EXTENDED DYNAMICAL SYSTEMS AND STABILITY THEORY

by

J. K. Hale[1] and E. F. Infante[2]

Center for Dynamical Systems
Brown University
Providence, Rhode Island

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The term dynamical system, as used in this note, is used to describe a one-parameter family of operators with certain properties defined in an appropriate space and is a natural generalization of differential equations, functional differential equations and certain partial differential equations. Zubov[1] has shown that the stability theorems of Liapunov as well as their converses are applicable to dynamical systems. These results play an important role in theoretical studies of stability but, unfortunately, are not easy to apply to particular problems.

For ordinary differential equations and functional differential equations LaSalle[2] and Hale[3] have shown that the limiting sets of trajectories which lie in a compact subset of the space are contained in the largest invariant set where the derivative of the Liapunov function V vanishes. The purpose of the present paper is to extend this result and other related stability results to dynamical systems. In this manner the invariance principle and the stability theorems obtained are also applicable to a large class of partial differential equations. The natural setting for the study of dynamical systems is a Banach space, which can be considered as the space of continuous functions over a finite interval in the case of functional differential equations, as the Euclidean space in the case of differential equations, and as a Sobolev space for certain hyperbolic partial differential equations.

Let R^+ denote the interval $[0,\infty)$ and $\mathscr{A}$ a Banach space with $\|\varphi\|_{\mathscr{A}}$ the norm of an element φ of this space.

Definition 1. We say u is a dynamical system on a Banach space $\mathscr{B}$ if u is a continuous mapping of $R^+ \times \mathscr{B}$ into $\mathscr{B}$, $u(t,\varphi)$ is uniformly continuous in t for t,φ in bounded sets, $u(0,\varphi) = \varphi$ and $u(t+\tau,\varphi) = u(t,u(\tau,\varphi))$ for all $t,\tau \geq 0$, φ in $\mathscr{B}$. The positive orbit $0^+(\varphi)$ through φ in $\mathscr{B}$ is defined as $0^+(\varphi) = \bigcup_{t\geq 0} u(t,\varphi)$. We say φ is an equilibrium point if $0^+(\varphi) = \varphi$.

Zubov[1] has discussed systems of this type, without the uniform continuity condition on bounded sets, and referred to them as generalized dynamical systems. In the theory of dynamical systems on n-dimensional vector spaces the concept of invariant sets is basic since the limits of orbits are invariant sets. Zubov defines an invariant set of his generalized dynamical system as a set M such that, for any φ in M, $0^+(\varphi)$ belongs to M. Since u is defined only on R^+ this appears at first sight to be a reasonable definition; however, this definition does not impart any special significance to the limit set of an orbit and appears unreasonable since it generally occurs that trajectories having limits can be used to define functions on $(-\infty,\infty)$. We shall therefore modify the definition of invariant set.

If u is a dynamical system on $\mathscr{B}$, then one can be assured that $0^+(\varphi)$ has a nonempty limit set if $0^+(\varphi)$ belongs to a compact subset of $\mathscr{B}$. In ordinary differential equations and functional differential equations it is possible to show that $0^+(\varphi)$ belonging to a bounded set implies $0^+(\varphi)$ belongs to a compact set (see, for example ref. 3) and thus the limit set is nonempty. However, for many partial differential equations, this is not the case. On the other hand, for certain partial differential equations bounded orbits in $\mathcal{B}$ will belong to a compact set of a larger Banach space $\mathcal{C}$.

It is this latter property which we wish to exploit in detail. More specifically, if we know that every bounded orbit in $\mathcal{B}$ belongs to a compact set in $\mathcal{C}$, then we can discuss the limit of the orbit in $\mathcal{C}$ (thus extending the dynamical system) and as a consequence hope to obtain more specific information about trajectories than would be possible by remaining only in $\mathcal{B}$. These remarks provide the motivation for the following discussion. The reader should contrast this approach with the one of Auslander and Seibert[4] in which it is assumed that the space $\mathcal{B}$ is locally compact.

Let $\mathcal{B}, \mathcal{C}$ be Banach spaces, $\mathcal{B} \subset \mathcal{C}$ and let there exist a constant $K > 0$ such that $\|\varphi\|_{\mathcal{C}} \leq K\|\varphi\|_{\mathcal{B}}$.

Definition 2. Let u be a dynamical system on $\mathcal{B}$. Let $\mathcal{B}^*$ be the set of φ in $\mathcal{C}$ such that there is a sequence φ_n in $\mathcal{B}$ and a function $u^*(t,\varphi)$ in $\mathcal{C}$ for t in R^+, such that $\|\varphi_n-\varphi\|_{\mathcal{C}} \to 0$, $\|u(t,\varphi_n)-u^*(t,\varphi)\|_{\mathcal{C}} \to 0$ as $n \to \infty$ uniformly on compact subsets of R^+. We refer to the function $u^*: R^+ \times \mathcal{B}^* \longrightarrow \mathcal{B}^*$ as the extension of the dynamical system u to $\mathcal{B}^*$ or simply as the extended dynamical system.

The function u^* is clearly an extension of u. In fact, if φ is in $\mathcal{B}$, then there exists a sequence φ_n in $\mathcal{B}$ such that $\|\varphi_n - \varphi\|_{\mathcal{B}} \to 0$ (and therefore $\|\varphi_n - \varphi\|_{\mathcal{C}} \to 0$) as $n \to \infty$. This fact and the continuity of u implies $\|u(t,\varphi_n) - u(t,\varphi)\|_{\mathcal{B}} \to 0$ as $n \to \infty$ and therefore $\|u(t,\varphi_n) - u(t,\varphi)\|_{\mathcal{C}} \to 0$ as $n \to \infty$. Thus $u^*(t,\varphi) = u(t,\varphi)$ for φ in $\mathcal{B}$. Furthermore it is easy to prove

Lemma 1. The function $u^*(t,\varphi)$ is continuous in t and $u^*(0,\varphi) = \varphi$, $u^*(t+\tau,\varphi) = u^*(t,u^*(\tau,\varphi))$ for t, τ in R^+ and φ in $\mathcal{B}^*$.

We now give a definition of invariance of a different nature from the one given by Zubov:

Definition 3: A set M in $\mathcal{B}^*$ is an invariant set of the dynamical system if for each φ in M there is a function $U(t,\varphi)$ defined and in M for t in $(-\infty,\infty)$ such that, for any σ in $(-\infty,\infty)$, $u^*(t,U(\sigma,\varphi)) = U(t+\sigma,\varphi)$ for all t in R^+.

Definition 4: For any φ in $\mathcal{B}$, the ω-limit set $\Omega(\varphi)$ of the orbit through φ is the set of ψ in $\mathcal{C}$ such that there is a nondecreasing sequence $\{t_n\}$, $t_n > 0$, $t_n \to \infty$ as $n \to \infty$ such that $\|u(t_n,\varphi) - \psi\|_{\mathcal{C}} \to 0$ as $n \to \infty$.

It should be noted that sets are invariant according to the above definition relative to the interval $(-\infty,\infty)$ and that the ω-limit set of an orbit is obtained relative to convergence in $\mathcal{C}$ and not in $\mathcal{B}$. With these definitions it is then possible to prove the fundamental

Lemma 2: Let φ in $\mathcal{B}$ be such that $O^+(\varphi)$ belongs to a bounded set of $\mathcal{B}$ and a compact subset of $\mathcal{E}$. Then the ω-limit set $\Omega(\varphi)$ of the orbit through φ is a nonempty, compact, connected set in $\mathcal{B}^*$, invariant with respect to the extended dynamical system and $\mathrm{dist}_{\mathcal{E}}\,(u(t,\varphi),\Omega(\varphi)) \to 0$ as $t \to \infty$.

Proof: Since $O^+(\varphi)$ belongs to a compact subset of $\mathcal{E}$, it is clear that $\Omega(\varphi)$ is nonempty and belongs to a compact subset of $\mathcal{E}$. We shall show below that it belongs to $\mathcal{B}^*$. Suppose that ψ in $\Omega(\varphi)$ is given and that $\{t_n\}$, nondecreasing, $t_n \geqq 0$, $t_n \to \infty$ as $n \to \infty$ satisfies $\|u(t_n,\varphi)-\psi\|_{\mathcal{E}} \to 0$ as $n \to \infty$. For a given τ in $[0,\infty)$ there exists an $n_o(\tau)$ such that $t_n - \tau \geqq 0$ for $n \geqq n_o(\tau)$ and it is therefore meaningful to consider the sequence $u(t+t_n,\varphi)$; $n \geqq n_o(\tau)$, t in $[-\tau,\tau]$. By hypothesis there exists an M such that $\|u(t,\varphi)\|_{\mathcal{B}} \leqq M$ for all t in $[0,\infty)$. Thus $\|u(t,\varphi)\|_{\mathcal{E}} \leqq KM$ for $n \geqq n_o(\tau)$, t in $[-\tau,\tau]$. Also, since $u(t,\varphi)$ is uniformly continuous in t for t,φ in bounded sets, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|u(t+s+t_n,\varphi)-u(t+t_n,\varphi)\|_{\mathcal{E}} \leqq K\|u(s,u(t+t_n,\varphi))-u(0,u(t+t_n,\varphi))\|_{\mathcal{B}} \leqq \epsilon$$

for $n \geqq n_o(\tau)$, $0 \leqq s \leqq \delta$. This proves that the sequence $\{u(t+t_n,\varphi)\}$, t in $[-\tau,\tau]$ is uniformly bounded and equicontinuous in $\mathcal{E}$. Since this sequence belongs by hypothesis to a compact subset of $\mathcal{E}$, Ascoli's theorem implies the existence of a subsequence which we again label by t_n such that it converges uniformly on $[-\tau,\tau]$; that is, there exists a function $U(t,\varphi)$ continuous in t such that $\lim_{n\to\infty} \|U(t,\varphi)-u(t+t_n,\varphi)\|_{\mathcal{B}} = 0$ uniformly on $[-\tau,\tau]$. Obviously $U(0,\varphi) = \psi$. Letting now $\tau = 1,2,\ldots$ successively and using the familiar triangularization procedure we determine a subsequence which is relabled by t_n and a continuous function $U(t,\varphi)$ defined for t in $(-\infty,\infty)$ such that $\lim_{n\to\infty} \|U(t,\varphi)-u(t+t_n,\varphi)\|_{\mathcal{B}} = 0$ uniformly on compact subsets of $(-\infty,\infty)$. Applying this in particular to $[0,\infty)$ we obtain that ψ belongs to $\mathcal{B}^*$. Furthermore, it is clear that $U(t,\varphi)$ is in $\Omega(\varphi)$.

Let now σ be an arbitrary real number in $(-\infty,\infty)$. We claim that $U(t+\sigma,\varphi) = u^*(t,U(\sigma,\varphi))$, $t \geqq 0$. For this particular σ We have $\lim_{n\to\infty} \|u(\sigma+t_n,\varphi)-U(\sigma,\varphi)\|_{\mathcal{B}} = 0$ and $\lim_{n\to\infty} \|u(t,u(\sigma+t_n,\varphi)) - U(t+\sigma,\varphi)\|_{\mathcal{B}} = 0$ uniformly on compact subsets of $[0,\infty)$. But this is precisely the manner in which $u^*(t,U(\sigma,\varphi))$ was defined. This shows that $\Omega(\varphi)$ is invariant with respect to the extended dynamical system. It is clear that $\Omega(\varphi)$ is connected.

We now show that $\Omega(\varphi)$ is closed. Let ψ_n in $\Omega(\varphi)$ be such that $\psi_n \to \psi$ as $n \to \infty$. Then for any ϵ-neighborhood of ψ in $\mathcal{B}$ there exists a t_ϵ, $t_\epsilon \to \infty$ as $\epsilon \to 0$ such that $\|u(t_\epsilon,\varphi) - \psi\|_{\mathcal{B}} < \epsilon$. Hence closure.

Finally, assume there exists a sequence $\{t_n\}$, nondecreasing, $t_n \to \infty$ as $n \to \infty$ and an $\alpha > 0$ such that $\|u(t_n,\varphi)-\psi\|_{\mathcal{B}} \geqq \alpha$

for all ψ in $\Omega(\varphi)$. By assumption $\{u(t_n,\varphi)\}$ belongs to a compact set of $\mathcal{B}$ and therefore there exists a subsequence which converges to $\bar{\psi}$ in $\mathcal{B}$. But then $\bar{\psi}$ belongs to $\Omega(\varphi)$ by definition, contradicting the assumption and the proof is complete.

We now define the concepts of stability with respect to these spaces:

Definition 5: If zero is an equilibrium point, then we say that zero is stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\varphi\|_{\mathcal{B}} < \delta$ implies $\|u(t,\varphi)\|_{\mathcal{B}} < \epsilon$ for all $t \geq 0$. If, in addition, there exists a $b > 0$ such that $\|\varphi\|_{\mathcal{B}} < b$ implies $\|u(t,\varphi)\|_{\mathcal{B}} \to 0$ as $t \to \infty$ then the origin is said to be asymptotically stable $(\mathcal{B}, \mathcal{B})$. The origin is called unstable if it is not stable.

It is remarked that asymptotic stability is defined by taking limits in $\mathcal{B}$, as is to be expected from the definition of ω-limit sets.

If V is a continuous scalar functional defined on $\mathcal{B}$, we define

$$\dot{V}(\varphi) = \overline{\lim_{t \to 0^+}} \frac{1}{t}[V(u(t,\varphi))-V(\varphi)].$$

Following LaSalle[5] we give

Definition 6: We say a scalar functional V is a Liapunov functional on a set G in $\mathcal{B}$ if V is continuous and bounded below on G and $\dot{V}(\varphi) \leq 0$ for φ in G. We define sets R, M as follows:

$$R = \{\varphi \text{ in } \mathcal{E} : \text{there exists } \{\varphi_n\} \text{ in } G \text{ with } \lim_{n\to\infty} \|\varphi_n - \varphi\|_{\mathcal{E}} = 0$$

$$\text{and } \lim_{n\to\infty} \dot{V}(\varphi_n) = 0\},$$

and M is the largest set in R which is invariant with respect to the extended dynamical system.

With the above definitions and with the fundamental Lemma 2 it is now possible to prove stability theorems which are direct generalizations of those given for functional differential equations and differential equations[3,4].

Theorem 1: Suppose every orbit $O^+(\varphi)$ which is in a bounded set in $\mathcal{B}$ also belongs to a compact set in $\mathcal{E}$. If V is a Liapunov functional on G and an orbit $O^+(\varphi)$ belongs to G and is in a bounded set in $\mathcal{B}$, then $u(t,\varphi) \to M$ in $\mathcal{E}$ as $t \to \infty$.

Corollary 1: Suppose that every orbit which belongs to a bounded set in $\mathcal{B}$ also belongs to a compact set in $\mathcal{E}$. Assume V is a continuous scalar functional defined on $\mathcal{B}$, $S_\rho = \{\varphi \text{ in } \mathcal{B}: V(\varphi) < \rho\}$ and let G be S_ρ or a component of S_ρ. If V is a Liapunov functional on G and any orbit remaining in G belongs to a bounded set in $\mathcal{B}$, then φ in G implies $u(t,\varphi) \to M$ in $\mathcal{C}$ as $t \to \infty$.

Note, in this corollary, that if zero is in G and M consists of only the point zero, then the origin is an "attractor" but we have not shown it to be stable. The following result gives conditions that insure stability. The part of the corollary which does not follow directly from Theorem 1 is proved as in the usual Liapunov theory.

Corollary 2: If the conditions of Corollary 1 are satisfied and V is a continuous positive definite functional on G, then zero is stable. If, in addition, $M = \{0\}$, then zero is asymptotically stable $(\mathcal{B}, \mathcal{C})$. If, in addition, $\dot{V}$ is negative definite, then zero is asymptotically stable $(\mathcal{B}, \mathcal{B})$.

The stronger form of asymptotic stability given in the last part of this corollary should be noted. Unfortunately, for any given system it is very difficult to construct a Liapunov functional with these characteristics.

Theorem 2: Suppose that every orbit which is in a bounded set in $\mathcal{B}$ also belongs to a compact set in $\mathcal{C}$. Let zero be an equilibrium point contained in the closure of an open set U and let N be a neighborhood of zero. Assume that

(i) V is a Liapunov functional on $G = U \cap N$,

(ii) $M \cap G$ is either the empty set or is zero,

(iii) $V(\varphi) < \eta$ on G when $\varphi \neq 0$

(iv) $V(0) = \eta$ and $V(\varphi) = \eta$ when φ is in that part of the boundary of G inside N.

Then zero is unstable. More precisely, if N_0 is a bounded neighborhood of zero properly contained in N, then $\varphi \neq 0$ in $G_0 = G \cap N_0$ implies that there exists a $\tau > 0$ such that $u(\tau, \varphi)$ belongs to the boundary of N_0.

The proofs of these theorems and corollaries follow closely those previously given for ordinary differential equations[5].

The lemmas and theorems displayed above are in terms of two spaces, $\mathcal{B}$ and $\mathcal{L}$. If the space $\mathcal{B}$ is a Hilbert space then a considerable simplification occurs.

Lemma 3: If $\mathcal{B}$ is a Hilbert space and $\mathcal{L}$ is a Banach space, $\mathcal{B} \subset \mathcal{L}$, $\|\varphi\|_{\mathcal{L}} \leq K\|\varphi\|_{\mathcal{B}}$ for some constant $K > 0$, then the unit ball in $\mathcal{B}$ is closed in $\mathcal{L}$.

This lemma is a direct consequence of the Banach-Saks Theorem.

It follows that if $\mathcal{B}$ and $\mathcal{L}$ are Hilbert spaces, then the set $\mathcal{B}^*$ in Definition 2 is the same as $\mathcal{B}$ and therefore the extended dynamical system is the same as the original dynamical system. Therefore, the ω-limit sets will belong to $\mathcal{B}$ but the convergence of $u(t, \varphi)$ to its ω-limit set is in the sense of the topology of $\mathcal{L}$ and not, in general in $\mathcal{B}$. These remarks play an important role in the applications to certain partial differential equations.