 14). Shown in Table VI are the values of σ and ω as well as the steady state amplitude equation for each case.

For free oscillations, P = 0, the exponent σ and the frequency ω are also obtained from equations (A. 2), (A. 6) and (A. 7). However, in this instance, the values of $x_{10} = 0$ and $x_{20} = x_0/2$ are found to yield the best results. The constant amplitude A_0 is zero. The amplitude A and phase angle ϕ are available from equation (A. 15). Table VII shows the values of σ and ω for free oscillations resulting from an initial displacement, x_0.

With the approximate solution known, equation (15), the envelope of the displacement, x, versus nondimensional time, τ, curve is obtained for each case. This is done by defining maximum and minimum values of x and the corresponding times for each. Connecting these points by a continuous curve shows the envelope of the transient response. Maximum and minimum values are shown for the step input in Table VIII and for the free oscillations in Table IX. For Case II

$$\left.\begin{aligned} \omega &= \omega_1 \\ A_0 &= A_{01} \end{aligned}\right\} \qquad (\operatorname{sgn} x)(\operatorname{sgn} \dot{x}) = +1$$

and

$$\left.\begin{aligned} \omega &= \omega_2 \\ A_0 &= A_{02} \end{aligned}\right\} \qquad (\operatorname{sgn} x)(\operatorname{sgn} \dot{x}) = -1 \quad .$$

For the velocity squared damping, Case III, the approximate analytical solution shows no decay and is not a good approximation for this case.

TABLE VI. SOLUTION EXPONENT, FREQUENCY, AND STEADY STATE AMPLITUDE - STEP INPUT

Case	σ	ω	S.S. Amplitude Equation
I	$\frac{-C_1}{2\sqrt{KM}}$	$\left(1 + 3\epsilon A_0^2 - \frac{C_1^2}{4KM}\right)^{1/2}$	$A_0 + \epsilon A_0^3 = \frac{P}{K}$
II	0	$\left[1 + 3\epsilon A_0^2 + \frac{C_0}{K}(\text{sgn } x)(\text{sgn } \dot{x})\right]^{1/2}$	$\frac{C_0 A_0}{K}(\text{sgn } x)(\text{sgn } \dot{x}) + A_0 + \epsilon A_0^3 = \frac{P}{K}$
III	0	$(1 + 3\epsilon A_0^2)^{1/2}$	$A_0 + \epsilon A_0^3 = \frac{P}{K}$
IV	$\frac{-C_3}{2\sqrt{KM}} A_0(\text{sgn } A_0)$	$\left(1 + 3\epsilon A_0^2 - \frac{C_3^2}{4KM} A_0^2\right)^{1/2}$	$A_0 + \epsilon A_0^3 = \frac{P}{K}$
V	$\frac{-C_1}{2\sqrt{KM}}(1 + \alpha A_0^2)$	$\left[1 + 3\epsilon A_0^2 - \frac{C_1^2}{4KM}(1 + \alpha A_0^2)^2\right]^{1/2}$	$A_0 + \epsilon A_0^3 = \frac{P}{K}$

TABLE VII. SOLUTION EXPONENT AND FREQUENCY-FREE OSCILLATION

Case	σ	ω
I	$\dfrac{-C_1}{2\sqrt{KM}}$	$\left(1 + \dfrac{3\epsilon x_0^2}{4} - \dfrac{C_1^2}{4KM}\right)^{1/2}$
II	0	$\left[1 + \dfrac{3\epsilon x_0^2}{4} + \dfrac{C_0}{K}(\mathrm{sgn}\ x)(\mathrm{sgn}\ \dot{x})\right]^{1/2}$
III	0	$\left(1 + \dfrac{3\epsilon x_0^2}{4}\right)^{1/2}$
IV	$\dfrac{-C_3}{2\sqrt{KM}}\ \dfrac{x_0}{2}(\mathrm{sgn}\ x_0)$	$\left[1 + \dfrac{3\epsilon x_0^2}{4} - \dfrac{C_3^2}{4KM}\left(\dfrac{x_0}{2}\right)^2\right]^{1/2}$
V	$\dfrac{-C_1}{2\sqrt{KM}}\left(1 + \dfrac{\alpha x_0^2}{4}\right)$	$\left[1 + \dfrac{3\epsilon x_0^2}{4} - \dfrac{C_1^2}{4KM}\left(1 + \dfrac{\alpha x_0^2}{4}\right)^2\right]^{1/2}$

TABLE VIII. MAXIMUM AND MINIMUM x VERSUS τ VALUES - STEP INPUT

Case	τ at x_{max} or x_{min}	x_{max} or x_{min}	n is
I, IV, V	$\frac{n\pi}{\omega}$	$x_{max} = A_0\left(e^{\frac{\sigma n\pi}{\omega}} + 1\right)$	odd
I, IV, V	$\frac{n\pi}{\omega}$	$x_{min} = A_0\left(1 - e^{\frac{\sigma n\pi}{\omega}}\right)$	even
II	$\frac{\pi}{\omega_1} + \frac{n\pi}{2}\frac{\omega_1 + \omega_2}{\omega_1\omega_2}$	$x_{max} = 2A_{01} + (A_{01} - A_{02})n$	even
II	$\frac{n\pi}{2}\frac{\omega_1 + \omega_2}{\omega_1\omega_2}$	$x_{min} = (A_{02} - A_{01})n$	even
III	-	$x_{max} = 2A_0$	-
III	-	$x_{min} = 0$	-

TABLE IX. MAXIMUM AND MINIMUM x VERSUS τ VALUES - FREE OSCILLATION

Cases	τ at x_{max} or x_{min}	x_{max} or x_{min}	n is
I, IV, V	$\frac{n\pi}{\omega}$	$x_{max} = x_0 e^{\frac{\sigma n\pi}{\omega}}$	even
I, IV, V	$\frac{n\pi}{\omega}$	$x_{min} = -x_0 e^{\frac{\sigma n\pi}{\omega}}$	odd
II	$\frac{n\pi}{2}\frac{\omega_1 + \omega_2}{\omega_1\omega_2}$	$x_{max} = x_0\left(\frac{\omega_2}{\omega_1}\right)^n$	even
II	$\frac{n\pi}{2}\frac{\omega_1 + \omega_2}{\omega_1\omega_2}$	$x_{min} = -x_0\left(\frac{\omega_2}{\omega_1}\right)^n$	odd
III	-	$x_{max} = x_0$	-
III	-	$x_{min} = -x_0$	-

Numerical results are presented in Figures 6 through 35. These results are presented for each of the five cases as two curves for two values of damping (approximately two percent and ten percent of critical) with the step input, plus two curves with the same two values of damping but with only an initial displacement input. Two additional curves are included for each case (with approximately ten percent of critical damping) showing the spring plus damper force versus displacement, or hysteresis loops, for the same two inputs. The solid curves represent the fourth-order Runge-Kuta numerical integration on the digital computer. The dashed curves are the results of the approximate analytical solution shown as an envelope of the transient response. Table X is included as a summary of Figures 6 through 35.

The analytical and numerical integration results for displacement versus nondimensional time are in good agreement. The one exception is the results for Case III. Horizontal deviations of maximum and minimum values, comparing analysis with numerical integration, indicate errors in the frequency ω. Vertical deviations indicate errors in the amplitude A and the exponent σ. Considerable difference in the response envelope is noted when the same coefficients are used with different analytical representations of damping.

CONCLUSIONS

In this study, a single-degree-of-freedom representation of the mass displacement permits the investigation of different analytical representations of damping forces. For different damping forces, the energy loss per cycle has different powers of amplitude and frequency. Transient energy losses are considerably different than steady state losses when the vibration decay is large and the frequency is small. For damping forces dependent on amplitude, the mean response amplitude affects the energy loss per cycle.

Different damping forces yield different hysteresis loops. It should be possible with a series of damping terms to approximate most experimentally determined hysteresis loops. Some of these hysteresis loops are dependent on mean vibration amplitude.

Differences occur in the amplitude versus time curve envelope using the same equation except for the dependence of damping on displacement and velocity. In some cases, the differences are very pronounced. A three-term exponential series solution of the nonlinear differential equation can be used to obtain these curves and their envelopes.

It appears that a single experimental amplitude versus time envelope cannot be attributed to one form of damping without further investigations. In other words, the magnitude of the damping coefficient and the damping function may be simultaneously changed to yield curves which are very similar.

Possible procedures for determining the true analytical representation of damping force by one term or a series of terms are as follows:

> Energy losses could be taken as a power series in amplitude and frequency with coefficients determined experimentally. The correlation of existing viscous damping information with energy loss ratios appears feasible. And, finally, the actual shape of the transient amplitude versus the response envelope can be correlated with those generated by different analytical representations of damping.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, October 17, 1966

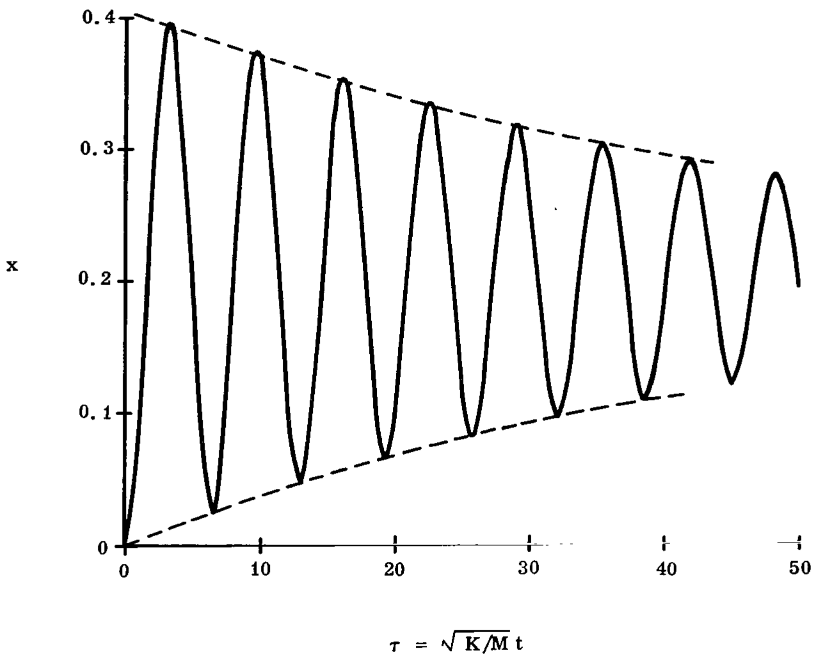


FIGURE 6. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE I

$$\left(\ddot{x} + 0.04\dot{x} + x - \frac{x^3}{3} = 0.2\right)$$

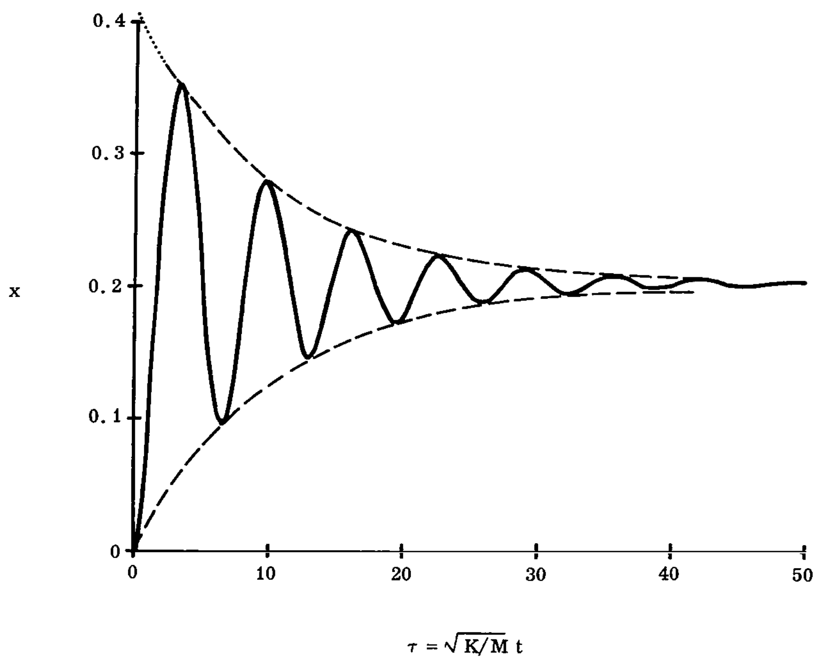


FIGURE 7. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE I

$$\left(\ddot{x} + 0.2\dot{x} + x - \frac{x^3}{3} = 0.2\right)$$

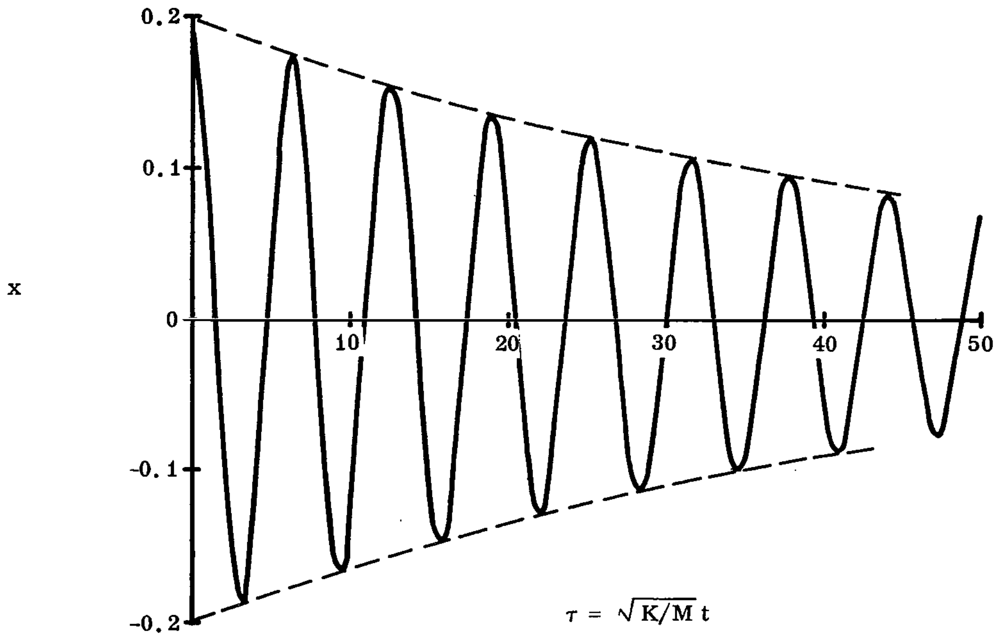


FIGURE 8. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE I

$$\left(\ddot{x} + 0.04\dot{x} + x - \frac{x^3}{3} = 0, \quad x(\tau = 0) = 0.2\right)$$

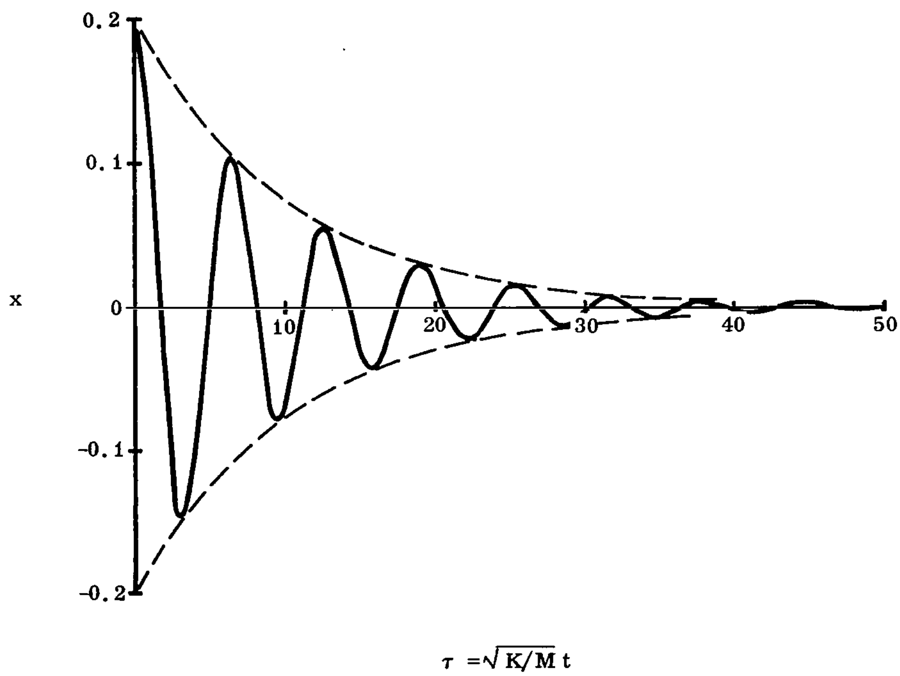


FIGURE 9. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE I

$$\left(\ddot{x} + 0.2\dot{x} + x - \frac{x^3}{3} = 0, \quad x(\tau = 0) = 0.2\right)$$

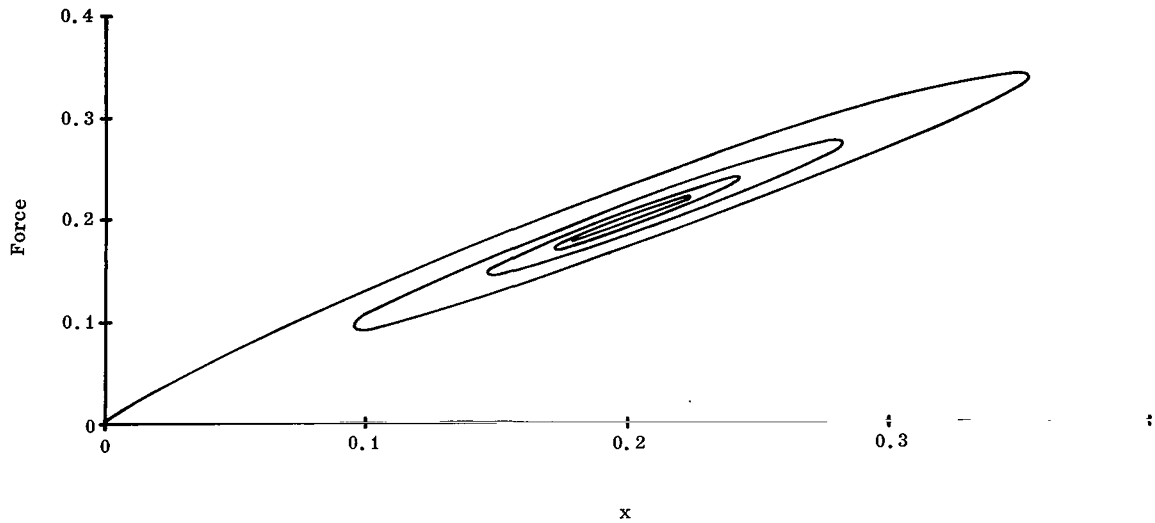


FIGURE 10. FORCE VERSUS DISPLACEMENT - CASE I

$$\left(\ddot{x} + 0.2\dot{x} + x - \frac{x^3}{3} = 0.2\right)$$

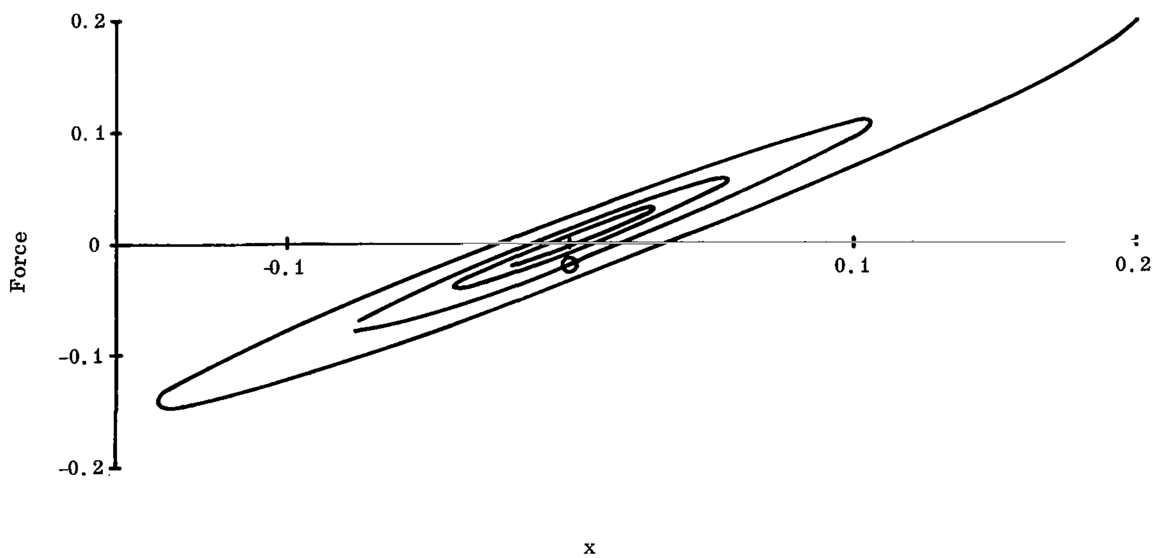


FIGURE 11. FORCE VERSUS DISPLACEMENT - CASE I

$$\left(\ddot{x} + 0.2\dot{x} + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

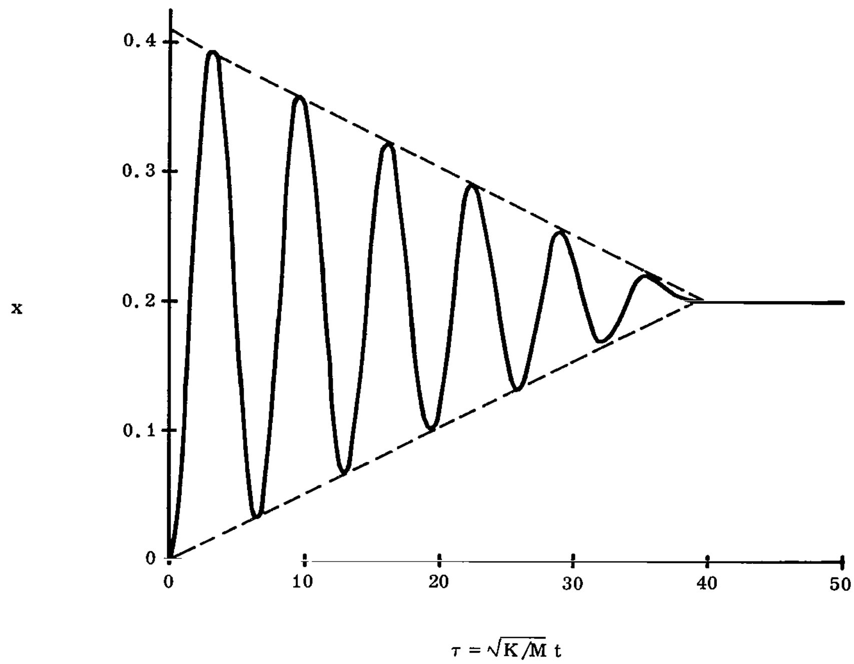


FIGURE 12. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE II

$$(\ddot{x} + 0.04|x|(\text{sgn}\,\dot{x}) + x - \frac{x^3}{3} = 0.2)$$

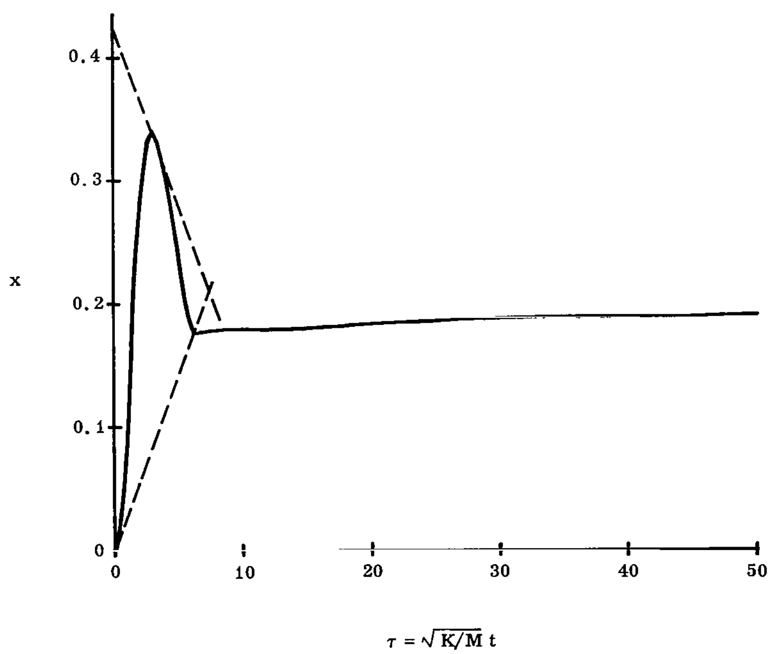


FIGURE 13. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE II

$$(\ddot{x} + 0.2|x|(\text{sgn}\,\dot{x}) + x - \frac{x^3}{3} = 0.2)$$

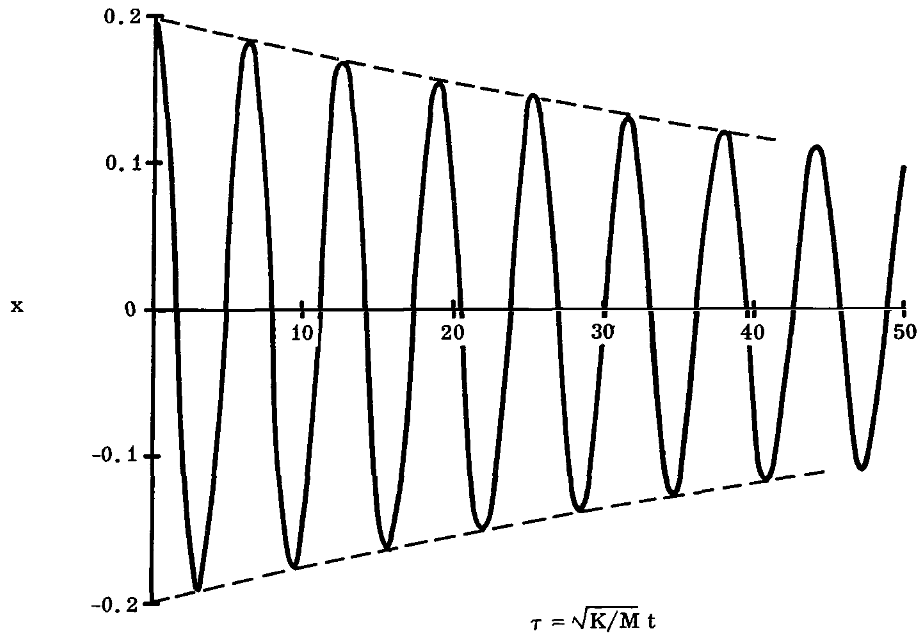


FIGURE 14. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE II

$$\left(\ddot{x} + 0.04|x|(\mathrm{sgn}\,\dot{x}) + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

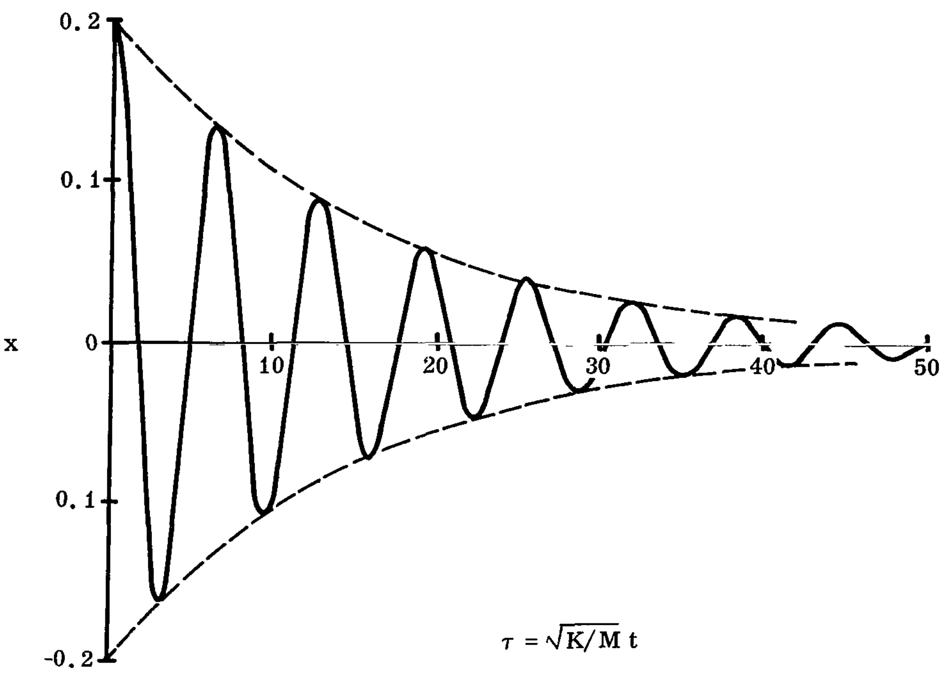


FIGURE 15. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE II

$$\left(\ddot{x} + 0.2|x|(\mathrm{sgn}\,\dot{x}) + x - \frac{x^3}{3} = 0,\ x(\tau = \dot{0}) = 0.2\right)$$

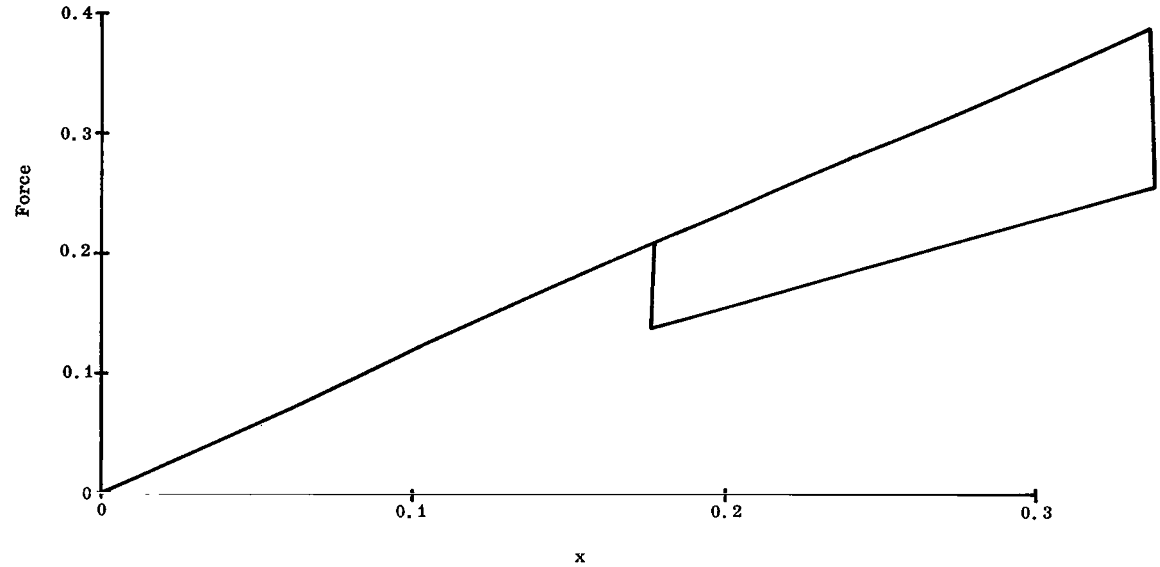


FIGURE 16. FORCE VERSUS DISPLACEMENT - CASE II

$$\left(\ddot{x} + 0.2|x|(\mathrm{sgn}\,\dot{x}) + x - \frac{x^3}{3} = 0.2\right)$$

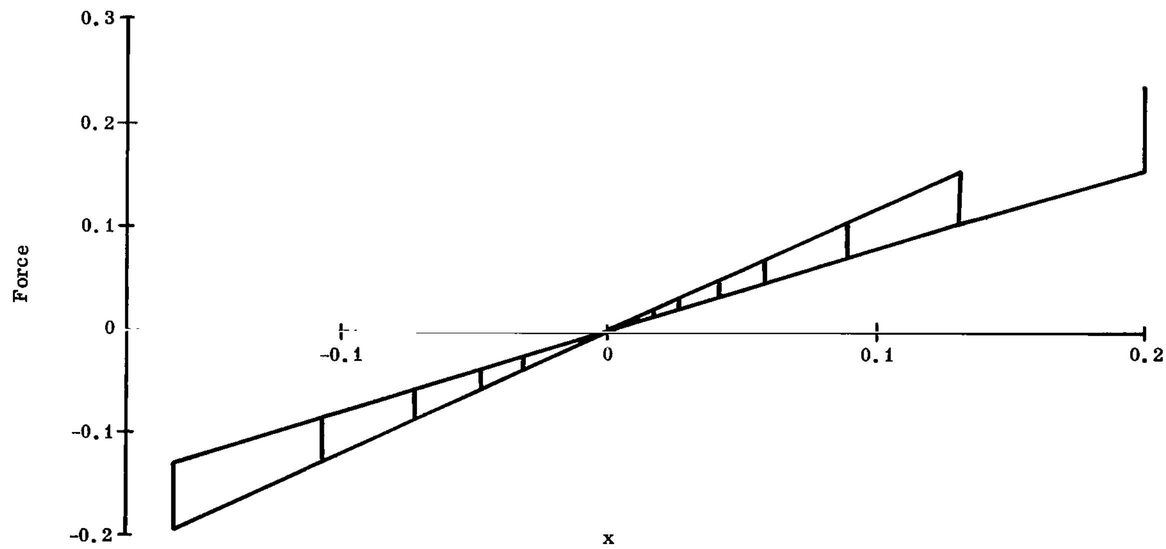


FIGURE 17. FORCE VERSUS DISPLACEMENT - CASE II

$$\left(\ddot{x} + 0.2|x|(\mathrm{sgn}\,\dot{x}) + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

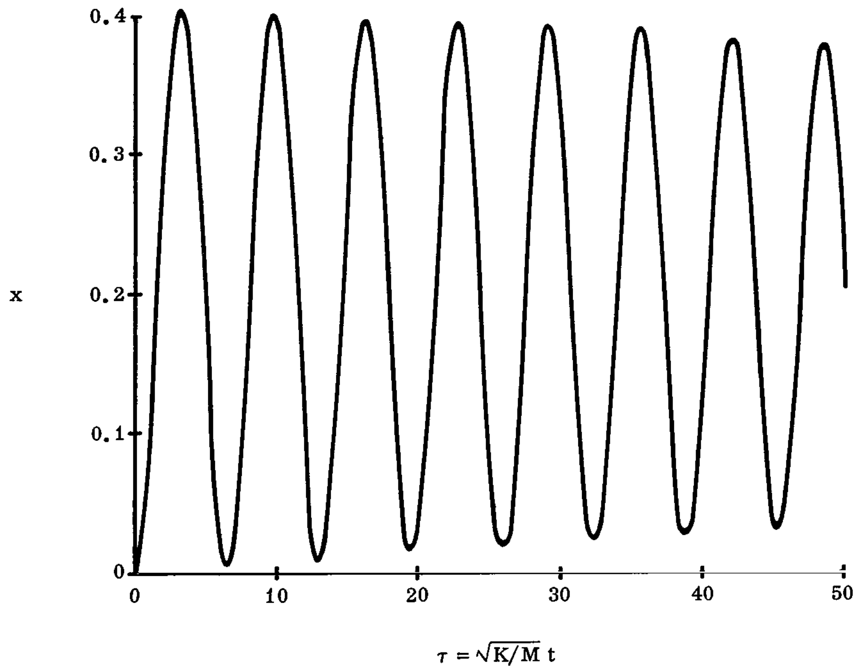


FIGURE 18. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE III

$$(\ddot{x} + 0.04\dot{x}^2 + x - \frac{x^3}{3} = 0.2)$$

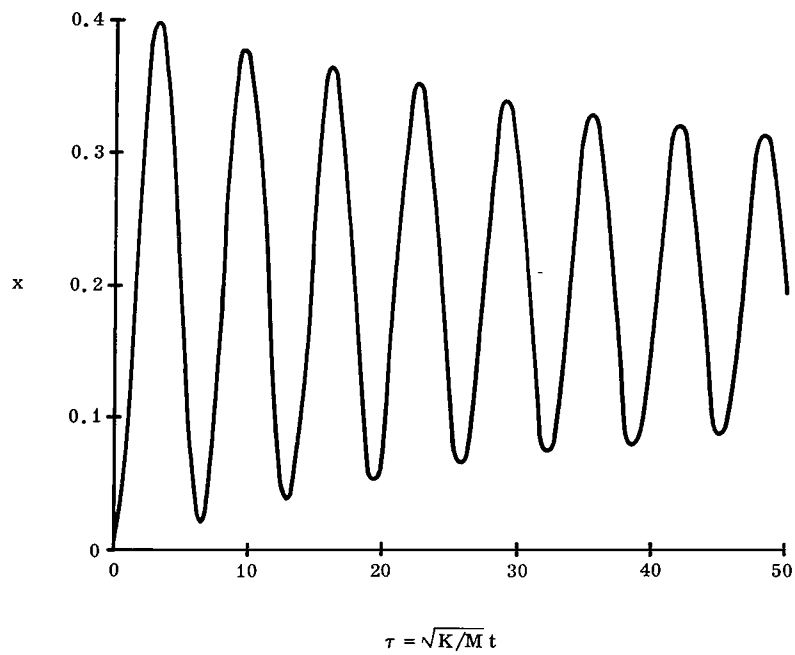


FIGURE 19. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE III

$$(\ddot{x} + 0.2\dot{x}^2 + x - \frac{x^3}{3} = 0.2)$$

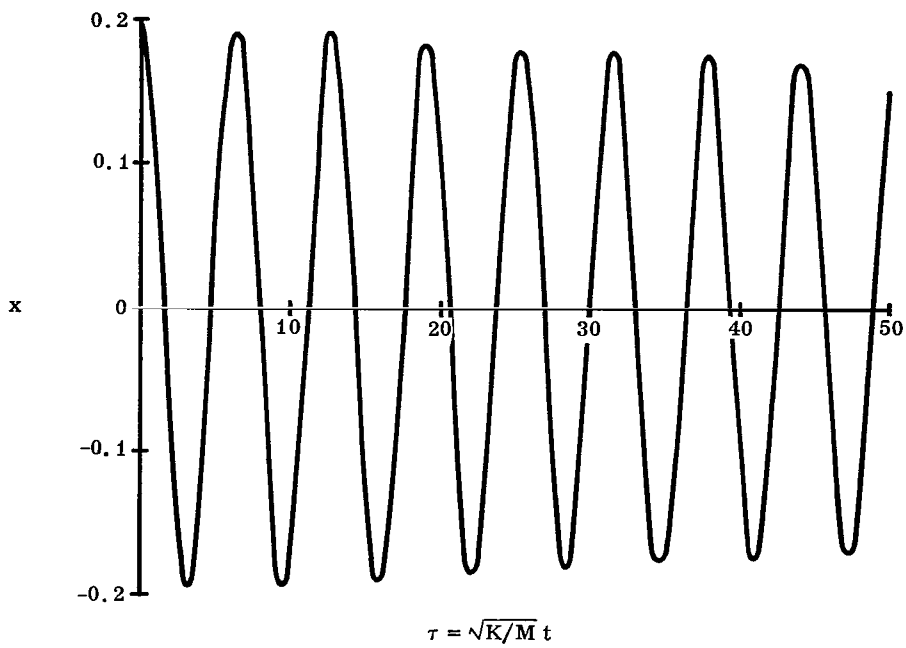


FIGURE 20. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE III

$$\left(\ddot{x} + 0.04\dot{x}^2 + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

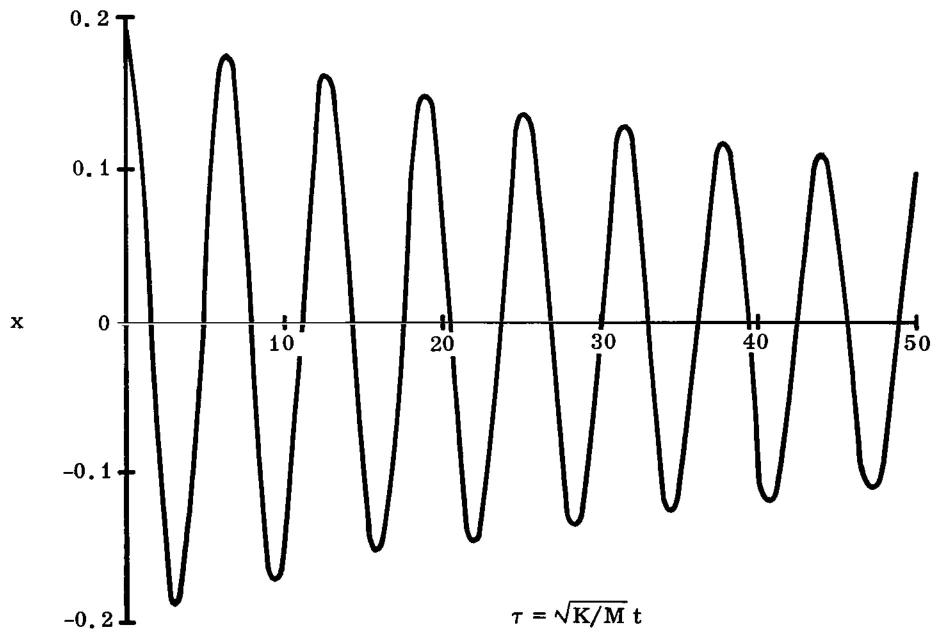


FIGURE 21. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE III

$$\left(\ddot{x} + 0.2\dot{x}^2 + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

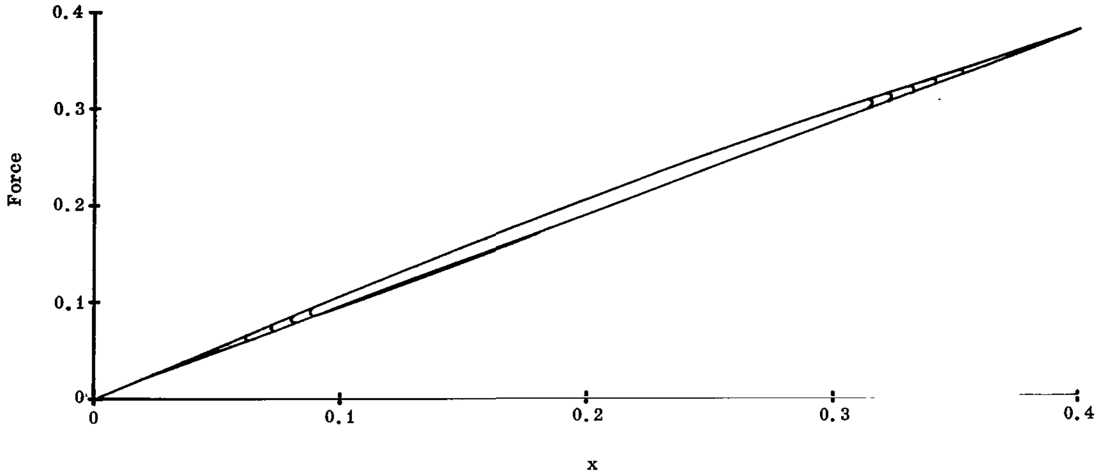


FIGURE 22. FORCE VERSUS DISPLACEMENT - CASE III

$$(\ddot{x} + 0.2\dot{x}^2 + x - \frac{x^3}{3} = 0.2)$$

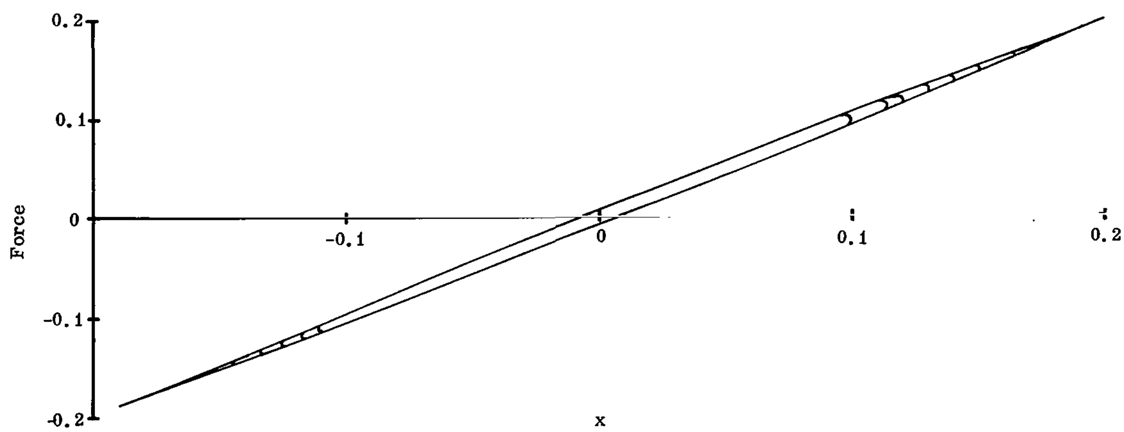


FIGURE 23. FORCE VERSUS DISPLACEMENT - CASE III

$$(\ddot{x} + 0.2\dot{x}^2 + x - \frac{x^3}{3} = 0, \; x(\tau = 0) = 0.2)$$

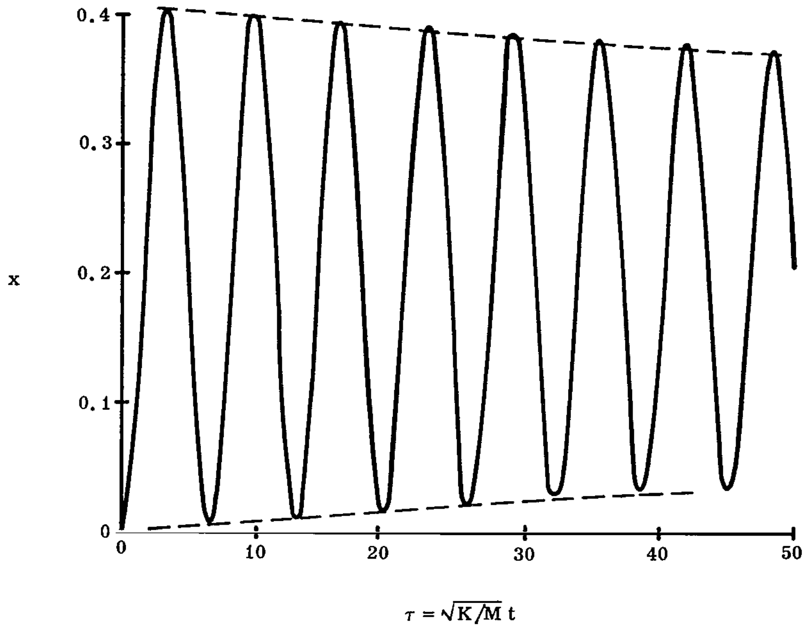


FIGURE 24. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE IV

$$(\ddot{x} + 0.04\,|x|\,\dot{x} + x - \frac{x^3}{3} = 0.2)$$

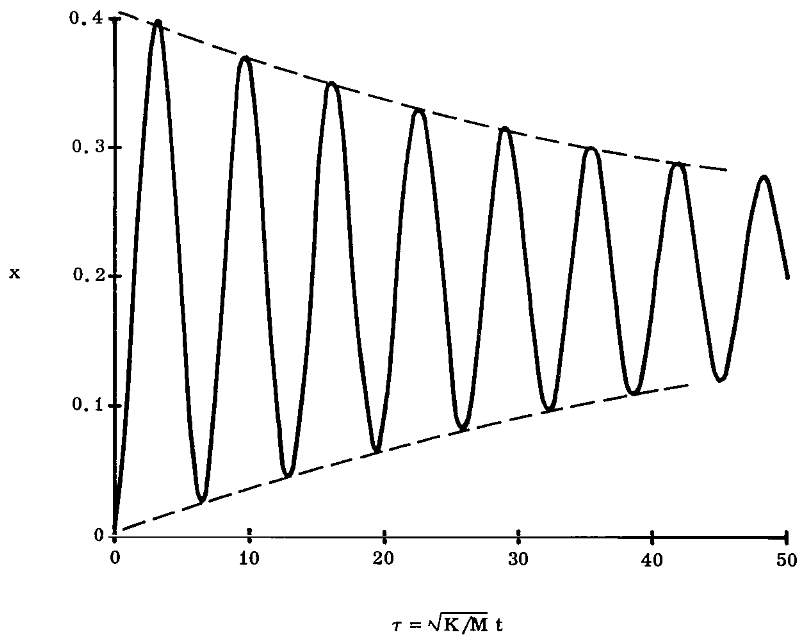


FIGURE 25. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE IV

$$(\ddot{x} + 0.2\,|x|\,\dot{x} + x - \frac{x^3}{3} = 0.2)$$

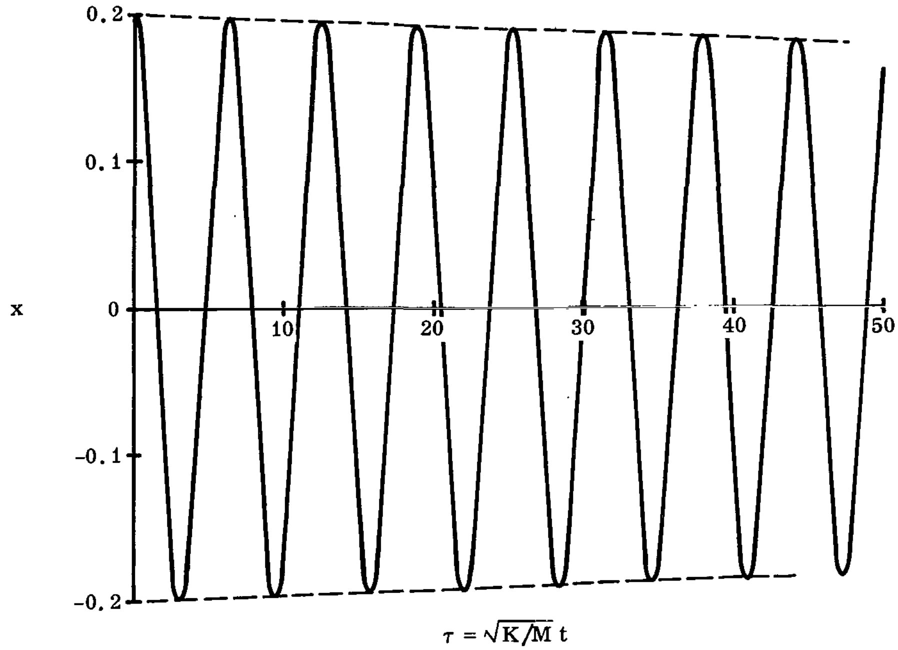


FIGURE 26. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE IV

$$\left(\ddot{x} + 0.04|x|\dot{x} + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

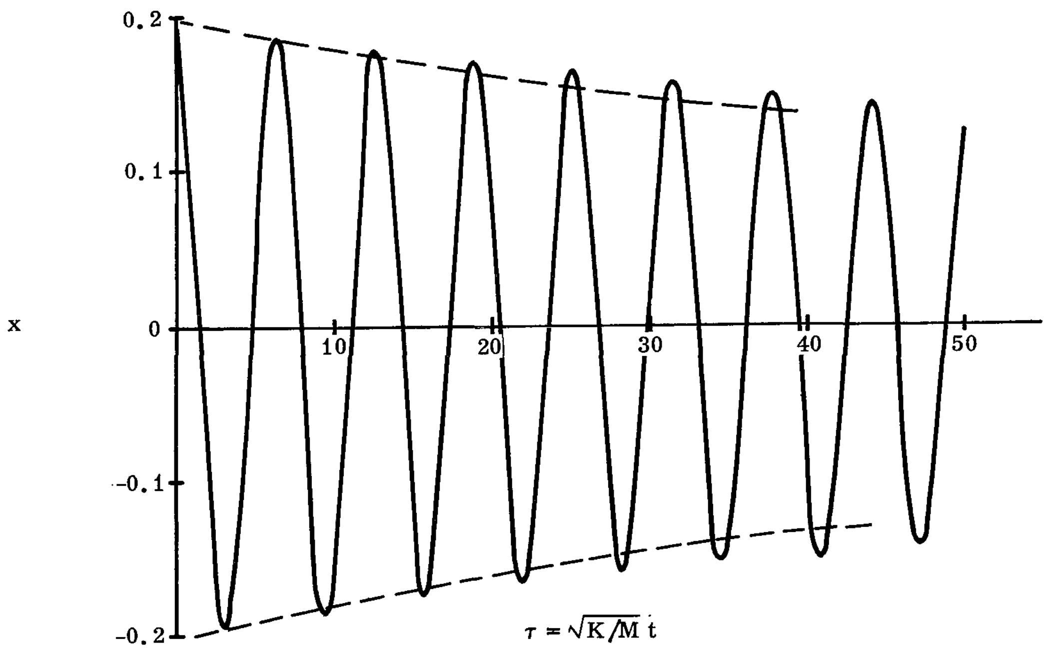


FIGURE 27. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE IV

$$\left(\ddot{x} + 0.2|x|\dot{x} + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2\right)$$

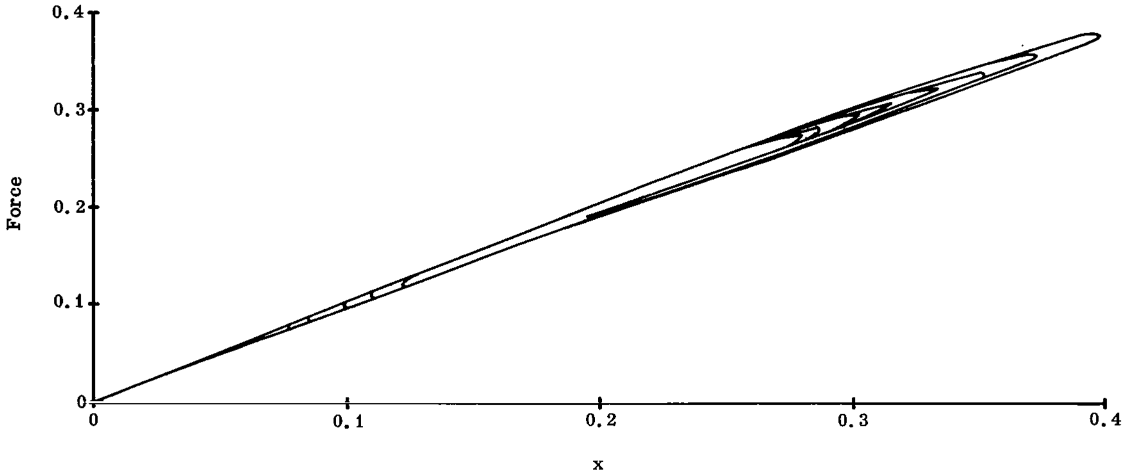


FIGURE 28. FORCE VERSUS DISPLACEMENT - CASE IV
$(\ddot{x} + 0.2|x|\dot{x} + x - \frac{x^3}{3} = 0.2)$

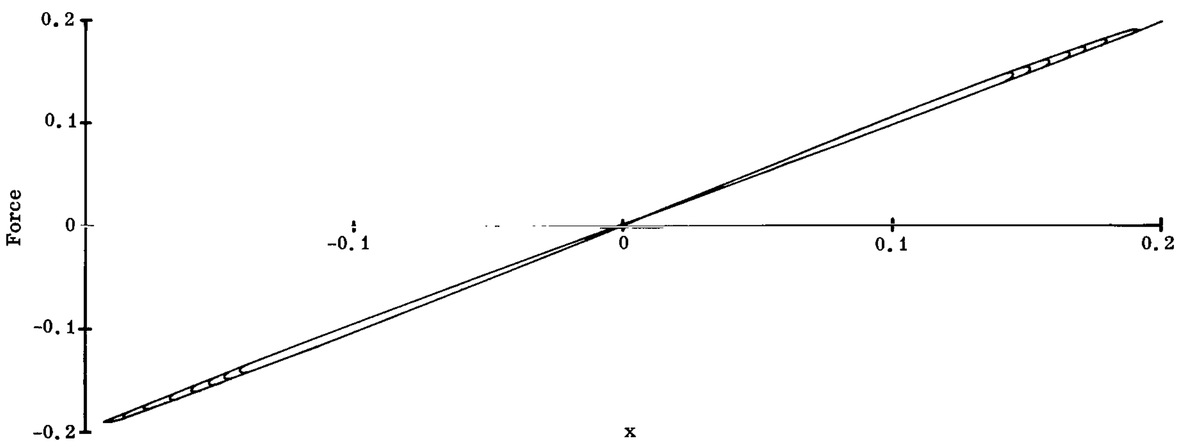


FIGURE 29. FORCE VERSUS DISPLACEMENT - CASE IV
$(\ddot{x} + 0.2|x|\dot{x} + x - \frac{x^3}{3} = 0,\ x(\tau = 0) = 0.2)$

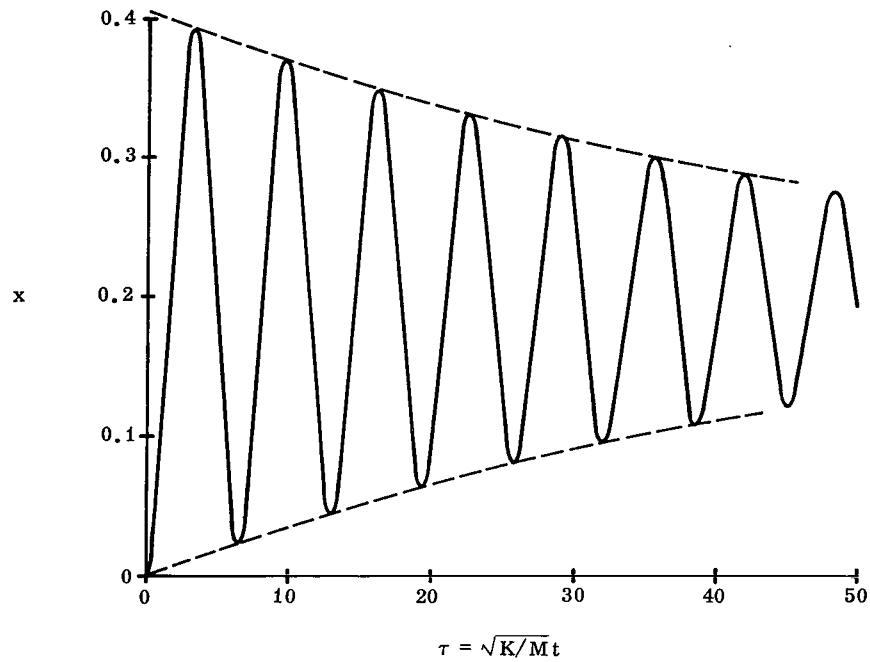


FIGURE 30. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE V

$$(\ddot{x} + 0.0392(1 + x^2)\dot{x} + x - \frac{x^3}{3} = 0.2)$$

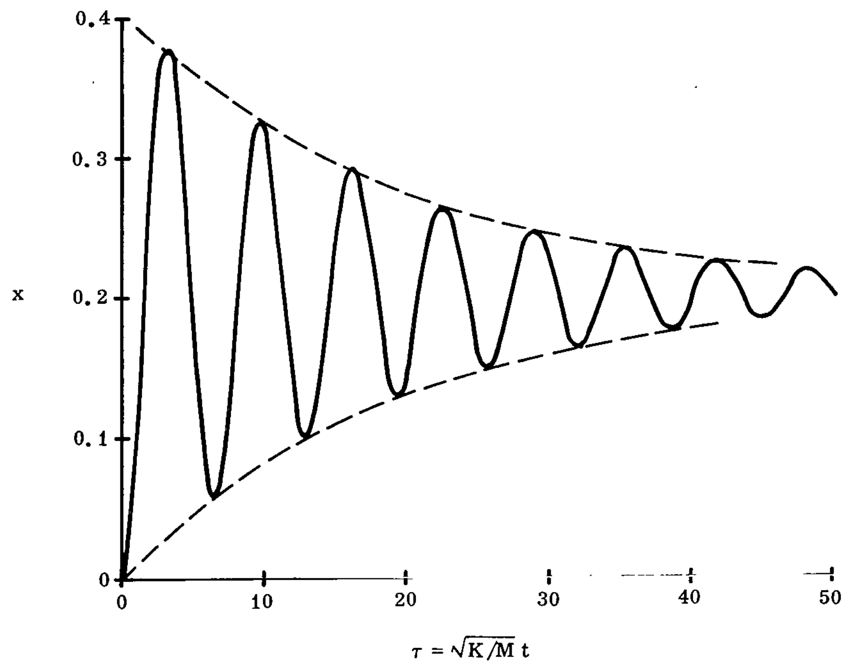


FIGURE 31. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE V

$$(\ddot{x} + 0.10(1 + x^2)\dot{x} + x - \frac{x^3}{3} = 0.2)$$

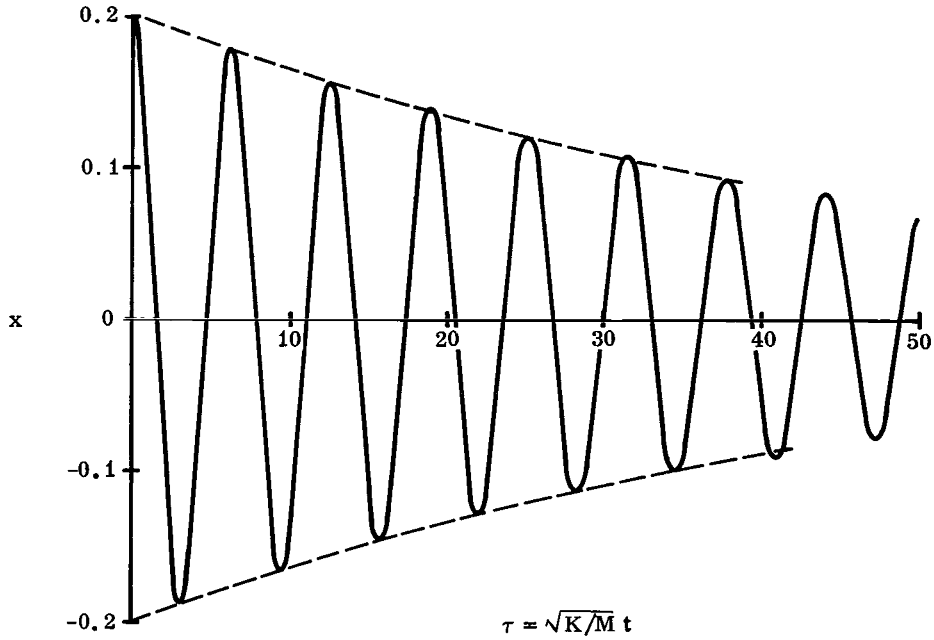


FIGURE 32. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE V

$$(\ddot{x} + 0.0392(1 + x^2)\dot{x} + x - \frac{x^3}{3} = 0, \; x(\tau = 0) = 0.2)$$

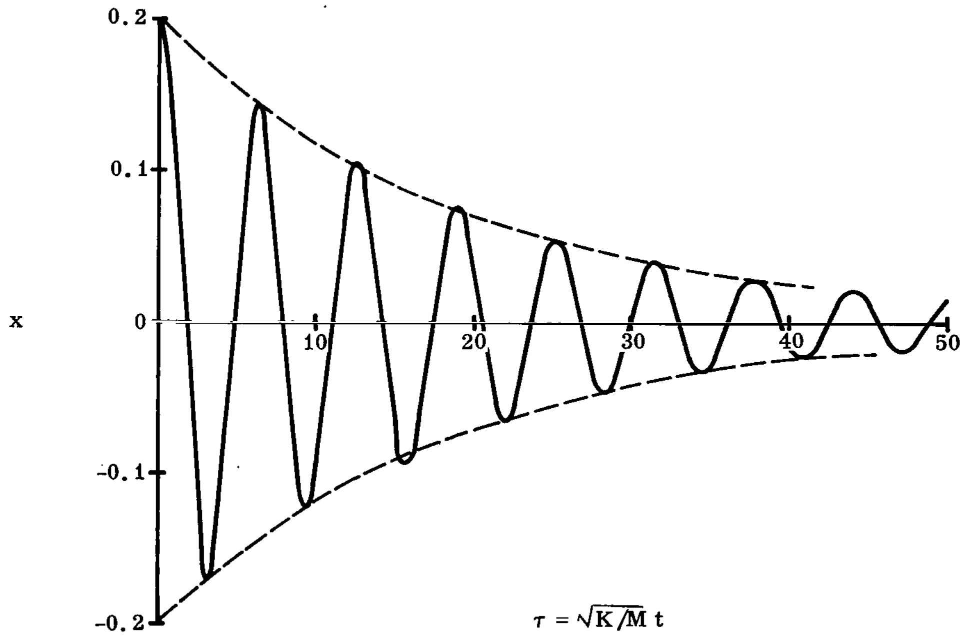


FIGURE 33. DISPLACEMENT VERSUS NONDIMENSIONAL TIME - CASE V

$$(\ddot{x} + 0.10(1 + x^2)\dot{x} + x - \frac{x^3}{3} = 0, \; x(\tau = 0) = 0.2)$$

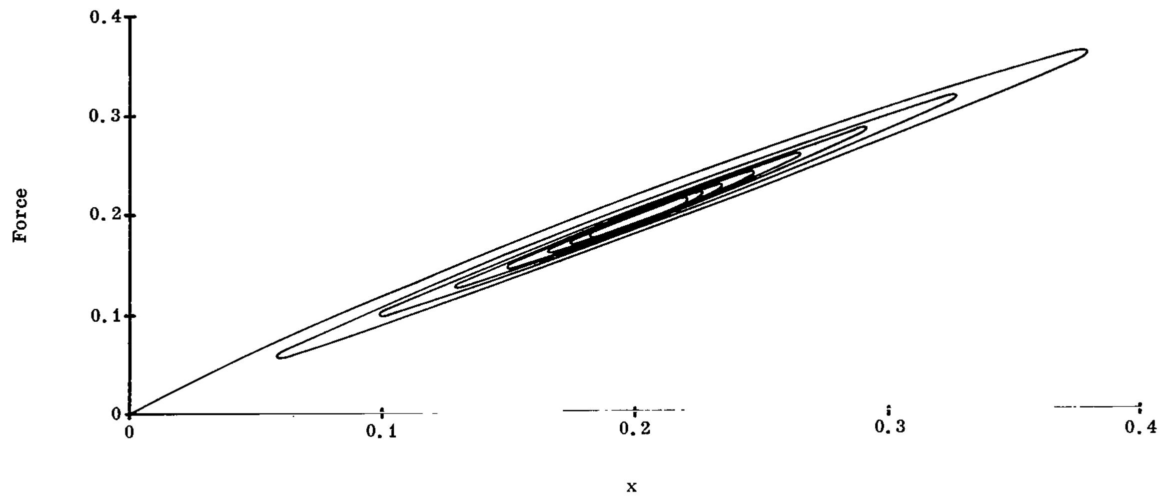


FIGURE 34. FORCE VERSUS DISPLACEMENT - CASE IV

$$\left(\ddot{x} + 0.1(1 + x^2)\dot{x} + x - \frac{x^3}{3} = 0.2\right)$$

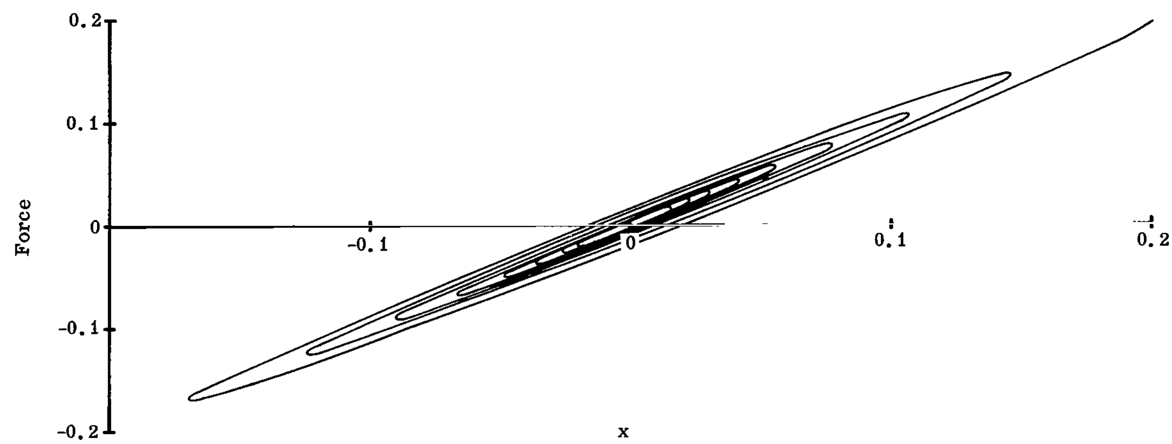


FIGURE 35. FORCE VERSUS DISPLACEMENT

$$\left(\ddot{x} + 0.1(1 + x^2)\dot{x} + x - \frac{x^3}{3} = 0,\ x\,(\tau = 0) = 0.2\right)$$

TABLE X. SUMMARY OF CURVES $\left(\ddot{x} + \frac{1}{K} g(\dot{x}, x) + x - \frac{x^3}{3} = \frac{P}{K}\right)$

Curve	Case	Plot[1]*	P/K	Damping Coefficient	Percent Critical	X_0
1	I	x vs τ	0.2	0.04	2	0
2	I	x vs τ	0.2	0.2	10	0
3	I	x vs τ	0	0.04	2	0.2
4	I	x vs τ	0	0.2	10	0.2
5	I	F vs x	0.2	0.2	10	0
6	I	F vs x	0	0.2	10	0.2
7	II	x vs τ	0.2	0.04	2	0
8	II	x vs τ	0.2	0.2	10	0
9	II	x vs τ	0	0.04	2	0.2
10	II	x vs τ	0	0.2	10	0.2
11	II	F vs x	0.2	0.2	10	0
12	II	F vs x	0	0.2	10	0.2
13	III	x vs τ	0.2	0.04	2	0
14	III	x vs τ	0.2	0.2	10	0
15	III	x vs τ	0	0.04	2	0.2
16	III	x vs τ	0	0.2	10	0.2
17	III	F vs x	0.2	0.2	10	0
18	III	F vs x	0	0.2	10	0.2
19	IV	x vs τ	0.2	0.04	2	0
20	IV	x vs τ	0.2	0.2	10	0
21	IV	x vs τ	0	0.04	2	0.2
22	IV	x vs τ	0	0.2	10	0.2
23	IV	F vs x	0.2	0.2	10	0
24	IV	F vs x	0	0.2	10	0.2
25	V	x vs τ	0.2	$\alpha = 1.0$ 0.0392	2	0
26	V	x vs τ	0.2	$\alpha = 1.0$ 0.100	10	0
27	V	x vs τ	0	$\alpha = 1.0$ 0.0392	2	0.2
28	V	x vs τ	0	$\alpha = 1.0$ 0.100	10	0.2
29	V	F vs x	0.2	$\alpha = 1.0$ 0.100	10	0
30	V	F vs x	0	$\alpha = 1.0$ 0.100	10	0.2

*

[1]F = spring plus damper force

APPENDIX
APPROXIMATE SOLUTION FOR TRANSIENTS

The approximate solution of the nonlinear differential equation of equation (4) is given as an exponential series

$$x = A_0 + \sum_{n=1}^{m} A_n e^{\alpha_n \tau} \quad . \tag{A. 1}$$

This series when substituted into equation (4) yields an equation with some terms which are single series and others which are multiple series. Equating coefficients of single series terms yields a set of basic exponents α_1 and α_2. Other exponents are linear combinations of the basic exponents. The constant term A_0 is obtained by letting $\tau \rightarrow \infty$ and equating the remaining terms. Remaining coefficients are obtained bv evaluating m initial conditions from equation (4) and setting these equal to the derivatives of equation (A.1) evaluated at $\tau = 0$. This method is discussed in much more detail by the author [4]. Only sufficient information is included to permit the approximate solution of the equations being studied.

Essentially the method outlined is equivalent to expressing the original equation, equation (4), as two first-order equations

$$\left.\begin{aligned} \dot{x}_1 &= -\frac{1}{K} g(x_1, x_2) - x_2 - \epsilon x_2^3 + \frac{P}{K} \\ &= f_1(x_1, x_2) \\ \dot{x}_2 &= x_1 \\ &= f_2(x_1, x_2) \end{aligned}\right\} \tag{A. 2}$$

where $x_2 = x$ and both x_1 and x_2 are expressible as a series. Equations (A. 2) can be expanded in a Taylor's series about some value $x_1 = x_{10}$, $x_2 = x_{20}$. Thus,

$$\dot{x}_i = f_i(x_{10}, x_{20}) + \left.\frac{\partial f_i}{\partial x_1}\right|_{x_{10}, x_{20}} (x_1 - x_{10})$$

$$+ \left.\frac{\partial f_i}{\partial x_2}\right|_{x_{10}, x_{20}} (x_2 - x_{20}) + \text{higher order terms}$$

$$i = 1, 2 \; . \tag{A. 3}$$

Since x_1 and x_2 are exponential series

$$\dot{x}_1 = \alpha' x_1 = \ddot{x}_2$$

$$\dot{x}_2 = \alpha'' x_2 = x_1$$

which are the variational equations or the linear approximations. Substituting the second into the first and realizing $\ddot{x}_2 = (\alpha')^2 x_2$ yields $\alpha' = \alpha''$. Therefore, in matrix notation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha' & 0 \\ 0 & \alpha' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; . \tag{A. 4}$$

Expressing equation (A. 3) in matrix notation and substituting equation (A. 4) into this equation gives

$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(x_{10}, x_{20}) \\ f_2(x_{10}, x_{20}) \end{bmatrix}
$$

$$
+ \begin{bmatrix} \frac{\partial f_1}{\partial x_1} - \alpha' & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} - \alpha' \end{bmatrix}_{x_{10}, x_{20}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

$$
- \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x_{10}, x_{20}} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}
$$

$$
+ \text{ higher order terms} \quad . \tag{A. 5}
$$

Terms not involving x_1 and x_2 should be identically satisfied by the proper choice of x_{10} and x_{20}. The final steady values $x_{10} = 0$ and $x_{20} = A_0$ form one such set. Therefore, the determinate of the coefficient matrices must be zero to satisfy equation (A. 5). Thus,

$$
\begin{vmatrix} \frac{\partial f_1}{\partial x_1} - \alpha' & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} - \alpha' \end{vmatrix}_{x_{10}, x_{20}} = 0
$$

which yields

$$(\alpha')^2 - \alpha'\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} = 0 \ , \tag{A. 6}$$

where partial derivatives are evaluated at x_{10}, x_{20}. Solving this equation yields two roots of α' which are the basic exponents

$$\left.\begin{aligned} &\alpha_1 = \sigma + j\omega \\ \text{and} \quad &\\ &\alpha_2 = \sigma - j\omega \ . \end{aligned}\right\} \tag{A. 7}$$

The roots of α' are complex when underdamped. Higher order exponents $\alpha_n (n > 2)$ are found to be linear combinations of α_1 and α_2, as the higher order terms of equation (A. 5), involve exponents $\alpha_1 + \alpha_2$ and so on.

For the purpose of this investigation, a three-term approximation ($m = 2$ in equation (A. 1)) is used so α_1 and α_2 are all that is required. Thus

$$x = A_0 + A_1 e^{\alpha_1 \tau} + A_2 e^{\alpha_2 \tau} \tag{A. 8}$$

where A_1 and A_2 are also complex conjugate

$$\left.\begin{aligned} &A_1 = a_1 + jb_1 \\ \text{and} \quad &\\ &A_2 = a_1 - jb_1 \end{aligned}\right\} \tag{A. 9}$$

to ensure a real solution.

The coefficient A_0 is found from the steady state conditions $\dot{x}_1 = \dot{x}_2 = 0$ in equation (A. 2). Thus,

$$\frac{1}{K} g(0, A_0) + A_0 + \epsilon A_0^3 = \frac{P}{K} \tag{A. 10}$$

becomes the steady state amplitude equation which must be solved for A_0. The coefficients a_1 and b_1 are obtained using initial conditions

$$\left.\begin{aligned} & x(\tau = 0) = x_0 \\ \text{and} \quad & \\ & \dot{x}(\tau = 0) = \dot{x}_0 \quad . \end{aligned}\right\} . \tag{A. 11}$$

Therefore, substituting equations (A. 7) and (A. 9) into equation (A. 8) and evaluating x and $\dot{x}$ at $\tau = 0$ yields

$$x_0 = A_0 + 2a_1$$

and

$$\dot{x}_0 = 2\sigma a_1 - 2\omega b_1 \quad .$$

That is,

$$a_1 = \frac{x_0 - A_0}{2}$$

$$b_1 = \frac{(x_0 - A_0)\sigma - \dot{x}_0}{2\omega}$$

and

$$x = A_0 + \left(\frac{x_0 - A_0}{2} + j\, \frac{(x_0 - A_0)\sigma - \dot{x}_0}{2\omega} \right) e^{(\sigma + j\omega)\tau}$$

$$+ \left(\frac{x_0 - A_0}{2} - j\, \frac{(x_0 - A_0)\sigma - \dot{x}_0}{2\omega} \right) e^{(\sigma - j\omega)\tau} \quad .$$

This three-term approximation may be expressed as a sinusoidal function

$$x = A_0 + Ae^{\sigma\tau}\sin(\omega\tau - \phi) \tag{A. 12}$$

where

$$\left.\begin{aligned} A &= -\frac{1}{\omega}\left[(x_0 - A_0)^2(\sigma^2 + \omega^2) - 2(x_0 - A_0)\dot{x}_0\sigma + \dot{x}_0^2\right]^{1/2} \\ \phi &= \tan^{-1}\frac{(x_0 - A_0)\omega}{(x_0 - A_0)\sigma - \dot{x}_0} \quad . \end{aligned}\right\} \tag{A. 13}$$

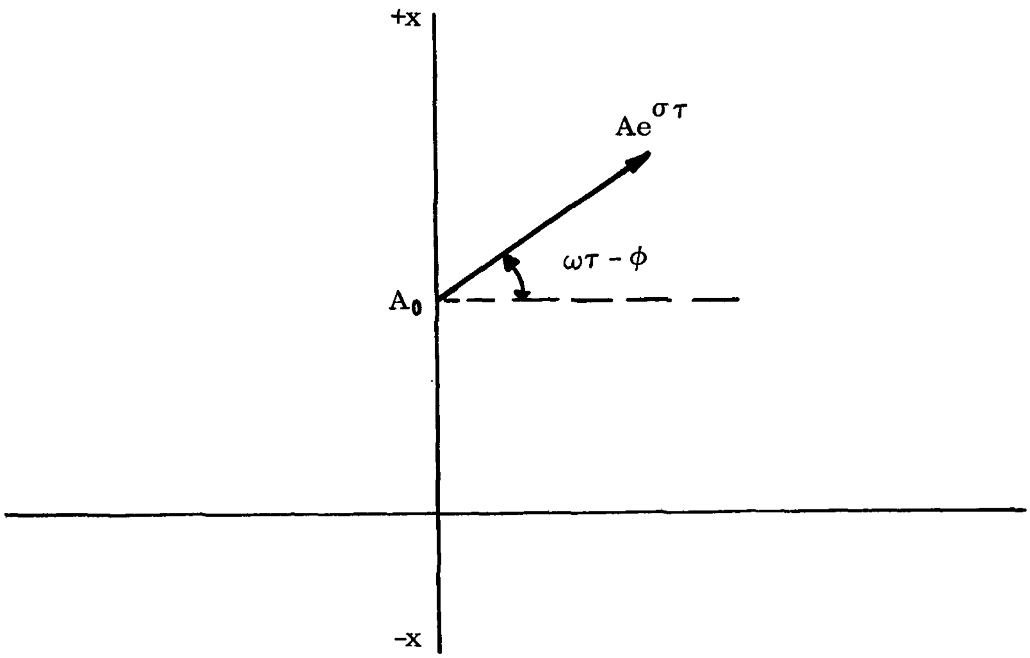
For the step input with zero initial conditions

$$\left.\begin{aligned} x &= A_0 + \frac{A_0}{\omega}(\sigma^2 + \omega^2)^{1/2} e^{\sigma\tau}\sin(\omega\tau - \phi) \\ \phi &= \tan^{-1}\frac{\omega}{\sigma} \end{aligned}\right\} \tag{A. 14}$$

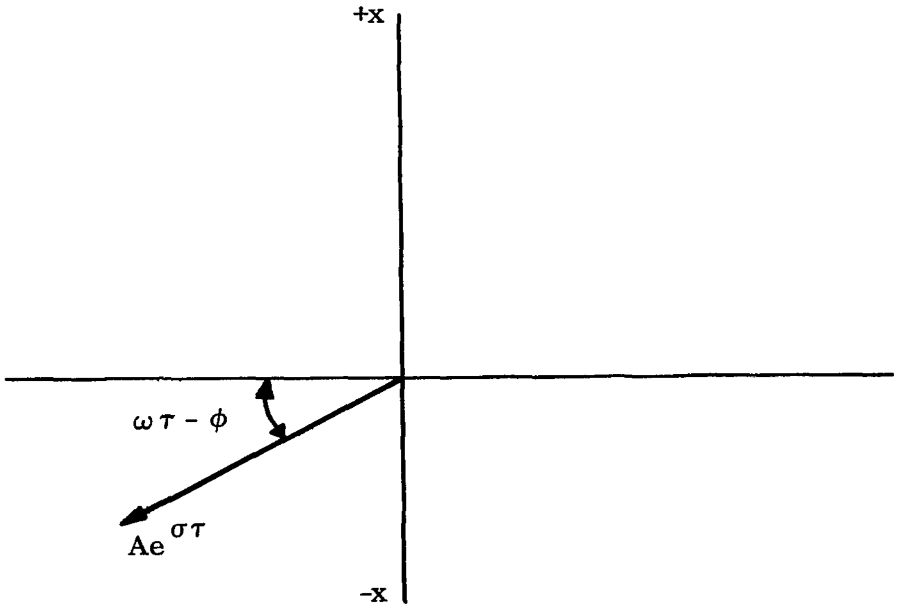
and for the free vibration with no initial velocity ($P = \dot{x}_0 = 0$)

$$\left.\begin{aligned} x &= -\frac{x_0}{\omega}(\sigma^2 + \omega^2)^{1/2} e^{\sigma\tau}\sin(\omega\tau - \phi) \\ \phi &= \tan^{-1}\frac{\omega}{\sigma} \end{aligned}\right\} \tag{A. 15}$$

where σ and ω are the real and imaginary parts of the roots of α' from equation (A. 6). Phasor representations of equations (A. 14) and (A. 15) are shown in Figure A. 1.



(a) Step Input



(b) Free Oscillations

FIGURE A. 1. PHASOR DIAGRAMS

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