N67-31716
(ACCESSION NUMBER) (THRU)
26 (PAGES) 1 (CODE)
CR-85977 (NASA CR OR TMX OR AD NUMBER) 32 (CATEGORY)

FACILITY FORM 602

WYLE LABORATORIES
TESTING DIVISION, HUNTSVILLE FACILITY

research

WYLE LABORATORIES - RESEARCH STAFF

REPORT WR 67-9

JOINT ACCEPTANCE FOR CYLINDERS

by

A. R. Wenzel

Work Performed Under Contract NAS 8-5384

Prepared by A. R. Wenzel
A. R. Wenzel

Approved by R. C. Potter
R. C. Potter
Program Manager

Approved by L. C. Sutherland
L. C. Sutherland
Director of Research Staff, Huntsville

Date June 1967

COPY NO. 2

ABSTRACT

A theoretical analysis of the response of a thin cylindrical shell to a random loading is given, and equations defining the joint acceptance of a cylinder are derived. The governing equations of motion are taken to be the Reissner shallow shell equations. The solution gives the spectral density function of both the radial displacement and stress function in terms of the normal modes of the cylinder and the spectral density function of the external pressure field. The final equations involve an external pressure-structural mode coupling term similar to the joint acceptance as defined by Powell, which then serves as the basis for defining the joint acceptance for cylinders. Using these equations, numerical results are obtained for the joint acceptance for a cylinder in a diffuse sound field.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

A_o, A_1	asymptotic expressions
a	radius of cylinder
a_{mn}, b_{mn}	coefficients
c_{mn}	coefficient vector $\begin{pmatrix} a_{mn} \\ b_{mn} \end{pmatrix}$
c	speed of sound in air
d	damping constant
d_{mn}	coefficient
D	$\dfrac{Eh^3}{12(1-\nu^2)}$
E	Young's modulus
$f_n(x)$	$\begin{cases} \dfrac{8}{n^2\pi^2} \dfrac{1-(-1)^n \cos n\pi x}{(x^2-1)^2} & \text{for } 0 \leq x < +\infty,\ x \neq 1 \\ 1 & \text{for } x = 1 \end{cases}$
f	$\begin{pmatrix} q \\ 0 \end{pmatrix}$
G, G_w, G_ϕ	Green's function
h	shell thickness
h_n	$(J_n \cos \gamma_n + N_n \sin \gamma_n)^2 - J_n^{\,2}$
$j_{mn}(\omega)$	joint acceptance for cylinder
J	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$J_{mn}^{m'n'}$	non-normalized cross-joint acceptance squared

J_n	Bessel function of order n
k	w/c
k_n	$\frac{n\pi a}{\ell}$
$K_m(x)$	$\epsilon_m h_m\left(ka\sqrt{1-(x/x_o)^2}\right)+2J_m^2\left(ka\sqrt{1-(x/x_o)^2}\right)$
ℓ	length of cylinder
L	differential operator matrix $\begin{pmatrix} \frac{D}{a^4}\nabla^4, & \frac{1}{a^3}\frac{\partial^2}{\partial\alpha^2} \\ \frac{1}{a^3}\frac{\partial^2}{\partial\alpha^2}, & -\frac{1}{Eha^4}\nabla^4 \end{pmatrix}$
N_n	Neumann function of order n
m, n	integers
p	cylinder surface pressure
q	time-independent cylinder surface pressure
P	pressure vector $\begin{pmatrix} p \\ 0 \end{pmatrix}$
$P^2(\omega)$	pressure power spectrum
S, S_p, S_w, S_ϕ	spectral density function
t	time
u, v	variables of integration
U	time-independent displacement vector $\begin{pmatrix} \hat{w} \\ \hat{\phi} \end{pmatrix}$
U_{mn}	$c_{mn}\,\phi_{mn}$
w	radial displacement of cylinder
W	$\begin{pmatrix} w \\ \phi \end{pmatrix}$
x	variable of integration
z	longitudinal cylinder coordinate

$Z_{mn}(\omega)$	$\lambda - \lambda_{mn}$
α	$\frac{z}{a}$, normalized longitudinal cylinder coordinate
β	circumferential cylinder coordinate
γ	variable of integration
γ_n	phase angle, defined by $\tan \gamma_o = -\frac{J_1}{N_1}$, $\tan \gamma_n = \frac{J_{n-1} - J_{n+1}}{N_{n+1} - N_{n-1}}$, $n = 1, 2, \ldots\ldots$
δ	$\theta - \theta'$
$\delta()$	Dirac delta-function
δ_{mn}	Kronecker delta (1 for $m = n$, 0 for $m \neq n$)
ϵ_n	Neumann factor (1 for $n = 0$, 2 for $n > 0$)
ζ	$z - z'$
θ	circumferential cylinder coordinate
κ	$\frac{k\ell}{\pi} \sin \gamma$
λ	$\sigma\omega^2 - i\omega d$
λ_{mn}	$\frac{1}{a^2}\left[\frac{D}{a^2}(m^2 + k_n^2)^2 + \frac{Ehk_n^4}{(m^2 + k_n^2)^2}\right]$
μ_{mn}	$-\frac{Ehak_n^2}{(m^2 + k_2^2)^2}$
μ	ℓ/a
ν	Poisson's ratio, integer
σ	cylinder surface density
ϕ	stress function
ϕ_{mn}	cylinder mode shape

ω	circular frequency
∇^2	$\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}$

INTRODUCTION

There is currently a growing interest in problems involving random vibration of thin cylindrical shells. Examples of such problems are the response of a launching vehicle structure or aircraft fuselage to random acoustic or aerodynamic loading, the response of submarine hulls to hydrodynamic turbulence etc.

A recent treatment of this problem was given by Cottis and Jasonides[1], who considered the correlation function of the radial displacement of a thin cylinder due to a purely random pressure field and to boundary layer pressure fluctuations. Most other work in the area of random vibration of structures has been restricted to strings [2,3], beams [4], or plates [4,5,6]. A general analysis of these problems has been given in a formal manner by Powell [7], who introduced the notion of joint acceptance.

The theoretical analysis which follows utilizes the Reissner shallow-shell equations [8], as applied to cylindrical shells by Cottis and Jasonides, except that no hysteretic damping term was included. The analysis departs from that of Cottis and Jasonides in that the two simultaneous differential equations of motion for the cylinder radial displacement and stress function are written as a single matrix equation, instead of obtaining a single differential equation for the radial displacement by differentiation and elimination. Also, the analysis was concerned with finding the spectral density function, rather than the correlation function, of the desired quantities.

The solution gives the spectral density functions of both the cylinder radial displacement and stress function in terms of the normal modes of the cylinder and the spectral density function of the external pressure field. The final equations involve an external pressure - structural mode coupling term similar to the joint acceptance as defined by Powell. The joint acceptance for the cylinder is then defined in terms of this quantity.

Using the equations derived in the text, numerical results were obtained for the joint acceptance of a simply supported cylinder in a diffuse sound field (i.e., a field in which sound waves impinge on the cylinder from all directions with random phase and with intensity independent of direction). The calculations utilized the formula for the surface pressure spectral density function for the cylinder in a diffuse field which was obtained by the author in a previous paper[9], and the results, for the first few cylinder mode shapes, are plotted in Figures (2) and (3). Also plotted were asymptotic expansions for the joint acceptance which were derived assuming large longitudinal mode number and large frequency.

THEORY

Consider a thin cylindrical shell of radius a and length ℓ. Letting $\alpha = \frac{z}{a}$ be the coordinate along the length of the cylinder and β the angular coordinate around the cylinder, the Reissner equations for the vibrating cylindrical shell can be written in the form

$$\sigma \ddot{w} + d\dot{w} + \frac{D}{a^4} \nabla^4 w + \frac{1}{a^3} \frac{\partial^2 \phi}{\partial \alpha^2} = - p(\alpha, \beta, t)$$
$$\frac{1}{a^3} \frac{\partial^2 w}{\partial \alpha^2} - \frac{1}{Eha^4} \nabla^4 \phi = 0, \qquad (1)$$

where w (α , β, t) is the lateral displacement of the middle surface, ϕ (α, β, t) is the stress function, p (α, β, t) is the loading on the surface of the cylinder, and the dot refers to differentiation with respect to t. Equations (1) can be written in matrix form as follows:

$$\sigma J \ddot{W} + d J \dot{W} + LW = - P \qquad (2)$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $W = \begin{pmatrix} w \\ \phi \end{pmatrix}$, $P = \begin{pmatrix} p \\ 0 \end{pmatrix}$,

and the differential operator L is given by

$$L = \begin{pmatrix} \frac{D}{a^4} \nabla^4, & \frac{1}{a^3} \frac{\partial^2}{\partial \alpha^2} \\ \frac{1}{a^3} \frac{\partial^2}{\partial \alpha^2}, & - \frac{1}{Eha^4} \nabla^4 \end{pmatrix} .$$

If p (α , β, t) is of the form

$$p(\alpha, \beta, t) = q(\alpha, \beta) e^{i\omega t},$$

then, assuming a solution of (2) of the form

$$W(\alpha, \beta, t) = U(\alpha, \beta) e^{i\omega t},$$

equation (2) becomes

$$(\lambda J - L) U = f, \qquad (3)$$

where $f = \begin{pmatrix} q \\ 0 \end{pmatrix}$ and $\lambda = \sigma \omega^2 - i\omega d$. Letting f = 0 we obtain the homogeneous form of (3):

$$(\lambda J - L) U = 0 . \qquad (4)$$

Assuming that the cylinder is simply supported at the ends, i.e., at z = o and z = ℓ, solutions λ_{mn}, U_{mn} of (4) can be obtained in the form

$$U_{mn} = c_{mn} \phi_{mn} (\alpha, \beta),$$

where $n = 1, 2, \ldots\ldots, m = 0, \pm 1, \pm 2, \ldots\ldots,$ $\phi_{mn}(\alpha, \beta) = \sqrt{\frac{a}{\pi \ell}} \sin k_n \alpha \; e^{im\beta}$,

$$c_{mn} = \begin{pmatrix} a_{mn} \\ b_{mn} \end{pmatrix} , \quad k_n = \frac{n\pi a}{\ell} ,$$

$$\lambda_{mn} = \frac{1}{a^2} \left[\frac{D}{a^2} (m^2 + k_n^2)^2 + \frac{Ehk_n^4}{(m^2 + k_n^2)^2} \right] ,$$

and a_{mn} and b_{mn} are solutions of

$$\left[\lambda_{mn} - \frac{D}{a^4} (m^2 + k_n^2)^2 \right] a_{mn} + \frac{k_n^2}{a^3} b_{mn} = 0$$

$$\frac{k_n^2}{a^3} a_{mn} + \frac{1}{Eha^4} (m^2 + k_n^2)^2 \, b_{mn} = 0 \quad . \tag{5}$$

the presence of longitudinal stiffeners and/or ring stiffners can be accounted for by additional restrictions on m and n. Here the functions $\phi_{mn}(\alpha, \beta)$ are chosen so that they are orthonormal i.e., so that the inner product

$$(\phi_{mn}, \phi_{m'n'}) = \int_0^{2\pi} \int_0^{\ell/a} \phi_{mn}^{*}(\alpha, \beta) \, \phi_{m'n'}(\alpha, \beta) \, d\alpha \, d\beta =$$

$$\begin{cases} 1 \text{ for } m = m', \; n = n' \\ 0 \text{ otherwise} \end{cases}$$

where the (*) denotes complex conjugate. Also a_{mn} and b_{mn} are chosen so that

$$a_{mn}^2 + b_{mn}^2 = 1 \; .$$

The general solution of (3) in terms of normal modes can now be obtained as follows: Assume a solution of the form

$$U = \sum_{m,n} d_{mn} U_{mn} = \sum_{m,n} d_{mn} \, c_{mn} \, \phi_{mn} , \tag{6}$$

when the d_{mn}'s are undetermined coefficients. Then

$$(\lambda J - L) U = \sum_{m,n} d_{mn} (\lambda J - L) U_{mn} =$$

$$\sum_{m,n} d_{mn} (\lambda J - \lambda_{mn} J + \lambda_{mn} J - L) U_{mn} =$$

$$\sum_{m,n} d_{mn} (\lambda - \lambda_{mn}) J U_{mn} \tag{7}$$

since

$$(\lambda_{mn} J - L) U_{mn} = 0 .$$

Now, assuming that the ϕ_{mn}'s form a complete ortho-normal sequence, we can set

$$q(\alpha, \beta) = \sum_{m,n} (\phi_{mn}, q) \phi_{mn} (\alpha, \beta) =$$

$$\sum_{m,n} \frac{1}{a_{mn}} (\phi_{mn}, q) a_{mn} \phi_{mn} (\alpha, \beta), \tag{8}$$

so that

$$f = \sum_{m,n} \frac{(\phi_{mn}, q)}{a_{mn}} J U_{mn} . \tag{9}$$

Equating (7) and (9) we obtain

$$d_{mn} = \frac{(\phi_{mn}, q)}{a_{mn} (\lambda - \lambda_{mn})} . \tag{10}$$

Setting $U(\alpha, \beta) = \begin{pmatrix} \hat{w} (\alpha, \beta) \\ \hat{\phi} (\alpha, \beta) \end{pmatrix}$, and substituting (10) into (6) gives

$$\hat{w} (\alpha, \beta) = \sum_{m,n} \frac{(\phi_{mn}, q)}{\lambda - \lambda_{mn}} \phi_{mn} (\alpha, \beta)$$

$$\hat{\phi} (\alpha, \beta) = \sum_{m,n} \frac{(\phi_{mn}, q)}{\lambda - \lambda_{mn}} \frac{b_{mn}}{a_{mn}} \phi_{mn} (\alpha, \beta) \tag{11}$$

where $\frac{b_{mn}}{a_{mn}}$ is obtained from (5).

For purposes of random analysis it is necessary to obtain the Green's function for equation (2), i.e., the solution G $(\alpha, \beta, \alpha', \beta', t)$ of

$$\sigma J \ddot{G} + d J \dot{G} + L G = \begin{pmatrix} -\delta(\alpha - \alpha', \beta - \beta')\,\delta(t) \\ 0 \end{pmatrix} \tag{12}$$

where δ refers to the Dirac delta-function. Letting $\hat{G}$ be the Fourier transform of G with respect to the variable ω, we obtain, by taking the Fourier transform of both sides of (12),

$$(\lambda J - L)\,\hat{G} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \delta(\alpha - \alpha', \beta - \beta') \\ 0 \end{pmatrix} \tag{13}$$

where $\lambda = \sigma\omega^2 - i\omega d$. Letting $G = \begin{pmatrix} G_w \\ G_\phi \end{pmatrix}$, $\hat{G} = \begin{pmatrix} \hat{G}_w \\ \hat{G}_\phi \end{pmatrix}$, we have, utilizing the previous solution given by (11),

$$\begin{aligned} \hat{G}_w &= \frac{1}{\sqrt{2\pi}} \sum_{m,n} \frac{\phi^*_{mn}(\alpha', \beta')}{\lambda - \lambda_{mn}}\, \phi_{mn}(\alpha, \beta) \\ \hat{G}_\phi &= \frac{1}{\sqrt{2\pi}} \sum_{m,n} \mu_{mn} \frac{\phi^*_{mn}(\alpha', \beta')}{\lambda - \lambda_{mn}}\, \phi_{mn}(\alpha, \beta), \end{aligned} \tag{14}$$

where

$$\mu_{mn} = \frac{b_{mn}}{a_{mn}} = -\frac{E h a k_n^2}{(m^2 + k_n^2)^2} .$$

Taking the inverse Fourier transform of (14) yields the desired Green's function. The solution of (1) is then written in terms of Green's function in the form

$$\begin{aligned} w(\alpha, \beta, t) &= \int_{-\infty}^{t} \int_{0}^{2\pi} \int_{0}^{\ell/a} G_w(\alpha, \beta, \alpha', \beta', t - t')\, p(\alpha', \beta', t')\, d\alpha'\, d\beta'\, dt' , \\ \phi(\alpha, \beta, t) &= \int_{-\infty}^{t} \int_{0}^{2\pi} \int_{0}^{\ell/a} G_\phi(\alpha, \beta, \alpha', \beta', t - t')\, p(\alpha', \beta', t')\, d\alpha'\, d\beta'\, dt' . \end{aligned} \tag{15}$$

Equations (15) can be used to obtain the cross spectral densities for the functions w and ϕ. Robson[10] exhibits a method for computing the cross spectral density of the response in terms of the cross spectral density of the forcing function and the associated Green's function, which, when applied to the present problem, gives the following formulas for the cross spectral densities $S_w(\alpha, \beta, \alpha', \beta', \omega)$ and $S_\phi(\alpha, \beta, \alpha', \beta', \omega)$ of w and ϕ:

$$S_w(\alpha, \beta, \alpha', \beta', \omega) =$$

$$2\pi \int_0^{2\pi}\int_0^{2\pi}\int_0^{\ell/a}\int_0^{\ell/a} \hat{G}_w^*(\alpha, \beta, \alpha_1, \beta_1, \omega)\, \hat{G}_w(\alpha', \beta', \alpha_2, \beta_2, \omega)\, S_p(\alpha_1, \beta_1, \alpha_2, \beta_2, \omega) \cdot d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2$$

$$S_\phi(\alpha, \beta, \alpha', \beta', \omega) = \qquad (16)$$

$$2\pi \int_0^{2\pi}\int_0^{2\pi}\int_0^{\ell/a}\int_0^{\ell/a} \hat{G}_\phi^*(\alpha, \beta, \alpha_1, \beta_1, \omega)\, \hat{G}_\phi(\alpha', \beta', \alpha_2, \beta_2, \omega)\, S_p(\alpha_1, \beta_1, \alpha_2, \beta_2, \omega) \cdot d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2,$$

where $S_p(\alpha_1, \beta_1, \alpha_2, \beta_2, \omega)$ is the cross spectral density of $p(\alpha, \beta, t)$. Substituting (14) into (16) we obtain finally

$$S_w(\alpha, \beta, \alpha', \beta', \omega) =$$

$$\sum_{m,n}\sum_{m',n'} \left[\int_0^{2\pi}\int_0^{2\pi}\int_0^{\ell/a}\int_0^{\ell/a} S_p(\alpha_1, \beta_1, \alpha_2, \beta_2, \omega)\, \phi_{mn}(\alpha_1, \beta_1)\, \phi^*_{m'n'}(\alpha_2, \beta_2)\, d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2\right] \cdot$$

$$\frac{\phi^*_{mn}(\alpha, \beta)\, \phi_{m'n'}(\alpha', \beta')}{Z^*_{mn}(\omega)\, Z_{m'n'}(\omega)} , \qquad (17)$$

$$S_\phi(\alpha, \beta, \alpha', \omega) =$$

$$\sum_{m,n}\sum_{m',n'} \left[\int_0^{2\pi}\int_0^{2\pi}\int_0^{\ell/a}\int_0^{\ell/a} S_p(\alpha_1, \beta_1, \alpha_2, \beta_2, \omega)\, \phi_{mn}(\alpha_1, \beta_1)\, \phi^*_{m'n'}(\alpha_2, \beta_2)\, d\alpha_1\, d\alpha_2\, d\beta_1\, d\beta_2\right] \cdot$$

$$\frac{\mu_{mn}\, \mu_{m'n'}}{Z^*_{mn}(\omega)\, Z_{m'n'}(\omega)}\, \phi^*_{mn}(\alpha, \beta)\, \phi_{m'n'}(\alpha', \beta') , \qquad (18)$$

where $Z_{mn}(\omega) = \lambda - \lambda_{mn}$. The quantity inside the square brackets in (17) and (18), when divided by the pressure power spectrum, is defined to be the cross-joint acceptance squared for the cylinder. Letting $J_{mn}^{m'n'}(\omega)$ be the quantity in square brackets in (17) and (18), we have, setting $\alpha = \frac{z}{a}$ and letting $\theta = \beta$,

$$J_{mn}^{m'n'}(\omega) = \frac{1}{a^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\ell}\int_0^{\ell} S(\theta, z, \theta', z', \omega)\phi_{mn}(\frac{z}{a}, \theta)\,\phi^*_{m'n'}(\frac{z'}{a}, \theta')\,dz\,dz'\,d\theta\,d\theta' \tag{19}$$

where $S(\theta, z, \theta', z', \omega) = S_p(\alpha, \beta, \alpha', \beta', \omega)$. Substituting for ϕ_{mn} and $\phi_{m'n'}$, (19) becomes

$$J_{mn}^{m'n'}(\omega) = \frac{1}{\pi a \ell}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\ell}\int_0^{\ell} S(\theta, z, \theta', z', \omega)\sin\frac{n\pi}{\ell}z\,\sin\frac{n'\pi}{\ell}z'\,e^{i(m\theta - m'\theta')}dz\,dz'\,d\theta\,d\theta'. \tag{20}$$

Cylinder in Diffuse Field

We assume now that the cylindrical shell in question is a section of an infinitely long cylinder which is subject to a three-dimensional diffuse sound field, i.e., one in which plane waves impinge on the cylinder from all directions with random phase, and with intensity independent of direction. The surface pressure spectral density function for this case has been obtained by the writer[9] in the form

$$S(\delta, \zeta, \omega) = \pi P^2(\omega)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\gamma\,\cos(k\zeta\sin\gamma)\cdot$$

$$\left\{J_0(2ka\cos\gamma\sin\tfrac{1}{2}\delta) + \sum_{\nu=0}^{\infty}\epsilon_\nu h_\nu(ka\cos\gamma)\cos\nu\delta\right\} d\gamma, \tag{21}$$

where $\delta = \theta - \theta'$, $\zeta = z - z'$, and $P^2(\omega)$ is the pressure power spectrum. Substituting (21) into (20), and setting $m = m'$ and $n = n'$ we obtain, after interchanging orders of integration,

$$J_{mn}^{mn}(\omega) = \frac{P^2(\omega)}{a\ell} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \left[\int_0^{2\pi} \int_0^{2\pi} \int_0^{\ell} \int_0^{\ell} \cos(k\zeta \sin\gamma) \left\{ J_0 (2ka\cos\gamma \sin\tfrac{1}{2}\delta) + \right. \right.$$

$$\left. \left. \sum_{\nu=0}^{\infty} \epsilon_\nu h_\nu (ka\cos\gamma) \cos\nu\delta \right\} \sin\frac{n\pi}{\ell} z \sin\frac{n\pi}{\ell} z' \, e^{im\delta} \, dz\, dz'\, d\theta\, d\theta' \right] d\gamma. \tag{22}$$

Assuming that the infinite series converges uniformly, this can be written

$$J_{mn}^{mn}(\omega) = \frac{P^2(\omega)}{a\ell} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \left[\int_0^{2\pi} \int_0^{2\pi} J_0 (2ka\cos\gamma \sin\tfrac{1}{2}\delta) e^{im\delta} \, d\theta\, d\theta' . \right.$$

$$\int_0^{\ell} \int_0^{\ell} \cos(k\zeta \sin\gamma) \sin\frac{n\pi}{\ell} z \sin\frac{n\pi}{\ell} z' \, dz\, dz' + \sum_{\nu=0}^{\infty} \epsilon_\nu h_\nu (ka\cos\gamma) .$$

$$\left. \int_0^{2\pi} \int_0^{2\pi} \cos\nu\delta \, e^{im\delta} \, d\theta\, d\theta' \int_0^{\ell} \int_0^{\ell} \cos(k\zeta \sin\gamma) \sin\frac{n\pi}{\ell} z \sin\frac{n\pi}{\ell} z' \, dz\, dz' \right] d\gamma. \tag{23}$$

Letting

$$I_m^{(1)} = \int_0^{2\pi} \int_0^{2\pi} J_0 (2ka\cos\gamma \sin\tfrac{1}{2}\delta) \, e^{im\delta} \, d\theta\, d\theta' ,$$

$$I_n^{(2)} = \int_0^{\ell} \int_0^{\ell} \cos(k\zeta \sin\gamma) \sin\frac{n\pi}{\ell} z \sin\frac{n\pi}{\ell} z' \, dz\, dz' ,$$

$$I_{\nu m}^{(3)} = \int_0^{2\pi} \int_0^{2\pi} \cos\nu\delta \, e^{im\delta} \, d\theta\, d\theta' ,$$

(23) can be written

$$J_{mn}^{mn}(\omega) = \frac{P^2(\omega)}{a\ell} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} I_n^{(2)} \cos\gamma \left[I_m^{(1)} + \sum_{\nu=0}^{\infty} \epsilon_\nu h_\nu (ka\cos\gamma) \, I_{\nu m}^{(3)} \right] d\gamma. \tag{24}$$

The three integrals $I_m^{(1)}$, $I_n^{(2)}$, and $I_{\nu m}^{(3)}$ can each be evaluated most readily using a coordinate transformation, the nature of which is best illustrated by an example. Choosing $I_n^{(2)}$ $(n > 0)$ as the integral to be evaluated, we make the following successive coordinate transformations:

$$v' = \frac{\pi}{\ell}(z' - z), \quad u' = \frac{\pi}{\ell}(z' + z),$$

and

$$u = u' - \pi \quad , \quad v = v'$$

which maps the square in the z, z' plane into the diamond shaped region in the u, v plane as shown in Figure 1.

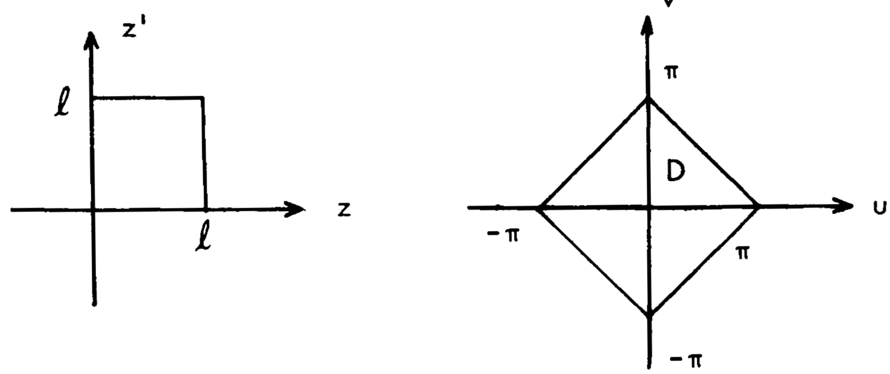

Figure 1. Transformation of Coordinates.

Then with respect to the u, v coordinates, $I_n^{(2)}$ can be written

$$I_n^{(2)} = \frac{\ell^2}{4\pi^2} \iint_D \cos(\kappa v) \left[\cos nv - (-1)^n \cos nu\right] du\, dv, \tag{25}$$

where $\kappa = \frac{k\ell}{\pi} \sin \gamma$. Integrating first with respect to u, (25) can be written

$$I_n^{(2)} = \frac{\ell^2}{4\pi^2} \int_{-\pi}^{\pi} \cos \kappa v \int_{-(\pi - |v|)}^{\pi - |v|} \left[\cos nv - (-1)^n \cos nu\right] du\, dv =$$

$$\frac{\ell^2}{\pi^2} \int_0^{\pi} \left[(\pi - v) \cos \kappa v \cos nv + \frac{1}{n} \cos \kappa v \sin nv\right] dv. \tag{26}$$

The second integral in (26) can be evaluated by means of tables to yield finally

$$I_n^{(2)} = \frac{2\ell^2}{\pi^2} \frac{n^2}{(\kappa^2 - n^2)^2} 2\left[1 - (-1)^n \cos \kappa\pi \right]. \tag{27}$$

By means of similar transformations, $I_m^{(1)}$ and $I_{\nu m}^{(3)}$ are evaluated to yield

$$I_m^{(1)} = 8\pi(-1)^m \int_0^{\frac{\pi}{2}} J_o(2\,ka\cos\gamma\cos\psi)\cos 2m\psi\, d\psi \tag{28}$$

$$I_{\nu m}^{(3)} = 2\pi^2 \delta_{\nu m}, \tag{29}$$

where $\delta_{\nu m}$ is the Kronecker delta. Making use of the identity[11]

$$\int_0^{\frac{\pi}{2}} J_o(2z\cos\theta)\cos 2m\theta\, d\theta = \frac{\pi}{2}(-1)^m J_m^2(z), \tag{30}$$

equation (28) can be written

$$I_m^{(1)} = 4\pi^2 J_m^2(ka\cos\gamma). \tag{31}$$

Substituting (27), (29), and (31) into (24) gives

$$J_{mn}^{mn}(\omega) = \frac{2n^2\ell^2 P^2(\omega)}{\pi^2 a\ell} \int_{\frac{-\pi}{2}}^{\frac{+\pi}{2}} \cos\gamma \frac{1-(-1)^n\cos\kappa\pi}{(\kappa^2-n^2)^2}\left[4\pi^2 J_m^2(ka\cos\gamma) + 2\pi^2 \epsilon_m h_m(ka\cos\gamma)\right] d\gamma, \tag{32}$$

which becomes, after substituting for $\kappa = \frac{k\ell}{\pi}\sin\gamma$ and setting $\mu = \ell/a$,

$$j_{mn}^2(\omega) = 8\pi^4 \mu n^2 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos\gamma \frac{1-(-1)^n\cos(\mu ka\sin\gamma)}{\left[(\mu ka\sin\gamma)^2 - n^2\pi^2\right]^2}.$$

$$\left[J_m^2 (k a \cos\gamma) + \frac{1}{2} \epsilon_m h_m (k a \cos\gamma) \right] d\gamma \tag{33}$$

where $j_{mn}^2(\omega) = \dfrac{J_{mn}^{mn}(\omega)}{P^2(\omega)}$ is the joint acceptance squared.

Using equation (33), an asymptotic expansion for j_{mn}^2 for large n and large ka can be obtained as follows: Making the substitution $x = (\mu k a / n\pi) \sin\gamma$, (33) can be written

$$j_{mn}^2(\omega) = \frac{\pi^2 \mu}{x_o} \int_o^{x_o} f_n(x) K_m(x)\, dx \tag{34}$$

where $x_o = \mu k a / n\pi$,

$$f_n(x) = \begin{cases} \dfrac{8}{n^2\pi^2} \dfrac{1 - (-1)^n \cos n\pi x}{(x^2 - 1)^2} & \text{for } 0 \leq x < \infty,\ x \neq 1 \\ 1 \quad \text{for } x = 1 \end{cases}$$

and

$$K_m(x) = \epsilon_m h_m \left(k a \sqrt{1 - (x/x_o)^2} \right) + 2 J_m^2 \left(k a \sqrt{1 - (x/x_o)^2} \right).$$

We note that the function $f_n(x)$ is continuous for $0 \leq x < +\infty$, $f_n(1) = 1$ for all n, and for all $x \neq 1$ $f_n(x) \rightarrow 0$ with increasing n. We expect, therefore, that for large n and for $x_o > 1$ only those values of x near x = 1 will contribute significantly to the integral of equation (34). Accordingly we write, for $x_o > 1$, and $0 < \epsilon < 1$,

$$\int_o^{x_o} f_n(x) K_m(x)\, dx = \int_o^{1-\epsilon} f_n(x) K_m(x)\, dx +$$

$$\int_{1-\epsilon}^{1+\epsilon} f_n(x) K_m(x)\, dx + \int_{1+\epsilon}^{x_o} f_n(x) K_m(x)\, dx . \tag{35}$$

Making use of some trigometric identities, we can write

$$\int_{1-\epsilon}^{1+\epsilon} f_n(x)\, k_m(x)\, dx = \frac{16}{n^2\pi^2} \int_{1-\epsilon}^{1+\epsilon} \frac{\sin^2 n\frac{\pi}{2}(x-1)}{(x^2-1)^2} K_m(x)\, dx,$$

which becomes, after a change of variable,

$$\int_{1-\epsilon}^{1+\epsilon} f_n(x) K_m(x)\, dx = \frac{4}{n^2\pi^2} \int_{-\epsilon}^{+\epsilon} \frac{\sin^2 n\frac{\pi}{2}u}{u^2(1+\frac{u}{2})^2} K_m(1+u)\, du. \tag{36}$$

Since $K_m(1+u)$ is analytic for $|u| \leq \epsilon < 1$, we can write

$$K_m(1+u) = a_o + a_1 u + a_2 u^2 + \ldots\ldots, \tag{37}$$

where $a_o = K_m(1)$ etc., while for $|u| \leq \epsilon$

$$\frac{1}{(1+\frac{u}{2})^2} = 1 - u + \frac{3}{4}u^2 + \ldots. \tag{38}$$

Substituting (37) and (38) into (36) gives

$$\int_{1-\epsilon}^{1+\epsilon} f_n(x) K_m(x)\, dx = \frac{4a_o}{n^2\pi^2} \int_{-\epsilon}^{+\epsilon} \frac{\sin^2 n\frac{\pi}{2}u}{u^2}\, du +$$

$$\frac{4}{n^2\pi^2}\left(\frac{3}{4}a_o - a_1 - a_2\right) \int_{-\epsilon}^{+\epsilon} \sin^2 n\frac{\pi}{2}u\, du + \frac{4}{n^2\pi^2} \int_{-\epsilon}^{+\epsilon} u^2 g(u) \sin^2 n\frac{\pi}{2}u\, du, \tag{39}$$

where $g(u)$ is regular in the interval $[-\epsilon, +\epsilon]$. Making another change of variable, the first term on the right hand side of (39) can be written

$$\frac{4a_o}{n^2\pi^2} \int_{-\epsilon}^{+\epsilon} \frac{\sin^2 n\frac{\pi}{2}u}{u^2}\, du = \frac{4a_o}{n\pi} \int_{o}^{\frac{n\pi\epsilon}{2}} \frac{\sin^2 u'}{u'^2}\, du' =$$

$$\frac{4a_o}{n\pi}\left[\int_{o}^{\infty} \frac{\sin^2 u'}{u'^2}\, du' - \int_{\frac{n\pi\epsilon}{2}}^{\infty} \frac{\sin^2 u'}{u'^2}\, du'\right] =$$

$$\frac{4a_o}{n\pi}\left[\frac{\pi}{2} - \int_{\frac{n\pi\epsilon}{2}}^{\infty} \frac{\sin^2 u'}{u'^2}\, du'\right]. \tag{40}$$

Now the second term in the brackets on the right hand side of (40) is of order $\frac{1}{n\epsilon}$, while the second and third terms on the right hand side of (39) are of order $\frac{\epsilon}{n^2}$; therefore we can write finally

$$\int_{1-\epsilon}^{1+\epsilon} f_n(x)\, K_m(x)\, dx = \frac{2a_o}{n} + \mathcal{O}\left(\frac{1}{n^2\epsilon}\right) + \mathcal{O}\left(\frac{\epsilon}{n^2}\right). \tag{41}$$

Turning now to the remaining two terms on the right hand side of (35), and noting that $K_m(x)$ is bounded for $x \geq 0$, we can write

$$\left|\int_0^{1-\epsilon} f_n(x)\, K_m(x)\, dx\right| \leq \frac{16}{n^2\pi^2} \int_0^{1-\epsilon} \frac{|K_m(x)|}{(x^2-1)^2}\, dx \leq$$

$$\frac{16M}{n^2\pi^2} \int_0^{1-\epsilon} \frac{dx}{(1-x)^2} \leq \frac{16M}{n^2\pi^2\epsilon},$$

where $M = \max_{x\geq 0} \left|K_m(x)\right|$. Therefore,

$$\int_0^{1-\epsilon} f_n(x) K_m(x)\, dx = \mathcal{O}\left(\frac{1}{n^2\epsilon}\right). \tag{42}$$

In a similar manner, it can be shown that

$$\int_{1+\epsilon}^{x_o} f_n(x) K_m(x)\, dx = \mathcal{O}\left(\frac{1}{n^2\epsilon}\right). \tag{43}$$

Substituting (41), (42), and (43) into (35), and taking ϵ to be fixed, we obtain finally

$$\int_0^{x_o} f_n(x) K_m(x)\, dx = \frac{2a_o}{n} + \mathcal{O}\left(\frac{1}{n^2}\right),$$

so that, from (34),

$$j_{mn}^2(\omega) = \frac{\pi^2\mu}{x_o}\left[\frac{2}{n} K_m(1) + \mathcal{O}\left(\frac{1}{n^2}\right)\right]. \tag{44}$$

Now for $x_o >> 1$,

$$K_m(1) = \epsilon_m h_m\left(ka\sqrt{1-\left(\frac{1}{x_o}\right)^2}\right) + 2J_m^2\left(ka\sqrt{1-\left(\frac{1}{x_o}\right)^2}\right) \cong$$

$$\epsilon_m h_m(ka) + 2J_m^2(ka), \tag{45}$$

while for large ka it can be shown, by substituting the asymptotic expansions

$$J_m(ka) \cong \sqrt{\frac{2}{\pi ka}}\cos\left(ka - m\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$N_m(ka) \cong \sqrt{\frac{2}{\pi ka}}\sin\left(ka - m\frac{\pi}{2} - \frac{\pi}{4}\right)$$

into the formula for $h_m(ka)$, that

$$h_m(ka) \cong \frac{2}{\pi ka}\sin^2\left(ka - m\frac{\pi}{2} - \frac{\pi}{4}\right). \tag{46}$$

Substituting (45), (46), and the asymptotic expansion for $J_m(ka)$ into (44) we obtain finally the following asymptotic expressions for $j_{mn}^2(\omega)$, valid for large n, large ka, and $\mu ka/n\pi >> 1$:

$$j_{0n}^2(\omega) \cong \left(\frac{2\pi}{ka}\right)^2\left[1 + \cos^2\left(ka - \frac{\pi}{4}\right)\right],$$

$$j_{mn}^2(\omega) \cong 2\left(\frac{2\pi}{ka}\right)^2 ; m > 0. \tag{47}$$

Using equation (33), numerical calculations of the joint acceptance were made for the first few modes of the cylinder, assuming $\mu = 2$, and the results plotted in Figures (2) and (3). Also plotted were the asymptotic expressions $A_o(ka)$ and $A_1(ka)$, where

$$A_o(ka) = \left(\frac{2\pi}{ka}\right)^2\left[1 + \cos^2\left(ka - \frac{\pi}{4}\right)\right]$$

$$A_1(ka) = 2\left(\frac{2\pi}{ka}\right)^2.$$

Referring to the figures, it can be seen that, although they were derived assuming that n is large, the asymptotic expressions $A_o(ka)$ and $A_1(ka)$ give a good approximation to $j^2_{mn}(\omega)$ even for $n = 1$ and $ka \geq 2$.

It also might be pointed out that the asymptotic behavior of a cylinder in a diffuse field differs from that of a flat plate in that only the cylinder radius, and not the length, appears in the asymptotic expression, while for a flat plate one would expect both the length and width of the plate to appear in the asymptotic expression in a more or less symmetric manner. This indicates that one should exercise a certain amount of caution when analyzing a cylinder by, in effect, unrolling it into a flat plate.

REFERENCES

1. M.G. Cottis and J.G. Jasonides, "The Response of a Finite Thin Cylindrical Shell to Random Pressure Fields," in "Acoustical Fatigue in Aerospace Structures," Proceedings of the Second International Conference, Syracuse University Press, Syracuse, New York, 1965.

2. R.H. Lyon, "Response of Strings to Random Noise Fields," J. Acoust. Soc. Am. 28, p. 391, (1956).

3. G. Maidanik and R.H. Lyon, "Response of Strings to Moving Noise Fields," J. Acoust. Soc. Am. 33, p. 1606, (1961).

4. A.C. Eringen, "Response of Beams and Plates to Random Loads," J. Appl. Mech. 24, p. 46, (1957).

5. I. Dyer, "Response of Plates to a Decaying and Convecting Random Pressure Field," J. Acoust. Soc. Am. 31, p. 922, (1959).

6. D.J. Bozich, "Spatial Correlation in Acoustic-Structural Coupling," J. Acoust. Soc. Am. 36, p. 52, (1964).

7. A. Powell, "On the Fatigue Failure of Structures due to Vibrations Excited by Random Pressure Fields," J. Acoust. Soc. Am. 30, p. 1130, (1958).

8. E. Reissner, "On Transverse Vibrations of Thin, Shallow Elastic Shells," Quart. Appl. Math. 13, 2, p. 169, (1955).

9. A. Wenzel, "Surface Pressure Correlation Function for a Cylinder in a Diffuse Reverberant Sound Field," Wyle Laboratories Research Report, WR 66-14, 1966.

10. J.D. Robson, "Random Vibration," Elsevier Publishing Company, New York, p. 89-92, 1964.

11. W. Magnus, F. Oberhettinger and R.P. Soni, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, New York Inc., p. 89, 1966.

12. M.G. Cottis, "Green's Function Technique in the Dynamics of a Finite Cylindrical Shell," J. Acoust. Soc. Am. 37, p. 31, (1965).

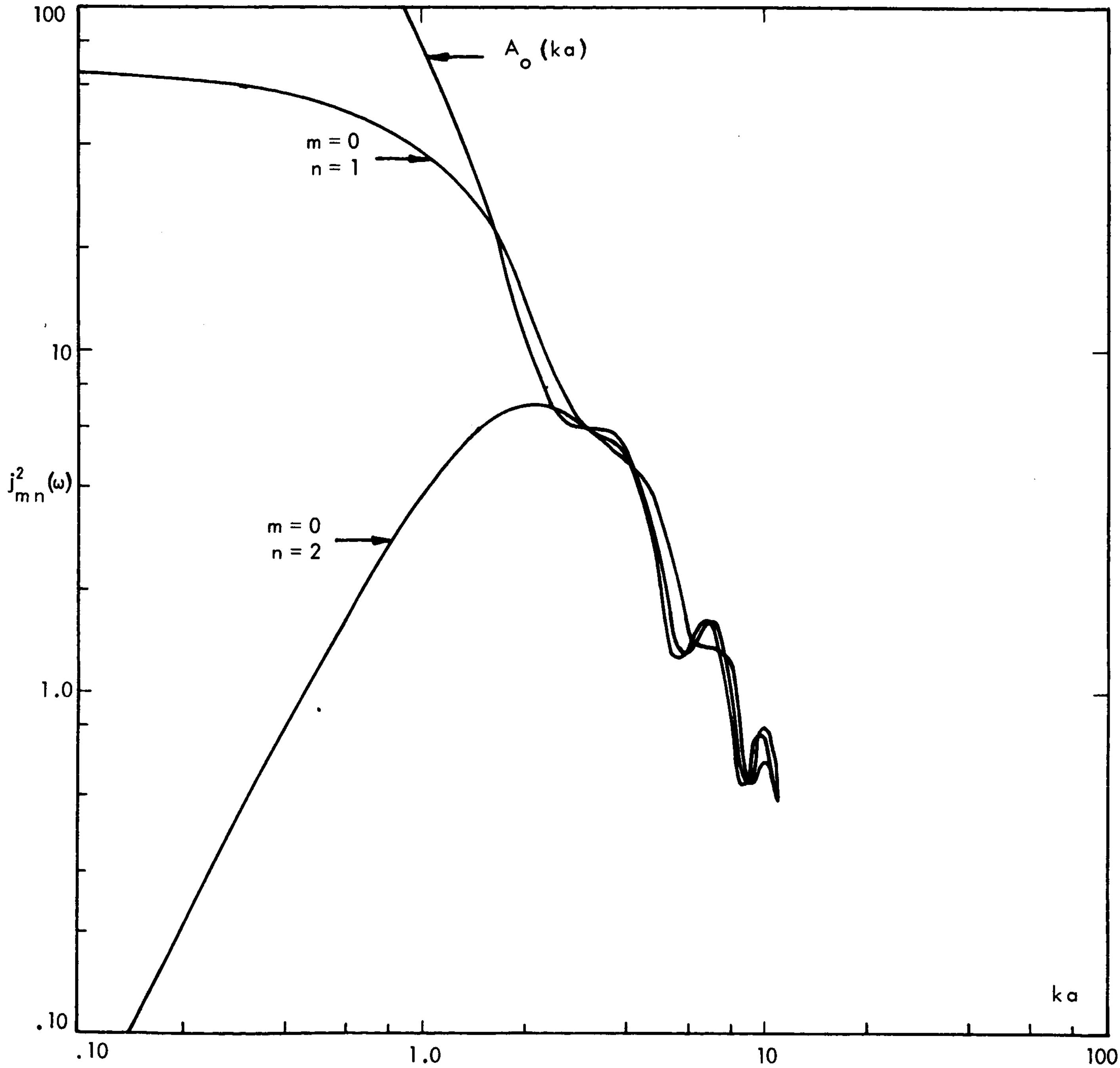

Figure 2. Joint Acceptance for Cylinder in Diffuse Field (m = 0, μ = 2)

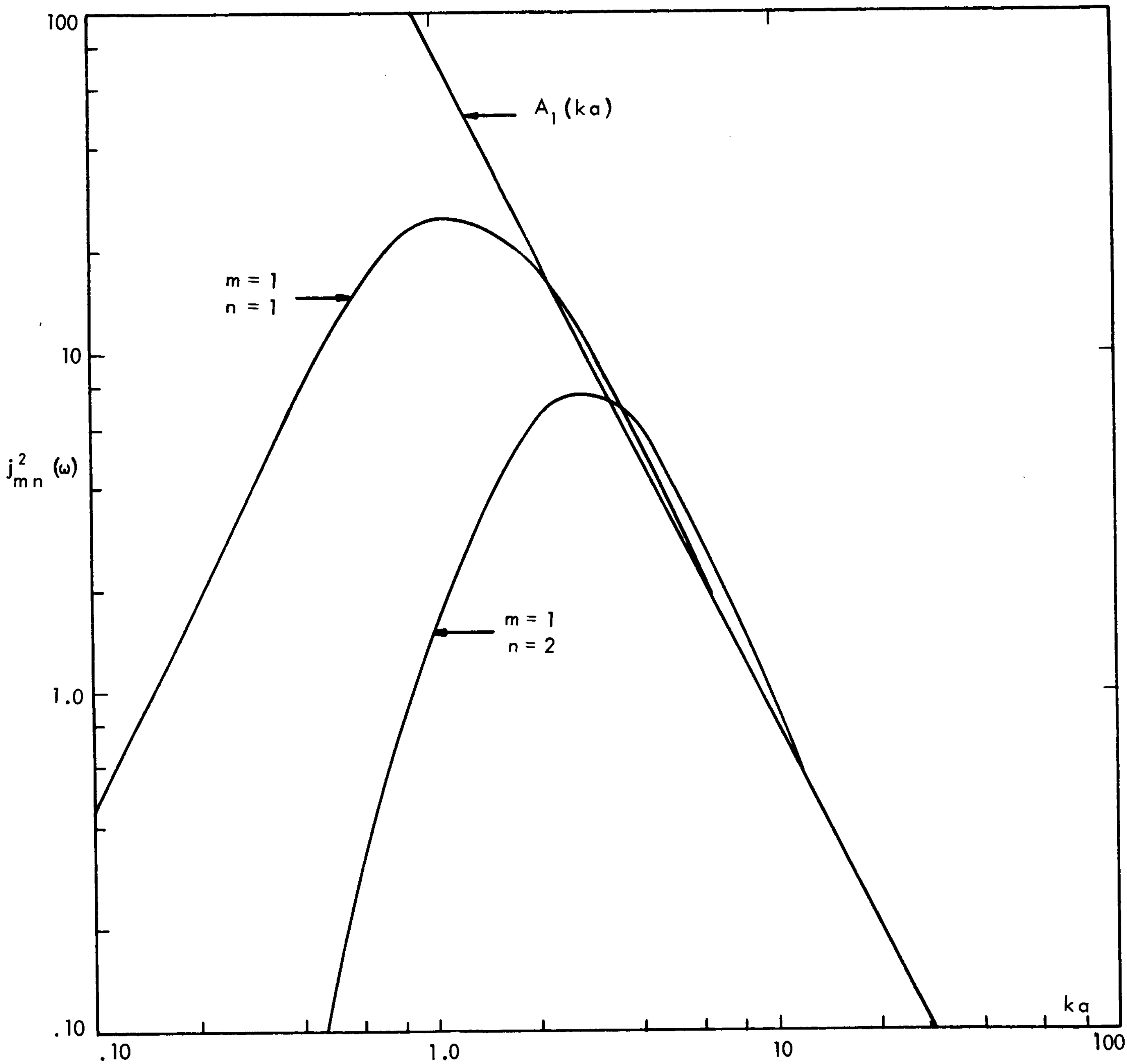

Figure 3. Joint Acceptance for Cylinder in Diffuse Field (m = 1, μ = 2).