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THE MAXIMUM RESPONSE OF SIMPLE BEAMS
TO RANDOM EXCITATION

By Richard L. Barnsoki
July 1967

The random inputs are described as stationary random excitations perfectly correlated in space and time. They should be defined as stationary random excitations perfectly correlated in the space and frequency domain. A more descriptive definition is to consider the random inputs as a pressure field correlated in time as white noise and correlated over the surface of the structure as a constant.

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THE MAXIMUM RESPONSE OF SIMPLE BEAMS TO RANDOM EXCITATION

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CONTENTS

1. INTRODUCTION

This report concerns an empirical solution to the single highest peak (SHP) problem for a simple beam subjected to stationary random excitation perfectly correlated in space and time. Both simply supported and rigidly clamped boundary conditions are considered. These time dependent stochastic solutions are obtained by examining the output statistics of an analog circuit which simulates the physical system. The results are presented in dimensionless form and find direct application in estimating the probability that the maximum response, within a finite time interval, remains below a preselected threshold level.

For completeness in this discussion, cursory reviews are made of basic theory common to the random vibration of beams and to the analog simulation of distributed structures. In this way, the reader can more fully understand and consequently appreciate the simulation procedure (and its limitations) used in this study.

2. BACKGROUND INFORMATION

The single highest peak (SHP) problem concerns the maximum response statistics of a physical system subjected to random excitation. Specifically, one seeks to predict the maximum response the system may experience within a finite time interval.

The statistics of the SHP problem are time varying and reflect a nontrivial exercise even for a simple mechanical configuration as a single degree-of-freedom system subjected to Gaussian bandlimited white noise. If the system is lightly damped with the natural frequency f_n and the bandwidth of the input excitation (1) is wide compared to the half-power bandwidth of the mechanical system and (2) includes f_n, then the output response is as shown in Figure 1. This motion appears as simple harmonic motion of frequency f_n with a randomly varying amplitude and phase.

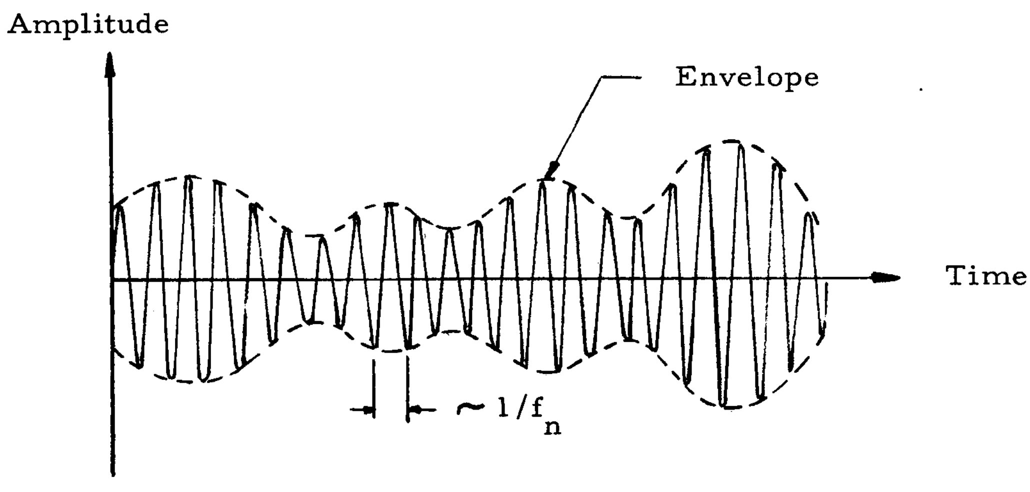

Figure 1. Response of a Lightly Damped Mechanical Oscillator to Broadband Stationary Random Excitation

The RMS displacement response to this stationary excitation is given by

$$\psi_x = \sqrt{\psi_x^2}$$

where the mean square displacement response is

$$\psi_x^2 = \int_0^{\omega_c} G_x(\omega)\, d\omega = \int_0^{\omega_c} \left|H_o(\omega)\right|^2 G_f(\omega)\, d\omega \tag{1}$$

The term $G_x(\omega)$ is the displacement response spectral density, $G_f(\omega)$ the spectral density of the applied excitation, ω_c the upper cutoff frequency of the bandlimited excitation, and $H_o(\omega)$ a displacement to force frequency response function of the system defined as

$$H_o(\omega) = \frac{1}{m\omega_n^2} \bullet \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\,2\zeta\frac{\omega}{\omega_n}} \tag{2}$$

For white noise, $G_f(\omega)$ reduces to the constant G_o and Eq. (1) becomes

$$\psi_x^2 = G_o \int_0^{\omega_c} \left|H_o(\omega)\right|^2 d\omega = \frac{\pi \cdot G_o Q}{2m^2 \omega_n^3} I_n \tag{3}$$

where the undamped natural frequency ω_n and the system damping Q are given by

$$\omega_n = 2\pi f_n$$
$$Q = \frac{1}{2\zeta} \tag{4}$$

The ζ term is the damping factor, m the mass of the system and I_n a dimensionless integral varying in value from zero to one. Similarly, the mean square velocity and acceleration responses are

$$\psi_{\dot{x}}^2 = G_o \int_0^{\omega_c} \omega^2 \left| H_o(\omega) \right|^2 d\omega = \frac{\pi G_o Q}{2 m^2 \omega_n} II_n \tag{5}$$

$$\psi_{\ddot{x}}^2 = G_o \int_0^{\omega_c} \omega^4 \left| H_o(\omega) \right|^2 d\omega = \frac{\pi G_o Q \omega_n}{2 m^2} III_n \tag{6}$$

where the dimensionless integrals I_n, II_n and III_n are found in References 2 and 9. The integral II_n, as with I_n, is bounded and ranges in value between zero and one. On the other hand, III_n generally becomes unbounded as $\omega_c \rightarrow \infty$ and tends to result in values substantially greater than one as $\omega_c/\omega_n > 1$. For these reasons, Eq. (6) finds limited practical application.

Other statistics regarding Figure 1 also are known. The probability distribution function is Gaussian for the instantaneous amplitudes and is essentially Rayleigh for the individual peaks. The probability distribution function for the SHP is neither Gaussian nor Rayleigh but of the form

$$\hat{P}_M(|\beta| \leq \beta_o, T) \simeq A_o\, e^{-\alpha_o \omega_n T Q} \tag{7}$$

where A_o is dependent upon the initial conditions and α_o is a stochastic parameter dependent upon the damping Q and the threshold level β_o.

The term β is a dimensionless ratio of the maximum response amplitude to the RMS response of the system to broadband stationary white noise. For many applications of practical interest, A_o may be assumed equal to unity with negligible error. Values of α_o may be found in Reference 8 as families of curves in β_o for both β^+ and $|\beta|$. An alternate and (perhaps) more descriptive manner of presenting the information contained in Eq. (7) is shown as Figure 2.

The form of the data presentation in Figure 2 represents a stochastic solution for a constant probability value and is originally based upon the results from an analog simulation study (Reference 3). This curve is for the constant probability $P_M(|\beta| \leq \beta_o) = 0.95$ and, as with the ordinate β, the abscissa is likewise a dimensionless ratio consisting of the natural frequency f_n, the system damping Q and the sampling time interval T. This time interval corresponds to the time duration during which the system response is observed. The product f_nT corresponds (approximately) to the number of response cycles so that f_nT/Q may be interpreted as the number of response cycles per Q of the system. For a specific single degree of freedom system and a preselected probability value (f_n, Q and $P_M(|\beta| \leq \beta_o)$ are thus fixed), the β response varies exponentially with the time T and theoretically approaches ∞ as $T \to \infty$. The greatest rate of increase in $|\beta|$ appears to occur within the range $0 \leq f_nT/Q \leq 3$. For each desired probability value of $P_M(|\beta| \leq \beta_o)$, a separate β curve is required. Each such β plot appears similar to Figure 2 wherein, as expected, the curves associated with the higher probability values yield correspondingly higher β values.

The response of a distributed elastic structure to random excitation is somewhat more involved than with the single degree-of-freedom system. For the single degree-of-freedom system, it is

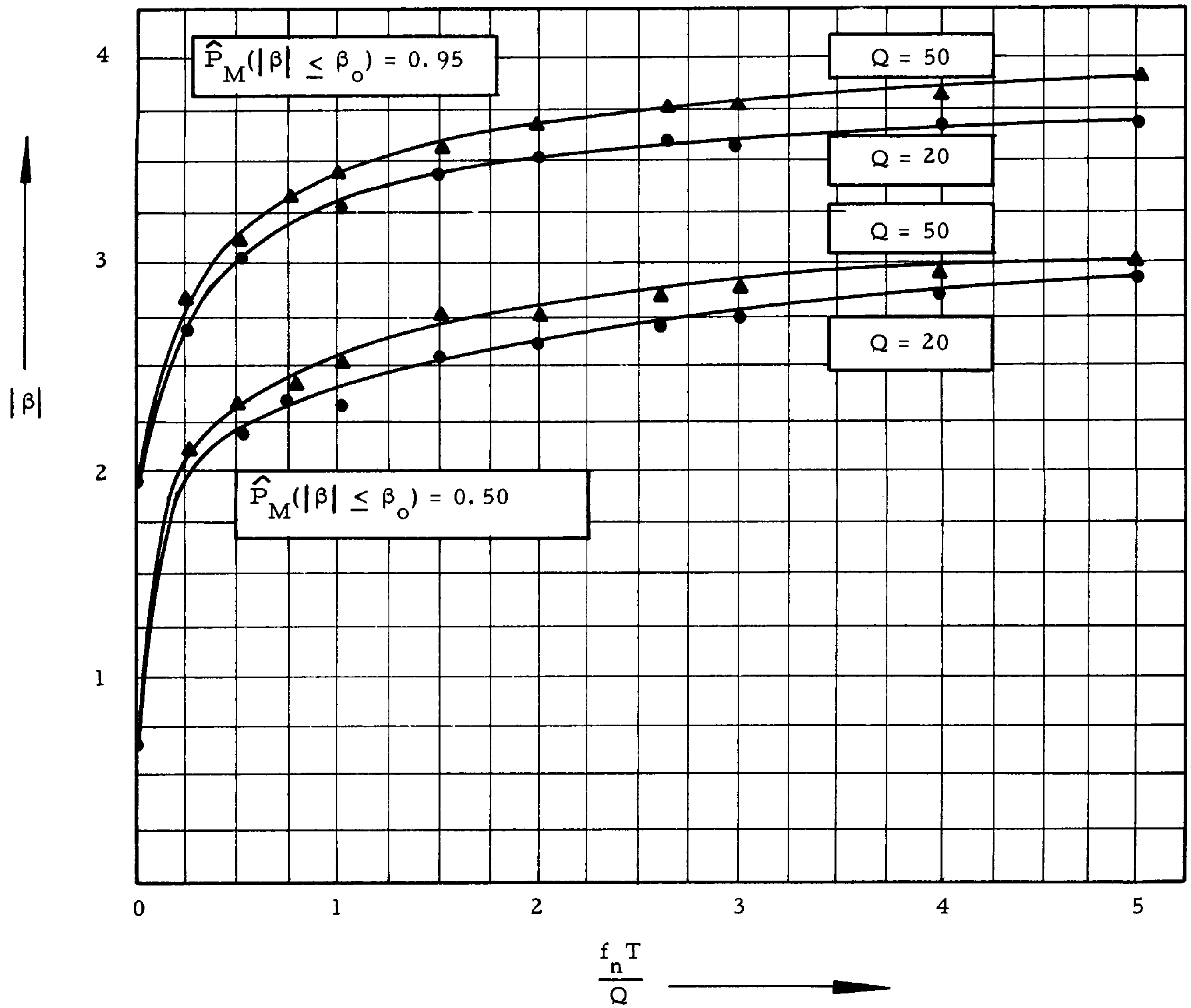

Figure 2. Maximum Response of a Single Degree-of-Freedom System to Stationary White Noise

recalled an ordinary differential equation is treated and the random excitation is functionally dependent only upon time. For a distributed structure, a partial differential equation must be examined wherein the external loading is random excitation and usually is correlated with the spatial dimensions as well as in time.

The structure has many (theoretically infinite in number) natural or modal frequencies and associated with each such frequency is a modal damping factor and a mode shape. These structural properties are used to define generalized quantities such as generalized mass, generalized damping, generalized stiffness and generalized force; all of which are convenient to a modal analysis of the system. For many engineering applications involving structures with nonuniform physical and geometric properties, it is frequently advantageous to represent such systems by discrete models. In this way, one treats sets of ordinary differential equations by matrix procedures rather than solve partial differential equations with variable coefficients. With either approach, concepts based upon modal solutions* are equally appropriate.

The response due to an arbitrary loading is dependent upon how well the loading couples with the structure. Such coupling is a function not only of the spectral frequency distribution of the applied excitation, but also upon the spatial frequency distribution of this loading. Consequently, to achieve a maximum coupling between the structure and the input excitation, coincidence or resonance in both space and time is required. Such phenomena naturally give rise to concepts of space-time spectral densities, space-time correlation functions and joint acceptances. These concepts are common to calculations for the RMS response of a distributed structure to random excitation.

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*For problems wherein the modal density (modal frequencies per excitation bandwidth) is $\gtrsim 6$, statistical energy procedures (Reference 10) become effective analysis tools for RMS calculations.

For the SHP problem of a distributed structure, a most salient question is what quantities does one use to categorize the maximum response characteristics? By deciding upon a simulation procedure (similar to Reference 3) to examine empirically the SHP statistics, the dimensionless ratios used in Figure 2 appear appealing. Since the RMS response of such a structure always can be estimated either theoretically and/or by measurement, this quantity is used to normalize the response maxima so that the basic definition of the β ratio is unchanged. In addition, by arbitrarily defining the dimensionless time ratio as $f_1 T/Q_1$, the basic form of the data presentation in Figure 2 remains intact. The quantity f_1 is the fundamental modal frequency of the structure and Q_1 is a measure of its modal damping. With these definitions, the maximum response results are presented in a useful form inasmuch as all of the various quantities in the dimensionless ratios may be plausibly calculated, measured, or estimated.

This report considers principally the β results for a Bernoulli-Euler beam with both simply supported and rigidly clamped boundaries where the random excitation is stationary white noise with a correlation of unity in both space and time. The beam and the excitation are simulated electrically and the β results then determined from measurements taken at the mid-span of the beam. Before examining these results in detail, it is judicious to review briefly the basic theory for (1) the beam dynamics and (2) the analog simulation.

3. BASIC BEAM THEORY

The equation of motion for a Bernoulli-Euler beam is of the form

$$m \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + EI \frac{\partial^4 y}{\partial x^4} = f(x, t) \tag{8}$$

where m is the mass per unit length, c a viscous damping coefficient, E the Young's modulus and I a cross-section area moment of inertia. The variable y represents the lateral displacement of the beam from the static equilibrium position and is a function of the spatial dimension x and time t so that $y = y(x, t)$. The quantity $f(x, t)$ is the applied excitation and may be likewise a function of x and t. By assuming the beam to be homogeneous and uniform, the coefficients of Eq. (8) reduce to constants and the complexity of mathematics required to calculate a solution is thereby decreased.

A modal solution of $y(x, t)$ may be written in the form

$$y(x, t) = \sum_j \phi_j(x)\, q_j(t) \tag{9}$$

where $\phi_j(x)$ is the jth normal mode of the system and $q_j(t)$ is the normal coordinate associated with the jth mode. The summation is implied to range from one to infinity, thus including all of the elastic modes of the distributed structure.

The normal modes are orthogonal functions in x and are sometimes called the eigenfunctions of the system. Physically, they may be interpreted as the spatial form of the free vibration of the system in the absence of damping and all external forces. If a body is thus distorted into one of the normal mode shapes, say $\phi_j(x)$, then released; the body will vibrate for all time in this jth mode with the modal frequency ω_j. The general form of the jth mode shape for a simple beam may be expressed as

$$\phi_j(x) = C_j \cos \lambda_j x + D_j \sin \lambda_j x + E_j \cosh \lambda_j x + F_j \sinh \lambda_j x \tag{10}$$

where the parameter λ_j is related to the jth modal frequency as

$$\omega_j^2 = \frac{EI}{m\ell^4} (\lambda_j \ell)^4 \tag{11}$$

The numerical values of the quantity $\lambda_j \ell$ are obtained from solutions to the frequency equation of the system, this equation being formed as a result of applying the boundary conditions to Eq. (10).

The normal coordinate $q_j(t)$ is obtained by solving the equation of motion

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{\overline{F}_j(t)}{\overline{M}_j} \tag{12}$$

where

$$\frac{\overline{C}_j}{\overline{M}_j} = 2\zeta_j \omega_j \tag{12a}$$

$$\frac{\overline{K}_j}{\overline{M}_j} = \omega_j^2 \tag{12b}$$

The term $\overline{M}_j$ is the generalized mass, $\overline{C}_j$ the generalized damping, $\overline{K}_j$ the generalized stiffness and $\overline{F}_j(t)$ the generalized force. These generalized quantities are related to the physical properties of the beam by

$$\overline{M}_j = \int_0^{\ell} m \,\phi_k(x)\, \phi_j(x)\, dx = m \int_0^{\ell} \phi_j^2(x)\, dx \tag{13}$$

$$\overline{C}_j = \int_0^{\ell} c \,\phi_k(x)\, \phi_j(x)\, dx = c \int_0^{\ell} \phi_j^2(x)\, dx \tag{14}$$

$$\overline{K}_j = \omega_j^2\, \overline{M}_j \tag{15}$$

$$\overline{F}_j(t) = \int_0^{\ell} \phi_j(x)\, f(x, t)\, dx \tag{16}$$

If initial conditions are quoted, these may be incorporated into the solution by evaluating the expressions

$$q_j(t) = \frac{\int_0^{\ell} m\, \phi_j(x)\, y(x,t)\, dx}{\overline{M}_j}$$

$$\dot{q}_j(t) = \frac{\int_0^{\ell} m\, \phi_j(x)\, \dot{y}(x,t)\, dx}{\overline{M}_j} \tag{17}$$

Since the form of Eq. (12) corresponds to that of a single degree-of-freedom system, $q_j(t)$ may be interpreted as the output response of a modal oscillator in the jth mode.

For forcing functions which are deterministic functions of space and time, Eq. (9) is an appropriate solution. For forcing functions which are random, the mean square response is a desired response solution. If the forcing field is isotropic, homogeneous and stationary, the mean square displacement response of an arbitrary linear elastic beam may be written as

$$\psi_y^2(x) = \sum_j \sum_k \phi_j(x)\, \phi_k(x) \int_0^{\infty} H_j(\omega)\, H_k^*(\omega)\, L_{jk}(\omega)\, d\omega \tag{18}$$

where $H_k^*(\omega)$ is the complex conjugate of $H_k(\omega)$ and $H_j(\omega)$ the jth modal magnification factor

$$H_j(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_j}\right)^2 + i\, 2\zeta_j \frac{\omega}{\omega_j}} \tag{19}$$

The quantity $L_{jk}(\omega)$ is expressed in terms of the spatial cross-spectral density function of the applied excitation $G_f(x, x', \omega)$ as

$$L_{jk}(\omega) = \frac{1}{\overline{M}_j \, \overline{M}_k \, \omega_j^2 \, \omega_k^2} \int_0^{\ell} \int_0^{\ell} \phi_j(x) \, \phi_k(x') \, G_f(x, x', \omega) \, dx \, dx' \tag{20}$$

or in terms of the joint acceptance $j_{jk}^2(\omega)$ as

$$L_{jk}(\omega) = \frac{G_f(x_o, \omega) \, \ell^2}{\overline{M}_j \, \overline{M}_k \, \omega_j^2 \, \omega_k^2} \, j_{jk}^2(\omega) \tag{21}$$

where

$$j_{jk}^2(\omega) = \frac{1}{G_f(x_o, \omega) \, \ell^2} \int_0^{\ell} \int_0^{\ell} \phi_j(x) \, \phi_k(x') \, G_f(x, x', \omega) \, dx \, dx' \tag{22}$$

The term $G_f(x_o, \omega)$ refers to a spectral density of the applied excitation at x_o wherein x_o is selected so that $G_f(x_o, \omega)$ is a maximum. In this way, $j_{jk}^2(\omega)$ will vary from zero to one.

By assuming the $j \neq k$ terms to be negligible in comparison with those for $j = k$, the mean square response becomes

$$\psi_y^2(x) = \sum_j \phi_j^2(x) \int_0^{\infty} \left| H_j(\omega) \right| L_j(\omega) \, d\omega \cdot \tag{23}$$

where the quantity $L_{jk}(\omega) \rightarrow L_j(\omega)$ and appears in the form

$$L_j(\omega) = \frac{1}{\overline{M}_j^2 \, \omega_j^4} \int_0^{\ell} \int_0^{\ell} \phi_j(x) \, \phi_j(x') \, G_f(x, x', \omega) \, dx \, dx' \tag{24}$$

For an excitation perfectly correlated in space and time, $G_f(x, x', \omega) \rightarrow G_o$ and the mean square response becomes

$$\psi_y^2(x) = G_o \sum_j \frac{\left[\int_0^{\ell} \phi_j(x) \, dx \right]^2}{\overline{M}_j^2 \, \omega_j^4} \phi_j^2(x) \int_0^{\infty} \left| H_j(\omega) \right|^2 d\omega \tag{25}$$

then reduces to

$$\psi_y^2(x) = \frac{2 G_o}{\pi m^2} \sum_{j=1, 3, 5 \ldots} \frac{Q_j}{j^2 \, \omega_j^3} \sin^2 \frac{j \pi x}{\ell} \tag{26}$$

for a simply supported beam.

4. BASIC ANALOG CONCEPTS

To conduct an analog simulation study, physically realizable electrical analogs depicting the dynamics of a distributed elastic beam are required. In addition, special attention must be given to procedures for applying random excitation with a specific space-time correlation function.

In this study, passive analog circuits are used to describe the beam. Such circuits appear topologically similar to the physical system; they correspond mechanically to a lumped parameter model and correspond mathematically to a finite-difference model. By requiring force $\sim$ current and velocity $\sim$ voltage, the network impedances become equivalent to mechanical mobility and the resultant analog is called categorically a mobility analog. These networks consist of capacitors, inductors, resistors and transformers — the latter component describing the geometry of the structure. For this particular simulation, capacitors $\sim$ mass, inductors $\sim$ flexibility and resistors $\sim$ damping.

Although various derivation methods may be used (Reference 7) one of the most efficient procedures employs energy relationships associated with both the structure and the circuit (Reference 1). By equating strain energy to the energy associated with an inductor and equating kinetic energy to the energy associated with a capacitor, minimum component circuits may be readily constructed. It is in describing the strain energy expression (which are spatial functions of the lateral deflection) that transformers are required. Such an analog circuit for a difference segment of a simple beam is shown as Figure 3. For notational convenience, albeit a slight departure from traditional circuit symbolisms, inductors L_n are noted by , transformers are represented by the conventional primary P_n to secondary S_n coding wherein the + signs indicate the transformer polarity, and electrical grounds are shown by .

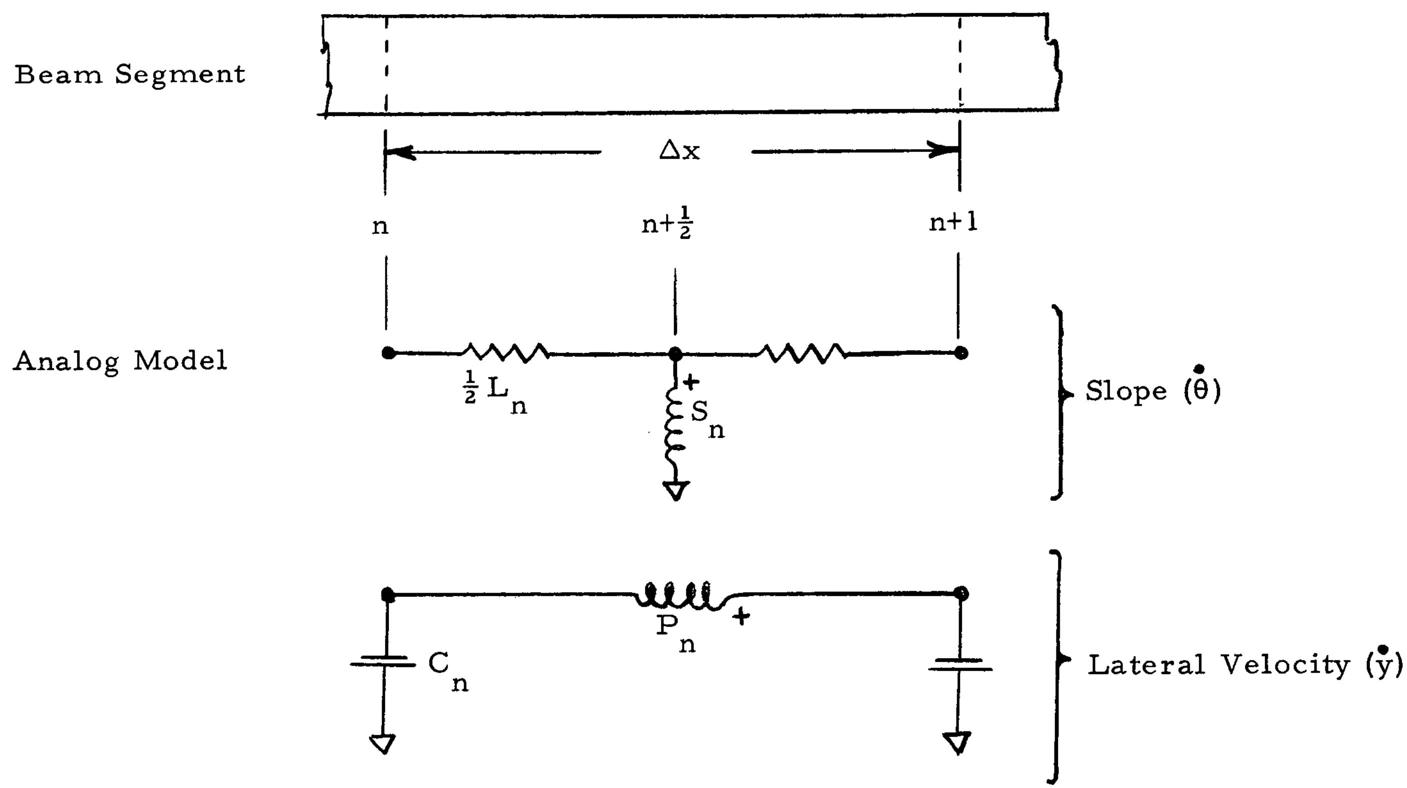


Figure 3. Passive Analog Model for a Difference Segment of Bernoulli-Euler Beam

This difference model consists of two principal circuits, (1) for the slope properties and (2) for the lateral deflection properties, and both are magnetically coupled by the transformer. Current flows in the slope circuit correspond to the internal bending moments between stations n to $n+\frac{1}{2}$ and $n+\frac{1}{2}$ to n. Similarly, current flow in the lateral velocity circuit corresponds to the internal shear between stations n to $n+1$. All $\dot{\theta}$ voltages denote slope velocities at particular spatial positions and, likewise, all $\dot{y}$ voltages denote lateral velocities at specific spatial locations.

By using the scale factor relationships given in Reference 7 as

$$F = \frac{k}{a} I$$

$$M = \frac{k P_\theta}{a} I_\theta$$

$$\dot{y} = \frac{ka}{N} e_{\dot{y}} \tag{27}$$

$$\dot{\theta} = \frac{ka}{N P_\theta} e_{\dot{\theta}}$$

$$t_m = N t_e$$

the individual circuit components become of the form

$$C_n = \left(\frac{a}{N}\right)^2 m\Delta x \Big|_{n-\frac{1}{2}}^{n+\frac{1}{2}}$$

$$L_n = \left(\frac{P_\theta}{a}\right)^2 \frac{\Delta x}{EI} \Big|_{n-\frac{1}{2}}^{n+\frac{1}{2}} \qquad (28)$$

$$T_n = \frac{P_n}{S_n} = \frac{\Delta x}{P_\theta}$$

The notation $\Big|_{n-\frac{1}{2}}^{n+\frac{1}{2}}$ implies the mass and flexibility distributions are integrated between stations $n-\frac{1}{2}$ to $n+\frac{1}{2}$. The scaling constants a, N and P_θ are selected such that the component values are consistent with setting values available on the analog computer.

By assuming $N = 1$ and then substituting the expressions of Eq. (28) into the frequency expression

$$\omega_j^2 = \frac{EI}{m\ell^4} (\lambda_j \ell)^4,$$

it follows the modal frequencies (electrical) in cycles per second are given by

$$f_j(\text{elec}) = \frac{1}{2\pi} \left(\frac{\lambda_j \ell}{n_o}\right)^2 \frac{1}{\frac{P_n}{S_n}\sqrt{L_n C_n}} \qquad (29)$$

where n_o denotes the number of difference segments into which the beam is subdivided. Since $\lambda_j = j\pi/\ell$ for a simply supported beam, the fundamental modal frequency (electrical) for this particular structure becomes

$$f_1(\text{elec}) = \frac{\pi^2}{4} \frac{1}{n_o^2} \frac{1}{\frac{P_n}{S_n}\sqrt{L_n C_n}} \tag{30}$$

By requiring the beam circuit to have its fundamental resonance at $f_1(\text{elec}) = 100$ cps, the various plausible interrelationships between n_o, P_n/S_n, L_n and C_n are shown in Figure 4.

By cascading circuits similar to that of Figure 3, an effective sixteen cell analog model of a simply supported beam is shown as Figure 5. By simply closing the switch in the $\dot{\theta}$ circuit at station 0, the beam boundary conditions convert to those for a simple beam rigidly clamped at both ends. Due to symmetry, only half of the beam is shown wherein station 0 corresponds to $x = 0$ and station 8 to $x = \ell/2$. The transformer whiffle-tree* allows the random excitation to be distributed with a correlation of unity in both space and time. For a sinusoidal input as the applied excitation, the loading would appear as a uniformly distributed harmonic forcing function. By applying such a harmonic excitation and recording the $\dot{y}$ response magnitude at station 8, the plots of Figure 6 are obtained. Since these plots denote the magnitude of velocity to force frequency response functions, they may appropriately be called plots of mobility magnitude.

*The transformers are all center-tapped.

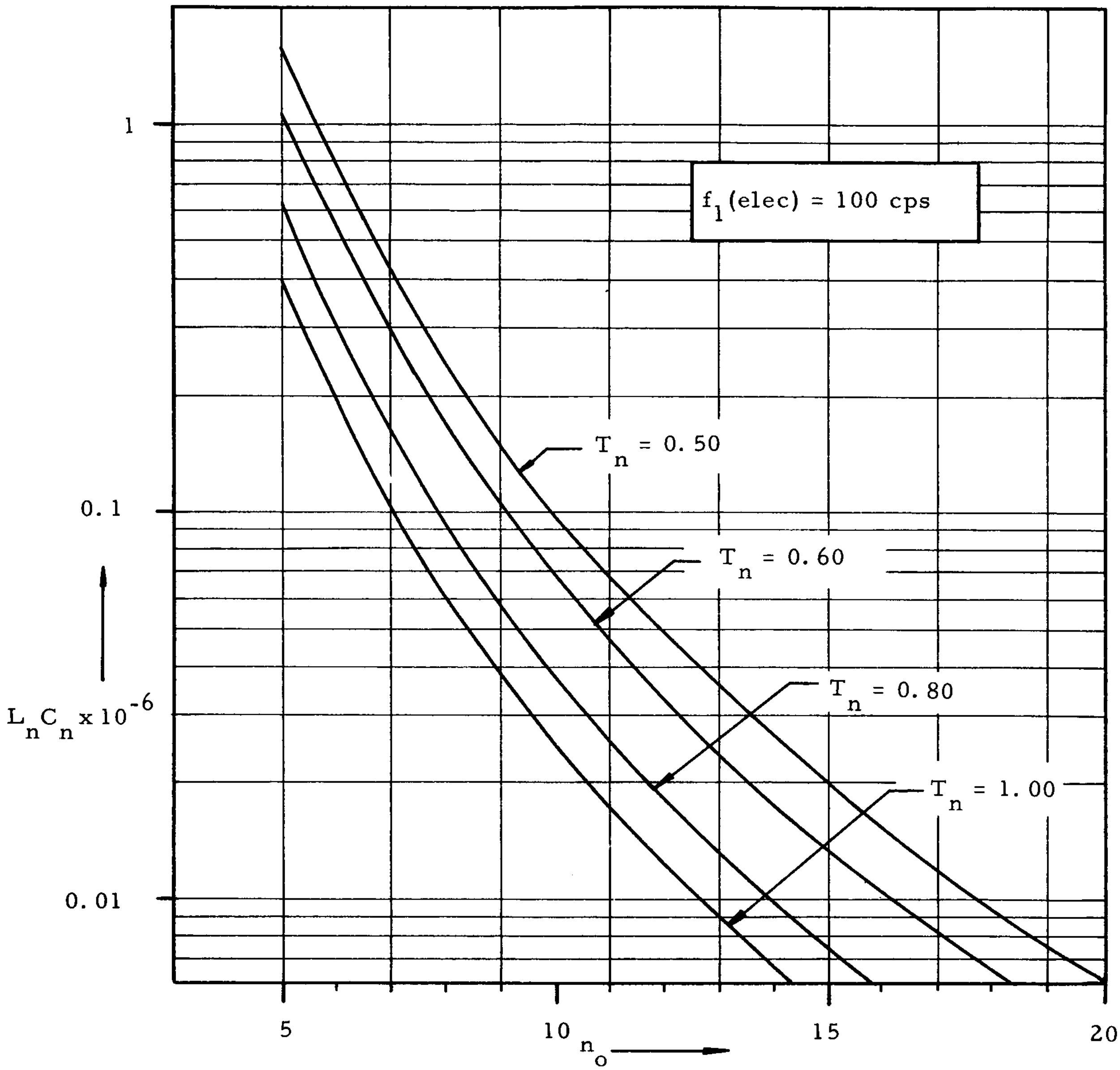


Figure 4. Relationships for the Analog Circuit of a Simply Supported Beam

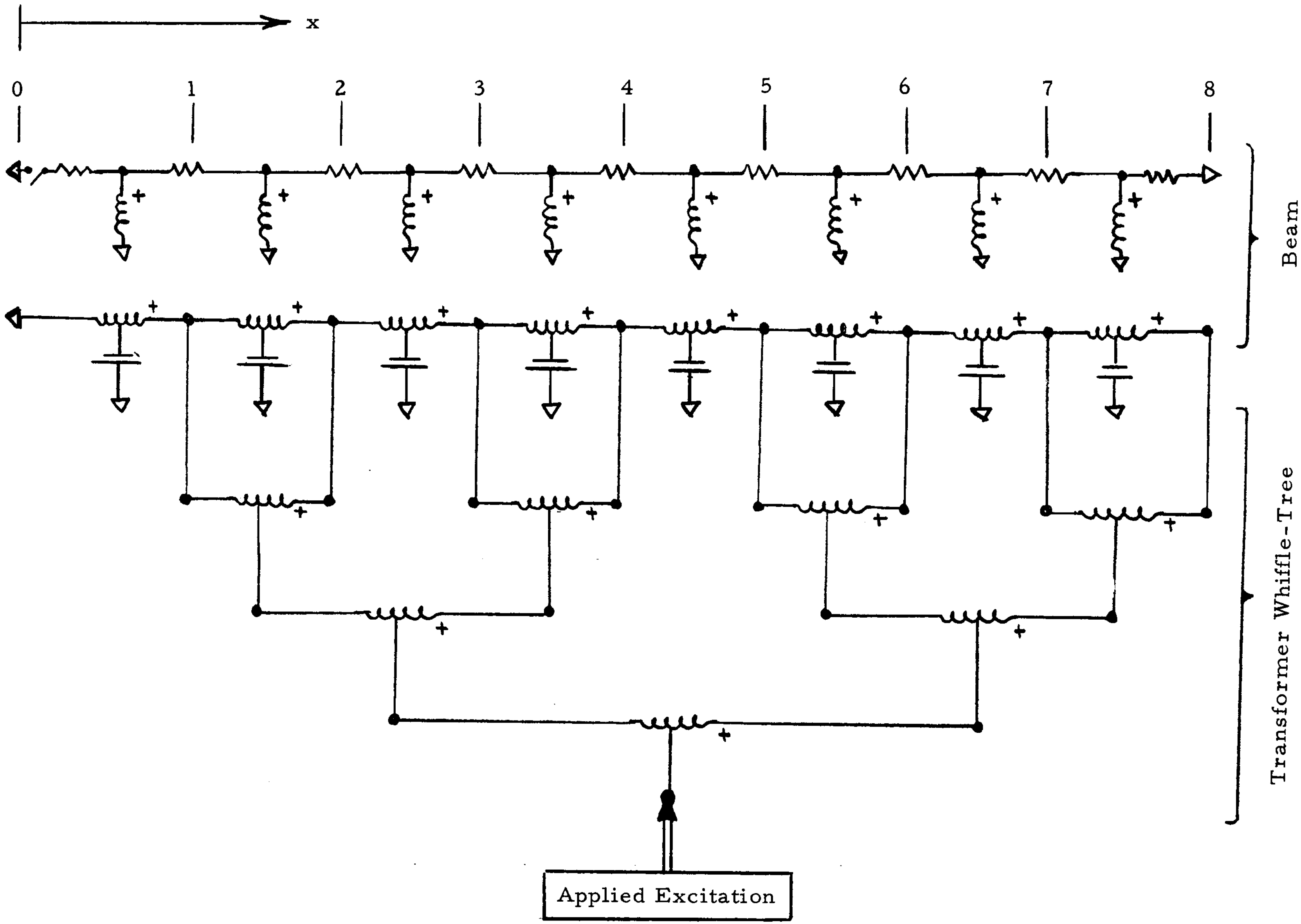


Figure 5. Analog Circuit for the Simple Beam Simulation Study

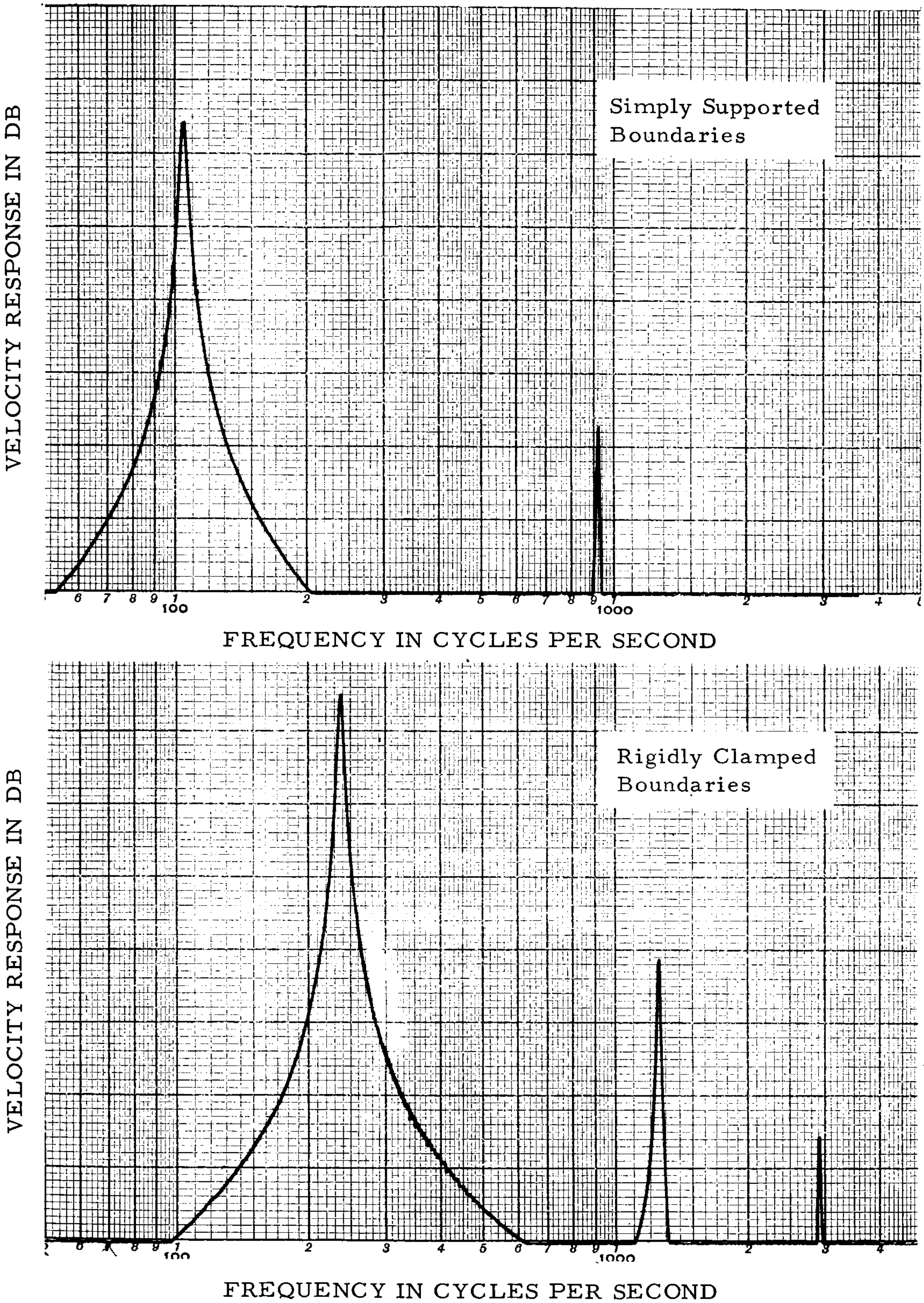


Figure 6. Mobility Magnitude Plots for a Simple Beam

5. ANALOG SIMULATION RESULTS

By using the circuits of Figure 6, three problems of interest are considered

1. the effect of a flexible boundary in bending on the fundamental modal frequency of a beam
2. the RMS profile of a beam subjected to random excitation perfectly correlated in both space and time
3. the SHP response of a beam subjected to random excitation perfectly correlated in both space and time

The first two problems may be treated efficiently by other analytical and/or computational methods whereas the last problem is particularly well adapted to this analog simulation.

The initial problem requires the first eigenvalue to the frequency equation for a simple beam with the following boundary conditions

$$\begin{aligned} y(0,t) &= 0 \qquad & y(\ell,t) &= 0 \\ M(0,t) &= k_\theta\,\theta(0,t) \qquad & M(\ell,t) &= -k_\theta\,\theta(\ell,t) \end{aligned} \tag{31}$$

By applying these boundary restraints to Eq. (10), there results the determinant

$$\begin{vmatrix} 1 & -\frac{2EI}{\ell k_\theta}\lambda_j\ell & 1 \\ \sin\lambda_j\ell & \cosh\lambda_j\ell - \cos\lambda_j\ell & \sinh\lambda_j\ell \\ \cos\lambda_j\ell - \frac{EI}{\ell k_\theta}\lambda_j\ell \sin\lambda_j\ell & \frac{EI}{\ell k_\theta}\lambda_j\ell(\cosh\lambda_j\ell + \cos\lambda_j\ell) + (\sin\lambda_j\ell + \sinh\lambda_j\ell) & \frac{EI}{\ell k_\theta}\lambda_j\ell\sinh\lambda_j\ell + \cosh\lambda_j\ell \end{vmatrix} = 0 \qquad (32)$$

which produces the frequency equation

$$\begin{aligned} &2(\lambda_j\ell)^2 \sin\lambda_j\ell \sinh\lambda_j\ell + \left(\frac{k_\theta\ell}{EI}\right)^2 (1 - \cos\lambda_j\ell \cosh\lambda_j\ell) \\ &+ 2\left(\frac{k_\theta\ell}{EI}\right)\lambda_j\ell(\sin\lambda_j\ell\cosh\lambda_j\ell - \cos\lambda_j\ell\sinh\lambda_j\ell) = 0 \end{aligned} \qquad (33)$$

Modal frequencies are then determined by substituting the $\lambda_j\ell$ solutions into the frequency expression

$$\omega_j^2 = \frac{EI}{m\ell^4}(\lambda_j\ell)^4$$

thus producing the results of Figure 7. Note the ordinate is normalized to the fundamental modal frequency of a simple supported beam, an easily calculable quantity. These same results are obtained from the analog circuit simply by tuning the oscillator to the fundamental resonant frequency of the network.

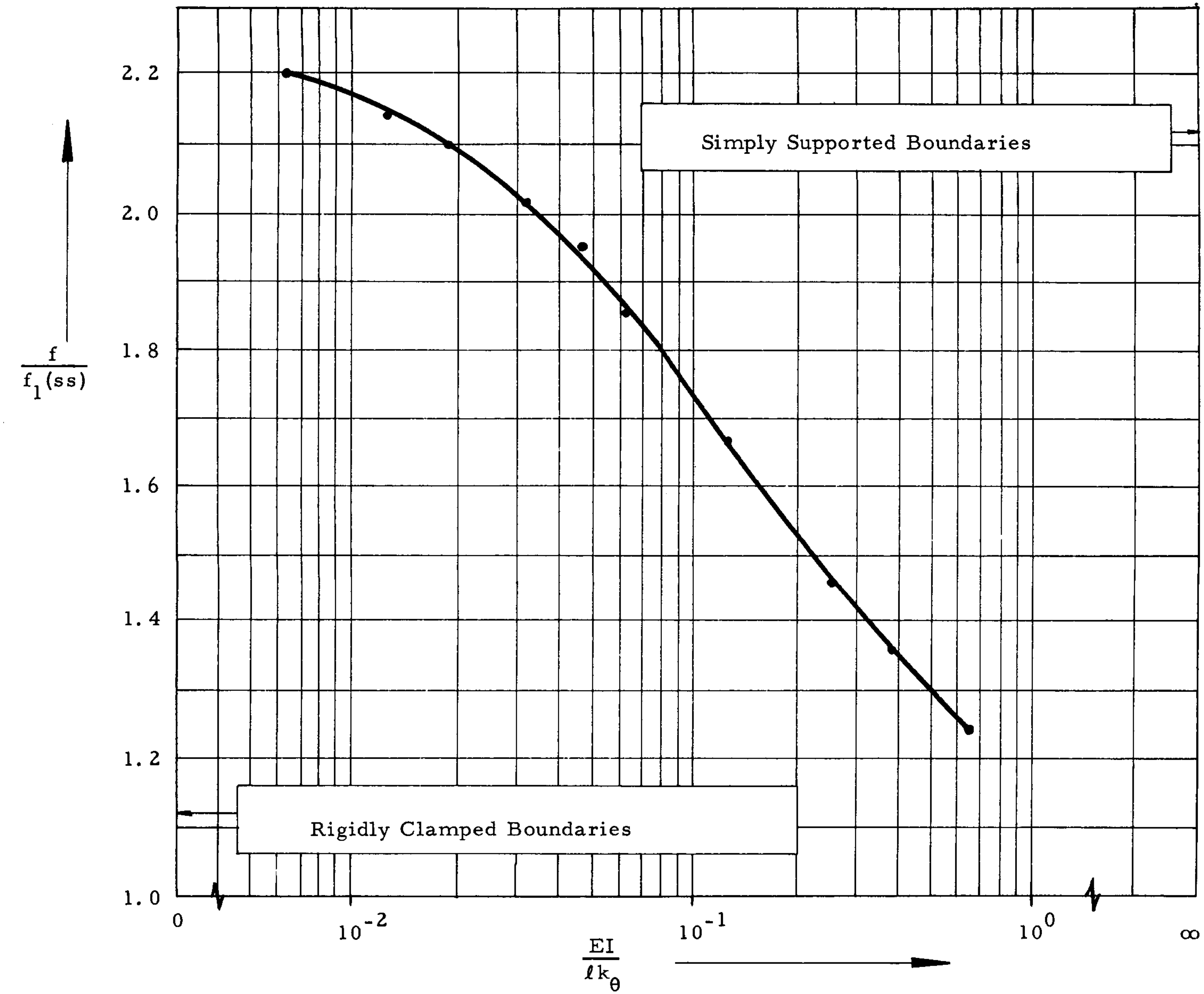


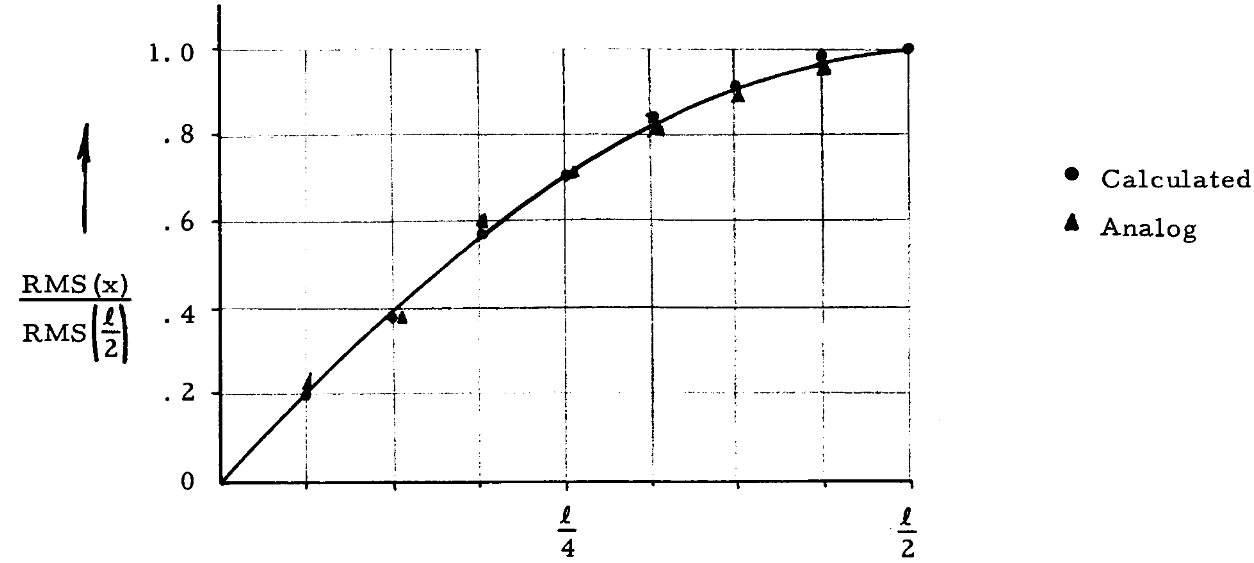
Figure 7. Effect of Flexible Restraint in Bending on the Fundamental Modal Frequency of a Simple Beam

The second problem resolves to evaluating the analytical expression for the mean square response given by Eq. (18) as

$$\psi_y^2(x) = \sum_j \sum_k \phi_j(x)\, \phi_k(x) \int_0^{\ell} H_j(\omega)\, H_k^*(\omega)\, L_{jk}(\omega)\, d\omega$$

For the random forcing function considered here, that is a forcing function correlated perfectly in space and time, this expression may be evaluated with relative ease. For forcing functions with variable spatial correlation functions and for structures with flexible boundaries, this expression is far more tedious to evaluate precisely and various approximations subsequently are made. In the analog simulation, one simply records the output response of a true RMS meter at the spatial position of interest. Such RMS results are shown as Figure 8 wherein the ordinate is normalized to the RMS response at the mid-span of the beam. Both analog and calculated values agree closely for both the simply supported and the rigidly clamped boundaries.

For the third problem, a purely analytical attempt is virtually intractable and a simulation study thus is in order. By means of the analog circuit shown in Figure 5 and using the format of Reference 3, peak response statistics to stationary random excitation may be collected and readily examined. Typical examples of the input excitation at any point on the beam and the response at $x = \ell/2$ are shown as Figure 9.



RMS Ratio for a Simply Supported Beam Subjected to a Stationary White Noise Force Excitation Perfectly Correlated in Space and Time

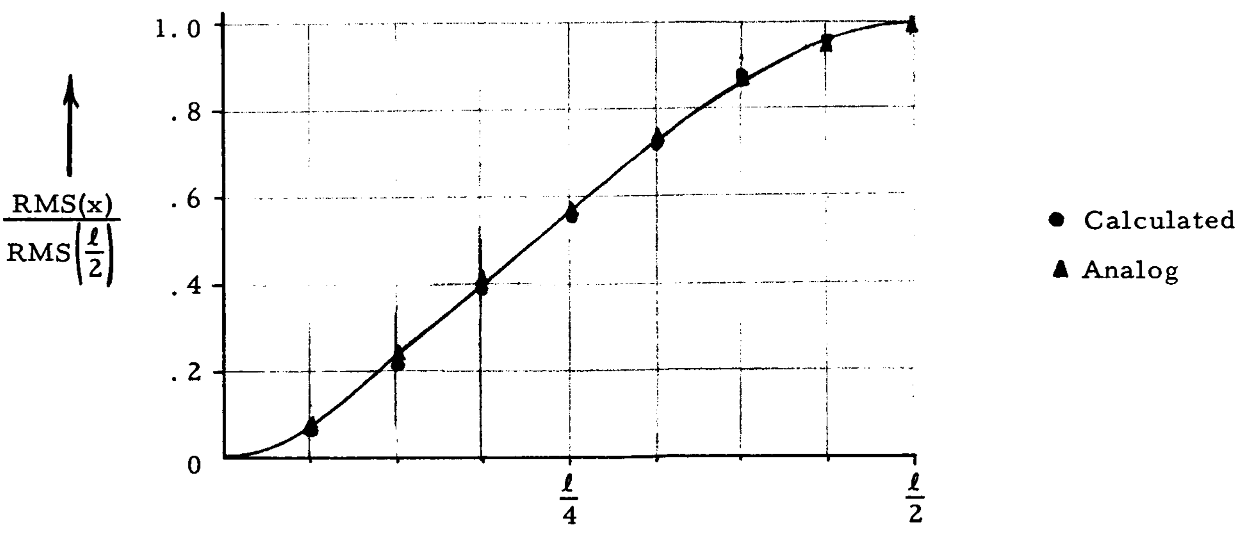
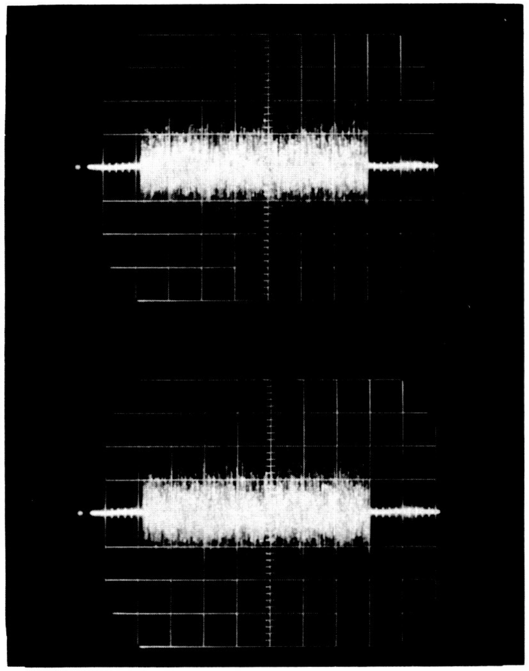
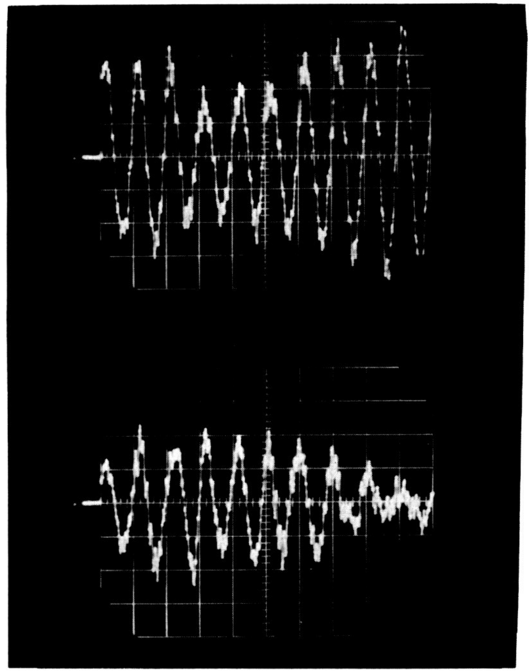


Figure 8. RMS Ratio for a Simply Supported Beam Subjected to a Stationary White Noise Force Excitation Perfectly Correlated in Space and Time



Stationary White Noise Excitation



Typical Response

Figure 9. Typical Response of a Simple Beam to Stationary White Noise Excitation Perfectly Correlated in Space and Time

For this SHP problem, it is remembered that the beam has achieved stationarity in its response and one measures the highest response (both positive and negative) of the system which occurs within the sampling time interval T. After accumulating 100 such readings, these data are arranged to form histograms such that $P_M(|\beta| \leq \beta_o)$ values can be estimated. By repeating this procedure for each preselected T value, plots similar to Figure 10 may be developed for any desirable probability value of $|\beta| \leq \beta_o$. The curve for the average absolute peak response is found to be approximately the same as $P_M(|\beta| \leq \beta_o) = 0.50$.

The peak response statistics are shown as β plots of $|\beta|$ versus the dimensionless time parameter $f_1 T/Q_1$. To form the β ratios, the absolute maximum response values are normalized by the RMS response of the system to stationary white noise. The curves in Figure 10, although consistently higher, are similar in form to those for a single degree-of-freedom system and graphically display the time dependency of $|\beta|$. Since there appears to be no consistent difference between the data for a simply supported beam and that for a rigidly clamped beam, one concludes the $|\beta|$ response essentially is independent of boundary conditions for a distributed elastic beam with rectangular geometry.

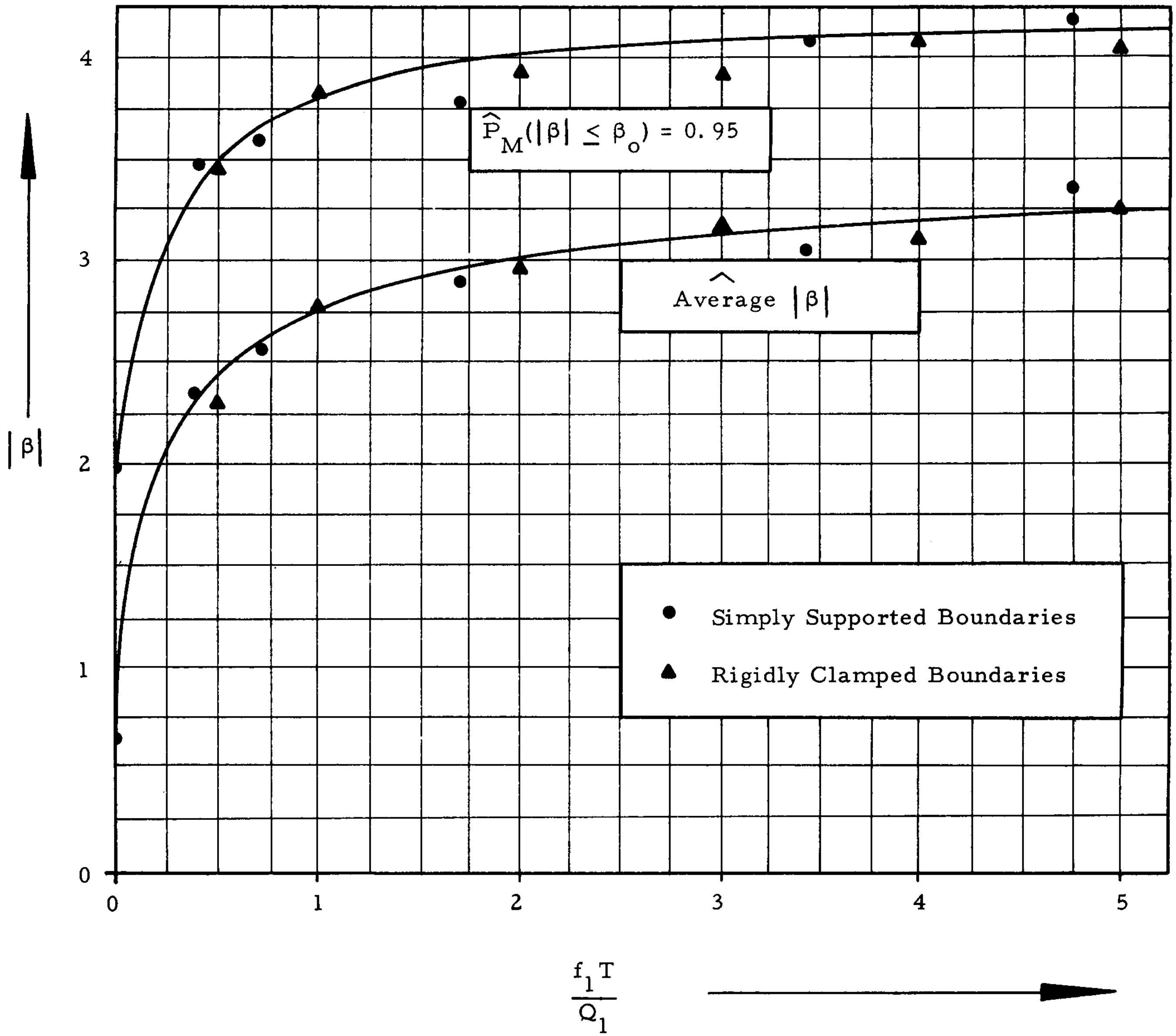


Figure 10. Maximum Response of a Simple Beam to Stationary White Noise Excitation Perfectly Correlated in Space and Time

6. CONCLUDING REMARKS

The analog simulation methods mentioned here are particularly well suited for parametric studies wherein the physical systems are defined by partial differential equations and the input excitation is stochastic. Typical problems categorically include those dealing with distributed structures and random excitation, transient thermal analyses of structures, control system—elastic vehicle dynamics, vibration attenuation characteristics of interstaging structure and viscoelastic response characteristics. By coupling these analog concepts with conventional analysis procedures common to structural dynamics and circuit analyses, impedance relationships naturally evolve so that additional insight into the dynamic behavior of multi-degree-of-freedom systems is obtained somewhat as a by-product.

The maximum response statistics presented as plots of $|\beta|$ versus $f_1 T/Q_1$ show the time dependency of the maximum response for an arbitrary distributed structure. A common application for SHP results involves structural design in a random environment wherein a maximum response criteria is appropriate. It is clear from such curves that the $|\beta|$ ratio, for a constant probability, changes as a function of exposure time in the random environment. This implies a maximum response solution (or an equivalent thereof) must be achieved in order to have a valid design for such an environment. The alternative is to employ far more costly experimental design procedures or use gross "over-design" factors.

By comparing the results of Figure 10 with equivalent results for a square plate (Reference 4), the $|\beta|$ plots for both the beam and the plate appear nearly identical. In both cases, moreover, the data are noted to be independent of boundary conditions. Since the practical implication of such results are noteworthy, additional selective experimental and theoretical work should be considered before general conclusions are definitively stated. In the absence of conflicting remarks, however, the empirical stochastic solutions of Figure 10 may be used in the design of both beams and plates of rectangular geometry subjected to random excitation.

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