Periodic Points of Diffeomorphisms

by

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I. Introduction:

In [1] Artin and Mazur prove that there is a dense set in the space of C^k mappings of a compact manifold into itself such that for each member of this set the number of fixed points under iteration grows at most exponentially. This estimate allows one to define an analytic ζ-function associated to the diffeomorphism that measures the number of fixed points of the diffeomorphism under iteration.

The theorem of Artin and Mazur gives no indication as to whether or not a specific diffeomorphism satisfies such an estimate.

In this note we announce (Theorem 1) that the number of fixed points of the general class of diffeomorphisms recently introduced by Smale [2,3] grows at most exponentially under iteration. It should be noted that this new theorem is neither contained in nor contains the theorem of Artin and Mazur.

The method of proof is quite simple. One need only show that the size of the domain where there is a unique fixed point of the diffeomorphism decreases at most exponentially by using an estimate on the domain of validity of the implicit function theorem. The complexity arises only from the necessity of checking uniformity at each step.

II. Notation and Theorem.

Let M be a compact C^2-Riemannian manifold and suppose that $f: M \to M$ is a diffeomorphism of M. A closed invariant set

$\Lambda \subset M$ is said to have a hyperbolic structure if the tangent bundle $T_\Lambda M$ of M restricted to Λ has a continuous invariant splitting $T_\Lambda M = E^u + E^s$ under df such that

$$df: E^u \to E^u \quad ; \quad df: E^s \to E^s$$

$$\|df^n(x)(u)\| < C\lambda^n\|u\|$$

$$\|df^n(x)(v)\| > C^{-1}\lambda^{-n}\|v\|$$

for some fixed constants $C > 0$, $0 < \lambda < 1$, where $x \in \Lambda$, $v \in E^s_x$, $u \in E^u_x$ and $n \in Z^+$.

If f is a diffeomorphism of M and $x \in M$ then x is called a wandering point if there exists a neighborhood U of x such that $\bigcap_{n \in Z} f^n(U) = \varphi$. A point x of M is called a non-wandering point if it is not a wandering point. Clearly the set of nonwandering points of f forms a compact invariant subset of M.

The class of diffeomorphisms introduced by Smale in [2,3] is the class of diffeomorphisms of M with a hyperbolic structure on the set of nonwandering points of f. This class of diffeomorphisms is sufficiently general to include all known examples of diffeomorphisms with globial stability properties (see [3] for a detailed discussion).

Let $N_n(f)$ be the number of fixed points of f^n. Then our main result is

Theorem 1. *If f is a C^2-diffeomorphism of M into itself with a hyperbolic structure on the set of nonwandering points of f then there exists a constant $k > 0$ such that*

$$N_n(f) \leq k^n \text{ \underline{for} } n \in Z^+.$$

III. Outline of the Proof:

In what follows $|\cdot|$ will denote the usual Euclidean norm in E^m with respect to a fixed basis. The following lemma follows easily from the implicit function theorem given in [4], page 12.

Lemma 1: Let φ_n, $n \in Z^+$, be a C^2 map from the closed ball B_a of radius a about the origin in E^m into itself with $\varphi_n(0) = 0$. Let the supremum of the modulus of the second partials of φ be less than b^n on B_a. Let $|(d\varphi_n(0)-I)| \leq c^n$ and $|(d\varphi_n(0)-I)^{-1}| \leq c^n$. Then there exists a constant $d = d(a,b,c)$ such that φ has a unique point in the sphere of radius d^n about the origin.

Let (V_i, y_i) and (U_i, x_i), $i = 1,\ldots r$ be a finite number of coordinate systems for M such that $V_i \supset \overline{U}_i$, $\bigcup_1^r U_i \supset M$ and $x_i = y_i|U_i$. Consider the sets $y_i(V_i)$ and $x_i(U_i)$ in E^m. There exists a constant $a > 0$ such that each point of $x_i(\overline{U}_i)$ is contained in a sphere of radius a completely contained in $y_i(V_i)$. We shall count the number of fixed points of f^n in each $x_i(\overline{U}_i)$.

Let $\|\cdot\|$ denote the norm induced in $y_i(V_i)$ by the metric on M.

Lemma 2: Let x_o be a fixed point of $f^n, x_i o f^n o x_i^{-1}$, and let A be the Jacobian matrix of f^n evaluated at x, then there exist constants N and $c > 0$ such that

$$|(A-I)| \leq c^n \text{ and } |(A-I)^{-1}| \leq c^n$$

for $n \geqq N$.

Comments on the Proof of Lemma 2.

At this point the strong uniformity of the hyperbolic structure on the set of nonwandering points is used. Because the set of nonwandering points is compact, and the splitting is continuous there exists a constant $e \geqq 1$ such that the norm $\|\cdot\|$ and the norm $|||\cdot|||$ defined by the coordinates such that A has the form $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ where A_j; $j = 1,2$, is the representation of the mapping df on $E^u_{x_0}$ and $E^s_{x_0}$ respectively satisfies the condition

$$e^{-1}|||\cdot||| \leqq \|\cdot\| \leqq e|||\cdot|||.$$

With this uniformity at hand the rest of the lemma follows by standard matrix methods.

Since the total volume of $x_i(\overline{U}_i)$ is finite and fixed points of f^n in $x_i(\overline{U}_i)$ can be covered by disjoint balls of radius $\frac{d^n}{3}$ the required estimate follows from the above two lemmas.

It seems likely that the general outline given above can be used to give a similar estimate for the number of periodic orbits for a flow on M having a hyperbolic structure on the set of nonwandering points. The author is presently working on this problem.

References

[1] Artin, M., and B. Mazur, On Periodic Points, _Ann. of Math._ 81 (1965), pp. 82-99.

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[3] Smale, S., Differential Dynamical Systems, to appear.

[4] Hartman, P., _Ordinary Differential Equations_, John Wiley, New York (1964).